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ON THE MODELING OF MIGRATION PHENOMENA ON SMALL NETWORKS

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This paper deals with the modeling of migration phenomena in a small network of nations, with the aim of investigating the influence that the wealth and the welfare policies have on this phenomena. The modeling approach is based on the kinetic theory of active particles, while individuals over the network are distinguished by a scalar variable (the activity) which expresses their social state. The dynamics is induced both by the communication of individuals over the network and by the welfare policy within each nation, which is expressed in terms of competitive and altruistic interactions. The evolution of the discrete probability distribution over the social state is described by a system of nonlinear ordinary differential equations. The existence and uniqueness of the solution is discussed and some specific case-studies are proposed in order to carry out simulations and to investigate the emerging behaviors.

Keywords: Kinetic theory; active particles; migration phenomena; population models.

1. Introduction

Since the very origins of human history, individuals were encouraged to overcome adversity and to get a better quality of life. In order to achieve these goals, migration has been so far expressed not only by individuals but also by entire communities. As it is stated in Ref. 33, today globalization, together with advances in communications and transportation, has greatly increased the number of people who have the desire, the necessity and the capacity to move to other places. This report by the United Nations, together with Ref. 25, are valuable references that describe many aspects of nowadays migration.

Although there is an emerging consensus that countries can cooperate to create triple wins, for migrants, for their countries of origin and for the societies that receive them, migration is still a major subject of debate in many senses. In this context, a better understanding of the phenomena and the evolution of the involved societies is needed.

This paper presents a modeling approach of migration phenomena in a complex network,[19] which can be defined as a collection of nodes connected by edges representing various complex interactions among the nodes. In this specific case, the network is composed by nations, where the dynamics is induced both by the communication of individuals over the network and by the welfare policies. A detailed description of the class of systems under consideration is given in Sec. 2.

The modeling approach is based on the tools of the so-called *kinetic theory of active particles*, briefly KTAP, whose essential features can be summarized as follows: the overall system is subdivided into *functional subsystems* constituted by entities, called *active particles*, whose individual state is called *activity*; the state of each functional subsystem is defined by a probability distribution over the activity variable; interactions are, in general delocalized and nonlinearly additive, and are modeled by stochastic games, where the state of the interacting particles and the output of the interactions can only be known in probability. Finally, the evolution of the probability distribution is obtained by a balance of particles within elementary volumes of the space of the microscopic states, where the dynamics of inflow and outflow of particles is related to interactions at the microscopic scale.

In this way, this approach provides a general mathematical framework that can be specialized into the derivation of a variety of models of complex systems in life sciences. For instance, theory of evolution,[7] opinion formation,[14] immune competition,[12] epidemics with virus mutation,[15,16] modeling of the dynamics of vehicles on roads and interacting individuals in a crowd,[8,11] among others, have been properly modeled and described by using this theory. In all of these applications, the heterogeneous behavior of individuals is modeled by using tools of game theory,[20,22] where the interactions usually involve competitive and altruistic outputs.[4,21,28] More specifically, models regarding to social systems and political competition have been proposed,[1–3,31] and the literature offers a great variety of different approaches such as population dynamics with structure[35] or supermacroscopic dynamical systems.[24] More recently in Ref. 9 a focus on the so-called *black swan* is introduced and the interested reader is referred to for additional useful references.

Some ideas of this type of dynamics applied to social systems were developed in Refs. 13 and 14, where a mathematical model was proposed in order to model the collective social behavior of large systems of interacting individuals within a certain population. The issue is also dealt with in various papers where methods derived from statistical mechanics are used to analyze political, economical and social competitions phenomena.[17,18,30]

In this present paper we consider social systems constituted by more than one interacting population. That is why further developments of the theory are needed to include the dynamics over a network. The migration dynamics is induced by the welfare policy within each nation and is related, as we shall see, to the government rules to control this social phenomena.

This paper is organized into five more sections. Section 2 describes the system to be studied and the involved phenomena. Section 3 introduces the general mathematical framework for the derivation of the model. Section 4 proposes a general migration model in a social network consisting on a small network of nations composed by individuals belonging to a number of social classes. This model, in view of the computational analysis, is then particularized to some specific case-studies. Section 5 deals with the computational analysis of the proposed model and reports some simulations addressed to show emerging behaviors. Finally, Sec. 6 develops a critical analysis and highlights some research perspectives.

2. Description of the System

Migration is a complex social phenomena generally related to the necessity to improve one's social condition. Generally, emigrants are individuals facing a social state that is not good at all, who move from one nation to the other looking for employments that are not available in their native country. In fact, the lure of a well-paid job in a wealthy country is one of the most powerful drivers of international migration. Moreover, many advanced and dynamic economies need migrant workers to fill jobs that cannot be outsourced and that do not find local workers willing to take them at going wages. Population ageing also underlies this growing demand, as it gives rise to deficits of workers relative to dependants. And as younger generations become better educated, fewer in their ranks are content with low-paid and physically demanding jobs.[33]

However, a proper model of migration phenomena must take into account that the typical division into "countries of origin" and "countries of destination" cannot be done so easily, since nowadays, to one degree or another, many nations can be regarded both as a source and as a receptor of migrants. For instance, as documented in Ref. 33, these distinctions, together with the perceived demarcation between the global "North" and "South", are being blurred, and in some cases have disappeared completely: countries such as Ireland, Italy and Spain, which not long ago sent millions of their citizens abroad, are now countries of destination, receiving thousands of newcomers each year.

In this context, the system to be studied consists on a small network of interacting nations or geographical regions constituted by individuals belonging to different social states. Each nation has its own social structure, that is expressed by a critical distance, which triggers either cooperation or competition among the classes. In this sense, the social behavior within each nation can be properly described by the existing models.[13,14]

The system to be studied in the present paper introduces the dynamics expressed by each pair of nations, in order to take into account that not only the social internal structure of each nation is considered, but also the foreign policies that are developed and that may arise on migration from one place to another.

3. Mathematical Tools

The overall social system is viewed, according to the description of Sec. 2, as a network of n interacting nations (or geographical regions) that are localized in nodes.

We study the dynamics of individuals established in each node and migrating from one node to another. Migration dynamics is somewhat related to the social state of the said individuals, who develop a strategy to improve their own state by interactions. More precisely, the modeling approach focuses on a state identified by a discrete variable $u \in I_u = \{u_1 = 0, \ldots, u_m = 1\}$, which represents the social state of the individuals, where $u_1 = 0$ and $u_m = 1$ are, respectively, the lowest and the highest values of the social state. In general, the study of the model seems to be more interesting when the number m of social classes is odd, because in such case a middle class $u_{\frac{m-1}{2}}$ can be assessed. In particular, when a certain individual is expressing a value of u greater than $u_{\frac{m-1}{2}}$, then it belongs to an affluent social class, while when it is expressing a value of u lower than $u_{\frac{m-1}{2}}$, then it belongs to a postponed one.

