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WEIGHTED A PRIORI ESTIMATES FOR THE SOLUTION OF THE DIRICHLET PROBLEM IN POLYGONAL DOMAINS IN \mathbb{R}^2

Abstract

Let Ω be a polygonal domain in \mathbb{R}^2 and let U be a weak solution of $-\Delta u = f$ in Ω with Dirichlet boundary condition, where $f \in L^p_{\omega}(\Omega)$ and ω is a weight in $A_p(\mathbb{R}^2)$, 1 . We give some estimates ofthe Green function associated to this problem involving some functions $of the distance to the vertices and the angles of <math>\Omega$. As a consequence, we can prove an a priori estimate for the solution u on the weighted Sobolev spaces $W^{2,p}_{\omega}(\Omega)$, 1 .

1 Introduction

Given a polygonal domain Ω in \mathbb{R}^2 , we consider the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$
(1)

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where $f \in L^p_{\omega}(\Omega)$ and ω is a weight in the Muckenhoupt class $A_p(\mathbb{R}^2)$.

Estimates for this solution in the classical Sobolev spaces were given by Grisvard in [6] where we can see a dependence of the angles of Ω . Therefore, it is a natural question whether weighted a priori estimates are valid also for the solution of the Dirichlet problem (1). In this paper we give a positive answer to this question, namely, we prove that for 1 ,

$$\|u\|_{L^{p}_{\omega}(\Omega)} + \sum_{|\beta|=1} \|\rho(x)D^{\beta}_{x}u\|_{L^{p}_{\omega}(\Omega)} + \sum_{|\alpha|=2} \|\sigma(x)D^{\alpha}_{x}u\|_{L^{p}_{\omega}(\Omega)} \le C\|f\|_{L^{p}_{\omega}(\Omega)},$$
(2)

where $\rho(x)$ and $\sigma(x)$ are suitable functions depending on the distance from x to the nearest vertex of Ω and the corresponding angle, and C is a constant depending only on Ω .

The paper is organized as follows: In Section 2 we remind the already known estimates for the Green function (and its derivatives) of the problem (1) when Ω is a disk, and we define the Schwarz-Christoffel mapping. These will be the main tools for the proof of our main result (2). In Section 3 and Section 4 we state the estimates for the Green function and its derivatives when Ω is a convex and a non-convex polygon respectively. Finally, in Section 5 we give the proof of the estimate in (2).

2 Preliminaries

The solution of (1) is given by

$$u(x) = \int_{\Omega} G_{\Omega}(x, y) f(y) \, dy, \qquad (3)$$

where G_{Ω} is the Green function for Ω which can be written as

$$G_{\Omega}(x,y) = \Gamma(x-y) + H_{\Omega}(x,y), \qquad (4)$$

where

$$\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$$

and $H_{\Omega}(x, y)$ satisfies, for each fixed $y \in \Omega$,

$$\begin{cases} \Delta_x H_\Omega(x,y) = 0 & \text{in } \Omega\\ H_\Omega(x,y) = -\Gamma(x-y) & \text{on } \partial\Omega. \end{cases}$$

For a conformal mapping h from the unit disc B to Ω it holds that $\Delta(u \circ h) = |h'|^2 (\Delta u) \circ h$, where $|h'|^2$ is the Jacobian of h. Then, $u \circ h$ satisfies

$$\begin{cases} -\Delta(u \circ h) = |h'|^2 (f \circ h) & \text{in } B\\ u \circ h = 0 & \text{on } \partial B, \end{cases}$$

and for $\xi \in B$ we have

$$(u \circ h)(\xi) = \int_B G_B(\xi, \eta) \left(f \circ h\right)(\eta) |h'|^2 d\eta$$

where

$$G_B(\xi,\eta) = \frac{1}{2\pi} \log |\eta - \xi|^{-1} - \frac{1}{2\pi} \log \left(|\xi| \left| \eta - \frac{\xi}{|\xi|^2} \right| \right)^{-1}$$

is the Green function in B.

Let $g: \Omega \to B$ be the inverse mapping of h, then

$$G_{\Omega}(x,y) = G_B(\xi,\eta), \tag{5}$$

and

$$H_{\Omega}(x,y) = H_B(\xi,\eta), \tag{6}$$

where $\xi = g(x)$ and $\eta = g(y)$.

From the known estimates

$$|D_{\xi}^{\alpha}G_{B}(\xi,\eta)| \le C |\xi-\eta|^{-|\alpha|} \min\left\{1, \frac{d_{B}(\eta)}{|\xi-\eta|}\right\} \quad \text{for } |\alpha|=1,2,$$

(see for example [4]), where $d_B(\eta)$ denotes the distance from η to the boundary of B we have

$$|D_{\xi}^{\alpha}G_{B}(\xi,\eta)| \le C |\xi-\eta|^{-|\alpha|} \min\left\{1, \frac{d_{B}(\xi)}{|\xi-\eta|}\right\} \quad \text{for } |\alpha| = 1, 2.$$
(7)

Observe that the letter C denotes a generic constant not necessarily the same at each occurrence. We will write $f \leq g$ if there exists a constant C > 0 such that $f \leq C g$.

The Schwarz-Christoffel mapping. Given a polygonal domain Ω with N sides, for $j = 1, \dots, N$, we denote by z_j and θ_j its vertices and corresponding interior angles respectively. Let $k_j \in \mathbb{R}$ be such that $k_j \pi + \theta_j = \pi$. Observe that $0 < k_j < 1$ corresponds to $0 < \theta_j < \pi$ while $-1 < k_j \leq 0$ to $\pi \leq \theta_j < 2\pi$. In particular, if Ω is convex, all the k_j are positive numbers.

Given complex numbers w_j such that $|w_j| = 1$ for $j = 1, \dots, N$, we define, for $\xi \in B$,

$$h'(\xi) := (\xi - w_1)^{-k_1} (\xi - w_2)^{-k_2} \dots (\xi - w_N)^{-k_N},$$

which is analytic in the interior of B.

