Polar factorization of conformal and projective maps of the sphere in the sense of optimal mass transport

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Abstract

Let M be a compact Riemannian manifold and let μ, d be the associated measure and distance on M. Robert McCann obtained, generalizing results for the Euclidean case by Yann Brenier, the polar factorization of Borel maps $S : M \to M$ pushing forward μ to a measure ν: each S factors uniquely a.e. into the composition $S = T \circ U$, where $U : M \to M$ is volume preserving and $T : M \to M$ is the optimal map transporting μ to ν with respect to the cost function $d^2/2$.

In this article we study the polar factorization of conformal and projective maps of the sphere S^n. For conformal maps, which may be identified with elements of $O_o(1, n+1)$, we prove that the polar factorization in the sense of optimal mass transport coincides with the algebraic polar factorization (Cartan decomposition) of this Lie group. For the projective case, where the group $GL_+(n+1)$ is involved, we find necessary and sufficient conditions for these two factorizations to agree.

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1 Introduction

Optimal mass transport and polar factorization. Given two spatial distributions of mass, the problem of Monge and Kantorovich (see for instance [15]) is to transport the mass from one distribution to the other as efficiently as possible. Here efficiency is measured against a cost function $c(x, y)$ specifying the transportation tariff per unit mass. More precisely, let X be a topological space, let $c : X \times X \to \mathbb{R}$ be a nonnegative cost function and let μ, ν be finite Borel measures on X with the same total mass. A map $T : X \to X$ that minimizes the functional

$$T \mapsto \int_X c(x, T(x))\, d\mu(x)$$

under the constraint that T pushes forward μ onto ν (that is, $\nu(B) = \mu(T^{-1}(B))$ for any Borel set in X, which is denoted by $T_{\#}\mu = \nu$) is called an *optimal transportation map*

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between μ and ν. In the following, when it is clear from the context, we call it simply optimal.

An important particular case is the following: Let M be a compact oriented Riemannian manifold and let $\mu = \text{vol}$ be the Riemannian measure on M and d the associated distance and consider the cost $c(p,q) = \frac{1}{2} d(p,q)^2$.

Let $S : M \to M$ be a Borel map pushing forward μ to a measure ν. Robert McCann proved in [12], generalizing results for the Euclidean case by Brenier [3], that S factors uniquely a.e. into the composition $S = T \circ U$, where $U : M \to M$ is volume preserving and $T : M \to M$ is the optimal map transporting μ to ν. This is called the *polar factorization of S in the sense of optimal mass transport.* For the sake of brevity we call it the Brenier-McCann polar factorization of S.

Polar factorization of conformal maps of the sphere. An orientation preserving diffeomorphism F of an oriented Riemannian manifold (M, g) of dimension $n \geq 2$ is said to be *conformal* if $F^* g = fg$ for some positive function f on M.

Let S^n be the unit sphere centered at the origin of $\mathbb{R}^{n+1}$ and let $S : S^n \to S^n$ be a conformal transformation of S^n. Let $G = O_o(1, n+1)$ be the identity component of the group preserving the symmetric bilinear form of signature $(1, n+1)$ on $\mathbb{R}^{n+2}$. The map S can be canonically identified with an element of G, thinking of S^n as the projectivization of the light cone in the Lorentz space $\mathbb{R}^{1,n+1}$. For $A \in G$ and $u \in S^n$ (in particular $(1, u)$ is a null vector) one defines $A \cdot u$ as the unique $u' \in S^n$ such that

$$A \begin{pmatrix} 1 \\ u \end{pmatrix} \in \mathbb{R} \begin{pmatrix} 1 \\ u' \end{pmatrix}. \tag{1}$$

This is well known; we refer for instance to [8] (see also [13]). By definition, the conformal transformations of the circle S^1 are given by the above action of $O_o(1,2)$ on it. They coincide with the Moebius maps of the circle, that is, the restrictions to S^1 of the Moebius maps of $\mathbb{C} \cup \{\infty\}$ preserving the unit disc.

The (algebraic) polar factorization of G is $\exp(\mathfrak{p})\, SO(n+1)$, where $\mathfrak{p}$ is the vector space of symmetric matrices in the Lie algebra $o(1, n+1)$ of G, that is

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & v^t \\ v & 0 \end{pmatrix} \mid v \in \mathbb{R}^n \right\}. \tag{2}$$

In this context, it is usually called the Cartan decomposition of G. We have the following result:

Theorem 1. *Let S be a conformal transformation of the sphere S^n. Then the Brenier-McCann polar factorization of S coincides with the algebraic polar factorization of S.*

Polar factorization of projective maps of the sphere. An orientation preserving diffeomorphism F of an oriented Riemannian manifold M of dimension $n \geq 2$ is said to be *projective* if for any geodesic γ of M, $F \circ \gamma$ is a reparametrization (not necessarily of constant speed) of a geodesic of M.

