Eur. Phys. J. B (2018) 91: 103
https://doi.org/10.1140/epjb/e2018-80624-9

THE EUROPEAN PHYSICAL JOURNAL B

Regular Article

Coupled Brownian motors*

Julián I. Peña Rosselló[1,a], Roberto R. Deza[1], and Horacio S. Wio[2,3]

[1] IFIMAR (UNMdP&CONICET), Univ. Nac. de Mar del Plata, Deán Funes 3350, B7602AYL Mar del Plata, Argentina
[2] IFISC (UIB&CSIC), Univ. de les Illes Balears, Carretera de Valldemosa, km 7,5, 07122 Palma de Mallorca, Spain
[3] IFCA (UC&CSIC), Univ. de Cantabria, Avda. de los Castros, s/n, 39005 Santander, Spain

Received 31 October 2017 / Received in final form 20 February 2018
Published online 6 June 2018

Abstract. A set of coupled Brownian motors—resulting from a noise-induced phase transition generated through an entropic mechanism, and formerly studied in mean-field approximation—is numerically integrated by Heun's method. The results add much insight to the mean-field ones, allowing interpretation of the underlying mechanisms. It turns out that the low-noise boundary of the ordered phase obtained in mean-field approximation lies near a region where the order parameter is negative. Hints of bistability and negative mobility around $F = 0$ are seen in two cases, and anomalous hysteresis is confirmed in one case.

1 Introduction

Technology has always relied on phase transitions. Although the steam engine alone suffices to justify this assertion, steel and ferromagnets are not less relevant examples. From a theoretical viewpoint, what defines a phase transition is ergodicity breakdown. For finite systems, this observation provides a sufficient criterion for discriminating phases.

Following the stochastic treatment of Brownian motion by Einstein [1] and Langevin [2], it was long expected that external noise could also trigger phase transitions [the term "noise-induced phase transitions" (NIPT) has been coined]. In fact, additive noise (equivalent to temperature in its effects, according to Einstein [1]) was shown in 1999 to induce a first-order NIPT [3]. But external noise can also be multiplicative, i.e. make the system's parameters into stochastic processes.

Despite our efforts to cast bifurcations into normal forms, the nonlinear world is highly diverse, and so is the repertoire of noise effects on nonlinear systems. As a rule, the (nonrigorous albeit effective in simpler cases) "mean-field approximation"—whereby one replaces a fluctuating variable by its mean—gives wrong results in more complex cases. It has become customary to speak of "constructive effects of the noise" or "noise-induced phenomena", a field which continues to amaze us with counterintuitive results after decades of intensive research, both in lumped and extended systems [4–6].

The evidence of (multiplicative) noise-induced transitions (NIT) in zero-dimensional stochastic systems [7] renewed the expectation that noise of this kind be capable of inducing phase transitions. A survey by Van den Broeck et al. yielded a surprising result: a (second-order, re-entrant with noise intensity) NIPT was obtained, but precisely for those systems for which NIT are ruled out [8,9]. The mechanism whereby these NIPT took place was the "freezing" by coupling of a short-time instability. So whereas for a zero-dimensional system

$$\dot{x} = f(x) + g(x)\,\xi(t), \quad \langle \xi(t)\xi(t')\rangle = \sigma^2\,\delta(t-t'),$$

undergoing NIT, the extrema $\bar{x}$ of the steady-state pdf must obey $f(\bar{x}) - \frac{\sigma^2}{2} g(\bar{x})g'(\bar{x}) = 0$, for the extended system

$$\dot{x}_i = f(x_i) + g(x_i)\,\xi_i(t) - \frac{D}{2d}\sum_{j\in n(i)}(x_i - x_j),$$
$$\langle \xi_i(t)\xi_j(t')\rangle = \sigma^2\,\delta_{ij}\,\delta(t-t'),$$

it is $\dot{\bar{x}} = f(\bar{x}) + \frac{\sigma^2}{2} g(\bar{x})g'(\bar{x})$. The foregoing results assumed the driving noise to be white. This is of course an idealization, its correlation function being a Dirac's delta function. For colored driving noise, the NIPT turned out to be also re-entrant with self-correlation time [10,11].