Then, the Schwarz-Christoffel mapping $h: B \to \Omega$ is defined as

$$h(\xi) = \int_{\xi_0}^{\xi} h'(s) \, ds,$$

where the integral is taken over the segment from a fixed $\xi_0 \in B$ to ξ . Note that h is analytic on the same region as h', continuous on B and maps the points inside the unit disk B to the points inside the simple closed polygon with vertex at $z_j = h(w_j)$. We will say that w_j are the pre-vertices of Ω . For more details about this mapping see, for example, [2].

We introduce $d_m = \min_{i \neq j} |w_i - w_j|$ and define $B_j = \overline{B(w_j, \frac{d_m}{4})} \cap B$, for $j = 1, \dots, N$, and $B_{N+1} = B \setminus \bigcup_{j=1}^N B_j$. Then, $\Omega_j = h(B_j)$ is a neighborhood of z_j and $\Omega = \bigcup_{j=1}^{N+1} \Omega_j$. We will analyze the behavior of the Green function G_{Ω} near each vertex z_j . The following remark outlines some useful observations.

Remark 1. For $\xi \in B_j$, with $j = 1, \dots, N$, we have

- 1. If $\eta \in B_j$ and s is in the segment from ξ to η , then $|s w_i| > \frac{d_m}{4}$ when $i \neq j$.
- 2. If $\eta \in B_i$ with $i \neq j$ and $i \neq N+1$, then $|\xi \eta| > \frac{d_m}{4}$.
- 3. If $\eta \in B_{N+1}$ and s is in the segment from ξ to η , then, either $|\xi \eta| > \frac{d_m}{8}$ or $|s - w_i| > \frac{d_m}{8}$, for all $i = 1, \dots, N$. For $\xi \in B_{N+1}$, we have
- 4. $|\xi w_i| > \frac{d_m}{4}$, for all $i = 1, \dots, N$.

3 The convex case

In this section we assume that Ω is a convex polygon. In this case the exponents defining the Schwarz-Christoffel mapping satisfy $0 < k_j < 1$.

Lemma 2. Let $\xi, \eta \in B_j$, with $j = 1, \dots, N$. Then if $k_j > 0$

$$|x-y| \leq |\xi-w_j|^{-k_j} |\xi-\eta|.$$

PROOF. By definition

$$h(\xi) - h(\eta) = \int_{\eta}^{\xi} h'(s) \, ds,$$
 (8)

where $h'(s) = (s - w_j)^{-k_j} \phi(s)$ for

$$\phi(s) = (s - w_1)^{-k_1} \dots (s - w_{j-1})^{-k_{j-1}} (s - w_{j+1})^{-k_{j+1}} \dots (s - w_N)^{-k_N}.$$

 ϕ is analytic in w_j and $|\phi(s)| \leq 1$. Moreover we can write

$$h'(s) = (s - w_j)^{-k_j} \phi(w_j) + (s - w_j)^{1 - k_j} \psi(s),$$

where ψ is analytic in B_j and $|\psi(s)| \leq 1$.

Then

$$|h(\xi) - h(\eta)| \leq |\eta - w_j|^{1-k_j} + |\xi - w_j|^{1-k_j} + |\xi - \eta|.$$

When $|\xi - w_j| \leq \frac{1}{2} |\eta - w_j|$ we have $\frac{1}{2} |\eta - w_j| \leq |\xi - \eta|$ and
 $|h(\xi) - h(\eta)| \leq |\eta - w_j|^{1-k_j} + |\xi - \eta|$
 $\leq |\xi - w_j|^{-k_j} |\xi - \eta|.$
When $|\xi - w_j| > \frac{1}{2} |\eta - w_j|$ and $|\xi - \eta| > \frac{1}{2} |\xi - w_j|$ we have

When $|\xi - w_j| > \frac{1}{2}|\eta - w_j|$ and $|\xi - \eta| > \frac{1}{2}|\xi - w_j|$ we have

$$|h(\xi) - h(\eta)| \leq |\xi - w_j|^{1-k_j} + |\xi - \eta|$$

 $\leq |\xi - w_j|^{-k_j} |\xi - \eta|.$

If $|\xi - \eta| \le \frac{1}{2} |\xi - w_j|$ we use that $|\xi - w_j| \le 2|s - w_j|$ for all s in the segment from ξ to η and then

$$|h(\xi) - h(\eta)| \leq \int_{\eta}^{\xi} |s - w_j|^{-k_j} \, ds \leq |\xi - w_j|^{-k_j} |\xi - \eta|$$

as we desire.

Remark 3. As a particular case of the previous lemma we obtain for $\xi \in B_j$ that

$$|x - z_j| \leq |\xi - w_j|^{1 - k_j},$$
(9)

with $j = 1, \dots, N$ and $k_j > 0$.

If Ω is a bounded domain, it was proved in [7] that

$$G_{\Omega}(x,y) \leq \log\left(1 + \frac{\min\{d_{\Omega}(x), d_{\Omega}(y)\}}{|x-y|}\right) \leq |x-y|^{-1},$$
 (10)

where $d_{\Omega}(x)$ denotes the distance from x to the boundary of Ω .

In order to have some estimates for the first and second order derivatives of $G_{\Omega}(x, y)$, using (5) we obtain

$$|D_x^{\alpha}G_{\Omega}(x,y)| \leq |D_{\xi}^{\alpha}G_B(\xi,\eta)| |g'(x)| \quad \text{for } |\alpha| = 1,$$
(11)

and

$$|D_{x}^{\alpha}G_{\Omega}(x,y)| \leq |D_{\xi}^{\alpha}G_{B}(\xi,\eta)| |g'(x)|^{2} + |D_{\xi}^{\beta}G_{B}(\xi,\eta)| |g''(x)| \quad \text{for } |\alpha| = 2,$$
(12)

where $|\beta| = 1$. We will use the following estimates for g:

$$|g'(x)| = \frac{1}{|h'(\xi)|} \leq |\xi - w_1|^{k_1} |\xi - w_2|^{k_2} \dots |\xi - w_N|^{k_N} \leq |\xi - w_j|^{k_j}$$
(13)

and

$$|g''(x)| \leq |\xi - w_1|^{k_j - 1} |g'(x)| \leq |\xi - w_j|^{2k_j - 1},$$
(14)

for $x \in \Omega_j$, with j = 1, ..., N.