It has to be acknowledged that it is technically reasonable to use a discrete variable related to various levels of social situation. Namely, social levels can be practically identified only within intervals corresponding to discrete states. In this way, for instance, empirical data obtained by measurements can be related precisely to ranges of variability. This argument is posed in Refs. 2, 13, 14.

As the modeling approach is based on the KTAP, it is worth identifying the components of the overall system. The individuals will be regarded as *active particles* and the value of u describing their state is the *activity* that they express. In this way, each *functional subsystem* corresponds to each nation.

The overall state of the system is described by the *discrete one-particle distribution function*

$$f_i^j = f_i(t; u_j), \quad (t; u_j) \in [0, T] \times I_u, \tag{3.1}$$

which maps, at each instant t, the space of the activity variable, into the half-line of non-negative real numbers. The subscript $i = 1, \ldots, n$ labels the functional subsystems and consequently, $f_i^j(t)$ denotes the number of active particles from functional subsystem i that, at time t, have the state u_j.

In this way, the total size of the ith functional subsystem at time t is $n_i(t) = \sum_{j=1}^{m} f_i^j(t)$, and consequently the total size $N(t)$ of the whole network is given by the sum of all the probability distribution functions:

$$N(t) = \sum_{i=1}^{n} n_i(t) = \sum_{i=1}^{n} \sum_{j=1}^{m} f_i^j(t). \tag{3.2}$$

Let us consider a closed conservative system, i.e. a system that does not interact with the outer environment and in which the time interval to be studied is short enough to neglect the development of proliferative or destructive events. In other words, the migration phenomena take place only within the nodes constituting the network, while the net output of birth and dead processes can be neglected for a time interval not large enough to make them significant, so that the total population $N(t)$ remains constant in time: $N(t) = N_0 = N(t = 0)$.

Therefore, all the probability distribution functions (3.1) can be divided by N_0 so that the terms $f_i^j(t)$ assume the meaning of a fraction of individuals in functional subsystem i with state u_j with respect to the total number of individuals. The calculations developed in what follows take into account this normalization which implies $N_0 = 1$, and consequently the functions defined in (3.1) can be regarded as *discrete probability densities*.

Interactions involve three types of particles:

- The *test* particle, which is representative of the system and whose distribution function is $f_i^j(t)$.
- *Candidate* particles whose distribution function is $f_h^p(t)$, which interact with field particles.
- *Field* particles whose distribution function is $f_k^q(t)$, which interact with test and candidate particles.

The mathematical structure for such a system should describe the evolution in time of the probability density functions f_i^j. It is obtained by equating the variation rate of particles, in the corresponding state u_j of functional subsystem i, with the difference between the inlet and outlet fluxes from this state. In this way, the balance equation can be summarized as follows:

$$\frac{d}{dt} f_i^j(t) = J_i^j[\mathbf{f}](t) = G_i^j[\mathbf{f}](t) - L_i^j[\mathbf{f}](t), \tag{3.3}$$

where J_i^j, G_i^j and L_i^j are suitable operators acting over the whole set of probability densities $\mathbf{f} = \{f_i^j\}_{i=1,\dots,n}^{j=1,\dots,m}$. Specifically:

- $J_i^j[\mathbf{f}](t)$ is the net flux, at time t, of particles that fall into the state u_j of the functional subsystem i.
- $G_i^j[\mathbf{f}](t)$ and $L_i^j[\mathbf{f}](t)$ denote, respectively, the inflow and outflow, at time t, into (and out) the state u_j of the functional subsystem i.

From now on, in order to simplify the technical description, let us denote by (i, j)-particle the particle of the ith functional subsystem with state u_j.

The mathematical framework to describe microscopic interactions between two subsystems can be described by means of two different functions:

- The *encounter rate* η_{hk}^{pq} that describes the rate of interactions between a candidate (or test) (h, p)-particle and a field (k, q)-particle.

- The *transition probability density* $\mathcal{B}_{hk}^{pq}(i,j)$ that describes the probability density that a candidate (h,p)-particle falls into the state u_j of the functional subsystem i after the interaction with a field (k,q)-particle.

The *transition probability density* is such that:

$$\forall\, u_p, u_q \in I_u, \forall\, h,k = 1,\ldots,n, \quad \sum_{i=1}^{n}\sum_{j=1}^{m} \mathcal{B}_{hk}^{pq}(i,j) = 1. \tag{3.4}$$

It is now possible to derive the equation that defines the evolution of the probability distribution over the microscopic state by the balance equation (3.3) of the inlet and outlet flows in the state u_j of functional subsystem i. Technical calculations lead to the following system of ordinary differential equations:

$$\frac{d}{dt} f_i^j(t) = \sum_{h,k=1}^{n}\sum_{p,q=1}^{m} \eta_{hk}^{pq}\mathcal{B}_{hk}^{pq}(i,j) f_h^p(t) f_k^q(t) - f_i^j(t)\sum_{k=1}^{n}\sum_{q=1}^{m} \eta_{ik}^{jq} f_k^q(t). \tag{3.5}$$

4. Modeling of Migration Phenomena

This section is devoted to the development of a modeling approach based on the mathematical frameworks proposed in Sec. 3. These are formally given by Eq. (3.3) and particularized by Eq. (3.5). The method is essentially based on a detailed modeling of the interaction terms, so that the mathematical structures can be fully characterized. Therefore, it is worth focusing on which specific mathematical functions can suitably describe the encounter rate and the transition probability density functions.

4.1. *Modeling the interaction terms*

The mathematical structure (3.5) can be specialized into a mathematical model when the interaction terms η_{hk}^{pq} and $\mathcal{B}_{hk}^{pq}$ are properly modeled according to the specific dynamics.

In order to model the *interaction rate*, a concept of distance needs to be defined for particles of the same or different functional subsystems. The simplest approach consists in assuming that it is identified only by the functional subsystem of the interacting particles, so that the encounter rate is a non-negative constant $\eta_{hk}^0 \leq 1$.

Alternatively, it can be supposed that the interaction rate depends not only on the geographical location of the interacting particles but also on the difference between their social states. In fact, it is reasonable to consider that the closer the states are of the two particles, the higher is the frequency in which they interact. In this way:

$$\eta_{hk}^{pq} = \eta_{hk}^0(1 - \varepsilon|u_p - u_q|), \tag{4.1}$$

where $0 \leq \varepsilon < 1$.