Shortly after, a second class of NIPT was discovered, supported by the so-called "entropic mechanism" [12–14]. Let $u(\mathbf{x}, t)$ be a relaxational flow with field-dependent kinetic coefficient $\Gamma(u)$ in a free-energy $\mathcal{F}[u]$. Then the noise term is made to fulfill the fluctuation–dissipation relation:

$$\partial_t u(\mathbf{x}, t) = -\Gamma(u)\,\frac{\delta \mathcal{F}[u]}{\delta u(\mathbf{x}, t)} + \Gamma^{1/2}(u)\,\eta(\mathbf{x}, t). \qquad (1)$$

* Contribution to the Topical Issue "The Physics of Micro-Energy Use and Transformation", edited by Luca Gammaitoni.

[a] e-mail: julian@ifimar-conicet.gob.ar

Again $\eta(\mathbf{x},t)$ is Gaussian, with $\langle\eta(\mathbf{x},t)\rangle = 0$ and

$$\langle\eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle = 2\sigma^2\,\delta^d(\mathbf{x}-\mathbf{x}')\,\delta(t-t').$$

The virtue of this proposal is that it provides an analytic way of finding the stationary probability distribution function for fields, namely

$$P_{\mathrm{st}}[u] \propto \exp\left\{-\frac{\mathcal{F}_{\mathrm{eff}}[u]}{\sigma^2}\right\}, \tag{2}$$

with an "effective potential" functional

$$\mathcal{F}_{\mathrm{eff}}[u] = \mathcal{F}[u] + \beta\int_R \mathrm{d}x \ln\Gamma(u(x)),$$

and $\beta \propto \sigma^2$. If the finite region R is a d-dimensional square lattice with mesh size Δx, then $\beta = \sigma^2/(2\Delta x^d)$ in Stratonovich's interpretation [12,13]. This proposal yields a handy tool for e.g. stabilizing nanopatterns [15–17] or inducing pinning and localized states [18,19], in adsorbed monolayers. Its compatibility with NITs has been shown experimentally in a recent work [20].

An innovative application within the active field of Brownian transport [21–23] was reported in [24]. A set of coupled rotators

$$\dot{x}_i = \left[-\frac{\partial U_i}{\partial x_i} - \frac{1}{N}\sum_{j=1}^{N} K(x_i - x_j)\right] + \sqrt{2T}\,\xi_i(t), \tag{3}$$

with

$$U_i(x,t) = [V(x) - Fx] + W(x)\sqrt{2Q}\,\eta_i(t), \tag{4}$$

$$V(x) = W(x) = -\cos x - A\cos 2x, \tag{5}$$

$$K(x) = K_0 \sin x \ (\text{with } K_0 > 0) \tag{6}$$

was shown to undergo a NIPT (driven by short-time instability) to a phase with nonzero mean particle current $\langle\dot{x}\rangle$, which thus features spontaneous (not built-in) ratchet-like properties. When $A > 0$ in equation (5), the direction of $\langle\dot{x}\rangle$ is opposite to that of symmetry breaking in the stationary probability distribution $P^{\mathrm{st}}(x)$, which leads in turn to negative zero-bias conductance and anomalous hysteresis.

The noises' self-correlation, addressed in [25–28], turned out to have the following effects:

- The re-entrance as a function of Q tends to disappear;
- The onset of the broken-symmetry phase occurs for larger Q and smaller K_0;
- The transition becomes re-entrant as a function of K_0;
- The "interaction-driven regime" (IDR), where anomalous hysteresis occurs, sets in for lower K_0 and becomes almost independent of Q;
- There appear new solutions to the mean-field equations in the IDR, associated with a more complex (anomalous) hysteretic behavior;
- The efficiency ε depends strongly on Q, K_0, the strength F of the load force, and τ; fixing the former three parameters, there is an optimal value of τ regarding ε.

The question of whether the entropic mechanism could give rise to similar features, was addressed in [29]. In Section 2, we briefly review the model and its mean-field solution. In Section 3, we undertake its numerical simulation by Heun's method and expose our results. In Section 4 we draw our conclusions.

2 The model and its mean-field solution

In [29], the set of N coupled stochastic equations

$$\dot{\varphi}_i = -\Gamma(\varphi_i)\left[G(\varphi_i) - F - \frac{K}{2}(\varphi_{i+1}+\varphi_{i-1}-2\varphi_i)\right] + \Gamma^{1/2}(\varphi_i)\,\xi_i(t), \tag{7}$$

(interpreted in the sense of Itô) was considered. The $\xi_i(t)$ are assumed to be Gaussian, with

$$\langle\xi_i(t)\rangle = 0 \text{ and } \langle\xi_i(t)\xi_k(t')\rangle = \sigma\,\delta_{ik}\delta(t-t').$$

The $\xi_i(t)$ are multiplicative: they couple to the function

$$\Gamma(\varphi_i) = 1 + \beta\cos^2\varphi_i,$$

so fluctuations are more pronounced at higher $\cos^2\varphi_i$ regions.

