

# The magnetic flow on the manifold of oriented geodesics of a three dimensional space form

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## Abstract

Let  $M$  be the three dimensional complete simply connected manifold of constant sectional curvature  $0, 1$  or  $-1$ . Let  $\mathcal{L}$  be the manifold of all (unparametrized) complete oriented geodesics of  $M$ , endowed with its canonical pseudo-Riemannian metric of signature  $(2, 2)$  and Kähler structure  $J$ . A smooth curve in  $\mathcal{L}$  determines a ruled surface in  $M$ .

We characterize the ruled surfaces of  $M$  associated with the magnetic geodesics of  $\mathcal{L}$ , that is, those curves  $\sigma$  in  $\mathcal{L}$  satisfying  $\nabla_{\dot{\sigma}}\dot{\sigma} = J\dot{\sigma}$ . More precisely: a time-like (space-like) magnetic geodesic determines the ruled surface in  $M$  given by the binormal vector field along a helix with positive (negative) torsion. Null magnetic geodesics describe cones, cylinders or, in the hyperbolic case, also cones with vertices at infinity. This provides a relationship between the geometries of  $\mathcal{L}$  and  $M$ .

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## 1 Introduction

For  $\kappa = 0, 1, -1$ , let  $M_\kappa$  be the three dimensional complete simply connected manifold of constant sectional curvature  $\kappa$ , that is,  $\mathbb{R}^3$ ,  $\mathbb{S}^3$  and the hyperbolic space  $\mathbb{H}^3$ . Let  $\mathcal{L}_\kappa$  be the manifold of all (unparametrized) complete oriented geodesics of  $M_\kappa$ . We may think of an element  $c$  in  $\mathcal{L}_\kappa$  as the equivalence class of unit speed geodesics  $\gamma : \mathbb{R} \rightarrow M_\kappa$  with image  $c$  such that  $\{\dot{\gamma}(s)\}$  is a positive basis of  $T_{\gamma(s)}c$  for all  $s$ .

Let  $\gamma$  be a complete unit speed geodesic of  $M_\kappa$  and let  $\mathcal{J}_\gamma$  be the space of all Jacobi fields along  $\gamma$  which are orthogonal to  $\gamma$ . There exists a well-defined canonical isomorphism

$$T_\gamma : \mathcal{J}_\gamma \rightarrow T_{[\gamma]}\mathcal{L}_\kappa, \quad T_\gamma(J) = \left. \frac{d}{dt} \right|_0 [\gamma_t], \quad (1)$$

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where  $\gamma_t$  is any variation of  $\gamma$  by unit speed geodesics associated with  $J$  (see [8]).

A pseudo-Riemannian metric of signature  $(2, 2)$  can be defined on  $\mathcal{L}_\kappa$  as follows [9]: For  $X \in T_{[\gamma]}\mathcal{L}_\kappa$ , the square norm  $\|X\| = \langle X, X \rangle$  is well defined by

$$\|X\| = \langle \dot{\gamma} \times J, J' \rangle, \quad (2)$$

where  $X = T_\gamma(J)$ , the cross product  $\times$  is induced by a fixed orientation of  $M_\kappa$  and  $J'$  denotes the covariant derivative of  $J$  along  $\gamma$ . Indeed, the right hand side of (2) is a constant function. In the following, for any vector  $X$ , we will denote  $\|X\| = \langle X, X \rangle$  and  $|X| = \sqrt{|\langle X, X \rangle|}$ . Recall that  $X$  is null, time-like or space-like if  $\|X\| = 0$ ,  $\|X\| < 0$  or  $\|X\| > 0$ , respectively.

Let  $[\gamma] \in \mathcal{L}_\kappa$  and let  $R_\gamma$  be the rotation in  $M_\kappa$  fixing  $\gamma$  through an angle of  $\pi/2$ . This rotation induces an isometry  $\tilde{R}_\gamma$  of  $\mathcal{L}_\kappa$  whose differential at  $[\gamma]$  is a linear isometry of  $T_{[\gamma]}\mathcal{L}_\kappa$  squaring to  $-\text{id}$ . This yields a complex structure  $J$  on  $\mathcal{L}_\kappa$ . With the metric defined above,  $\mathcal{L}_\kappa$  is Kahler.

A *magnetic geodesic*  $\sigma$  of  $\mathcal{L}_\kappa$  is a curve satisfying  $\nabla_{\dot{\sigma}}\dot{\sigma} = J\dot{\sigma}$ . These curves have constant speed, so they will be null, time-like or space-like.

A smooth curve in  $\mathcal{L}_\kappa$  determines a ruled surface in  $M_\kappa$ . For  $\kappa = 0, -1$ , a generic geodesic of  $\mathcal{L}_\kappa$  describes a helicoid in  $M_\kappa$  [5, 4, respectively]. Our purpose is to characterize the ruled surfaces in  $M_\kappa$  associated with the magnetic geodesics of  $\mathcal{L}_\kappa$ . For  $v \in TM_\kappa$ ,  $\gamma_v$  denotes the geodesic of  $M_\kappa$  with initial velocity  $v$ .

**Theorem 1** *A generic magnetic geodesic  $\sigma$  of  $\mathcal{L}_\kappa$  describes the ruled surface in  $M_\kappa$  given by the binormal vector field of a helix. More precisely,  $\sigma$  is a time-like (space-like) magnetic geodesic of  $\mathcal{L}_\kappa$  if and only if  $\sigma$  has the form*

$$\sigma(t) = [\gamma_{B(t)}], \quad (3)$$

where  $B$  is the binormal vector field of a helix in  $M_\kappa$  with curvature  $k$ , speed  $1/k$  and positive (negative) torsion, for some  $k > 0$ .

Now we study null magnetic geodesics in  $\mathcal{L}_{-1} = \mathcal{L}(\mathbb{H}^3)$ . We recall some concepts related with the hyperbolic space (see for instance [3]).

Two unit speed geodesics  $\gamma$  and  $\alpha$  of  $\mathbb{H}^3$  are said to be asymptotic if there exists a positive constant  $C$  such that  $d(\gamma(s), \alpha(s)) \leq C, \forall s \geq 0$ . Two unit vectors  $v, w \in T^1\mathbb{H}^3$  are said to be asymptotic if the corresponding geodesics  $\gamma_v$  and  $\gamma_w$  have this property.

A point at infinity for  $\mathbb{H}^3$  is an equivalence class of asymptotic geodesics of  $\mathbb{H}^3$ . The set of all points at infinity for  $\mathbb{H}^3$  is denoted by  $\mathbb{H}^3(\infty)$  and has a canonical differentiable structure diffeomorphic to the 2-sphere. The equivalence class represented by a geodesic  $\gamma$  is denoted by  $\gamma(\infty)$ , and the equivalence class represented by the oppositely oriented geodesic  $s \mapsto \gamma(-s)$  is denoted by  $\gamma(-\infty)$ .

Given  $v \in T^1\mathbb{H}^3$ , the *horosphere*  $H(v)$  is the limit of metric spheres  $\{S_n\}$  in  $\mathbb{H}^3$  that pass through the foot point of  $v$  as the centers  $\{p_n\}$  of  $\{S_n\}$  converge to  $\gamma_v(\infty)$ . Below we present a more precise definition.