A second step consists in modeling the *transition probability densities*. The modeling of this term can be developed by tools from game theory. With this purpose,

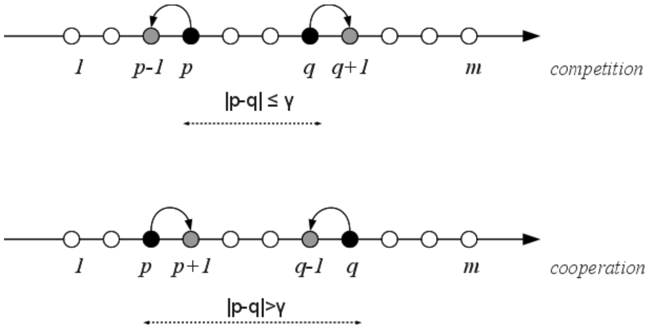

Fig. 1. Sketch of altruistic and competitive behaviors.

let us consider the interaction between a candidate (h, p)-particle and a field (k, q)-particle.

We move from the observation that the interactions between individuals belonging to the same nation can modify their social state but without producing a migration phenomena. These interactions may have the character of conflicts or collaborations, as visualized in Fig. 1. On the other hand, interactions between individuals belonging to different nations are those which may arise on the decision to migrate, taking into account that an immigrant will — in principle — face the same social state immediately after migrating, like illustrated in Fig. 2. These dynamics are summarized in Table 1.

More in detail, let us consider the different kinds of interactions that can take place between a candidate (h, p)-particle and a field (k, q)-particle.

Particles from the same functional subsystem, $h = k$:

♦ Same social states, $p = q$:

$$\mathcal{B}_{hh}^{pp}(h, p) = 1. \tag{4.2}$$

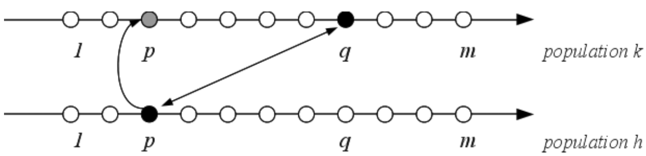

Fig. 2. Interaction between individuals from different populations may arise on the decision to migrate.

Table 1. Summary of the possible functional subsystems and activity values that may occur after an interaction between a candidate (h, p)-particle and a field (k, q)-particle.

	Functional subsystem after the interaction	Activity after the interaction
$h = k$	$i = h$	$j = p,\ p - 1\ \ or\ p + 1$
$h \neq k$	$i = h\ or\ k$	$j = p$

If the particles have the same social state, then both of them remain in the same state.

- Different but close social states, $p \neq q$ and $|p-q| \leq \gamma_h$:

When two individuals from the same nation have different but close social states, then a competition occurs having as a result the fact that the individual placed in the higher social position improves its situation, while the one in a lower position faces a further decrease. The parameter γ_h is a critical distance between social classes.

  - If $p = 1$ or $p = m$, then

$$\mathcal{B}_{hh}^{pq}(h,p) = 1. \tag{4.3}$$

An individual of the class 1 cannot decrease its state, being already at the lowest possible level. Similarly, an individual of the class m cannot increase its state, which is already at the highest possible level. In this way, if the amount of wealth of the two individuals has not to be altered, it is imposed that the co-interacting individuals continue to stay in its own class too.

  - If $p \neq 1$ and $p \neq m$, then

$$p < q : \begin{cases} q \neq m : \begin{cases} \mathcal{B}_{hh}^{pq}(h,p-1) = \alpha_h(1-|u_q-u_p|), \\ \mathcal{B}_{hh}^{pq}(h,p) = 1-\alpha_h(1-|u_q-u_p|), \end{cases} \\ q = m : \mathcal{B}_{hh}^{pq}(h,p) = 1, \end{cases} \tag{4.4}$$

and

$$p > q : \begin{cases} q \neq 1 : \begin{cases} \mathcal{B}_{hh}^{pq}(h,p+1) = \alpha_h(1-|u_q-u_p|), \\ \mathcal{B}_{hh}^{pq}(h,p) = 1-\alpha_h(1-|u_q-u_p|), \end{cases} \\ q = 1 : \mathcal{B}_{hh}^{pq}(h,p) = 1. \end{cases} \tag{4.5}$$

Not every individual belonging to the class p, interacting with an individual from the class q, will change its state moving to the class $p+1$ or $p-1$, depending on the interaction kind the two are undergoing. Eventually, only a portion of individuals could change class, while a portion of individuals will remain in the class p. The non-negative coefficient $\alpha_h < 1$ identifies the portion of individuals that change class and the presence of the terms $(1-|u_q-u_p|)$ implies that the closer is the social state of the interacting individuals, the deeper is the competitive behavior.

- Far social states, $|p-q| > \gamma_h$:

$$p < q : \begin{cases} \mathcal{B}_{hh}^{pq}(h,p+1) = \alpha_h(|u_q-u_p|), \\ \mathcal{B}_{hh}^{pq}(h,p) = 1-\alpha_h(|u_q-u_p|), \end{cases} \tag{4.6}$$

and

$$p > q : \begin{cases} \mathcal{B}_{hh}^{pq}(h,p-1) = \alpha_h(|u_q-u_p|), \\ \mathcal{B}_{hh}^{pq}(h,p) = 1-\alpha_h(|u_q-u_p|). \end{cases} \tag{4.7}$$

When the social state of the individuals is sufficiently far away, the opposite behavior (altruistic behavior) occurs. In contrast to the competitive case, now the altruistic effects will be greater when the further are the social states.

Particles from different functional subsystems, $h \neq k$:

♦ Lowest social state, $p = 1$:

$$\mathcal{B}_{hk}^{1q}(h, 1) = 1. \tag{4.8}$$

At the point of origin, deeper poverty does not lead automatically to higher migration. The least wealthy people generally do not have the resources to bear the costs and risks of international migration. International migrants are usually drawn from low- and middle-income households, but not from the lowest.

♦ Higher social states, $p > 1$:

$$\begin{aligned} p < q &: \begin{cases} \mathcal{B}_{hk}^{pq}(k, p) = \beta_{hk}|u_q - u_p|, \\ \mathcal{B}_{hk}^{pq}(h, p) = 1 - \beta_{hk}|u_q - u_p|, \end{cases} \\ p \geq q &: \mathcal{B}_{hk}^{pq}(h, p) = 1. \end{aligned} \tag{4.9}$$

An individual from nation h may decide to migrate to nation k only after an interaction with an individual from the latter that holds a better social state. Of course, not every individual will decide to migrate. The coefficient β_{hk} is introduced to denote the portion of individuals from nation h which migrate to nation k after an interaction with an individual from the latter and the factor $|u_q - u_p|$ in the previous expressions means that the further the social classes are, the greater is the probability to migrate. On the other hand, if the social state of the candidate particle is greater than the one of the field particle, then no transition occurs and the candidate remains in the same population and with the same social state.