In the deterministic part, function $\Gamma(\varphi)$ multiplies the sum of:

- A periodic force $G(\varphi_i) = \sin\varphi_i + 2A\sin 2\varphi_i$ (with $A > 0$);
- A (nearest neighbor) diffusive term with strength K; and
- A constant F, included for the analysis of Brownian transport.

The mean-field approach proceeds—as usual—by replacing the fluctuating φ_{i+1} and φ_{i-1} by a constant field value M, playing the role of an order parameter ($M := \langle\varphi\rangle$). The system (7) then reduces to a single equation

$$\dot{\varphi} = -\Gamma(\varphi)\frac{\partial\mathcal{V}_{\mathrm{eff}}(\varphi,M)}{\partial\varphi} + \Gamma^{1/2}(\varphi)\,\xi(t), \tag{8}$$

with the potential

$$\mathcal{V}_{\mathrm{eff}}(\varphi,M) = -[\cos\varphi + A\cos 2\varphi + F\varphi] + \frac{K}{2}(\varphi - M)^2. \tag{9}$$

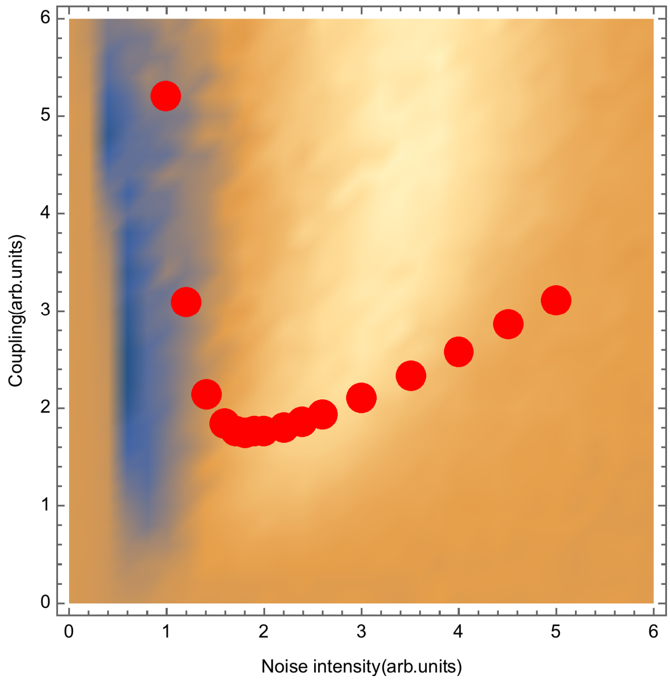

Fig. 1. Color-code density plot of the order parameter $\langle\varphi\rangle$ as a function of σ and K, for $F = 0$. Clear: $\langle\varphi\rangle > 0$; dark: $\langle\varphi\rangle < 0$. The superimposed dots belong to the mean-field phase diagram (Fig. 2 in [29]).

The associated Fokker–Planck equation in Itô's prescription is [6]

$$\partial_t P(\varphi,t;M) = \frac{\partial}{\partial\varphi}\left[\Gamma(\varphi)\frac{\partial\mathcal{V}_{\rm eff}(\varphi,M)}{\partial\varphi}\,P(\varphi,t;M)\right] + \frac{1}{2}\frac{\partial^2}{\partial\varphi^2}\left[\sigma\Gamma(\varphi)\,P(\varphi,t;M)\right]. \tag{10}$$

For periodic variables, $J = 0$ is not the only stationary possibility. The solution for $J = \mathrm{const}(M) \neq 0$ is essentially

$$\exp\left\{-\frac{2}{\sigma}[\mathcal{V}_{\rm eff}(\varphi,M) - \mathcal{V}_{\rm eff}(-\pi,M)]\right\},$$

times a prefactor [29],[1] and so is $J(M)$. The unknown mean-field value M can be extracted from the stationary probability using the self-consistency relation

$$M = \langle\varphi\rangle = \int_{-\pi}^{\pi} d\varphi\,\varphi\,P_{\rm st}(\varphi,M). \tag{11}$$

In the case of a short time instability [24], the phase diagram in $\sigma - K$ space was obtained by solving

$$1 = \int_{-\pi}^{\pi} d\varphi\ \varphi\ \frac{\partial}{\partial M}P_{\rm st}(\varphi,M)\bigg|_{M=0}. \tag{12}$$

[1] See [30] and Appendix 1 in [25] for a detailed derivation.