Let  $\psi^\pm : \mathcal{L}(\mathbb{H}^3) \rightarrow \mathbb{H}^3(\infty)$  be the smooth functions given by  $\psi^\pm([\gamma]) = \gamma(\pm\infty)$  and let  $\mathcal{D}^\pm$  be the distributions on  $\mathcal{L}(\mathbb{H}^3)$  given by  $\mathcal{D}_{[\gamma]}^\pm = \text{Ker}(d\psi_{[\gamma]}^\pm)$ . These distributions are called *the horospherical distributions* on  $\mathcal{L}(\mathbb{H}^3)$ .

*Cones with vertices at infinity:* Let  $x \in \mathbb{H}^3(\infty)$  and let  $v_o \in T^1\mathbb{H}^3$  such that  $\gamma_{v_o}(\pm\infty) \in x$ . Let  $t \mapsto v(t)$  be a curve in  $T^1\mathbb{H}^3$  such that  $v(0) = \pm v_o$ ,  $v(t)$  is asymptotic to  $\pm v_o$  for all  $t \in \mathbb{R}$  and the foot points of  $v(t)$  lie on a circle of geodesic curvature  $\pm k$  (with  $k > 0$ ) and speed  $1/k$  in the horosphere determined by  $\pm v_o$ . Under these conditions we say that the curve in  $\mathcal{L}(\mathbb{H}^3)$  given by  $t \mapsto [\gamma_{\pm v(t)}]$  describes a *forward cone with vertex at  $x$  (for  $+$ )* or a *backward cone with vertex at  $x$  (for  $-$ )*. These cones can be better visualized in the upper half space model of  $\mathbb{H}^3$  (in particular  $\mathbb{H}^3(\infty) = \{z = 0\} \cup \{\infty\}$ ): Let  $\gamma_t^\pm(s) = (\frac{1}{k} \cos(t), \pm \frac{1}{k} \sin(t), e^{\pm s})$ . A curve  $\sigma$  in  $\mathcal{L}(\mathbb{H}^3)$  describes a cone with forward (respectively, backward) vertex at  $\infty$  if it is  $Sl(2, \mathbb{C})$ -congruent to  $t \mapsto [\gamma_t^+]$  (respectively, to  $t \mapsto [\gamma_t^-]$ ).

**Theorem 2** *A null magnetic geodesic of  $\mathcal{L}(\mathbb{H}^3)$  describes in  $\mathbb{H}^3$  a cylinder, a cone with vertex at  $p \in \mathbb{H}^3$  or a cone with vertex at infinity. More precisely, if  $\sigma$  is a curve in  $\mathcal{L}(\mathbb{H}^3)$ , then*

- a)  $\sigma$  is a null magnetic geodesic with  $\dot{\sigma}(0) \in \mathcal{D}_{\sigma(0)}^\pm$  if and only if  $\sigma$  describes a cone with vertex at  $\sigma(0)(\pm\infty)$  (forward for  $+$  and backward for  $-$ );
- b)  $\sigma$  is a null magnetic geodesic with  $\dot{\sigma}(0) \notin \mathcal{D}_{\sigma(0)}^\pm$  if and only if  $\sigma$  either has the form

$$\sigma(t) = [\gamma_{B(t)}], \quad (4)$$

where  $B$  is the binormal vector field of a helix  $h$  in  $\mathbb{H}^3$  with curvature  $k$ , speed  $1/k$  and zero torsion (in particular,  $h$  is contained in a totally geodesic surface  $S$  and  $B$  is normal to  $S$  and parallel along  $h$ ), or  $\sigma$  has the form

$$\sigma(t) = [\gamma_{v(t)}], \quad (5)$$

where  $v$  is a curve with geodesic curvature  $k$  and speed  $1/k$  in  $T_p^1\mathbb{H}^3$ , for some  $p \in \mathbb{H}^3$ , for certain  $k > 0$ .

**Theorem 3** *The ruled surfaces associated with null magnetic geodesics of  $\mathcal{L}_\kappa$  for  $\kappa = 0, 1$  are described in an analogous manner as in the previous theorem, except that case a) is empty. Besides, for  $\kappa = 1$ , a null magnetic geodesic has simultaneously the forms (4) and (5).*

## 2 Preliminaries

For the simultaneous analysis of the three cases  $\kappa = 0, 1, -1$ , we consider the standard presentation of  $M_\kappa$  as a submanifold of  $\mathbb{R}^4$ . That is,  $\mathbb{R}^3 = \{(1, x) \in \mathbb{R}^4 \mid x \in \mathbb{R}^3\}$ ,  $\mathbb{S}^3 = \{x \in \mathbb{R}^4 \mid |x|^2 = 1\}$  and  $\mathbb{H}^3 = \{x \in \mathbb{R}^4 \mid -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1 \text{ and } x_0 > 0\}$ .

Let  $G_\kappa$  be the identity component of the isometry group of  $M_\kappa$ , that is,  $G_0 = SO_3 \times \mathbb{R}^3$ ,  $G_1 = SO_4$  and  $G_{-1} = O_o(1, 3)$ . We consider the usual presentation of  $G_0$  as a subgroup of  $GL_4(\mathbb{R})$ . The group  $G_\kappa$  acts on  $\mathcal{L}_\kappa$  as follows:  $g \cdot [\gamma] = [g \circ \gamma]$ . This action is transitive and smooth.

If we denote by  $\mathfrak{g}_\kappa$  the Lie algebra of  $G_\kappa$  we have that

$$\mathfrak{g}_\kappa = \left\{ \begin{pmatrix} 0 & -\kappa x^t \\ x & B \end{pmatrix} \mid x \in \mathbb{R}^3, B \in so_3 \right\}.$$

Let  $\gamma_o$  be the geodesic in  $M_\kappa$  with  $\gamma_o(0) = e_0$  and initial velocity  $e_1 \in T_{e_0}M_\kappa$ , where  $\{e_0, e_1, e_2, e_3\}$  is the canonical basis of  $\mathbb{R}^4$ . For  $A, B \in \mathbb{R}^{2 \times 2}$ , let  $\text{diag}(A, B) = \begin{pmatrix} A & 0_2 \\ 0_2 & B \end{pmatrix}$ , where  $0_2$  denotes the  $2 \times 2$  zero matrix. Then the isotropy subgroup of  $G_\kappa$  at  $[\gamma_o]$  is

$$H_\kappa = \{ \text{diag}(R_\kappa(t), B) \mid t \in \mathbb{R}, B \in SO_2 \},$$

where

$$R_0(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad R_1(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad R_{-1}(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}. \quad (6)$$

Let  $j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . The Lie algebra of  $H_\kappa$  is

$$\mathfrak{h}_\kappa = \{ \text{diag}(r_\kappa(t), sj) \mid s, t \in \mathbb{R} \},$$

where  $r_\kappa(t) = \begin{pmatrix} 0 & -\kappa t \\ t & 0 \end{pmatrix}$ . We may identify  $\mathcal{L}_\kappa$  with  $G_\kappa/H_\kappa$  via the diffeomorphism

$$\phi : G_\kappa/H_\kappa \rightarrow \mathcal{L}_\kappa, \quad \phi(gH_\kappa) = g \cdot [\gamma_o]. \quad (7)$$

For  $x, y \in \mathbb{R}^2$  we denote  $Z(x, y) = \begin{pmatrix} 0_2 & (-\kappa x, -y)^t \\ (x, y) & 0_2 \end{pmatrix}$ . Let

$$\mathfrak{p}_\kappa = \{ Z(x, y) \in \mathfrak{g}_\kappa \mid x, y \in \mathbb{R}^2 \},$$

which is an  $\text{Ad}(H_\kappa)$ -invariant complement of  $\mathfrak{h}_\kappa$ .