Remark 4.1. Substitution of expressions (4.2)–(4.9) into the formal structure (3.5) generates specific models corresponding to the table of games. These models preserve, in addition to the total number of individuals N_0, the total wealth, which is defined as:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} u_j f_i^j(t) = \sum_{i=1}^{n}\sum_{j=1}^{m} u_j f_i^j(t = 0) = L_0. \tag{4.10}$$

Remark 4.2. The modeling approach considered in this present paper considers linear interactions, in which η_{hk}^{pq} and $\mathcal{B}_{hk}^{pq}(i, j)$ are conditioned only by the state of the interacting particles for each pair of functional subsystems. It is worth mentioning that nonlinear interactions can also be considered, i.e. interactions in which particles are not simply subject to the superposition of binary actions but are also affected by the global current state of the system. In this way, the interacting terms may be conditioned by the moments of the distribution functions. This one is a very important issue and some perspectives about it are discussed in the last section.

Remark 4.3. The threshold γ_h characterizes in each nation the welfare policy of the government. It can be either a constant,[13] when the government imposes it, or it can be determined by the dynamics of the system when the policy of the government leaves it to the social competition within each nation.[9] Although in this paper γ_h is taken to be constant, a further development of the model can consider it as depending on the instantaneous wealth distribution, in order to account for nonlinearly additive interactions among the active particles.

4.2. *Some specific models*

It is useful, in view of the computational analysis developed in the next section, presenting some specific models selected as case-studies with the aim of putting well in evidence a variety of specific phenomena and, in particular, the influence of the model parameters on the overall output of the dynamics.

Case-study 1: *Two interacting nations*

Let us first consider a small network of nations characterized by the following features:

- Two nations are involved, labeled by $i = 1, 2$. The first one is a wealthy country which is the basing of attractions for the second one, that is less wealthy and constitute a broad field of individuals ready to migration.
- Nine social classes, labeled by $j = 1, \ldots, 9$, are selected to take into account a broad variety of simulations involving, for instance, different critical distances within each country. In this way, the state of the individuals is described by a scalar discrete variable u which can attain values on the set $I_u = \{u_1 = 0, \ldots, u_5 = 1/2, \ldots, u_9 = 1\}$ and $j = 5$ separates the affluent from the postponed social classes.

It is assumed that interactions between individuals of the same population occur more frequently than interactions between individuals of different populations. In general, the terms η^0_{hk} are assumed to be equal to unity if $h = k$ and equal to $1/2$ if $h \neq k$, and ε is assumed to be equal to $1/2$.

The model is thus characterized by the following parameters:

— $\theta = N_{20}/N_{10}$ is the ratio between Nation's 2 and Nation's 1 initial populations, N_{20} and N_{10}, respectively.
— $\phi = L_{20}/L_{10}$ is the ratio between the initial wealths of both populations, denoted by L_{10} and L_{20}, respectively.
— γ_1 and γ_2 are the thresholds which, within each population, separate competitive from altruistic dynamics.
— α_1 and α_2 are parameters that characterize the probability of changing class within Nations 1 and 2, respectively.

— β_{12} and β_{21} are parameters that characterize the probability of migrating from Nation 1 to Nation 2 and vice versa, respectively.

Case-study 2: *Three interacting nations*

Let us now consider a network constituted by three nations or geographical zones, characterized by the following features:

- Three nations are involved, labeled by $i = 1, 2, 3$.
- As well as in the first case-study, nine social classes, labeled by $j = 1, \ldots, 9$, are considered.

Regarding to the interactions between individuals, the interaction rates are subject to the same assumptions as in the previous case.

The model is thus characterized by the following parameters:

— $\theta_2 = N_{20}/N_{10}$ and $\theta_3 = N_{30}/N_{10}$ are the ratios between the corresponding initial populations.
— $\phi_2 = L_{20}/L_{10}$ and $\phi_3 = L_{30}/L_{10}$ are the ratios between the corresponding initial wealths.
— γ_1, γ_2 and γ_3 are the thresholds which, within each population, separate competitive from altruistic dynamics.
— α_1, α_2 and α_3 are parameters that characterize the probability of changing class within each nation.
— β_{12}, β_{21}, β_{13}, β_{31}, β_{23} and β_{32} are parameters that characterize the probability of migrating from one nation to another.

Note that the parameters β_{hk} can be regarded as entries of a non-symmetric matrix with null diagonal, as they involve the facilities/disadvantages that can make one of the nations more attractive for immigration than the other and consequently $\beta_{hk} \neq \beta_{kh}$.

5. Simulations and Emerging Phenomena

The mathematical structure proposed in (3.5) generates a model stated in terms of a system of ordinary differential equations. Mathematical problems are obtained by linking system (3.5) to the initial conditions. The statement of the initial value problem is as follows:

$$\begin{cases} \dfrac{d}{dt} f_i^j(t) = J_i^j[\mathbf{f}](t), \\ f_i^j(0) = f_{0i}^j \in [0,1], \quad i = 1, \ldots n, \quad j = 1, \ldots, m, \\ \|\mathbf{f}_0\|_1 = \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{m} |f_{0i}^j| = 1, \end{cases} \tag{5.1}$$

where $\mathbf{f}_0 = \{f_{0i}^j\}_{i=1,\ldots,n}^{j=1,\ldots,m}$ and $\| \cdot \|_1$ is the 1-norm in $\mathbb{R}^{n.m}$ to be adopted.

It is worth mentioning that this problem is well-posed and that it has a unique large-time solution $\mathbf{f}(t)$ such that

$$\mathbf{f}(t) \geq 0, \quad \sum_{i=1}^{n}\sum_{j=1}^{m} f_i^j(t) = 1 \quad \forall\, t \in [0, T], \tag{5.2}$$

for an arbitrary large time T.

This result follows from the use of classical fixed-point theorems and the interested reader is referred to Ref. 16 for a detailed proof. It is basically based on the Lipschitz property of the right-hand side of the first equation in (5.1), which gives local existence, and on its L^1-norm preserving property, that leads to existence for large times.