For $n \geq 2$, the projective transformations of S^n are exactly those of the form

$$p \longmapsto Ap / \|Ap\| \tag{3}$$

for some $A \in GL_+(n+1)$, the group of linear automorphisms of $\mathbb{R}^{n+1}$ with positive determinant. By definition, the projective transformations of the circle S^1 are given by the action of $GL_+(2)$ on it as above.

Theorem 2. *Let S be a projective transformation of the sphere S^n. Then the Brenier-McCann polar factorization of S coincides with the algebraic polar factorization PO of S (that is, with positive definite self adjoint P and orthogonal O) if and only if P has at most two distinct eigenvalues.*

We would like very much to know explicitly, if possible, the Brenier-McCann polar factorization of the projective map of S^2 induced by, say, diag $(1, 2, 3)$.

Next we give the main arguments in the proofs of the theorems. A projective map of S^n induced by a positive definite self adjoint transformation of $\mathbb{R}^{n+1}$ with at most two distinct eigenvalues preserves meridians of the sphere through points lying in two fixed orthogonal subspaces. It turns out to be a particular case of the optimal maps considered in Theorem 5 in the preliminaries, that we prove using the characterization by McCann of the optimal maps on Riemannian manifolds involving c-convex potentials. These are a powerful theoretical tool, in general arduous to deal with in concrete cases, but the fact that the optimal maps of the circle are well known and the symmetries of our problem allowed us to apply them.

Conformal maps on S^n induced by symmetric transformations of Euclidean space behave similarly. We prove Theorem 1 by verifying that the conformal map induced by the symmetric part of the polar algebraic decomposition satisfies the hypotheses of Theorem 6, which is analogous to Theorem 5.

We comment on the proof of Theorem 2. When P has at most two distinct eigenvalues we check that the projective map induced by P satisfies the hypotheses of Theorem 5 using that conformal maps of the circle double cover the projective maps of the circle. In the case that P has at least three distinct eigenvalues, we resort to a necessary condition for a map on a Riemannian manifold M to be optimal: that its graph be a.e. a Lagrangian submanifold with respect to a certain symplectic form defined a.e. on $M \times M$ in terms of the cost function.

In [13] and [14] (see also [7]) the second author et al. studied force free conformal and projective motions of S^n. In particular, they found some geodesics σ of the groups $SL\,(n+1, \mathbb{R})$ and $O_o\,(1, n+1)$ endowed with the not invariant (as it happens with non-rigid motions) Riemannian metric given by the kinetic metric. The curves σ induce curves of measures $t \mapsto (\sigma_t)_\#\,(\mathrm{vol})$. We wonder whether this is related, with conformal or projective constraints, with the survey paper [4] on transportation problems in which a given mass dynamically moves from an initial configuration to a final one (see also [1] and [2]).

2 Preliminaries

Let (M, g), $\mu = \mathrm{vol}_M$ and $c = d^2/2$ be as in the introduction. We recall the result by McCann in [12], which gives an expression for the optimal map transporting μ to a measure ν. Given a lower semi-continuous function $\phi : M \to \mathbb{R}$, the supremal convolution of ϕ is the function $\phi^c : M \to \mathbb{R}$ defined by

$$\phi^c(p) = \sup_{q \in M} \{-c(p, q) - \phi(q)\}. \tag{4}$$

(We adopt the notation of [11], using supremal instead of infimal convolution.)

Theorem 3 (McCann)**.** *The optimal map T between μ and $T_{\#}(\mu)$ can be expressed as a gradient map, that is,*

$$T(p) = \mathrm{Exp}_p \left(\mathrm{grad}_p \phi\right)$$

a.e., where Exp *is the geodesic exponential map of M and ϕ is a c-convex potential, that is, $\phi : M \to \mathbb{R}$ is a lower semi-continuous function satisfying $\phi^{cc} = \phi$.*

Cordero-Erausquin characterized in [5] (see also [12]) the optimal transportation maps on tori. We write below an equivalent statement.