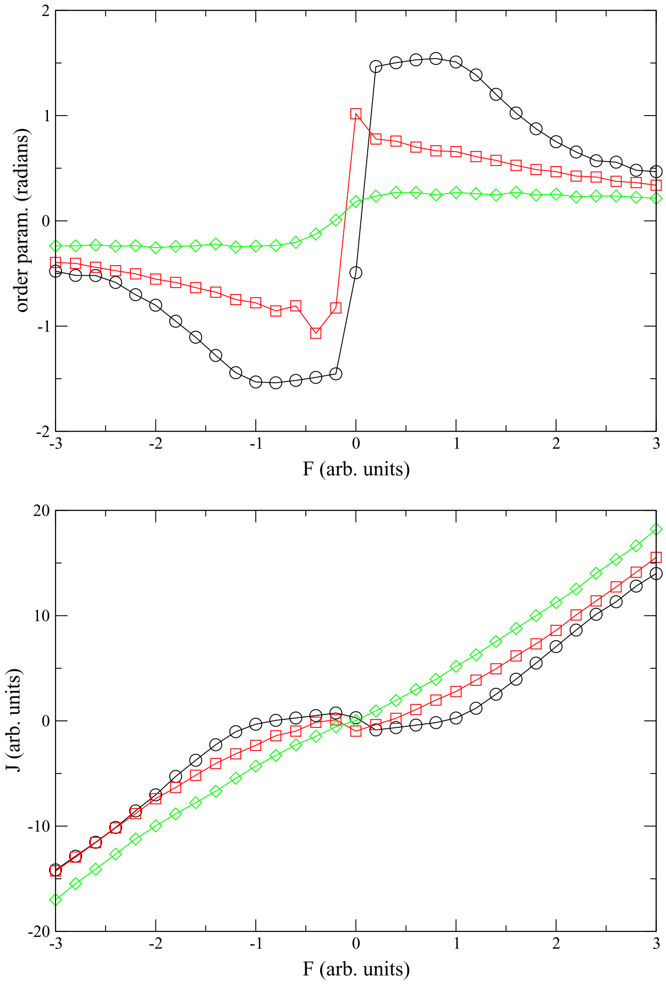

Fig. 2. (a) $\langle\varphi\rangle$ vs. F, (b) $\langle J\rangle$ vs. F, for $K = 2.5$ and $\sigma = 1$ (circles), 3 (squares), and 5 (rhombs), to allow comparison with Fig. 3 in [29]. Random initial condition.

In the present case [29], the equation to solve is

$$1 = \int_{-\pi}^{\pi} d\varphi\ \varphi\ \frac{\partial}{\partial M}P_{\rm st}(\varphi,M)\bigg|_{M=M^*}, \tag{13}$$

where $M^* \neq 0$ is the solution of

$$M^* = \langle\varphi\rangle = \int_{-\pi}^{\pi} d\varphi\,\varphi\,P_{\rm st}(\varphi,M^*). \tag{14}$$

Hence, two equations have to be simultaneously solved. The current was calculated in [29] as

$$J = -\frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi\,\Gamma(\varphi)\frac{\partial\mathcal{V}_{\rm eff}(\varphi,M)}{\partial\varphi}\,P_{\rm st}(\varphi,M). \tag{15}$$

We refer the reader to [29] for the detailed mean-field results (here we have just reproduced the dots in Fig. 2 of that work).