For  $\kappa = 0, 1$ , we consider on  $\mathfrak{g}_\kappa$  the inner product such that  $\mathfrak{h}_\kappa \perp \mathfrak{p}_\kappa$ ,  $\|Z(x, y)\| = \det(x, y)$  and

$$\|\text{diag}(r_\kappa(t), sj)\| = -ts.$$

(for  $\kappa = 0$ , we have learnt of this inner product from [6, page 499]). On  $\mathfrak{g}_{-1}$  we consider the Killing form ( $\mathfrak{h}_\kappa \perp \mathfrak{p}_\kappa$  also holds). For  $\kappa = 0, 1, -1$ , this inner product on  $\mathfrak{g}_\kappa$  induces on  $G_\kappa$  a bi-invariant metric. Thus, there exists a unique pseudo-Riemannian metric on  $\mathcal{L}_\kappa \simeq G_\kappa/H_\kappa$  such that  $\pi : G_\kappa \rightarrow G_\kappa/H_\kappa$  is a pseudo-Riemannian submersion. For  $\kappa = 0, 1$ , this metric on  $\mathcal{L}_\kappa$  coincides with the given in (2), see Lemma 5 b). For  $\kappa = -1$ , the metric on  $\mathcal{L}_{-1}$  associated with the Killing form is different from the one in (2).

However, the magnetic geodesics of either metric on  $\mathcal{L}_{-1}$  are the same. This follows since the geodesics are the same (see [8]), so the Levi-Civita connections coincide.

Let us call  $A = \text{diag}(0_2, j)$ , which is in the center of  $\mathfrak{h}_\kappa$ . We have that  $\text{ad}_A$  is orthogonal and  $\text{ad}_A^2 = -\text{id}$  in  $\mathfrak{p}_\kappa$ . Hence,  $\text{ad}_A$  induces a complex structure on  $G_\kappa/H_\kappa$ . A straightforward computation shows that it coincides, via  $\phi$  in (7), with the complex structure given in the introduction. With the metric above and this complex structure,  $\mathcal{L}_\kappa$  is a Hermitian symmetric space.

As a direct application of a result by Adachi, Maeda and Udagawa in [1] (see also [2], Remark 1) we have

**Theorem 4** *Let  $\sigma$  be a magnetic geodesic of  $G_\kappa/H_\kappa$  with initial conditions  $\sigma(0) = H_\kappa$  and  $\dot{\sigma}(0) = X \in \mathfrak{p}_\kappa$ . Then  $\sigma(t) = \pi(\exp t(X + A))$ .*

As we saw in (1),  $\mathcal{J}_{\gamma_o}$  is isomorphic to  $T_{[\gamma_o]}\mathcal{L}_\kappa \cong \mathfrak{p}_\kappa$ . In the next Lemma we relate  $\mathfrak{p}_\kappa$  and  $\mathcal{J}_{\gamma_o}$  explicitly, involving the matrix  $A$ .

**Lemma 5** *Let  $Z = Z(x, y) \in \mathfrak{p}_\kappa$ .*

- a) *The Jacobi field  $J(s) = \frac{d}{dt}\Big|_0 \exp t(Z + A) \cdot \gamma_o(s)$  in  $\mathcal{J}_{\gamma_o}$  is the unique one that satisfies  $J(0) = (0, 0, x)^t$  and  $J'(0) = (0, 0, y)^t$ .*
- b)  *$T_{\gamma_o}(J) = d(\phi \circ \pi)Z$  and its norm is  $\|d(\phi \circ \pi)Z\| = \det(x, y)$ .*

Proof. For each  $\kappa$ , we consider the following parameterization of  $\gamma_o$ :

$$\begin{aligned} \gamma_o(s) &= (1, s, 0, 0), & \text{if } \kappa = 0; \\ \gamma_o(s) &= (\cos s, \sin s, 0, 0), & \text{if } \kappa = 1; \\ \gamma_o(s) &= (\cosh s, \sinh s, 0, 0), & \text{if } \kappa = -1. \end{aligned}$$

Given  $Z = Z(x, y) \in \mathfrak{p}_\kappa$ , the Jacobi field along  $\gamma_o$  defined by  $J(s) = \frac{d}{dt}\Big|_0 \exp t(Z + A) \cdot \gamma_o(s)$  belongs to  $\mathcal{J}_{\gamma_o}$ , because for all  $s \in \mathbb{R}$ ,

$$\langle J(s), \dot{\gamma}_o(s) \rangle = \langle (Z + A)(\gamma_o(s)), \dot{\gamma}_o(s) \rangle = 0,$$

since  $(Z + A)(\gamma_o(s))$  is orthogonal to  $e_0$  and  $e_1$ , while  $\dot{\gamma}_o(s)$  has non zero components only in these two directions.

One verifies easily that  $J(0) = (Z + A)(e_0) = (0, 0, x)^t$ . On the other hand,

$$\begin{aligned} J'(0) &= \frac{D}{ds}\Big|_0 \frac{\partial}{\partial t}\Big|_0 \exp t(Z + A) \cdot \gamma_o(s) \\ &= \frac{D}{\partial t}\Big|_0 \exp t(Z + A)(e_1) = (Z + A)(e_1) = (0, 0, y)^t. \end{aligned}$$

Besides,

$$\begin{aligned} T_{\gamma_o}(J) &= \frac{d}{dt}\Big|_0 [\exp t(Z + A) \cdot \gamma_o] = \frac{d}{dt}\Big|_0 \phi(\exp t(Z + A)H_\kappa) \\ &= \frac{d}{dt}\Big|_0 \phi(\pi(\exp t(Z + A))) = d\phi \circ d\pi Z, \end{aligned}$$

where the last equality holds since  $A \in \mathfrak{h}_\kappa$ . Finally, the norm (2) of  $d(\phi \circ \pi)Z$  equals

$$\|d(\phi \circ \pi)Z\| = \langle \dot{\gamma}_o(0) \times J(0), J'(0) \rangle = \det(x, y)$$

and the assertions of b) are verified.  $\square$

Let  $Z(x, y) \in \mathfrak{p}_\kappa$  and let  $h = \text{diag}(R_\kappa(t), B) \in H_\kappa$ , where  $B \in SO_2$  and

$$R_\kappa(t) = \begin{pmatrix} c_\kappa(t) & -\kappa s_\kappa(t) \\ s_\kappa(t) & c_\kappa(t) \end{pmatrix}$$

is as in (6). Then  $\text{Ad}(h)Z(x, y) = Z(Bx_t, By_t)$ , where

$$x_t = c_\kappa(t)x - s_\kappa(t)y, \quad y_t = \kappa s_\kappa(t)x + c_\kappa(t)y.$$

We denote by  $\epsilon_1$  and  $\epsilon_2$  the vectors of the canonical basis of  $\mathbb{R}^2$ .