Simulations can be developed with the aim of visualizing the predictive ability of the model by analyzing the influence of the parameters on the collective migration phenomena. The analysis is well-addressed since the well-posedness of the mathematical problem associated to the particular model is guaranteed.

As stated in the previous section, specific cases need to be considered to avoid overgeneralization. Bearing all above in mind, let us consider the case-studies proposed in Sec. 4 consisting the first one on two nations and the second one on three nations, each of them with nine social classes. These two cases generate systems of 18 and 27 ordinary differential equations, respectively.

The specific aim of the simulations presented in the following subsections is to investigate the evolution of the system and its dependence on some of the parameters. In particular, we consider that the initial distribution of the wealth of each nation, in relationship with the other ones, is a key parameter which determines the dynamics of the migration phenomena. That is why different choices of initial wealth distributions, together with the total initial size within each nation, are taken into account and their influence over the emerging behavior is discussed.

For the second case-study, already introduced in the previous section, we also consider that the migration policy that each country establishes is also a key parameter that could determine the development of different strategies by the individuals. That is why simulations for this case-study aim to provide an insight on the importance of this parameter over the dynamics.

5.1. *Case-study* 1: *Two interacting nations*

Let us study the migration phenomena between a wealthy nation and a postponed one. The first step is to translate the general model to this specific case, considering all the possible interactions between individuals. As it was stated in Sec. 4, the interaction rate between any individuals in the network can be modeled by

$$\eta_{hk}^{pq} = \eta_{hk}^{0}\left(1 - \frac{1}{2}|u_p - u_q|\right), \tag{5.3}$$

where $\eta_{11}^0 = \eta_{22}^0 = 1$ and $\eta_{12}^0 = \eta_{21}^0 = 1/2$.

The next step is to detail the table of games described by Eqs. (4.2)–(4.9) for this particular case, in order to obtain the entire formulation of the initial value problem (5.1).

Interactions between individuals in Nation 1

First of all, according to Eq. (4.2), we have that for $p = 1, \dots, 9$, $\mathcal{B}_{11}^{pp}(1, p) = 1$.

Now, if $|p-q| \le \gamma_1$, by Eqs. (4.3)–(4.5), we get that $\mathcal{B}_{11}^{1q}(1, 1) = 1$, $\mathcal{B}_{11}^{9q}(1, 9) = 1$,

$$1 < p < q \le 9 : \begin{cases} q \neq 9 : \begin{cases} \mathcal{B}_{11}^{pq}(1, p-1) = \alpha_1(1 - |u_q - u_p|), \\ \mathcal{B}_{11}^{pq}(1, p) = 1 - \alpha_1(1 - |u_q - u_p|), \end{cases} \\ q = 9 : \mathcal{B}_{11}^{p9}(1, p) = 1, \end{cases}$$

and

$$1 \le q < p < 9 : \begin{cases} q \neq 1 : \begin{cases} \mathcal{B}_{11}^{pq}(1, p+1) = \alpha_1(1 - |u_q - u_p|), \\ \mathcal{B}_{11}^{pq}(1, p) = 1 - \alpha_1(1 - |u_q - u_p|), \end{cases} \\ q = 1 : \mathcal{B}_{11}^{p1}(1, p) = 1. \end{cases}$$

Finally, if $|p-q| > \gamma_1$, by Eqs. (4.6)–(4.7), we have

$$p < q : \begin{cases} \mathcal{B}_{11}^{pq}(1, p+1) = \alpha_1(|u_q - u_p|), \\ \mathcal{B}_{11}^{pq}(1, p) = 1 - \alpha_1(|u_q - u_p|), \end{cases}$$

and

$$p > q : \begin{cases} \mathcal{B}_{11}^{pq}(1, p-1) = \alpha_1(|u_q - u_p|), \\ \mathcal{B}_{11}^{pq}(1, p) = 1 - \alpha_1(|u_q - u_p|). \end{cases}$$

Interactions between individuals in Nation 2

According to Eq. (4.2), for $p = 1, \dots, 9$ we have that $\mathcal{B}_{22}^{pp}(2, p) = 1$.

Now, if $|p-q| \le \gamma_2$, by Eqs. (4.3)–(4.5) we obtain $\mathcal{B}_{22}^{1q}(2, 1) = 1$, $\mathcal{B}_{22}^{9q}(2, 9) = 1$,

$$1 < p < q \le 9 : \begin{cases} q \neq 9 : \begin{cases} \mathcal{B}_{22}^{pq}(2, p-1) = \alpha_2(1 - |u_q - u_p|), \\ \mathcal{B}_{22}^{pq}(2, p) = 1 - \alpha_2(1 - |u_q - u_p|), \end{cases} \\ q = 9 : \mathcal{B}_{22}^{p9}(2, p) = 1, \end{cases}$$

and

$$1 \le q < p < 9 : \begin{cases} q \neq 1 : \begin{cases} \mathcal{B}_{22}^{pq}(2, p+1) = \alpha_2(1 - |u_q - u_p|), \\ \mathcal{B}_{22}^{pq}(2, p) = 1 - \alpha_2(1 - |u_q - u_p|), \end{cases} \\ q = 1 : \mathcal{B}_{22}^{p1}(2, p) = 1. \end{cases}$$

Finally, if $|p-q| > \gamma_2$, by Eqs. (4.6)–(4.7), we have

$$p < q : \begin{cases} \mathcal{B}_{22}^{pq}(2, p+1) = \alpha_2(|u_q - u_p|), \\ \mathcal{B}_{22}^{pq}(2, p) = 1 - \alpha_2(|u_q - u_p|), \end{cases}$$

and

$$p > q : \begin{cases} \mathcal{B}_{22}^{pq}(2, p-1) = \alpha_1(|u_q - u_p|), \\ \mathcal{B}_{22}^{pq}(2, p) = 1 - \alpha_1(|u_q - u_p|). \end{cases}$$

Interactions between an individual belonging to Nation 1 with state $\boldsymbol{u_p}$ and an individual belonging to Nation 2 with state $\boldsymbol{u_q}$

By Eqs. (4.8) and (4.9) we obtain:

$$p = 1 : \mathcal{B}_{12}^{1q}(1, 1) = 1,$$

$$p < q : \begin{cases} \mathcal{B}_{12}^{pq}(2, p) = \beta_{12}|u_q - u_p|, \\ \mathcal{B}_{12}^{pq}(1, p) = 1 - \beta_{12}|u_q - u_p|, \end{cases}$$

$$p \geq q : \mathcal{B}_{12}^{pq}(1, p) = 1.$$

Interactions between an individual belonging to Nation 2 with state $\boldsymbol{u_p}$ and an individual belonging to Nation 1 with state $\boldsymbol{u_q}$