Lemma 4. Let $x, y \in \Omega$ and $|\alpha| = 1$. Then we have

$$|D_x^{\alpha}G_{\Omega}(x,y)| \leq |x-y|^{-1}.$$

PROOF. Consider first $x \in \Omega_j$, with j = 1, ..., N. For $y \in \Omega_j$ we have that

$$|D_{x}^{\alpha}G_{\Omega}(x,y)| \leq |D_{\xi}^{\alpha}G_{B}(\xi,\eta)| |g'(x)| \leq |\xi-\eta|^{-1} |\xi-w_{j}|^{k_{j}} \leq |x-y|^{-1}$$

by (11), (7), (13) and Lemma 2.

For $y \in (\Omega_j \cup \Omega_{N+1})^c$, recalling that $|\xi - \eta| > \frac{d_m}{4}$, we have

$$|D_x^{\alpha}G_{\Omega}(x,y)| \leq |\xi - \eta|^{-1} |\xi - w_j|^{k_j} \leq 1.$$

For $y \in \Omega_{N+1}$, it only remains to see the case when $\frac{d_m}{8} < |s - w_i| \le 1$, for i = 1, ..., N and s is in the segment from ξ to η . But there $|g'(x)| \le 1$ and $|h'(x)| \le 1$, then

$$|D_x^{\alpha}G_{\Omega}(x,y)| \leq |\xi - \eta|^{-1} \leq |x - y|^{-1}.$$

Finally, if $x \in \Omega_{N+1}$, we have $\frac{d_m}{4} < |\xi - w_i| \le 1$ for all i = 1, ..., N. Therefore $|x - y| \le |\xi - \eta|$ and

$$|D_x^{\alpha}G_{\Omega}(x,y)| \leq |D_{\xi}^{\alpha}G_B(\xi,\eta)| |g'(x) \leq |x-y|^{-1}.$$

In the following two lemmas we analyze separately each term of (12) to obtain estimates for the second order derivatives of $G_{\Omega}(x, y)$.

Lemma 5. Let $x \in \Omega_j$, with j = 1, ..., N and $|\beta| = 1$. Then we have:

1.
$$|x - z_j|^{1-a} |D_{\xi}^{\beta} G_B(\xi, \eta)| |g''(x)| \leq |x - y|^{-1-a}, \text{ if } y \in \Omega_j \text{ and } 0 \leq a < 1.$$

2. $|D_{\xi}^{\beta} G_B(\xi, \eta)| |g''(x)| \leq 1, \text{ if } y \in (\Omega_j \cup \Omega_{N+1})^c.$

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3. $|D_{\xi}^{\beta}G_{B}(\xi,\eta)| |g''(x)| \leq |x-y|^{-1}, \text{ if } y \in \Omega_{N+1}.$

PROOF. (1) If $y \in \Omega_j$ and $|x - y| \le |x - z_j|$, we have for any $a \ge 0$

$$|D_{\xi}^{\beta}G_{B}(\xi,\eta)| |g''(x)| \leq |\xi - \eta|^{-1} |\xi - w_{j}|^{2k_{j}-1}$$
$$\leq |x - y|^{-1-a} |x - z_{j}|^{a} |\xi - w_{j}|^{k_{j}-1}$$
$$\leq |x - y|^{-1-a} |x - z_{j}|^{a-1},$$

by (7), (14), Lemma 2 and (9).

On the other hand, if $|x - y| > |x - z_j|$, we have for $0 \le a < 1$

$$|D_{\xi}^{\beta}G_{B}(\xi,\eta)| |g''(x)| \leq \frac{d_{B}(\xi)}{|\xi-\eta|^{2}} |\xi-w_{j}|^{2k_{j}-1}$$
$$\leq |x-y|^{-2}$$
$$\leq |x-z_{j}|^{-1+a} |x-y|^{-1-a}.$$

by (7), (14) and Lemma 2.

(2) If $y \in (\Omega_j \cup \Omega_{N+1})^c$, since $|\xi - \eta| > \frac{d_m}{4}$, we obtain

$$|D_{\xi}^{\beta}G_{B}(\xi,\eta)| |g^{''}(x)| \leq \frac{d_{B}(\xi)}{|\xi-\eta|^{2}} |\xi-w_{j}|^{2k_{j}-1} \leq |\xi-w_{j}|^{2k_{j}} \leq 1.$$

(3) For $y \in \Omega_{N+1}$, it remains to consider the case when $\frac{d_m}{8} < |s - w_i| \le 1$, for i = 1, ..., N and s is in the segment from ξ to η . But there $|g''(x)| \le 1$ and

$$|D_{\xi}^{\beta}G_{B}(\xi,\eta)| |g''(x)| \leq |\xi-\eta|^{-1} \leq |x-y|^{-1}.$$

Lemma 6. Let $x \in \Omega_j$, with j = 1, ..., N and $|\alpha| = 2$. Then we have:

1.
$$|D_{\xi}^{\alpha}G_{B}(\xi,\eta)| |g'(x)|^{2} \leq \frac{d_{\Omega}(x)}{|x-y|^{3}}, \text{ if } y \in \Omega_{j} \cup \Omega_{N+1}.$$

2.
$$|D_{\xi}^{\alpha}G_B(\xi,\eta)| |g'(x)|^2 \leq 1, \text{ if } y \in (\Omega_j \cup \Omega_{N+1})^c.$$

PROOF. (1) If $y \in \Omega_j$ we have that

$$|D_{\xi}^{\alpha}G_{B}(\xi,\eta)| |g'(x)|^{2} \leq \frac{d_{B}(\xi)}{|\xi-\eta|^{3}} |\xi-w_{j}|^{2k_{j}} \leq \frac{d_{B}(\xi)}{|x-y|^{3}} |\xi-w_{j}|^{-k_{j}},$$

by (7), (13) and Lemma 2.