Theorem 4 (Cordero-Erausquin)**.** *Let μ and ν be two probability measures on the torus $\mathbb{R}^n/2\pi\mathbb{Z}^n$ with the same total mass which are absolutely continuous with respect to the Lebesgue measure. Then the optimal transportation map T between μ and ν lifts a.e. to a map $\widetilde{T} : \mathbb{R}^n \to \mathbb{R}^n$ of the form $\widetilde{T} =$ grad ψ, where ψ is a convex function on $\mathbb{R}^n$.*

Conformal or projective maps of S^n induced by self adjoint transformations of $\mathbb{R}^{n+1}$ (with at most two distinct eigenvalues, in the projective case) have a particular behavior encompassed in the following two theorems. Besides, for $n = 1$, we have a particular case of the theorem above. We denote by $\{e_0, \cdots, e_n\}$ the canonical basis of $\mathbb{R}^{n+1}$.

Theorem 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth π-periodic odd function with $f' + 1 > 0$ and write $\mathbb{R}^{n+1} = \mathbb{R}^k \oplus \mathbb{R}^\ell$. Then $T : S^n \to S^n$,*

$$T\left(\cos x\ u, \sin x\ v\right) = \left(\cos\left(x + f\left(x\right)\right) u, \sin\left(x + f\left(x\right)\right) v\right), \tag{5}$$

where $x \in \mathbb{R}$, $u \in \mathbb{R}^k$, $v \in \mathbb{R}^\ell$ and $\|u\| = \|v\| = 1$, is well defined and is the optimal transportation map among all maps sending vol_{S^n} to $T_{\#}\left(\mathrm{vol}_{S^n}\right)$.

Proof. Straightforward computations show that T is well defined. Now we prove that T is optimal. We consider first the case $k = \ell = 1$ ($n = 1$). We may suppose $u = e_0$ and $v = e_1$. Then, after the canonical identification $\mathbb{R}^2 = \mathbb{C}$ we have that $T\left(e^{ix}\right) = e^{i(x+f(x))}$, whose lift $\widetilde{T} : \mathbb{R} \to \mathbb{R}$ with $\widetilde{T}(0) = 0$ (notice that $T(1) = 1$ since f is odd) is given by $\widetilde{T}(x) = x + f(x)$. Now, a primitive of $\widetilde{T}$ is convex since $1 + f' > 0$. By the result of Cordero-Erausquin in Theorem 4 for $n = 1$, T is optimal.

We return to the general case. We consider the vector field W on S^n defined by

$$W(p) = f(x)\left(-\sin x\ u, \cos x\ v\right)$$

for $p = (\cos x\ u, \sin x\ v)$. One can check as above that W is well defined. We have that

$$T(p) = \mathrm{Exp}_p\left(W(p)\right).$$

By the result of McCann in Theorem 3 it suffices to show that W is the gradient of a function ϕ with $\phi^{cc} = \phi$. We propose

$$\phi\left(\cos x\ u, \sin x\ v\right) = \int_0^x f(s)\ ds,$$

which is well defined since $\int_a^{a+\pi} f(s)\ ds = 0$ for any $a \in \mathbb{R}$ (indeed, the fact that f is π-periodic and continuous implies that $\int_a^{a+\pi} f(s)\ ds = \int_0^{\pi} f(s)\ ds$ for any $a \in \mathbb{R}$, and also as f is odd we have that the last integral vanishes).

We compute

$$d\phi_p\left(-\sin x\ u,\cos x\ v\right) = \left.\frac{d}{dt}\right|_0 \phi\left(\left(\cos\left(x+t\right)u,\sin\left(x+t\right)v\right)\right) = \left.\frac{d}{dt}\right|_0 \int_0^{x+t} f\left(s\right)\ ds = f\left(x\right).$$

Hence $\left\langle \text{grad}_p\phi, \left(-\sin x\ u, \cos x\ v\right)\right\rangle = f\left(x\right)$.

Let $X \in T_pS^n = p^{\perp}$ with $X \perp \left(-\sin x\ u, \cos x\ v\right)$. Suppose that $X = \left(u', v'\right)$ (in particular, $u' \perp u$ and $v' \perp v$). Let α and β be smooth curves on S^{k-1} and $S^{\ell-1}$ with $\alpha\left(0\right) = u, \alpha'\left(0\right) = u'/\cos x, \beta\left(0\right) = v$ and $\beta'\left(0\right) = v'/\sin x$. We compute

$$d\phi_p\left(X\right) = \left.\frac{d}{dt}\right|_0 \phi\left(\cos x\ \alpha\left(t\right), \sin x\ \beta\left(t\right)\right) = -\left.\frac{d}{dt}\right|_0 \int_0^{x} f\left(s\right)\ ds = 0.$$

Therefore grad $\phi = W$.