Using Eqs. (4.8) and (4.9) again, we obtain:

$$p = 1 : \mathcal{B}_{21}^{1q}(1, 1) = 1,$$

$$p < q : \begin{cases} \mathcal{B}_{21}^{pq}(1, p) = \beta_{21}|u_q - u_p|, \\ \mathcal{B}_{21}^{pq}(2, p) = 1 - \beta_{21}|u_q - u_p|, \end{cases}$$

$$p \geq q : \mathcal{B}_{21}^{pq}(2, p) = 1.$$

5.1.1. *The wealthier nation is less populated*

Let us first consider the situation in which Nation 2 has a population that duplicates that of Nation 1 (fix $\theta = 2$), while it has a much postponed social structure, with a greater threshold between social classes ($\gamma_1 = 3$, $\gamma_2 = 5$), which gives to the residents less opportunities of improving their own situation. In addition, the attractiveness of migrating from Nation 2 to Nation 1 is expressed (in terms of probability) as twice the one of migrating in the opposite direction ($\beta_{12} = 1/20$, $\beta_{21} = 1/10$). The other parameters are taken to be $\alpha_1 = \alpha_2 = 0.2$.

Simulations are developed for values $\phi = 0.2$ and $\phi = 0.6$ in order to evaluate how migration affects to the social structure of the nations for different ratios between the initial wealths.

Figure 3 shows the evolution of each nation's size over time. It can be seen that the higher the initial wealth of Nation 2 is, the stronger is the migratory flux. This

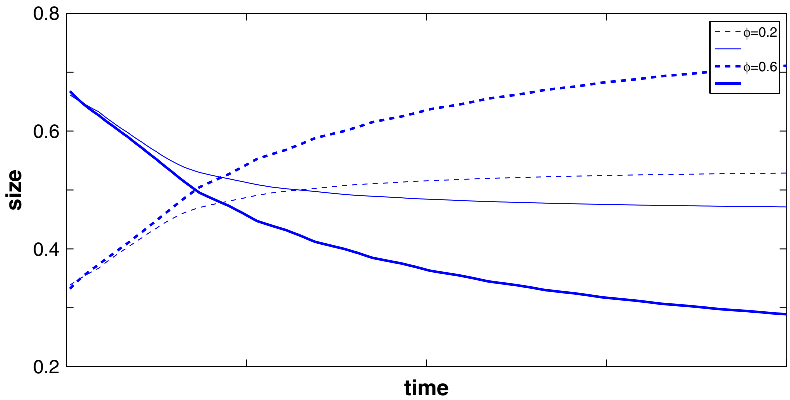

Fig. 3. *Case-study* 1: *Two interacting nations*, $\theta = 2$. Evolution of the size of each nation versus dimensionless time for different values of ϕ. The dashed lines refer to Nation 1 and the continuous lines refer to Nation 2.

Table 2. Summarizing table of the qualitatively observed behaviors for case-study 1. The number of circles represents the initial size of each nation and the number of diamonds represents the corresponding initial wealths.

	Size	Wealth	General behavior
Nation 1	◯◯	◇◇◇◇◇	A migratory flux from Nation 2 to Nation 1 is observed. The higher the wealth of Nation 2 is with respect to Nation 1, the stronger is this flux
Nation 2	◯◯◯◯	◇◇	
Nation 1	◯◯◯◯	◇◇◇◇◇	A migratory flux from Nation 2 to Nation 1 is observed. Nation 2 can benefit from this migratory phenomena, increasing its relative wealth
Nation 2	◯◯	◇◇	

is because for lower values of ϕ, the situation of Nation 2 does not allow its people to leave the country. This behavior is summarized in the first part of Table 2.

Figure 4 shows how each nation's absolute wealth varies with time. It is clear that, since the total wealth is conserved, one of the nations increases its total wealth, while the other gets worse. However, as shown in Fig. 5, note that the relative wealth, expressed as the relationship between the nation's wealth and its population, decreases.

5.1.2. *The wealthier nation is more populated*

If we now consider the situation in which the less wealthy nation is also less populated, a different behavior is observed. Figures 6 and 7 show the evolution of the wealth and the relative wealth for a fixed value $\phi = 0.5$ and different values of θ. Note that in this case, according to Fig. 7, Nation 2 could clearly benefit from the emigration of a group of individuals. Figure 6 shows that the total wealth of Nation 1 can be increased for a certain bounded period of time, and then starts

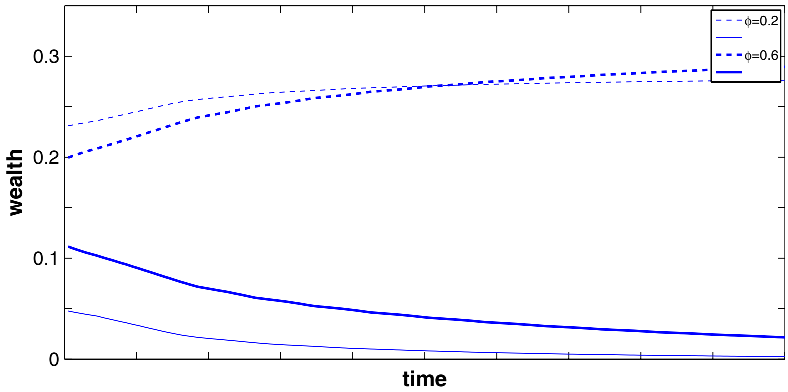

Fig. 4. *Case-study* 1: *Two interacting nations*, $\theta = 2$. Evolution of the absolute wealth of each nation versus dimensionless time for different values of ϕ. The dashed lines refer to Nation 1 and the continuous lines refer to Nation 2.

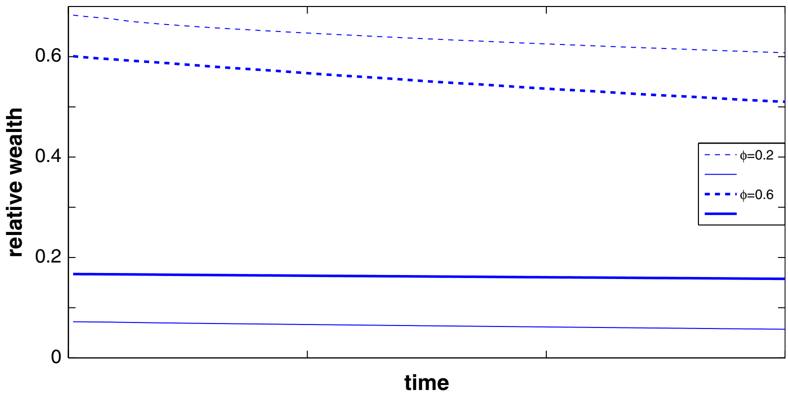

Fig. 5. *Case-study* 1: *Two interacting nations*, $\theta = 2$. Evolution of the relative wealth of each nation versus dimensionless time for different values of ϕ. The dashed lines refer to Nation 1 and the continuous lines refer to Nation 2.

to decrease, leading the way that countries should change their migration policies over time. The overall behavior for this case is summarized in the second part of Table 2.