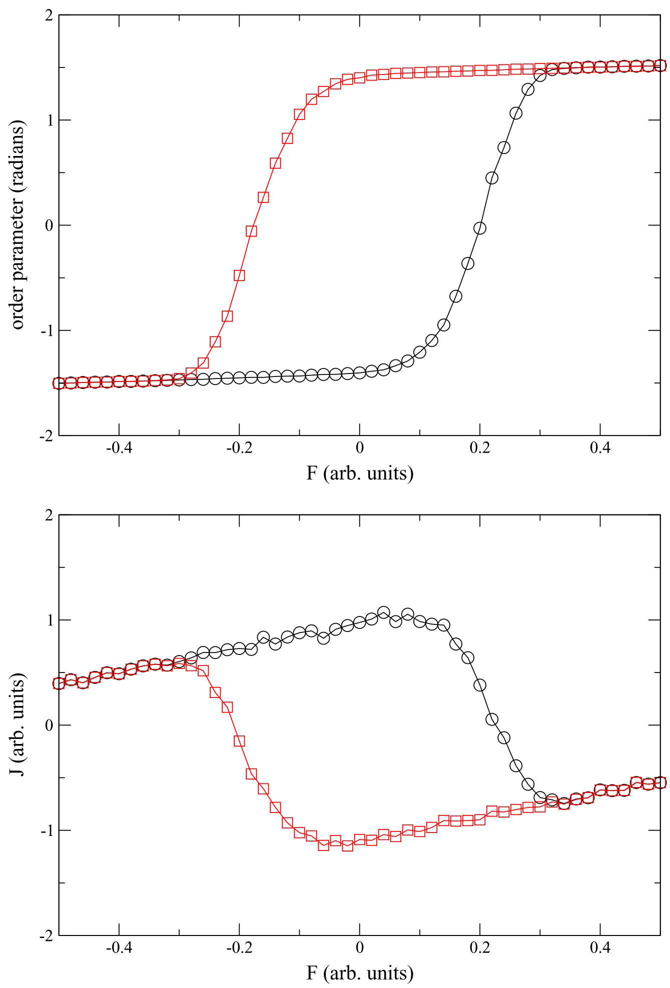

Fig. 3. Zoom of Figure 2, for different initial conditions: $-\pi/2$ (circles) and $+\pi/2$ (squares).

3 Numerical results

In this work, we undertake the numerical simulation of system (7) by Heun's method, in Itô's interpretation. This implies adding $\frac{\sigma\beta}{4}\sin 2\varphi_i$ to the drift, in the algorithm described in e.g. [31]. The fixed parameters are consistent with those adopted in [29], namely $A = 0.15$ and $\beta = 20$. Moreover, we have chosen $N = 64$ and $\Delta t = 0.01$. We have monitored the order parameter $\langle\varphi\rangle$, and the current $J := \langle -\Gamma(\varphi_i)\left[G(\varphi_i) - F - \frac{K}{2}(\varphi_{i+1} + \varphi_{i-1} - 2\varphi_i)\right]\rangle$, averaged over 1000 noise realizations.

Figure 1 is a color-code density plot of $\langle\varphi\rangle$ as a function of σ and K, for $F = 0$. The $\langle\varphi\rangle > 0$ (lighter) region qualitatively supports the phase diagram obtained in mean-field approximation (Fig. 2 in [29], to which the superimposed dots belong). A feature not detected in mean-field approximation, namely the relatively narrow region with $\langle\varphi\rangle < 0$ (dark), now becomes apparent.

Figure 2 plots $\langle\varphi\rangle$ and J as functions of F, for $K = 2.5$ and $\sigma = 1$, 3, and 5 (as in Fig. 3 of [29]). The large jumps in $\langle\varphi\rangle$ and negative mobilities (slopes of J) are a hint of bistability, and thus hysteresis. These features are also qualitatively consistent with the corresponding plots in Figure 3 of [29]. We now learn that the aforementioned effects are most evident where $\langle\varphi\rangle < 0$.

Figure 3 is a zoom of Figure 2, for different initial conditions: $-\pi/2$ (circles) and $+\pi/2$ (squares). The occurrence of hysteresis in both $\langle\varphi\rangle$ and J is apparent. Moreover, hysteresis in J is anomalous, as in [24] (the cycle is toured around clockwise).

4 Conclusions

We have undertaken the numerical simulation of a system studied formerly in mean-field approximation [29]: a set of coupled Brownian motors, generated through a NIPT produced through an entropic mechanism. The results of the present work are not only consistent with those in [29], but they add much insight. It now becomes clear that the point at $K = 2.5$, $\sigma = 1$ is well inside a region where $\langle\varphi\rangle < 0$, and that the entropic NIT also gives rise to negative mobility and anomalous hysteresis, as in [24].

The fact that the system can transit from a pretty linear regime to one displaying negative mobility and anomalous hysteresis just by decreasing the noise intensity, may find interesting applications in nanotechnology. If, for instance, dynamical states are used to encode logical states, then one might devise logical gates that perform different logical functions, depending on the noise intensity (noise-controlled multipurpose logic gates [32–34]). The results reported in this work must be regarded as preliminary, and requiring more refined study.

Support by CSIC, UC and UIB of Spain, and CONICET and UNMdP of Argentina, is acknowledged.

Author contribution statement

This work was proposed by H.S. Wio. Its main author, J.I. Peña Rosselló, was supervised by R.R. Deza.