**Lemma 6** *Let  $Z(x, y) \neq 0$  in  $\mathfrak{p}_\kappa$ .*

- a) *If  $\{x, y\}$  is a linearly independent set of  $\mathbb{R}^2$ , then there exists  $h \in H_\kappa$  such that  $\text{Ad}(h)Z(x, y) = Z(a\epsilon_1, b\epsilon_2)$ , with  $a > 0$  and  $b \neq 0$ , for  $\kappa = 0, \pm 1$ .*
- b) *If  $\kappa = 0, 1$  and  $\{x, y\}$  is a linearly dependent set of  $\mathbb{R}^2$ , then there exists  $h \in H_\kappa$  such that either  $\text{Ad}(h)Z(x, y) = Z(0, b\epsilon_2)$ , with  $b \neq 0$ , or  $\text{Ad}(h)Z(x, y) = Z(a\epsilon_1, 0)$ , with  $a > 0$ . This is true for  $\kappa = -1$  if in addition  $|x| \neq |y|$ .*
- c) *For  $\kappa = 1$ , there exists  $h \in H_\kappa$  such that  $\text{Ad}(h)Z(\epsilon_1, 0) = Z(0, \epsilon_2)$ .*

*Proof.* For the proof of a), as  $\{x, y\}$  is a linearly independent set, then for  $\kappa = 0, \pm 1$  there exists  $t \in \mathbb{R}$  such that  $\langle x_t, y_t \rangle = 0$ . Indeed, for each  $\kappa$ , this is equivalent to fact that the equation

$$\begin{aligned} c_3 - c_2 t &= 0 && \text{if } \kappa = 0; \\ \frac{1}{2}(c_1 - c_2) \sin(2t) + c_3 \cos(2t) &= 0 && \text{if } \kappa = 1; \\ -\frac{1}{2}(c_1 + c_2) \sinh(2t) + c_3 \cosh(2t) &= 0 && \text{if } \kappa = -1 \end{aligned}$$

has a real solution, where  $c_1 = \langle x, x \rangle$ ,  $c_2 = \langle y, y \rangle$  and  $c_3 = \langle x, y \rangle$ . But the linear independence of  $x$  and  $y$  determines the existence of the solution in each case. Then, we can take  $B \in SO_2$  such that  $Bx_t = a\epsilon_1$ , with  $a > 0$  and  $By_t = b\epsilon_2$ , with  $b \neq 0$ . Therefore the isometry  $h = \text{diag}(R_\kappa(t), B) \in H_\kappa$  satisfies  $\text{Ad}(h)Z(x, y) = Z(a\epsilon_1, b\epsilon_2)$ .

For the proof of b), first we suppose that  $x = 0$  or  $y = 0$  (but not both zero since  $Z(x, y) \neq 0$ ). Let  $B \in SO_2$  such that  $Bx = a\epsilon_1$  with  $a > 0$ , if  $x \neq 0$ , and in the case that  $y \neq 0$ , let  $B \in SO_2$  such that  $By = b\epsilon_2$ , with  $b \neq 0$ . Then we can take  $h = \text{diag}(I, B) \in H_\kappa$ .

Now, let  $x \neq 0$  and  $y \neq 0$ . So  $x = \lambda y$  or  $y = \lambda x$ , with  $\lambda \neq 0$ . We suppose that  $y = \lambda x$  (for  $x = \lambda y$  the argument is similar). In the cases  $\kappa = 0, 1$  there exists  $t \in \mathbb{R}$  such that  $x_t = 0$ . In fact, from the hypothesis and some computations,  $t \in \mathbb{R}$  is obtained by solving

$$1 - \lambda t = 0, \quad \text{if } \kappa = 0 \quad \text{and} \quad \cos t - \lambda \sin t = 0, \quad \text{if } \kappa = 1.$$

Thus, taking  $B \in SO_2$  such that  $By_t = b\epsilon_2$  (with  $b \neq 0$  as  $y_t \neq 0$ ), we have that  $h = \text{diag}(R_\kappa(t), B) \in H_\kappa$  satisfies  $\text{Ad}(h)Z(x, y) = Z(0, b\epsilon_2)$ .

For  $\kappa = -1$ , as in the cases  $\kappa = 0, 1$ , we find  $t \in \mathbb{R}$  such that either  $x_t = 0$  or  $y_t = 0$  by solving

$$\cosh t - \lambda \sinh t = 0, \quad \text{and} \quad -\sinh t + \lambda \cosh t = 0,$$

respectively. But these equations have a solution if and only if  $\lambda \neq \pm 1$ . That is, if and only if  $|x| \neq |y|$ . Hence, taking  $B \in SO_2$  such that either  $By_t = b\epsilon_2$  or  $Bx_t = a\epsilon_1$  (with  $a > 0$ ; here again we have that  $x_t \neq 0$ ), as appropriate. Then  $h = \text{diag}(R_{-1}(t), B) \in H_{-1}$  is as desired in this case.

For part c), we observe that  $h = \text{diag}(R_1(\pi/2), B) \in H_1$ , where  $B \in SO_2$  takes  $\epsilon_1$  to  $\epsilon_2$ , satisfies  $\text{Ad}(h)Z(\epsilon_1, 0) = Z(0, \epsilon_2)$ .  $\square$

Remark. The previous lemma corresponds, geometrically, with the fact of finding  $s \in \mathbb{R}$  at which the Jacobi field associated with  $Z(x, y)$  (given by Lemma 5) and its covariant derivative are orthogonal.

Recall that if  $h$  is a regular curve in  $M_\kappa$  of constant speed  $a$ , then the Frenet frame of  $h$  is

$$T(t) = \frac{1}{a} \dot{h}(t), \quad N(t) = \dot{h}'(t) / \left| \dot{h}'(t) \right|, \quad B(t) = T(t) \times N(t) \quad (8)$$

(here the prime denotes the covariant derivative along  $h$ ), and its curvature and torsion are given by

$$k(t) = \frac{1}{a^2} \left| \dot{h}'(t) \right|, \quad \tau(t) = -\frac{1}{a} \langle B'(t), N(t) \rangle. \quad (9)$$

For each  $g \in G_\kappa$  we have that  $g$  is an isometry of  $\mathcal{L}_\kappa$  and preserves the Hermitian structure. Hence,  $g$  takes magnetic geodesics to magnetic geodesics.

### 3 Time- and space-like magnetic geodesics

Proof of Theorem 1. Let  $Z \in \mathfrak{p}_\kappa$  be the initial velocity of  $\sigma$ , with  $\|Z\| \neq 0$ . First, we consider the case  $Z = Z(a\epsilon_1, b\epsilon_2)$ , with  $a > 0$  and  $b \neq 0$ .

For each  $t \in \mathbb{R}$ , let  $\alpha(t) = \exp t(Z + A)$ . By Theorem 4 and the diffeomorphism  $\phi$  in (7), we know that  $\sigma(t) = \alpha(t) \cdot [\gamma_o]$ , that is,  $\sigma(t) = [\alpha(t) \cdot \gamma_o]$ .

Let  $h$  be the curve in  $M_\kappa$  given by  $h(t) = \alpha(t)(e_0)$ . As  $\alpha$  is a one-parameter subgroup of isometries of  $M_\kappa$ , we have that  $h$  is a curve with constant curvature and torsion, thus  $h$  is a helix in  $M_\kappa$ .