Finally, we have to verify that ϕ is a c-convex potential. For this, we resort to the case $n = 1$ and use $SO\left(n\right)$-invariance. Since the sphere is compact, given $p \in S^n$ there exists $q_o \in S^n$ such that

$$\phi^c(p) = \sup_{q \in S^n}\{-c_p(q) - \phi(q)\} = -c_p(q_o) - \phi(q_o),$$

where $c_p(q) = c(p,q)$. Next we observe that p and q_o lie in the same great circle through $\left(\mathbb{R}^k \times \{0\}\right) \cap S^n$ and $\left(\{0\} \times \mathbb{R}^k\right) \cap S^n$. In fact, since q_o is the maximum of the map

$$q \in S^n \mapsto -c_p(q) - \phi(q) \in \mathbb{R},$$

it follows that $(dc_p)_{q_o} = -(d\phi)_{q_o}$. So, their kernels coincide. Suppose $q_o = (\xi, \eta)$. We saw above that Ker $(d\phi)_{q_o} = \xi^{\perp} \times \eta^{\perp}$.

Also, a straightforward computation yields that Ker $(dc_p)_{q_o} = q_o^{\perp} \cap p^{\perp}$. If $q_o \neq \pm p$, then $p \perp q_o^{\perp} \cap p^{\perp} = \xi^{\perp} \times \eta^{\perp}$, and so

$$p \in \left(\xi^{\perp} \times \eta^{\perp}\right)^{\perp} = \text{span }\left\{\left(\xi, 0\right), \left(0, \eta\right)\right\}.$$

We suppose first that p is in the circle $S := \text{span}\{e_0, e_{k+1}\} \cap S^n$ and, by the above observation, we see that

$$\phi^c(p) = \sup_{q \in S^n}\{-c_p(q) - \phi(q)\} = \max_{q \in S}\ \{-c_p(q) - \phi(q)\} = \phi_o^c(p),$$

where $\phi_o = \phi|_S$. Now, if $p \in S_p := \text{span }\left\{\left(u, 0\right), \left(0, v\right)\right\} \cap S^n$ with u, v unit vectors, let $R \in SO\left(k\right) \times SO\left(\ell\right)$ such that $R(u,0) = e_0$, $R\left(0, v\right) = e_{k+1}$. Since

$$c(p,q) = c(R(p), R(q)) \qquad \text{and} \qquad \phi \circ R(q) = \phi(q)$$

for all $p, q \in S^n$, we have that

$$\phi^c(p) = \max_{q \in S_p}\ \{-c_p(q) - \phi(q)\} = \max_{q \in S}\ \{-c_{R(p)}(q) - \phi(q)\} = \phi_o^c(R(p)),$$

where the first equality holds again by the above observation. In the same way, $\phi^{cc}(p) = \phi_o^{cc} \circ R(p)$. We recall that at the beginning of the proof we saw that $T|_S$ is optimal, so $\phi_o^{cc} = \phi_o$. Then, for $p \in S^n$ we obtain that

$$\phi^{cc}(p) = \phi_o^{cc} \circ R(p) = \phi_o \circ R(p) = \phi(p).$$

Therefore, ϕ is a c-convex potential, as we wanted to see, and the proof of the theorem is complete. □

We have a statement similar to the one of the theorem above. The proof is essentially the same.

Theorem 6. *Let $g:\mathbb{R}\to\mathbb{R}$ be a smooth 2π-periodic odd function with $g'+1>0$. Then $T:S^n\to S^n$,*

$$T(\cos x,\sin x\ v)=(\cos(x+g(x)),\sin(x+g(x))\,v)\,, \tag{6}$$

where v is a unit vector in $\mathbb{R}^n$, is well defined and is the optimal transportation map among all maps sending vol_{S^n} to $T_\#(\mathrm{vol}_{S^n})$.

Graphs of optimal maps as Lagrangian submanifolds. Let M be a Riemannian manifold and $c=\frac{1}{2}d^2$, as in the introduction. Kim, McCann and Warren found in [10] a necessary condition for a map of M to be optimal, which will be useful in the proof of Theorem 2. They proved that the graph of the optimal map is calibrated a.e. by a certain split special Lagrangian calibration on an open dense subset of $M\times M$ endowed with a certain neutral metric. In particular, the graph turns out to be Lagrangian with respect to the symplectic form ω on an open dense subset of $M\times M$ defined by $\omega=d\alpha$, with α the 1-form on the same subset of $M\times M$ given by

$$\alpha(Z)=dc(X,0)=(X,0)(c)\,,$$

where $Z_{(p,q)}=(X_p,Y_q)$ after the natural identification $T_{(p,q)}(M\times M)\approx T_pM\times T_qM$ (see [9]). It is clear that $\omega((u,0),(v,0))=\omega((0,u),(0,v))=0$ for any pair of vector fields u,v on M.