Let now $X_0 \in \partial \Omega$ such that $d_{\Omega}(x) = |x - X_0|$ and $Q_0 \in \partial B$ with $g(X_0) = Q_0$. Then there exists ξ_0 in the segment from x to X_0 and $\eta_0 = g(\xi_0)$ such that

$$d_B(\xi) \le |g'(\xi_0)| |x - X_0| \le |\eta_0 - w_j|^{k_j} d_\Omega(x).$$
(15)

Therefore

$$|D_{\xi}^{\alpha}G_{B}(\xi,\eta)| |g'(x)|^{2} \leq \frac{d_{\Omega}(x)}{|x-y|^{3}} |\eta_{0} - w_{j}|^{k_{j}} |\xi - w_{j}|^{-k_{j}},$$

and for each $1 \leq i \leq M$, there exists ξ_i in the segment from ξ_{i-1} to z_j such that

$$|\eta_{i-1} - w_j|^{k_j} = |g(\xi_{i-1}) - g(z_j)| \le |g'(\xi_i)| |\xi_i - z_j| \le |\eta_i - w_j|^{k_j} |\xi_i - z_j|.$$

By iterating, we have

$$\begin{aligned} |\eta - w_j| &\leq |\eta_1 - w_j|^{k_j} |\xi_0 - z_j| \\ &\leq |\eta_2 - w_j|^{k_j^2} |\xi_1 - z_j|^{k_j} |\xi_0 - z_j| \\ &\leq |\eta_3 - w_j|^{k_j^3} |\xi_2 - z_j|^{k_j^2} |\xi_1 - z_j|^{k_j} |\xi_0 - z_j| \\ &\cdots \\ &\leq |\eta_M - w_j|^{k_j^M} \cdots |\xi_2 - z_j|^{k_j^2} |\xi_1 - z_j|^{k_j} |\xi_0 - z_j| \\ &\leq |x - z_j|^{k_j^M} \cdots |x - z_j|^{k_j^2} |x - z_j|^{k_j} |x - z_j|, \end{aligned}$$

where we used that $|\xi_i - z_j| \leq |x - z_j|$ and $|\eta_i - w_j| \leq |x - z_j|$.

Note that the implicit constant involved in \preceq above does not depend on M. In fact, by (13) and (9)

$$|g'(\xi_i)| \leq |\eta_i - w_j|^{k_j} \left(\frac{d_m}{4}\right)^p,$$

where $p = \sum_{k_j < 0} k_j$ and we have that

$$\left(\frac{d_m}{4}\right)^{p\sum_{n=0}^M k_j^n} \le \left(\frac{d_m}{4}\right)^{p\sum_{n=0}^\infty k_j^n} < \infty.$$

Therefore

$$|D_{\xi}^{\alpha}G_{B}(\xi,\eta)| |g'(x)|^{2} \leq \frac{d_{\Omega}(x)}{|x-y|^{3}} |x-z_{j}|^{\beta} |\xi-w_{j}|^{-k_{j}},$$

where $\beta = \sum_{n=1}^{M+1} k_j^n = k_j \left(\frac{1-k_j^{M+2}}{1-k_j}\right)$. Taking $\gamma = \frac{k_j}{1-k_j}$, by (9), it follows that

$$|x - z_j|^{-\beta + \gamma} |D_{\xi}^{\alpha} G_B(\xi, \eta)| |g'(x)|^2 \preceq \frac{d_{\Omega}(x)}{|x - y|^3}.$$

Then, given $\varepsilon > 0$ there exists M large enough such that $-\beta + \gamma < \varepsilon$ and taking ε tending to zero

$$|D^{\alpha}_{\xi}G_{B}(\xi,\eta)| |g'(x)|^{2} \leq \frac{d_{\Omega}(x)}{|x-y|^{3}}.$$

For $y \in \Omega_{N+1}$, we consider only the case when $\frac{d_m}{8} < |s - w_i| \le 1$, for i = 1, ..., N and s is in the segment from ξ to η (the other case will be considered in (2)). In this case, $|x - y| \le |\xi - \eta|$ and

$$|D_{\xi}^{\alpha}G_{B}(\xi,\eta)| |g'(x)|^{2} \leq \frac{d_{B}(\xi)}{|\xi-\eta|^{3}} |\xi-w_{j}|^{2k_{j}} \leq \frac{d_{\Omega}(x)}{|x-y|^{3}} |\eta-w_{j}|^{k_{j}} \leq \frac{d_{\Omega}(x)}{|x-y|^{3}}.$$

(2) If $y \in (\Omega_j \cup \Omega_{N+1})^c$, since $|\xi - \eta| > \frac{d_m}{4}$, we obtain

$$|D_{\xi}^{\alpha}G_{B}(\xi,\eta)| |g'(x)|^{2} \leq |\xi-\eta|^{-2} |\xi-w_{j}|^{2k_{j}} \leq |\xi-w_{j}|^{2k_{j}} \leq 1.$$

4 The non-convex case

In this section we assume that Ω is a nonconvex polygon. In this case the exponents defining the Schwarz-Christoffel mapping can be negative, i.e. there exists at least one j = 1, ..., N such that $-1 < k_j \leq 0$.

Lemma 7. Let $\xi, \eta \in B_j$, with j = 1, ..., N. Then if $k_j \leq 0$

$$|x-y| \preceq |u-v|.$$

PROOF. As $k_j \leq 0$ we have $|s - w_j|^{-k_j} \leq 1$ and by (8)

$$|h(u) - h(v)| \leq \int_{v}^{u} |s - w_j|^{-k_j} |\phi(s)| \, ds$$
$$\leq |u - v|.$$

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To complete the study of the first and second order derivatives of $G_{\Omega}(x, y)$ for the non-convex case we need to obtain estimates when $-1 < k_j \leq 0$. To do this, we use (11), (12), (13) and (14) as in the convex case.