Let us see that  $\sigma(t) = [\gamma_{B(t)}]$ , where  $B(t)$  is the binormal field of  $h$ . For each  $t \in \mathbb{R}$ , the initial velocity of the geodesic  $\alpha(t) \cdot \gamma_o$  is  $d(\alpha(t))(e_1)$ , hence  $\sigma(t) = [\gamma_{d(\alpha(t))(e_1)}]$ . Then, we have to verify that  $B(t) = d(\alpha(t))(e_1)$ , for all  $t \in \mathbb{R}$ . Since  $\alpha(t)$  is an isometry that preserves the helix and takes the Frenet frame at  $t = 0$  to the Frenet frame at  $t$ , it suffices to show that  $B(0) = e_1$ .

By the usual identifications, since  $\alpha(t)$  is a linear transformation, we can write  $d(\alpha(t))(e_1) = \alpha(t)(e_1)$ , so

$$\dot{h}(t) = \alpha(t)((Z + A)e_0) \quad \text{and} \quad \dot{h}'(t) = [\alpha(t)((Z + A)^2 e_0)]^T,$$

where  $T$  denotes the tangent projection. Since

$$\dot{h}(0) = (Z + A)e_0 = ae_2,$$

$$\dot{h}'(0) = [(Z + A)^2 e_0]^T = [-\kappa a^2 e_0 + ae_3]^T = ae_3$$

and  $\alpha(t)$  is an isometry, we have  $|\dot{h}(t)| = a = |\dot{h}'(t)|$ . By the computation before and (8) we obtain

$$B(0) = \frac{1}{a^2} \dot{h}(0) \times \dot{h}'(0) = e_1.$$

Since  $B'(t) = [\alpha(t)((Z + A)e_1)]^T$  (recall that we have just proven that  $B(t) = \alpha(t)(e_1)$ ), then  $B'(0) = be_3$ . Besides, using (8) and the previous computations it follows that  $N(0) = e_3$ . Therefore, by (9) we have that the curvature and torsion of  $h$  are equal to

$$k = 1/a, \quad \tau = -b/a. \quad (10)$$

The assertion regarding the sign of the torsion is immediate from Lemma 5 b) and (10). Thus, the theorem is proved in this particular case.

Now, let  $\sigma$  be a magnetic geodesic with  $\sigma(0) = [\gamma]$  and initial velocity with non zero norm. Since  $G_\kappa$  acts transitively on  $\mathcal{L}_\kappa$ , there is an isometry  $g$  such that  $g \cdot [\gamma] = [\gamma_0]$ . So, the magnetic geodesic  $g \cdot \sigma$  also has initial velocity with non zero norm and  $g \cdot \sigma(0) = [\gamma_0]$ . By Lemma 5 b), if  $d(\phi \circ \pi)Z(x, y)$  is the initial velocity of  $g \cdot \sigma$ , we have that the vectors  $\{x, y\}$  are linearly independent. Then, by Lemma 6 a), there exists  $h \in H_\kappa$  such that  $\text{Ad}(h)Z(x, y) = Z(ae_1, be_2)$ , with  $a > 0$  and  $b \neq 0$ . Since  $((h \circ g) \cdot \sigma)'(0) = d(\phi \circ \pi)(\text{Ad}(h)Z(x, y))$ , the curve  $(h \circ g) \cdot \sigma$  is a magnetic geodesic of the type studied above. Therefore,  $\sigma$  has the form (3).

Conversely, let  $h$  be a helix in  $M_\kappa$  with curvature  $k > 0$ , non zero torsion  $\tau$  and speed  $1/k$ . Let  $\{T, B, N\}$  be the Frenet frame of  $h$ . As  $M_\kappa$  is a simply connected manifold of constant curvature, we have that there exists an isometry  $g$  of  $M_\kappa$  preserving the orientation such that  $g(h(0)) = e_0$  and its differential at  $h(0)$  takes  $B(0)$  to  $e_1$ ,  $T(0)$  to  $e_2$  and  $N(0)$  to  $e_3$ .

Let  $a = 1/k$  and  $b = -\tau/k$ . Let  $Z = Z(ae_1, be_2) \in \mathfrak{p}_\kappa$ . We consider, for each  $t \in \mathbb{R}$ ,  $\alpha(t) = \exp t(Z + A)$ . According to computations from the first part of the proof, both helices have initial position  $e_0$ , curvature  $k$ , torsion  $\tau$ , speed  $1/k$  and the same Frenet frame at  $t = 0$ . Hence  $(g \circ h)(t) = \alpha(t)e_0$ . So, if we call  $\bar{B}$  the binormal field of  $g \circ h$ , we have that  $\bar{B}(t) = d(\alpha(t))e_1$ , for all  $t$ . Finally, since the curve  $[\gamma_{\bar{B}(t)}]$  is a magnetic geodesic in  $\mathcal{L}_\kappa$  and

$$[\gamma_{B(t)}] = [\gamma_{dg^{-1}\bar{B}(t)}] = g^{-1} \cdot [\gamma_{\bar{B}(t)}],$$

we obtain that  $[\gamma_{B(t)}]$  is a magnetic geodesic. □



## 4 Null magnetic geodesics

We deal first with the hyperbolic case. We use the notation given in the introduction and we recall from [3] certain properties of horospheres and related concepts. To simplify the notation we omit the subindex  $\kappa = -1$ .

Let  $\gamma$  be a geodesic of  $\mathbb{H}^3$ . Then, for each  $p \in \mathbb{H}^3$  there exists a unique unit speed geodesic  $\alpha$  of  $\mathbb{H}^3$  such that  $\alpha(0) = p$  and  $\alpha$  is asymptotic to  $\gamma$ . Let  $v \in T^1\mathbb{H}^3$ . If  $p$  is any point of  $\mathbb{H}^3$ , then  $v(p)$  denotes the unique unit tangent vector at  $p$  that is asymptotic to  $v$ . The Busemann function  $f_v : \mathbb{H}^3 \rightarrow \mathbb{R}$  is defined by

$$f_v(p) = \lim_{t \rightarrow +\infty} d(p, \gamma_v(t)) - t,$$

and satisfies  $\text{grad}_p(f_v) = -v(p)$ . The *horosphere* determined by  $v$  is given by

$$H(v) = \{q \in M : f_v(q) = 0\}.$$

The Jacobi vector fields orthogonal to  $\dot{\gamma}_o$  have the form

$$J(s) = e^s U(s) + e^{-s} V(s), \quad (11)$$

where  $U$  and  $V$  are parallel vector fields along  $\gamma_o$  and orthogonal to  $\dot{\gamma}_o$ .