In the proposition below we describe explicitly ω for the case where M is the sphere S^n. We will use it in Proposition 9.

Proposition 7. *Let p,q be a pair of non-antipodal distinct points on S^n and let $\gamma:[0,d]\to S^n$ be the unique unit speed shortest geodesic joining p with q. Suppose that $\mathcal{B}=\{u_1,\dots,u_n\}$ is an orthonormal basis of T_pS^n with $u_1=\gamma'(0)$ and denote by v_i the parallel transport of u_i along γ from 0 to d (in particular, $v_1=\gamma'(d)$ and $\bar{\mathcal{B}}=\{v_1,\dots,v_n\}$ is an orthonormal basis of T_qS^n). Let $\mathcal{C}$ be the oriented basis of $T_{(p,q)}(S^n\times S^n)\approx T_pS^n\times T_qS^n$ obtained by juxtaposing $\mathcal{B}$ with $\bar{\mathcal{B}}$. Then*

$$\left[\omega_{(p,q)}\right]_{\mathcal{C}}=\begin{pmatrix}0 & A\\ -A & 0\end{pmatrix},$$

where $A=\mathrm{diag}\left(1,\frac{d}{\sin d}\ I_{n-1}\right)$, with I_k the $k\times k$ identity matrix.

Proof. Let U,V be vector fields defined on open neighborhoods of p and q in S^n, respectively. Denote $u=U_p$, $v=V_q$. If σ and τ are curves in S^n with $\sigma(0)=p$, $\sigma'(0)=u$, $\tau(0)=q$ and $\tau'(0)=v$, then

$$\begin{aligned}\omega_{(p,q)}((u,0),(0,v)) &= d\alpha_{(p,q)}((u,0),(0,v))=-(0,v)_{(p,q)}(dc(U,0))\\ &= -\left.\frac{d}{dt}\right|_0 dc_{(p,\tau(t))}(U_p,0)=-\left.\frac{d}{dt}\right|_0\left.\frac{d}{ds}\right|_0 c(\sigma(s),\tau(t))\,,\end{aligned}$$

where the second equality follows from the fact that $\alpha(0,V)=0$ and $[(U,0),(0,V)]=0$.

Since S^n is two-point homogeneous, we may suppose without loss of generality that $p=e_0$, $q=\cos d\ e_0+\sin d\ e_1$ with $0<d<\pi$ (so $\gamma(t)=\cos t\ e_0+\sin t\ e_1$) and $u_i=e_i$

$(i = 1, \ldots, n)$. Let σ_i, τ_j be the geodesics in S^n with $\sigma_i(0) = p$ and $\sigma_i'(0) = u_i$, $\tau_j(0) = q$ and $\tau_j'(0) = v_j$. We have $\sigma_1 = \gamma$, $\tau_1(t) = \gamma(t + d)$ and

$$\sigma_i(s) = \cos s\ e_0 + \sin s\ e_i, \qquad \tau_2(t) = \cos t\,(\cos d\ e_0 + \sin d\ e_1) + \sin t\ e_2.$$

We have $A_{ij} = \omega((u_i, 0), (0, v_j))$. By the $SO(n-1)$-symmetries of S^n fixing the trajectory of γ it suffices to show that $A_{11} = 1$, $A_{12} = A_{21} = A_{32} = 0$ and $A_{22} = \frac{d}{\sin d}$. This follows from standard facts about Jacobi fields on the sphere along γ (see for instance [6]). □

3 Proofs of the theorems

Proof of Theorem 1. Let $S \in G$. Thus, $S = \exp(A)O$, where $A \in \mathfrak{p}$ (see (2)) and $O \in SO(n+1)$. Clearly, O preserves vol_{S^n}. So, by the uniqueness of the Brenier-McCann polar factorization (see the introduction), we have to prove that $\exp(A)$ is the optimal transportation map between vol_{S^n} and $S_{\#}(\mathrm{vol}_{S^n})$. Without lost of generality we can take $A = a\begin{pmatrix} 0 & e_0^t \\ e_0 & 0 \end{pmatrix} \in \mathfrak{p}$, where $a > 0$. In fact, if $R \in SO(n)$ satisfies $R(ae_0) = v$, then A and $\begin{pmatrix} 0 & v^t \\ v & 0 \end{pmatrix}$ are conjugate by the isometry $\mathrm{diag}\,(1, R)$ of S^n, and isometries preserve optimality.