5.2. *Three interacting nations*

Let us now consider the case of three interacting nations, where two of them are wealthier and consequently attractive for individuals from the third one, which exhibits a postponed situation.

The interaction rates are taken again to be $\eta_{hk} = 1/2$ when $h \neq k$ and $\eta_{hh} = 1$. The corresponding table of games follows immediately from Eqs. (4.2)–(4.9) in the same way as that for the previous case-study.

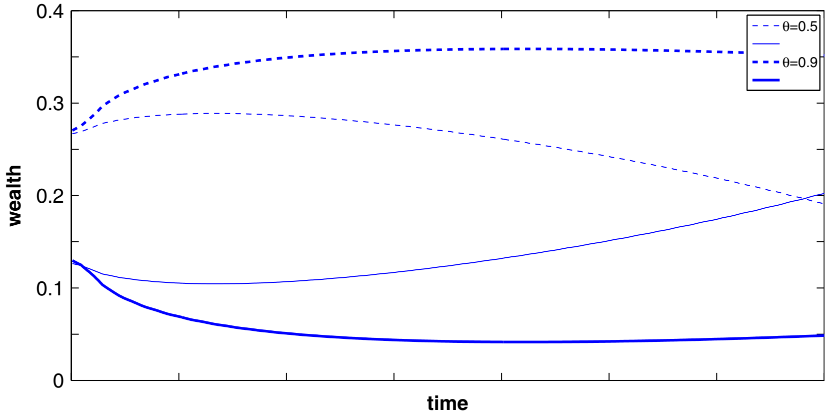

Fig. 6. *Case-study* 1: *Two interacting nations*, $\phi = 0.5$. Evolution of the absolute wealth of each nation versus dimensionless time for different values of θ. The dashed lines refer to Nation 1 and the continuous lines refer to Nation 2.

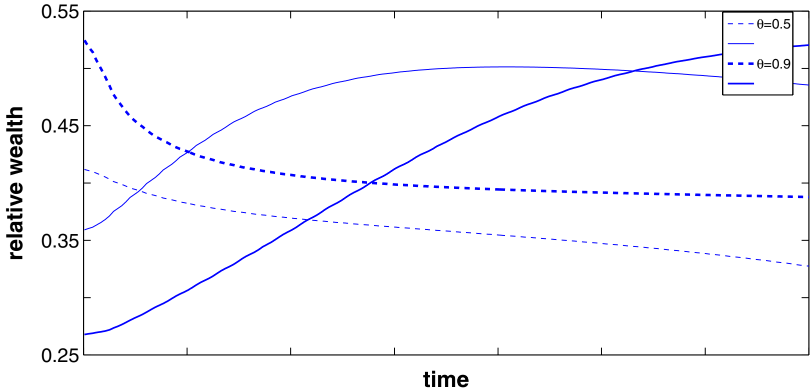

Fig. 7. *Case-study* 1: *Two interacting nations*, $\phi = 0.5$. Evolution of the relative wealth of each nation versus dimensionless time for different values of θ. The dashed lines refer to Nation 1 and the continuous lines refer to Nation 2.

The aim of the following simulations is to analyze the evolution of the system according to two key parameters: the threshold for the least wealthy nation and the migration's facilities that one of the wealthier nations gives or refuses to give to the more postponed one.

5.2.1. *Different levels of restrictions in one of the wealthy countries*

Let nations labeled by 1 and 2 be two wealthier and less populated countries with respect to Nation 3. The following parameters are adopted: $\theta_2 = 1$, $\theta_3 = 2$, $\phi_2 = 0.7$,

$\phi_3 = 0.3$, $\gamma_1 = 1$, $\gamma_2 = 3$, $\gamma_3 = 5$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0.3$. The components of matrix β represent the probabilities of transition from one nation to the other:

$$\beta = \begin{pmatrix} 0 & 1/20 & 1/400 \\ 1/20 & 0 & 1/400 \\ \beta_{31} & 1/200 & 0 \end{pmatrix},$$

where β_{31} takes different values in the simulations.

Figures 8 and 9 show, respectively, the evolution of the size and the evolution of the wealth of the three nations. It is clear that for a lower value of β_{31}, the flux of individuals from Nation 3 decreases, becoming more significative the migration phenomena between Nations 1 and 2, which practically have *free transit* between them. However, as time evolves, those individuals from Nation 3 who were able to cross to Nation 2 might eventually end up in Nation 1, evading the restrictions.

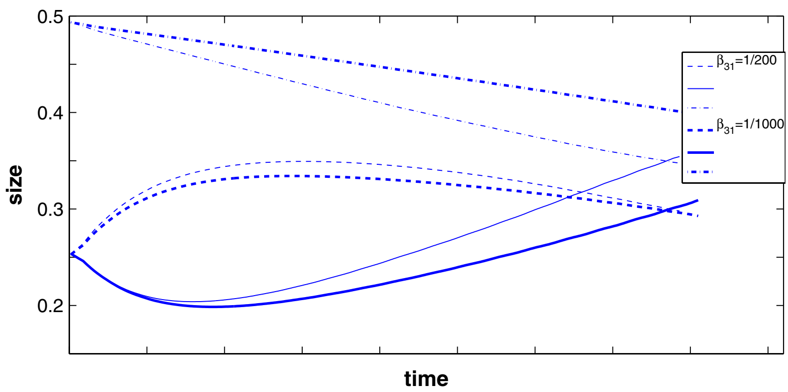

Fig. 8. *Case-study* 2: *Three interacting nations*. Evolution of the size of each nation for different values of β_{31}. Dashed lines refer to Nation 1, continuous lines refer to Nation 2 and dotted lines refer to Nation 3.