A Jacobi vector field  $Y$  along a geodesic  $\gamma$  of  $\mathbb{H}^3$  is said to be *stable* (*unstable*) if there exists a constant  $c > 0$  such that

$$|Y(s)| \leq c \quad \forall s \geq 0 \quad (\forall s \leq 0).$$

In what follows we shall denote by  $\hat{\pi}$  the canonical projection from  $T\mathbb{H}^3$  onto  $\mathbb{H}^3$ . We recall that in the introduction we have defined the smooth maps  $\psi^\pm : \mathcal{L}(\mathbb{H}^3) \rightarrow \mathbb{H}^3(\infty)$  by  $\psi^\pm[\gamma] = \gamma(\pm\infty)$  and the distributions  $\mathcal{D}^\pm$  in  $\mathcal{L}(\mathbb{H}^3)$  given by  $\mathcal{D}_{[\gamma]}^\pm = \text{Ker}(d\psi_{[\gamma]}^\pm)$ . We need to relate the distributions  $\mathcal{D}^\pm$  with distributions  $\bar{\mathcal{E}}^\pm$  and  $\mathcal{E}^\pm$  on  $G$  and  $T^1\mathbb{H}^3$ , respectively.

Let  $\bar{\mathcal{E}}^\pm$  be the left invariant distribution on  $G$  defined at  $I \in G$  by

$$\bar{\mathcal{E}}_I^\pm = \{Z(u, \mp u) \in \mathfrak{p} \mid u \in \mathbb{R}^2\}.$$

As the canonical action of  $G$  on  $T^1\mathbb{H}^3$  is transitive, the projection  $\bar{p} : G \rightarrow T^1\mathbb{H}^3$  given by  $\bar{p}(g) = dg_{e_0}e_1$  is a submersion. Since given  $v \in T^1\mathbb{H}^3$  there exists  $g \in G$  such that  $\bar{p}(g) = v$ , we define:

$$\mathcal{E}^\pm(v) = (d\bar{p} \bar{\mathcal{E}}^\pm)(\bar{p}(g)) = d\bar{p}_g(\bar{\mathcal{E}}_g^\pm).$$

We have that  $\mathcal{E}^\pm$  determines a well defined distribution on  $T^1\mathbb{H}^3$ , which is called the *horospherical distribution* on  $T^1\mathbb{H}^3$ . This distribution has the following property: if  $t \mapsto v(t)$  is a curve in  $T^1\mathbb{H}^3$  tangent to the distribution  $\mathcal{E}^\pm$ , then  $\hat{\pi}(v(t))$  is in the horosphere  $H(\pm v(0))$ .

**Lemma 7** *Let  $Z \in \bar{\mathcal{E}}_I^\pm$ . For each  $t \in \mathbb{R}$ , let  $\gamma_t^\pm(s) = \exp t(Z + A) \cdot \gamma_o(\pm s)$ . Then the geodesics  $\gamma_t^\pm$  are asymptotic to each other for all  $t \in \mathbb{R}$ .*

Proof. Let  $J$  be the Jacobi vector field associated with the variation by geodesics  $t \mapsto \gamma_t^\pm$ . By Lemma 5 a),  $J(0) = -J'(0)$ . Hence, by (11) we have that  $J(s) = e^{-s}U(s)$ , where  $U$  is a parallel vector field along  $\gamma_o$  orthogonal to  $\dot{\gamma}_o$ . Thus,  $J$  is a stable vector field, that is, there exists  $c > 0$  such that  $|J(s)| \leq c \forall s \geq 0$ .

We have to show that given  $t_0, t_1 \in \mathbb{R}$  with  $t_0 < t_1$ , there exists  $N > 0$  such that

$$d(\gamma_{t_0}^\pm(s), \gamma_{t_1}^\pm(s)) \leq N \quad \forall s \geq 0.$$

For fixed  $s$ ,

$$d(\gamma_{t_0}^\pm(s), \gamma_{t_1}^\pm(s)) \leq \text{length}([t_0, t_1] \ni t \mapsto \gamma_t^\pm(s)) = \int_{t_0}^{t_1} \left| \frac{d}{dt} \gamma_t^\pm(s) \right| dt.$$

For each  $t \in \mathbb{R}$ , let  $J_t(s) = \frac{d}{dt} \gamma_t^\pm(s)$ . We observe that  $J_{t'+t}(s) = d \exp(t'Z) J_t(s)$  for all  $t, t'$ . Since  $\exp(t'Z)$  is an isometry, we have  $|J_t(s)| = |J(s)|$ . Therefore,

$$\int_{t_0}^{t_1} |J_t(s)| dt = \int_{t_0}^{t_1} |J(s)| dt \leq c(t_1 - t_0)$$

for all  $s \geq 0$ . Then, we may take  $N = c(t_1 - t_0) > 0$ .  $\square$

We consider the projection  $p : T^1\mathbb{H}^3 \rightarrow \mathcal{L}(\mathbb{H}^3)$ ,  $p(v) = [\gamma_v]$ . We call  $\bar{\mathcal{D}}^\pm$  the distribution on  $\mathcal{L}(\mathbb{H}^3)$   $p$ -related with  $\mathcal{E}^\pm$  (well defined). More specifically, given  $[\gamma] \in \mathcal{L}(\mathbb{H}^3)$  and  $v \in T^1\mathbb{H}^3$  such that  $p(v) = [\gamma]$ ,

$$\bar{\mathcal{D}}^\pm([\gamma]) = dp_v \mathcal{E}_v^\pm.$$

**Proposition 8** *Let  $\mathcal{D}^\pm$  and  $\bar{\mathcal{D}}^\pm$  be the distributions on  $\mathcal{L}(\mathbb{H}^3)$  defined above. Then  $\mathcal{D}^\pm = \bar{\mathcal{D}}^\pm$ .*

Proof. Since  $\mathcal{D}^\pm$  and  $\bar{\mathcal{D}}^\pm$  are  $G$ -invariant, it is enough to show  $\mathcal{D}_{[\gamma_o]}^\pm = dp_{(e_0, e_1)}(\mathcal{E}_{(e_0, e_1)}^\pm)$  (we observe that  $\bar{p}(I) = (e_0, e_1)$  and  $p(e_0, e_1) = [\gamma_o]$ ).

Let  $Z \in \bar{\mathcal{E}}_I^\pm$ . We take the curve in  $\mathcal{L}(\mathbb{H}^3)$  given by  $\alpha(t) = \exp tZ \cdot [\gamma_o]$ . As  $\alpha(t) = p \circ \bar{p}(\exp tZ)$ , we have that  $\alpha(0) = [\gamma_o]$  and  $\dot{\alpha}(0) = d(p \circ \bar{p})_I Z$ . That is,  $\dot{\alpha}(0) \in dp_{(e_0, e_1)}(\mathcal{E}_{(e_0, e_1)}^\pm)$ . Besides,

$$\left. \frac{d}{dt} \right|_0 \exp tZ \cdot \gamma_o(s) = \left. \frac{d}{dt} \right|_0 \exp t(Z + A) \cdot \gamma_o(s), \quad (12)$$

since both Jacobi fields have the same initial conditions. Hence, Lemma 7 applies to the geodesics  $\gamma_t^\pm(s) = \exp tZ \cdot \gamma_o(\pm s)$ . Thus,  $\psi^\pm \circ \alpha$  is constant. Then  $(d\psi^\pm)_{[\gamma_o]}(\dot{\alpha}(0)) = 0$ , that is,  $\dot{\alpha}(0) \in \mathcal{D}_{[\gamma_o]}^\pm$ .