Next we compute $\exp(A)$ as a map of the sphere. Putting

$$H_t = \begin{pmatrix} \cosh at & \sinh at \\ \sinh at & \cosh at \end{pmatrix}$$

we have that $\exp(tA) = \mathrm{diag}\,(H_t, I_n)$.

Let $q = (u_0, \ldots, u_n) \in S^n$ and let $\gamma(t) = \exp(tA)(q)$. By (1),

$$\gamma(t) = (\sinh at + u_0 \cosh at, u_1, \ldots, u_n) / h(u_0, t)$$

with $h(u, t) = u \sinh at + \cosh at$. Now, γ is the integral curve through q of the vector field V on the sphere defined by

$$V(p) = \left.\frac{d}{dt}\right|_0 \exp(tA)(p) = ae_0 - \langle ae_0, p\rangle\, p.$$

Hence,

$$\left\|\gamma'(t)\right\| = \|V(\gamma(t))\| = \frac{a\sqrt{1 - u_0^2}}{h(u_0, t)}.$$

Since γ lies on a meridian through e_0, which is a geodesic, and V vanishes at $\pm e_0$, the distance $D(q)$ from q to $\exp(A)(q)$ is the length of $\gamma|_{[0,1]}$. Hence, $\exp(A)(q) = \mathrm{Exp}_q(U(q))$, where $U(q)$ is a tangent vector pointing in the direction of $\gamma'(0)$ with $\|U(q)\| = D(q)$. We have

$$D(\cos x, \sin x\ v) = \int_0^1 \frac{a\,|\sin x|}{h(\cos x, t)}\, dt.$$

The appropriate choice of sign yields that $\exp(A)$ equals T as in (6) with

$$g(x) = -\int_0^1 \frac{a \sin x}{h(\cos x, t)}\, dt. \tag{7}$$

Now we apply Theorem 6 to prove that $\exp(A)$ is optimal. Since g is odd and 2π-periodic, we have to verify only that $g' + 1 > 0$. We compute

$$g'(x) = -\int_0^1 \frac{d}{dx}\frac{a \sin x}{h(\cos x, t)}\,dt = -\int_0^1 \frac{h_t(\cos x, t)}{h(\cos x, t)^2}\,dt = \frac{1}{\cos x \sinh a + \cosh a} - 1,$$

where $h_t = \partial h / \partial t$. Hence $g' + 1 > 0$, as desired. □

Proof of Theorem 2. The polar decomposition of $S \in GL_+(n+1)$ is $S = PO$, where $O \in SO(n+1)$ and P is a positive definite self adjoint linear transformation. As in the conformal case, we may suppose that P is diagonal and the question is when the induced operator $T(q) = P(q) / \|P(q)\|$ on S^n is optimal.

We consider first the case when P has at least three distinct eigenvalues. As a consequence of the necessary condition for optimality stated in the preliminaries we have that T is not optimal, since otherwise the graph of T would be Lagrangian a.e., contradicting Proposition 9 below.

So now we assume that P has exactly two distinct eigenvalues, say λ and μ, with respective eigenspaces of dimensions k and ℓ (the case when P is a multiple of the identity is trivial). We may suppose that

$$P = \operatorname{diag}\left(e^{\frac{a}{2}} I_k, e^{-\frac{a}{2}} I_\ell\right) = \exp B,$$

where $B = \frac{a}{2}\operatorname{diag}(I_k, -I_\ell)$ for some $a \in \mathbb{R}$. In fact, T does not change if we take instead of P a positive multiple cP of it (we chose $c = 1/\sqrt{\lambda\mu}$ and $a = \log(\lambda/\mu)$). We compute

$$T(\cos x\ u, \sin x\ v) = \frac{\left(e^{a/2}\cos x\ u, e^{-a/2}\sin x\ v\right)}{\left(e^a \cos^2 x + e^{-a}\sin^2 x\right)^{1/2}}, \tag{8}$$

where $u \in S^{k-1}$ and $v \in S^{\ell-1}$. We see that T preserves the meridian $S^n \cap$ span $\{u, v\}$ and moreover it has the form (5) for some π-periodic odd function f. Now, identifying span $\{u, v\}$ with $\mathbb{C}$ $(u = 1, v = i)$ we have by Lemma 8 below that

$$e^{i2(x+g(x))} = e^{i(2x+f(2x))}$$

where g is the 2π-periodic odd function in (7). Since $f(0) = g(0) = 0$, we have that $f(x) = 2g\left(\frac{1}{2}x\right)$ for all x. Now, $g' + 1 > 0$ by the proof of Theorem 1, hence $f' + 1 > 0$. Therefore, T is optimal by Theorem 5. □