Lemma 8. Let $x \in \Omega_j$, with j = 1, ..., N, $y \in \Omega$ and $|\alpha| = 1$. Then we have

$$|x-z_j|^{1-\frac{\pi}{\theta_j}}|D_x^{\alpha}G_{\Omega}(x,y)| \leq |x-y|^{-1}.$$

PROOF. For $y \in \Omega_j$ we have that

$$|D_x^{\alpha}G_{\Omega}(x,y)| \leq |g'(x)||\xi - \eta|^{-1} \leq |\xi - w_j|^{k_j}|\xi - \eta|^{-1} \leq |\xi - w_j|^{k_j}|x - y|^{-1},$$

by (11), (7), (13) and Lemma 7. Taking $\gamma := \frac{-k_j}{(1-k_j)} = 1 - \frac{\pi}{\theta_j} > 0$ it follows from (9) that $|x - z_j|^{\gamma} \leq |\xi - w_j|^{(1-k_j)\gamma}$ and

$$|x-z_j|^{\gamma}|D_x^{\alpha}G_{\Omega}(x,y)| \leq |x-y|^{-1},$$

as we wanted to prove.

For $y \in (\Omega_j \cup \Omega_{N+1})^c$ we have

$$|D^{\alpha}_{\xi}G_B(\xi,\eta)| |g'(x)| \leq |\xi - w_j|^{k_j}$$

and we obtain the desired inequality as before.

For $y \in \Omega_{N+1}$ the proof is analogous to the case $0 < \theta_j < \pi$.

Analogously to the convex case, we analyze separately each term of (12) to obtain estimates for the second order derivatives of $G_{\Omega}(x, y)$.

Lemma 9. Let $x \in \Omega_j$ with j = 1, ..., N and $|\beta| = 1$. Then we have:

1.
$$|x - z_j|^{2 - \frac{\pi}{\theta_j}} |D_{\xi}^{\beta} G_B(\xi, \eta)| |g''(x)| \leq |x - y|^{-1}, \text{ if } y \in \Omega_j.$$

2. $|x - z_j|^{2 - \frac{\pi}{\theta_j}} |D_{\xi}^{\beta} G_B(\xi, \eta)| |g''(x)| \leq 1, \text{ if } y \in (\Omega_j \cup \Omega_{N+1})^c$
3. $|D_{\xi}^{\beta} G_B(\xi, \eta)| |g''(x)| \leq |x - y|^{-1}, \text{ if } y \in \Omega_{N+1}.$

PROOF. (1) If $y \in \Omega_i$ we have that

$$|D_{\xi}^{\beta}G_{B}(\xi,\eta)||g''(x)| \leq |\xi-\eta|^{-1}|\xi-w_{j}|^{2k_{j}-1} \leq |x-y|^{-1}|\xi-w_{j}|^{2k_{j}-1},$$

by (7), (14) and Lemma 7. Taking $\gamma = \frac{1-2k_j}{(1-k_j)} = 2 - \frac{\pi}{\theta_j}$ it follows from (9) that $|x - z_j|^{\gamma} \leq |\xi - w_j|^{-2k_j+1}$ and

$$|x - z_j|^{\gamma} |D_{\xi}^{\beta} G_B(\xi, \eta)| |g''(x)| \leq |x - y|^{-1}.$$

(2) If
$$y \in (\Omega_j \cup \Omega_{N+1})^c$$
,
 $|D_{\xi}^{\beta}G_B(\xi,\eta)| |g''(x)| \leq |\xi - \eta|^{-1} |\xi - w_j|^{2k_j - 1} \leq |\xi - w_j|^{2k_j - 1}$

and the result follows in the same way as above.

(3) For $y \in \Omega_{N+1}$ and ξ and η are at a distance from the pre-vertex of Ω greater than $\frac{d_m}{8}$, $|x - y| \leq |\xi - \eta|$ and

$$|D_{\xi}^{\beta}G_{B}(\xi,\eta)| |g''(x)| \leq |\xi-\eta|^{-1} |\xi-w_{j}|^{2k_{j}-1} \leq |x-y|^{-1}.$$

Lemma 10. Let $x \in \Omega_j$ with j = 1, ..., N and $|\alpha| = 2$. Then we have:

1.
$$|x - z_j|^{2 - \frac{\pi}{\theta_j}} |D_{\xi}^{\alpha} G_B(\xi, \eta)| |g'(x)|^2 \leq \frac{d_{\Omega}(x)}{|x - y|^3}, \text{ if } y \in \Omega_j \cup \Omega_{N+1} \text{ and } x$$

such that $d_{\Omega}(x) \leq \frac{1}{2}|x - z_j|.$

2. $|x-z_j|^{a+2-2\frac{\pi}{\theta_j}} |D_{\xi}^{\alpha}G_B(\xi,\eta)| |g'(x)|^2 \leq |x-y|^{-2+a}, \text{ if } y \in \Omega_j \cup \Omega_{N+1}, x$ such that $\frac{1}{2}|x-z_j| < d_{\Omega}(x) \leq |x-y|$ and a > 0.

3.
$$|x - z_j|^{2 - \frac{\pi}{\theta_j}} |D_{\xi}^{\alpha} G_B(\xi, \eta)| |g'(x)|^2 \leq 1, \text{ if } y \in (\Omega_j \cup \Omega_{N+1})^c.$$

PROOF. (1) If $y \in \Omega_j$ and $d_{\Omega}(x) \leq \frac{1}{2}|x-z_j|$, we have that

$$|D_{\xi}^{\alpha}G_{B}(\xi,\eta)| |g'(x)|^{2} \leq \frac{d_{B}(\xi)}{|\xi-\eta|^{3}} |\xi-w_{j}|^{2k_{j}} \leq \frac{d_{\Omega}(x)}{|x-y|^{3}} |\eta-w_{j}|^{k_{j}} |\xi-w_{j}|^{2k_{j}},$$
(16)

by (7), (13), Lemma 7 and (15), where $h(\eta) = \xi$ is in the segment form x to X_0 .