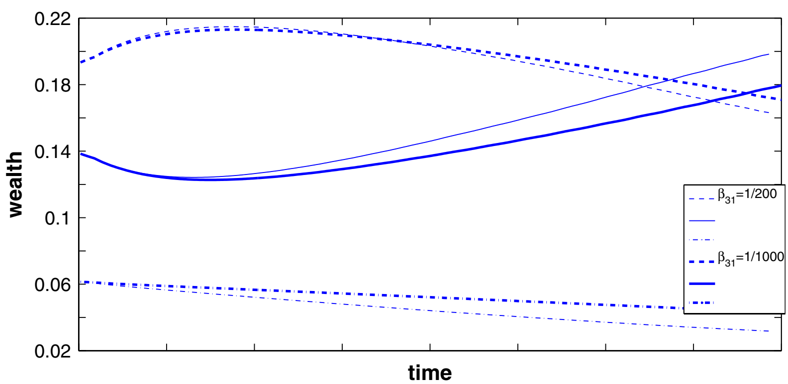

Fig. 9. *Case-study* 2: *Three interacting nations*. Evolution of the wealth of each nation for different values of β_{31}. Dashed lines refer to Nation 1, continuous lines refer to Nation 2 and dotted lines refer to Nation 3.

Table 3. Summarizing table of the qualitatively observed behaviors for case-study 2. The number of circles represents the initial size of each nation, the number of diamonds represents the corresponding initial wealths, and the circle-backslash symbols represent the level of restrictions that Nations 1 and 2 put over potential immigrants from Nation 3.

	Size	Wealth	Restrictions	General behavior
Nation 1	◯◯	♢♢♢♢♢	⊘⊘⊘⊘⊘	Strong migratory fluxes from Nation 2 to Nation 1 and from Nation 3 to Nation 2 are observed. Nation 2 becomes a transit country that may let citizens from Nation 3 end up in Nation 1
Nation 2	◯◯	♢♢♢♢	⊘	
Nation 3	◯◯◯◯	♢♢		
Nation 1	◯◯	♢♢♢♢♢	⊘	Migratory flux from Nation 3 to Nations 1 and 2 is observed. Also, individuals from Nation 2 migrate to Nation 1, and in the absence of evolutionary dynamics Nation 2 could become wealthier than Nation 1
Nation 2	◯◯	♢♢♢♢	⊘	
Nation 3	◯◯◯◯	♢♢		

This could be the case, for instance, of two European countries with a similar social organization but where both of them put different levels of restrictions for individuals from a less wealthy country.[23] The results are summarized in Table 3.

5.2.2. *Different critical distances in the less wealthy country*

Let us now analyze how the system evolves when Nation 3 changes, for instance via some action from the government, its framework from a competitive to a cooperative structure. In the following, the parameters are taken to be equal to the ones above, but fixing $\beta_{31} = 1/800$ and taking different values for γ_3. In particular, we consider the extremal cases of $\gamma_3 = 1$ and $\gamma_3 = 7$.

Figure 10 shows the evolution of the sizes of the three nations. Surprisingly, a cooperative framework in the least wealthy country, that according to previous

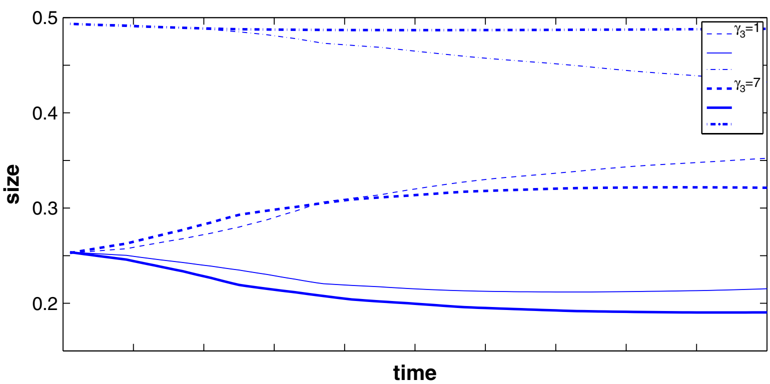

Fig. 10. *Case-study* 2: *Three interacting nations.* Evolution of the size of each nation for different values of γ_3. Dashed lines refer to Nation 1, continuous lines refer to Nation 2 and dotted lines refer to Nation 3.

works would be recommendable to avoid a heavier concentration of lower classes, conduces to a higher migratory flux. It is evident that the medium classes, as long as they are constituted, tend to emigrate.

6. Looking Ahead

A mathematical model of migration dynamics in small networks has been presented in this paper. This model studies the interplay between welfare policies and migration phenomena. It is plain that this preliminary investigation needs to be followed by additional ones, motivated also by a further enrichment of the model. Focusing on this latter perspective, let us refer to some features which can be a subject of further developments.

The study presented in Ref. 9 shows that interplay with the welfare policy can have an important influence on the overall dynamics of a nation, hence also on migration phenomena. Even though the model proposed in this paper considers that the threshold is a constant output of the welfare policy, it is plain that a more detailed interplay can be analyzed by studying how migration phenomena can influence over the welfare dynamics and vice versa. In such a context, learning abilities and evolution are essential features of the modeling strategies. Therefore, the model can be extended by using tools of the so-called evolutionary games theory[26,27,29,32] in order to consider evolutionary welfare policies, which may vary in time according to the necessity of the countries to adapt themselves to novel situations.

Another additional topic worth of future analysis refers to space dynamics. More precisely, the approach of this paper focuses on the dynamics over a small network, by models that define the time evolution of the probability distribution over the activity variable. Therefore, the model does not include a space dynamics, which might however be important to study clustering phenomena. This one is a challenging topic where the derivation of equations at the macroscopic scale should be properly related to the underlying description at the lower micro-scale. Also in this case, there exists an interesting literature that can offer a useful methodological basis to be properly generalized to the specific case under consideration. Let us specifically mention the papers[31,34] which focus on criminality issues and macroscopic models with space structure modeling pattern formation, respectively. In addition, recent studies on biological tissues[5,6] provide the methodology to derive macroscopic equations from the underlying description at the micro-scale.

Finally, analogous reasonings can be developed concerning to the modeling of the interactions. As stated in Remark 4.2, it is worth providing a deeper insight into sources of nonlinearly additive rules. The greatest part of the literature is based on linearly additive interactions. However, the relevance of the topic under consideration needs a critical analysis on the nonlinearity features of interactions. More precisely, the following, selected among various ones, can be indicated: (i) candidate active particles can be influenced not only by the information coming from a fixed distance, but also by a fixed amount of information independently of the distance; (ii) the output of the interactions depends not only on the state (activity) of the interacting particles, but also on the moments of the probability distribution function. In general, it depends on low-order moments such as the mean value of the field particles in their interaction domain.

These issues have been put into a detailed framework in Ref. 10 for models with space dynamics, in the specific case of swarms. Possibly, this approach can be also extended to the system under consideration.

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