Next we state the lemma we used in the proof above. It is well known that there is a double covering morphism $PSL(2,\mathbb{R}) \to O_o(1,2)$. We only need the morphism restricted to some subgroups isomorphic to the circle. Let

$$A = \frac{a}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad B = a\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $a > 0$. Then $\exp A$ and $\exp B$ induce a projective and a conformal map of S^1 as in (3) and (1), respectively. The following lemma asserts that the first one double covers the second one.

Lemma 8. *Let A and B be as above and let $\rho : S^1 \to S^1$, $\rho(x,y) = \left(x^2 - y^2, 2xy\right)$ (that is, $\rho(z) = z^2$ after the identification $\mathbb{R}^2 = \mathbb{C}$). Then the following diagram commutes*

$$\begin{array}{ccc} S^1 & \stackrel{\exp A}{\longrightarrow} & S^1 \\ \downarrow \rho & & \downarrow \rho \\ S^1 & \stackrel{\exp B}{\longrightarrow} & S^1, \end{array}$$

where we are considering the induced conformal and projective maps of the circle.

Proof. The statement is well-known; we sketch the proof for the sake of completeness. We have that the projective map on S^1 induced by $\exp(A)$ applied to $(\cos x, \sin x)$ equals the right hand side of (8) with $u = v = e_1 = 1$. We also have

$$\exp(B) \cdot (\cos x, \sin x) = \frac{(\cosh a \cos x + \sinh a, \sin x)}{\sinh a \cos x + \cosh a}.$$

Now, a straightforward computation yields the commutativity of the diagram. □

We used the following proposition in the proof of Theorem 2. It involves the symplectic form ω considered in the preliminaries.

Proposition 9. *Let P be a positive definite self adjoint operator of $\mathbb{R}^{n+1}$ with at least three distinct eigenvalues and let T be the projective map induced by P on S^n. Then there exists an open dense subset W of S^n such that* graph (dT_p) *is* not *a Lagrangian subspace of $T_pS^n \times T_{T(p)}S^n$ with respect to $\omega_{(p,T(p))}$ for any $p \in W$. In particular* graph (T) *is not a Lagrangian submanifold a.e. of $S^n \times S^n$.*

Proof. We may suppose without lost of generality that $Pe_i = \lambda_i e_i$ for $i = 0, \ldots, n$ and that λ_0, λ_1, λ_2 are positive and pairwise different. Let W be the open subset of S^n consisting of all points whose coordinates are all different from zero. Let $p \in W$ and let $q = T(p) \in W$ (which is different from p and $-p$). Let $\mathcal{B}$ and $\bar{\mathcal{B}}$ be as in Proposition 7. Let $\mathcal{B}'$ be the ordered basis consisting of all the elements of the basis $\bar{\mathcal{B}}$, except that v_1 is substituted for $V_1 = \frac{\sin d}{d} v_1$, and let $\mathcal{C}'$ be the juxtaposition of $\mathcal{B}$ and $\mathcal{B}'$. Then the matrix of $\omega_{(p,q)}$ with respect to $\mathcal{C}'$ is a multiple of $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. It is well-known that the graph of a linear transformation $L : T_pS^n \to T_qS^n$ is Lagrangian for $\omega_{(p,q)}$ if and only if the matrix of L with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$ is symmetric.