 X_0 . Taking $\gamma = \frac{-2k_j}{1-k_j}$ and $\beta = \frac{-k_j}{1-k_j}$, by (9) it follows that

$$|\xi - z_j|^{\beta} |x - z_j|^{\gamma} |D_{\xi}^{\alpha} G_B(\xi, \eta)| |g'(x)|^2 \preceq \frac{d_{\Omega}(x)}{|x - y|^3}.$$

Since $\gamma + \beta < 2 - \frac{\pi}{\theta_j}$ it is enough to prove that $|x - z_j| \leq |\xi - z_j|$ provided that $d_{\Omega}(x) \leq \frac{1}{2}|x - z_j|$.

We will consider the following two cases:

If $|x - \xi| \leq \frac{1}{4}|x - z_j|$ the result follows directly.

If $|x - \xi| > \frac{1}{4}|x - z_j|$ we also have that $\frac{1}{2}|x - z_j| \le |X_0 - z_j|$. Then $\frac{1}{2}|x - z_j| \le |X_0 - z_j| \le d_B(\xi) + |\xi - z_j| \le 2|\xi - z_j|$ as we desire.

(2) If
$$y \in \Omega_j$$
 and $\frac{1}{2}|x - z_j| < d_{\Omega}(x) < |x - y|$, we have for any $a > 0$
 $|D_u^{\alpha}G_B(\xi,\eta)| |g'(x)|^2 \leq |\xi - \eta|^{-2} |\xi - w_j|^{2k_j} \leq |x - y|^{-2+a} |x - z_j|^{-a} |\xi - w_j|^{2k_j}$,
(17)

by (7), (13) and Lemma 7. Taking $\gamma = \frac{-2k_j}{1-k_j}$, by (9) it follows that

$$|x - z_j|^{a + \gamma} |D_u^{\alpha} G_B(\xi, \eta)| |g'(x)|^2 \leq |x - y|^{-2 + a}$$

For $y \in \Omega_{N+1}$ and ξ and η at a distance from the pre-vertex of Ω greater than $\frac{d_m}{8}$ (the other case will be considered in (3)), $|x-y| \leq |\xi-\eta|$ and consider again the previous two cases using that $|\xi-w_j|^{2k_j}$ in (16) and (17) is bounded. (3) Since $|\xi-\eta| > \frac{d_m}{4}$ we obtain

$$|D_u^{\alpha}G_B(\xi,\eta)| |g'(x)|^2 \leq |\xi-\eta|^{-2} |\xi-w_j|^{2k_j} \leq |\xi-w_j|^{2k_j}$$

and the result follows in the same way that (16).

To complete the study of the behavior of the second order derivatives of the Green function G_{Ω} , it suffices to consider $x \in \Omega_{N+1}$. In this case there is no relation to the vertex of Ω as we prove in the following lemma:

Lemma 11. Let $x \in \Omega_{N+1}$ and $y \in \Omega$. Then we have:

1. For $|\beta| = 1$ $|D_u^{\beta}G_B(\xi, \eta)| |g^{''}(x)| \leq |x - y|^{-1}.$ 2. For $|\alpha| = 2$ and $d_{\Omega}(x) \leq |x - y|$

$$|D_{u}^{\alpha}G_{B}(\xi,\eta)| |g'(x)|^{2} \preceq \frac{d_{\Omega}(x)}{|x-y|^{3}}.$$

PROOF. (1) For $|\beta| = 1$ we have that

$$|D_u^{\beta} G_B(\xi,\eta)| |g''(x)| \leq |\xi - \eta|^{-1},$$

by (14) and using that $\frac{d_m}{4} < |\xi - w_i| \le 1$ for i = 1, ..., N. Moreover, we have by Lemma 2 that $|x - y| \le |\xi - \eta|$ and the result follows directly.

(2) For $|\alpha| = 2$ we have that

$$|D_u^{\alpha}G_B(\xi,\eta)| |g'(x)|^2 \preceq \frac{d_B(\xi)}{|\xi-\eta|^3} |\xi-w_j|^{k_j} \preceq \frac{d_{\Omega}(x)}{|x-y|^3} |\eta-w_j|^{k_j},$$

where we are assuming as in (15) that w_j is the pre-vertex closest to η .

If $k_j > 0$ we have $|\eta - w_j|^{k_j} \leq 1$ as we desired.

If $k_j \leq 0$ we can follow the proof of (2) of Lemma 6 and we consider the cases $d_{\Omega}(x) \leq \frac{1}{2}|x-z_j|$ and $\frac{1}{2}|x-z_j| < d_{\Omega}(x) < |x-y|$ respectively using also that, when $x \in \Omega_{N+1}$, $|x-z_j| > \frac{d_m}{4}$.

5 Main result

Let us recall that the solution of the problem (1) is given by

$$u(x) = \int_{\Omega} G_{\Omega}(x, y) f(y) \, dy,$$

where $G_{\Omega}(x, y) = \Gamma(x, y) + H_{\Omega}(x, y)$.

In order to prove our main result (2) we also need to establish some estimates for the second order derivatives of H_{Ω} and Γ .

Applying the same ideas used in the previous section for G_{Ω} , by (6), we have for $|\beta| = 1$ and $|\alpha| = 2$

$$|D_x^{\alpha} H_{\Omega}(x,y)| \leq |D_{\xi}^{\alpha} H_B(\xi,\eta)| |g'(x)|^2 + |D_{\xi}^{\beta} H_B(\xi,\eta)| |g''(x)|.$$

Moreover, as B is a smooth bounded domain, we have

$$|D_{\xi}^{\beta}H_B(\xi,\eta)| \le C \, d_B(\xi)^{-1} \text{ and } |D_{\xi}^{\alpha}H_B(\xi,\eta)| \le C \, d_B(\xi)^{-2},$$
 (18)

(see Lemma 2.1 in [5]) and we have the following lemma:

Lemma 12. Let $y \in \Omega$ and $|\alpha| = 2$. Then

1. For $x \in \Omega_j$, with j = 1, ..., N we have:

(a)
$$|D_x^{\alpha} H_{\Omega}(x,y)| \leq d_{\Omega}(x)^{-2}, \text{ if } 0 < k_j < 1.$$