On the other hand, let  $\varphi : T_{e_0}^1\mathbb{H}^3 \rightarrow \mathcal{L}(\mathbb{H}^3)$ ,  $\varphi(v) = [\gamma_v]$ , be the submanifold whose image  $\mathcal{L}_{e_0}(\mathbb{H}^3)$  consists of all the oriented geodesics passing through  $e_0$ . Besides,  $H(\infty)$  is a manifold with the differentiable structure (well defined) such that  $F_{e_0} : T_{e_0}^1\mathbb{H}^3 \rightarrow$

$H(\infty)$  given by  $F_{e_0}(v) = \gamma_v(\infty)$  is a diffeomorphism. Then, since  $\psi^+|_{\mathcal{L}_{e_0}(\mathbb{H}^3)} \circ \varphi = F_{e_0}$ , we have that  $(d\psi^+)_{[\gamma_o]}$  is surjective. Now,  $(d\psi^-)_{[\gamma_o]}$  is also surjective because  $\psi^-$  is the composition of  $\psi^+$  with the diffeomorphism of  $\mathcal{L}(\mathbb{H}^3)$  assigning  $[\gamma^{-1}]$  to  $[\gamma]$ . Therefore,  $\dim \mathcal{D}_{[\gamma_o]}^\pm = \dim \bar{\mathcal{D}}_{[\gamma_o]}^\pm$  and equality follows.  $\square$

The word *cylinder* in the statement of Theorem 2 refers to a ruled surface determined by a parallel vector field along a curve  $c$  of constant geodesic curvature  $k$  contained in a totally geodesic surface in  $M_\kappa$  (and normal to it), as explained. For  $\kappa = -1$ , this ruled surface is diffeomorphic to  $S^1 \times \mathbb{R}$  if  $|k| > 1$ ; otherwise it is diffeomorphic to a plane.

Proof of Theorem 2 a). By Lemma 5 b), we have that every element of  $\mathcal{D}_{[\gamma]}^\pm$  is null. As  $G$  acts transitively on  $\mathcal{L}(\mathbb{H}^3)$  and by the  $G$ -invariance of the horospherical distributions, we may suppose without loss of generality that  $\sigma(0) = [\gamma_o]$ , hence  $\dot{\sigma}(0) \in \mathcal{D}_{[\gamma_o]}^\pm$ . By Proposition 8, there exists  $Z \in \bar{\mathcal{E}}_I^\pm$  such that  $\dot{\sigma}(0) = (dp)_{(e_0, e_1)}(d\bar{p})_I Z$ . Thus, by Theorem 4,  $\sigma(t) = [\exp t(Z + A) \cdot \gamma_o]$ .

We assume that  $Z \in \bar{\mathcal{E}}_I^+$ . Let us show that  $\sigma$  describes a forward cone with vertex at  $\gamma_o(+\infty)$ . In a similar way, if  $Z \in \bar{\mathcal{E}}_I^-$ , then  $\sigma$  describes a backward cone with vertex at  $\gamma_o(-\infty)$ .

We consider the geodesics  $\gamma_t(s) = \exp t(Z + A) \cdot \gamma_o(s)$  of  $\mathbb{H}^3$ . As  $Z \in \bar{\mathcal{E}}_I^+$ , by Lemma 7, we have that the geodesics  $\gamma_t$  are asymptotic to each other for all  $t$ . Hence,  $z(t) = \dot{\gamma}_t(0)$  is a curve in  $T^1\mathbb{H}^3$  of asymptotic vectors to  $e_1$ .

Let  $c(t) = \hat{\pi}(z(t)) = \exp t(Z + A)(e_0)$ . In order to see that  $c(t) \in H(e_1)$  for all  $t$ , we observe that

$$\frac{d}{dt} f_{e_1}(c(t)) = (df_{e_1})_{c(t)} \dot{c}(t) = \langle \text{grad}_{c(t)}(f_{e_1}), \dot{c}(t) \rangle. \quad (13)$$

Since  $\text{grad}_p(f_v) = -v(p)$  we have that

$$\text{grad}_{c(t)}(f_{e_1}) = -z(t) = -d(\exp t(Z + A))e_1.$$

On the other hand,

$$\dot{c}(t) = d(\exp t(Z + A))(Z + A)e_0.$$

Since  $\exp t(Z + A)$  is an isometry and observing that  $(Z + A)e_0$  and  $e_1$  are perpendicular ( $Z \in \bar{\mathcal{E}}_I^+$ ), it follows that the expression in (13) is equal to  $-\langle e_1, (Z + A)(e_0) \rangle = 0$ . Then,  $f_{e_1}(c(t)) = f_{e_1}(e_0) = 0$  for all  $t$ , that is,  $c(t) \in H(e_1)$  for all  $t$ .

Now, as  $c$  is the orbit through  $e_0$  of a one-parameter subgroup of isometries of  $G$  preserving  $H(e_1)$ , its geodesic curvature and speed are constant. If  $Z = Z(u, -u)$  for certain  $0 \neq u \in \mathbb{R}^2$ , we obtain that the speed of  $c$  is  $|u|$ . For each  $v \in T^1H^3$  we consider on  $H(v)$  the orientation given by  $-\text{grad } f_v$ . The geodesic curvature of  $c$  is then

$$k = \langle -\text{grad}_{e_0}(f_{e_1}), \dot{c}(0) \times \dot{c}'(0) \rangle / |u|^3 = 1/|u|,$$

since  $\dot{c}(0) = (Z + A)e_0$  and  $\dot{c}'(0) = ((Z + A)^2 e_0)^\top$ . As for each  $v \in T^1\mathbb{H}^3$ ,  $H(v)$ , with the induced metric of  $\mathbb{H}^3$ , is isometric to  $\mathbb{R}^2$ , we have that  $c(t)$  runs along a circle on  $H(e_1)$  of geodesic curvature  $k = 1/|u| > 0$  and speed  $1/k = |u|$ .

Besides,  $\sigma(t) = [\gamma_{z(t)}]$ . Thus we have that all conditions are satisfied in order to assert that  $\sigma$  describes a forward cone with vertex at  $\gamma_o(+\infty)$ .

Conversely, let  $\sigma$  be a curve in  $\mathcal{L}(\mathbb{H}^3)$  that describes a forward cone with vertex at infinity. As  $G$  acts transitively on the positively oriented frame bundle, and also each element of  $G$  takes horospheres to horospheres, preserving their orientation, we may suppose that  $\sigma(t) = [\gamma_{v(t)}]$ , where  $v(t)$  is a curve in  $T^1\mathbb{H}^3$  of asymptotic vectors to  $v(0) = e_1$  and  $c(t) = \hat{\pi}(v(t))$  is a curve of geodesic curvature  $k$  and speed  $1/k$  in  $H(e_1)$  with  $\dot{c}(0) = \frac{1}{k}e_2$ , for some  $k > 0$ . Let  $Z = Z(\frac{1}{k}\epsilon_1, -\frac{1}{k}\epsilon_1) \in \bar{\mathcal{E}}_I^+$ . We define

$$\bar{c}(t) = \exp t(Z + A)(e_0) \quad \text{and} \quad \bar{v}(t) = d(\exp t(Z + A))(e_1).$$

We showed above that  $\bar{c}(t)$  is a curve of geodesic curvature  $k$  and speed  $\frac{1}{k}$  in  $H(e_1)$ . Moreover,  $\bar{c}(0) = e_0$  and the initial velocity of  $\bar{c}$  is  $\frac{1}{k}e_2$ . So, we obtain that  $\bar{c} = c$ . This implies, together with the identities  $\hat{\pi} \circ \bar{v} = \bar{c}$  and  $\hat{\pi} \circ v = c$ , that  $\hat{\pi} \circ \bar{v} = \hat{\pi} \circ v$ .