References

1. A. Einstein, Ann. Phys. (Leipzig) **17**, 549 (1905)
2. P. Langevin, C.R. Acad. Sci. (Paris) **146**, 530 (1908)
3. A.A. Zaikin, J. García-Ojalvo, L. Schimansky-Geier, Phys. Rev. E **60**, R6275 (1999)
4. J. García-Ojalvo, J.M. Sancho, *Noise in spatially extended system* (Springer, New York, 1999)
5. F. Sagués, J.M. Sancho, J. García-Ojalvo, Rev. Mod. Phys. **79**, 829 (2007)
6. H.S. Wio, R.R. Deza, J.M. López, *An introduction to stochastic processes and nonequilibrium statistical physics*, revised edn. (World Scientific, Singapore, 2012)
7. W. Horsthemke, R. Lefever, *Noise induced transitions* (Springer, Berlin, 1984)
8. C. Van den Broeck, J.M.R. Parrondo, R. Toral, Phys. Rev. Lett. **73**, 3395 (1994)
9. C. Van den Broeck, J.M.R. Parrondo, R. Toral, R. Kawai, Phys. Rev. E **55**, 4084 (1997)
10. S.E. Mangioni, R.R. Deza, H.S. Wio, R. Toral, Phys. Rev. Lett. **79**, 2389 (1997)

11. S.E. Mangioni, R.R. Deza, H.S. Wio, R. Toral, Phys. Rev. E **61**, 223 (2000)
12. M. Ibañes, J. García-Ojalvo, R. Toral, J.M. Sancho, Phys. Rev. Lett. **87**, 020601 (2001)
13. O. Carrillo, M. Ibañes, J. García-Ojalvo, J. Casademunt, J.M. Sancho, Phys. Rev. E **67**, 046110 (2003)
14. J. Buceta, M. Ibañes, J.M. Sancho, K. Lindenberg, Phys. Rev. E **67**, 021113 (2003)
15. S.E. Mangioni, Physica A **389**, 1799 (2010)
16. S.E. Mangioni, R.R. Deza, Phys. Rev. E **82**, 042101 (2010)
17. S.E. Mangioni, R.R. Deza, Physica A **391**, 4191 (2012)
18. S.E. Mangioni, R.R. Deza, Phys. Rev. E **92**, 032116 (2015)
19. S.E. Mangioni, R.R. Deza, Acta Phys. Pol. B **48**, 849 (2017)
20. G.A. Zarza, S.E. Mangioni, J. Fernández Acevedo, R.R. Deza, Phys. Rev. E **95**, 052143 (2017)
21. P. Reimann, Phys. Rep. **290**, 149 (1997)
22. P. Hänggi, F. Marchesoni, Rev. Mod. Phys. **81**, 387 (2009)
23. J. Um, H. Hong, F. Marchesoni, H. Park, Phys. Rev. Lett. **108**, 060601 (2012)
24. P. Reimann, R. Kawai, C. Van den Broeck, P. Hänggi, Europhys. Lett. **45**, 545 (1999)
25. S.E. Mangioni, R.R. Deza, H.S. Wio, Phys. Rev. E **63**, 041115 (2001)
26. S.E. Mangioni, R.R. Deza, H.S. Wio, Phys. Rev. E **66**, 051106 (2002)
27. H.S. Wio, S.E. Mangioni, R.R. Deza, Physica D **168–169**, 184 (2002)
28. S.E. Mangioni, R.R. Deza, H.S. Wio, in *Instabilities and nonequilibrium structures IX*, edited by O. Descalzi, J. Martínez, S. Rica (Kluwer, Dordrecht, 2004), pp. 185–194
29. H.S. Wio, in *22nd International Conference on Noise and Fluctuations (ICNF)* (IEEE, Piscataway, NJ, 2013)
30. C.W. Gardiner, *Handbook of stochastic methods*, 4th edn. (Springer, Berlin, 2009)
31. M. San Miguel, R. Toral, in *Instabilities and nonequilibrium structures VI*, edited by E. Tirapegui, J. Martínez, R. Tiemann (Springer, Dordrecht, 2000), pp. 35–127
32. M. Das, D. Mondal, D.S. Ray, Phys. Rev. E **86**, 041112 (2012)
33. D.N. Guerra, A.R. Bulsara, W.L. Ditto, S. Sinha, K. Murali, P. Mohanty, Nano Lett. **10**, 1168 (2010)
34. G.S. Snider, P.J. Kuekes, IEEE Trans. Nanotech. **5**, 129 (2006)