(b) $|x - z_j|^{2 - \frac{\pi}{\theta_j}} |D_x^{\alpha} H_{\Omega}(x,y)| \leq d_{\Omega}(x)^{-2}, \text{ if } -1 < k_j \leq 0.$

2. For $x \in \Omega_{N+1}$ we have:

$$|D_x^{\alpha} H_{\Omega}(x,y)| \leq d_{\Omega}(x)^{-2}.$$

PROOF. (1) For $x \in \Omega_j$, let $X_0 \in \partial \Omega$ such that $g(X_0) = \xi_0$, with $d_B(\xi) = |\xi - \xi_0|$. Then there exists η in the segment from ξ to ξ_0 such that

$$d_{\Omega}(x) \le |h'(\eta)| |\xi - \xi_0| \le |\eta - w_j|^{-k_j} d_B(\xi).$$
(19)

Consider first $0 < k_j < 1$. It follows from (18), (13), (14) and (19) that

$$|D_{\xi}^{\alpha}H_{B}(\xi,\eta)| |g'(x)|^{2} \leq d_{B}(\xi)^{-2}|\xi - w_{j}|^{2k_{j}} \leq |\eta - w_{j}|^{-2k_{j}} d_{\Omega}(x)^{-2}|\xi - w_{j}|^{2k_{j}}$$
(20)

and

$$|D_{\xi}^{\beta}H_{B}(\xi,\eta)| |g^{''}(x)| \leq d_{B}(\xi)^{-1} |\xi - w_{j}|^{2k_{j}-1} \leq |\eta - w_{j}|^{-k_{j}} d_{\Omega}(x)^{-1} |\xi - w_{j}|^{2k_{j}-1}.$$
(21)

If we also consider $|\eta - w_j| > \frac{1}{2} |\xi - w_j|$ we have

$$|D_{\xi}^{\alpha}H_{B}(\xi,\eta)||g'(x)|^{2} + |D_{\xi}^{\beta}H_{B}(\xi,\eta)||g''(x)| \leq d_{\Omega}(x)^{-2} + d_{\Omega}(x)^{-1}|x-z_{j}|^{-1} \leq d_{\Omega}(x)^{-2},$$

by (20), (21) and (9).

If $|\eta - w_j| \leq \frac{1}{2}|\xi - w_j|$, we can see that $d_B(\xi) \geq \frac{1}{2}|\xi - w_1|$. Then we have

$$|D_{\xi}^{\alpha}H_{B}(\xi,\eta)| |g'(x)|^{2} + |D_{\xi}^{\beta}H_{B}(\xi,\eta)| |g''(x)| \leq |x-z_{j}|^{-2} \leq d_{\Omega}(x)^{-2},$$

by (20), (21) and (9).

Now, consider $-1 < k_j \leq 0$. By (19) we obtain $d_{\Omega}(x) \leq d_B(\xi)$ and taking $\gamma_1 := \frac{-2k_j}{1-k_j} < 2 - \frac{\pi}{\theta_j}$ and $\gamma_2 := \frac{1-2k_j}{1-k_j} = 2 - \frac{\pi}{\theta_j}$, it follows from (20), (21) and (9) that

$$|x - z_j|^{\gamma_1} |D_{\xi}^{\alpha} H_B(\xi, \eta)| |g'(x)|^2 \leq d_{\Omega}(x)^{-2}$$

and

$$|x-z_j|^{\gamma_2}|D_{\xi}^{\beta}H_B(\xi,\eta)||g''(x)| \leq d_{\Omega}(x)^{-1},$$

as we desired.

(2) If $x \in \Omega_{N+1}$ we have $|g'(x)| \leq 1, |g''(x)| \leq 1$ and $|D_x^{\alpha} H_B(x, y)| \leq d_B(\xi)^{-2} + d_B(\xi)^{-1}.$

In order to prove that $d_{\Omega}(x) \leq d_B(\xi)$ we consider two cases depending on the k_j associated with the pre-vertex w_j closest to η given by (19).

If $k_j \leq 0$ the proof follows directly and if $k_j > 0$ we have to consider $|\eta - w_j| > \frac{1}{2} |\xi - w_j| > \frac{d_m}{8}$ and $|\eta - w_j| \leq \frac{1}{2} |\xi - w_j|$ as above. \Box

With respect to Γ , since $|D_x^{\beta}\Gamma(x)| \leq C|x|^{1-n}$ for $|\beta| = 1$, we have

$$D_x^\beta \int_\Omega \Gamma(x-y) f(y) \, dy = \int_\Omega D_x^\beta \Gamma(x-y) f(y) \, dy.$$

However, for $|\alpha| = 2$, $D_x^{\alpha}\Gamma$ is not an integrable function and we cannot interchange the order between second derivatives and integration. A known standard argument shows that for $|\delta| = |\beta| = 1$

$$D_x^{\delta} \int_{\Omega} D_x^{\beta} \Gamma(x-y) f(y) \, dy = K f(x) + c(x) f(x),$$

where c is a bounded function and

$$Kf(x) = \lim_{\epsilon \to 0} \int_{|x-y| > \epsilon} D_x^{\alpha} \Gamma(x-y) f(y) \, dy$$

is a Calderón-Zygmund operator. Indeed, since $D_x^{\beta}\Gamma \in C^{\infty}(\mathbb{R}^2 \setminus \{0\})$ and it is a homogeneous function of degree -1, it follows that $D_x^{\alpha}\Gamma_{\Omega}(x-y)$ is homogeneous of degree -2 and has vanishing average on the unit sphere (see Lemma 11.1 in [1, page 152]). Then, it follows from the general theory given in [3] that K is a bounded operator in L^p for 1 .

Moreover, the maximal operator

$$\widetilde{K}f(x) = \sup_{\epsilon>0} \left| \int_{|x-y|>\epsilon} D_x^{\alpha} \Gamma_{\Omega}(x-y) f(y) \, dy \right|$$

is also bounded in L^p for 1 .