According to the first part of the proof,  $\bar{v}$  and  $v$  are curves of asymptotic vectors to  $e_1$ . Hence,  $-\bar{v}(t) = \text{grad}_{\bar{c}(t)}(f_{e_1}) = -v(t)$ . Therefore,  $[\gamma_{\bar{v}(t)}] = [\gamma_{v(t)}]$ , which is a null magnetic geodesic with initial velocity in the horospherical distribution since  $[\gamma_{v(t)}] = [\exp t(Z + A) \cdot \gamma_o]$ .  $\square$

Proof of Theorem 2 b). We suppose first that  $\sigma$  is a null magnetic geodesic such that  $\sigma(0) = [\gamma_o]$  and  $\dot{\sigma}(0) = d(\phi \circ \pi)Z(a\epsilon_1, 0)$ , with  $a > 0$ . The expression (4) and the relation between the speed and curvature of  $h$  are obtained as in the prove of Theorem 1. By (10) we know that the torsion of  $h$  is  $\tau = -b/a = 0$  (since  $b = 0$ ). Thus  $h$  is contained in a totally geodesic surface  $S$  of  $\mathbb{H}^3$  and  $B$  is normal to  $S$ .

Now, we suppose that  $\dot{\sigma}(0) = d(\phi \circ \pi)Z$ , where  $Z = Z(0, b\epsilon_2)$  with  $b \neq 0$ . By Theorem 4 we have that  $\sigma(t) = [\alpha(t) \cdot \gamma_o]$ , where  $\alpha(t) = \exp t(Z + A)$ . Since  $Z + A$  is in the Lie algebra of the isotropy subgroup  $H$  of  $G$  at  $e_0 \in \mathbb{H}^3$ , we get that  $\alpha(t)$  fixes  $e_0$ . Moreover, if  $v$  is the curve in  $T_{e_0}^1\mathbb{H}^3$  given by  $v(t) = d(\alpha(t))e_1$ , then

$$\sigma(t) = [\alpha(t) \cdot \gamma_o] = [\gamma_{v(t)}],$$

since the initial velocity of the geodesic  $\alpha(t) \cdot \gamma_o$  is  $v(t)$ , for each  $t \in \mathbb{R}$ .

Furthermore, as  $v$  is the orbit through  $e_1$  of a one-parameter subgroup of  $H$  (the canonical differential action of  $G$  on  $T_{e_0}^1\mathbb{H}^3$ ), then  $v$  has constant speed and constant geodesic curvature in  $T_{e_0}^1\mathbb{H}^3 \cong \mathbb{S}^2$ . Easy computations yield

$$\dot{v}(0) = (0, 0, b)^t \quad \text{and} \quad \ddot{v}(0) = (-b^2, -b, 0)^t.$$

So, the speed of  $v$  is  $|b|$  and its geodesic curvature is

$$k = \langle v(0), \dot{v}(0) \times \ddot{v}(0) \rangle / |b|^3 = 1/|b|$$

(we consider the orientation of the sphere given by the unit normal field pointing outwards). Thus,  $v$  is a curve in  $T_{e_0}^1\mathbb{H}^3$  of geodesic curvature  $k > 0$  and speed  $1/k$ . Consequently,  $\sigma$  has the form (5).

Now, let  $\sigma$  be a null magnetic geodesic such that  $\sigma(0) = [\gamma]$  and  $\dot{\sigma}(0) \notin \mathcal{D}_{[\gamma]}^\pm$ . As  $G$  acts transitively on  $\mathcal{L}(\mathbb{H}^3)$  and by the  $G$ -invariance of the horospherical distributions, we may suppose that  $\sigma(0) = [\gamma_o]$  and  $\dot{\sigma}(0) \notin \mathcal{D}_{[\gamma_o]}^\pm$ . Let  $Z = Z(x, y) \in \mathfrak{p}$  such that  $\dot{\sigma}(0) = d(\phi \circ \pi)Z$ . By Lemma 5 b), as the norm of the initial velocity of  $\sigma$  is zero, we have that  $x$  and  $y$  are linearly dependent, and since  $d(\phi \circ \pi)Z \notin \mathcal{D}_{[\gamma_o]}^\pm$ , we also have  $|x| \neq |y|$ . Now, the isometries in Lemma 6 b) take  $\sigma$  to magnetic geodesics of the particular types studied above. Therefore,  $\sigma$  has the form (4) or has the form (5), as desired.

Conversely, given a helix  $h$  in  $\mathbb{H}^3$  with curvature  $k$ , speed  $1/k$  and torsion  $\tau = 0$ , the proof that the expression (4) is a magnetic geodesic is identical to the proof of the converse of Theorem 1. As  $h$  has zero torsion, the initial velocity of the magnetic geodesic in (4) is not in the distributions  $\mathcal{D}^\pm$ .

Let  $v$  be a curve in  $T_p^1\mathbb{H}^3$  with geodesic curvature  $k > 0$  and speed  $1/k$ . Let  $g$  be the isometry of  $\mathbb{H}^3$  preserving the orientation such that  $g(p) = e_0$ ,  $dg(v(0)) = e_1$  and  $dg(\dot{v}(0)) = be_3$ , for certain  $b > 0$ . Hence,  $g \cdot v$  is a curve in  $T_{e_0}^1\mathbb{H}^3$  having the same geodesic curvature and the same speed as  $v$ , and also  $b = 1/k$ . As we showed above,  $\bar{v}$  is a curve in  $T_{e_0}^1\mathbb{H}^3$  with  $\bar{v}(0) = g \cdot v(0)$  and with the same initial velocity and geodesic curvature that  $g \cdot v$ . By uniqueness, we have that  $\bar{v} = g \cdot v$ . To complete the proof we observe that  $g \cdot [\gamma_{v(t)}] = [\gamma_{g \cdot v(t)}] = [\gamma_{\bar{v}(t)}]$ .  $\square$

Proof of Theorem 3. Lemma 6 b) implies that the analogue of Theorem 2 a) is empty for the cases  $\kappa = 0, 1$ . The proof of the fact that every curve  $\sigma$  in  $\mathcal{L}_\kappa$  is a null magnetic geodesic if and only if  $\sigma$  has the form (4) or (5) is similar to that of Theorem 2 b).

We check the last statement of the theorem. Without loss of generality, we consider only null magnetic geodesics passing through  $[\gamma_o]$  at  $t = 0$ . We observe that if, in particular,  $\sigma$  is a magnetic geodesic with initial velocity  $d(\phi \circ \pi)Z(ae_1, 0)$ , with  $a > 0$ , (that is,  $\sigma$  has the form (4)), then by Lemma 6 c) there exists  $h \in H_1$  such that  $\text{Ad}(h)Z(ae_1, 0) = Z(0, ae_2)$ . Hence,  $h \cdot \sigma$  is a null magnetic geodesic with initial velocity  $d(\phi \circ \pi)Z(0, ae_2)$ , and then it has the form (5). So,  $\sigma$  also has this form.  $\square$

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