A straightforward computation yields

$$dT_p(v) = \frac{1}{\|P(p)\|} \left(P(v) - \langle P(v), q \rangle q\right) = \frac{1}{\|P(p)\|} \mathrm{pr}_{q^\perp}(P(v)) \tag{9}$$

for any $p \in S^n$ and $v \in T_pS^n$, where $\mathrm{pr}_{q^\perp}$ is the orthogonal projection onto $q^\perp$. The vectors $u_1 \in T_pS^n$ and $v_1 \in T_qS^n$ as in Proposition 7 are

$$u_1 = \frac{q - \langle p, q \rangle p}{\|q - \langle p, q \rangle p\|} \qquad \text{and} \qquad v_1 = \frac{\langle p, q \rangle q - p}{\| \langle p, q \rangle q - p\|}.$$

Now we write $p = (x, y)$ with $x \in \mathbb{R}^3$ and $P(x, y) = (P_1(x), P_2(y))$, where $P_1 = \operatorname{diag}(\lambda_0, \lambda_1, \lambda_2)$. It is easy to verify that we can take vectors $u_2 \in T_pS^n$ and $v_2 \in T_qS^n$ as in Proposition 7 as follows:

$$u_2 = v_2 = (P_1(x) \times x, 0) / \|P_1(x) \times x\|.$$

Now we verify that the matrix of dT_p with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$ (recall that $V_1 = \frac{\sin d}{d} v_1$) is not symmetric. We call

$$a = \|P(p)\|, \qquad b = \|q - \langle p, q\rangle p\| = \|\langle p, q\rangle q - p\| \qquad \text{and} \qquad c = \|P_1(x) \times x\|.$$

Straightforward computations using (9) yield that

$$\begin{aligned} \langle dT_p(u_1), v_2\rangle &= \frac{1}{abc} \left\langle P_1^2(x), P_1(x) \times x \right\rangle \\ \langle dT_p(u_2), V_1\rangle &= \frac{\sin d}{d} \frac{1}{abc} \langle q, p\rangle \left\langle P_1^2(x), P_1(x) \times x \right\rangle. \end{aligned}$$

We compute $\left\langle P_1^2(x), P_1(x) \times x \right\rangle = (\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2) x_0 x_1 x_2 \neq 0$. Hence $\langle dT_p(u_1), v_2\rangle = \langle dT_p(u_2), V_1\rangle$ if and only if $\langle q, p\rangle \sin d = d$, or equivalently

$$\sin(2d) = 2 \cos d \sin d = 2d,$$

which holds only for $d = 0$. Therefore the matrix of dT_p with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$ is not symmetric for all $p \in W$, as desired. □

References

[1] J. D. Benamou, Y. Brenier, A computational Fluid Mechanics solution to the Monge-Kantorovich mass transfer problem, Numer. Math. 84 (2000) 375–393.

[2] L. Brasco, A survey on dynamical transport distances, J. Math. Sci. (N.Y.) 181 (2012) 755–781.

[3] Y. Brenier, Décomposition polaire et réarrangement monotone des champs de vecteurs, C.R. Acad. Sci. Paris Sér. I Math. 305 (1987) 805–808.

[4] G. Buttazzo, Evolution models for mass transportation problems, Milan J. Math. 80 (2012) 47–63.

[5] D. Cordero-Erausquin, Sur le transport de mesures périodiques, Comptes Rendus de l'Académie des Sciences-Series I-Mathematics 329 (1999) 199–202.

[6] D. Cordero-Erausquin, R. McCann, M. Schmuckenschläger, A Riemannian interpolation inequality à la Borell, Brascamp and Lieb, Invent. Math. 146 (2001) 219–257.

[7] D. Emmanuele, M. Salvai, Force free Moebius motions of the circle, J. Geom. Symmetry. Phys. 27 (2012) 59–65.

[8] U. Hertrich-Jeromin, Introduction to Möbius differential geometry. London Mathematical Society Lecture Note Series, 300. Cambridge University Press, Cambridge, 2003.

[9] Y. H. Kim, R. McCann, Continuity, curvature, and the general covariance of optimal transportation, J. Eur. Math. Soc. 12 (2010) 1009–1040.

[10] Y. H. Kim, R. McCann, M. Warren, Pseudo-Riemannian geometry calibrates optimal transportation, Math. Res. Lett. 17 (2010) 1183–1197.

[11] G. Loeper, Regularity of optimal maps on the sphere: The quadratic cost and the reflector antenna, Arch. Rational Mech. Anal. 199 (2011) 269–289.

[12] R. McCann, Polar factorization of maps on Riemannian manifolds, Geom. Funct. Anal. 11 (2001) 589–608.

[13] M. Salvai, Force free conformal motions of the sphere, Differential Geom. Appl. 16 (2002) 285–292.

[14] M. M. Lazarte, M. Salvai, A. Will, Force free projective motions of the sphere, J. Geom. Phys. 57 (2007) 2431–2436.

[15] C. Villani, Optimal transport. Old and new. Grundlehren der Mathematischen Wissenschaften, 338. Springer-Verlag, Berlin, 2009.