Our main result is a consequence of the next proposition which follows the same ideas of Lemma 2.3 in [5].

Proposition 13. Let u be a solution of (1) and let ρ and σ be the functions given by

$$\rho(x) := \begin{cases} |x - z_j|^{1 - \frac{\pi}{\theta_j}} & \text{for } x \in \Omega_j \text{ and } \pi \le \theta_j < 2\pi \\ 1 & \text{for either } x \in \Omega_j \text{ and } 0 < \theta_j < \pi \text{ or } x \in \Omega_{N+1}, \end{cases}$$

and

$$\sigma(x) := \begin{cases} |x - z_j|^{2 - \frac{\pi}{\theta_j}} & \text{for } x \in \Omega_j \text{ and } \pi \le \theta_j < 2\pi \\ |x - z_j|^{1 - a} & \text{for } x \in \Omega_j \text{ and } 0 < \theta_j < \pi \\ 1 & \text{for } x \in \Omega_{N+1}, \end{cases}$$

with $0 \leq a < 1$.

Then for any $x \in \Omega$, $|\beta| = 1$ and $|\alpha| = 2$ we have

$$|u(x)| + |\rho(x)D_x^{\beta}u(x)| \leq Mf(x),$$

$$|\sigma(x)D_x^{\alpha}u(x)| \preceq \widetilde{K}f(x) + Mf(x) + |f(x)|,$$

where Mf(x) is the usual Hardy-Littlewood maximal function of f.

PROOF. Calling δ the diameter of Ω

$$|u(x)| \preceq \int_{|x-y| \le \delta} \frac{|f(y)|}{|x-y|^{-1}} \, dy = \sum_{k=0}^{\infty} \int_{\{2^{-(k+1)}\delta \le |x-y| \le 2^{-k}\delta\}} \frac{|f(y)|}{|x-y|} \, dy$$

by (3) and (10). Then, it follows that

$$|u(x)| \preceq Mf(x)$$

(see Lemma 2.8.3 in [9, page 85] for details).

Analogously, from Lemma 4 and Lemma 8 we obtain

$$|\rho(x)D_x^\beta u(x)| \preceq Mf(x).$$

On the other hand, by (3) and (4) we obtain

$$\begin{split} \sigma(x)D_x^{\alpha}u(x) &= \lim_{\epsilon \to 0} \int_{\epsilon < |x-y| \le d_{\Omega}(x)} \sigma(x) D_x^{\alpha} \Gamma(x-y) f(y) \, dy + cf(x) \\ &+ \int_{|x-y| \le d_{\Omega}(x)} \sigma(x) D_x^{\alpha} H_{\Omega}(x,y) f(y) \, dy \\ &+ \int_{|x-y| > d_{\Omega}(x)} \sigma(x) D_x^{\alpha} G_{\Omega}(x,y) f(y) \, dy \\ &:= I + II + III + IV. \end{split}$$

Now, we have

$$|I| \le |Kf(x)| + \widetilde{K}f(x) \le 2\widetilde{K}f(x).$$

Since c is a bounded function we have $|II| \leq f(x)$. Therefore, we only need to estimate the last two terms. By Lemma 12 and as $\sigma(x) \leq 1$ for $x \in \Omega_j$ with $0 < k_j < 1$ it holds that

$$\int_{|x-y| \le d_{\Omega}(x)} \sigma(x) D_x^{\alpha} H_{\Omega}(x,y) f(y) \, dy \preceq d_{\Omega}(x)^{-2} \int_{|x-y| \le d_{\Omega}(x)} |f(y)| \, dy$$
$$\preceq M f(x).$$

Finally, by the results given by Lemma 5, Lemma 6, Lemma 9, Lemma 10 and Lemma 11, it follows that

$$\int_{|x-y| > d_{\Omega}(x)} \sigma(x) D_x^{\alpha} G_{\Omega}(x,y) f(y) \, dy \preceq M f(x)$$

and the proposition is proved.

We can now state and prove our main result. First we recall the definition of the $A_p(\mathbb{R}^2)$ class for 1 . A non-negative locally integrable function $<math>\omega$ belongs to $A_p(\mathbb{R}^2)$ if there exists a constant C such that

$$\left(\frac{1}{|Q|} \int_Q \omega(x) \, dx\right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)} \, dx\right)^{p-1} \le C$$

for every cube $Q \subset \mathbb{R}^2$.

For any weight $\omega,\,L^p_\omega(\Omega)$ is the space of measurable functions f defined in Ω such that

$$||f||_{L^p_{\omega}(\Omega)} = \left(\int_{\Omega} |f(x)|^p \,\omega(x) \,dx\right)^{1/p} < \infty$$

and $W^{k,p}_{\omega}(\Omega)$ is the space of functions such that

$$\|f\|_{W^{k,p}_{\omega}(\Omega)} = \left(\sum_{|\alpha| \le k} \|D^{\alpha}f\|_{L^p_{\omega}(\Omega)}^p\right)^{1/p} < \infty.$$

Theorem 14. Let Ω be a polygonal domain in \mathbb{R}^2 . Let u be a solution of (1) with $f \in L^p_{\omega}(\Omega), 1 .$

Then, for $\omega \in A_p(\mathbb{R}^2)$, we have

$$\|u\|_{L^p_{\omega}(\Omega)} + \sum_{|\beta|=1} \|\rho(x)D^{\beta}_x u\|_{L^p_{\omega}(\Omega)} + \sum_{|\alpha|=2} \|\sigma(x)D^{\alpha}_x u\|_{L^p_{\omega}(\Omega)} \leq \|f\|_{L^p_{\omega}(\Omega)},$$

where $\rho(x)$ and $\sigma(x)$ are the functions defined in Proposition 13.

PROOF. Taking $\Omega = \bigcup_{j=1}^{N+1} \Omega_j$, since M and \widetilde{K} are bounded operators in L^p_{ω} (see [8, Chapter V]), the proof is a consequence of Proposition 13.

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