On the global dynamic behaviour for a generalized haematopoiesis model with almost periodic coefficients and oscillating circulation loss rate

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Abstract

A generalized nonlinear non-autonomous model for the hematopoiesis (cell production) with several delays and an oscillating circulation loss rate is studied. We prove a fixed point theorem in abstract cones, from which different results on existence and uniqueness of positive almost periodic solutions are deduced. Moreover, some criteria are given in order to guarantee that the obtained positive almost periodic solution is globally exponentially stable.

Keywords— Non-linear nonautonomous delay differential equations, Existence and uniqueness of almost periodic solutions, Global exponential stability, Fixed point theorems, Haematopoiesis.

1 Introduction

Nonlinear delay differential equations have numerous applications to economics, physics, statistics, biology and many other fields. An example of such applications is the autonomous delay differential equation proposed by Mackey and Glass to study the regulation of hematopoiesis, namely

$$x'(t) = \frac{\lambda x^m(t-\tau)}{1+x^n(t-\tau)} - \gamma x(t), \tag{1}$$

where $m = 0$ or 1 and n, γ and τ are positive constants (for more details see [22, 23]). Often, the environment is not temporally constant; thus, it is intuitive to assume that this fact influences many biological dynamical systems and suggests the need of considering time-dependent parameters (see e.g. [21, 22, 24]). The following extension of (1) with several delays

$$x'(t) = \sum_{k=1}^{M} \lambda_k r_k(t) \frac{x^m(t-\tau_k(t))}{1+x^n(t-\tau_k(t))} - b(t)x(t), \tag{2}$$

where $m \geq 0, n, \lambda_k > 0$, and b, r_k, τ_k are positive and continuous functions for $k = 1, 2, \ldots, M$, was studied for example in [2, 9, 15, 21, 32–34].

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It is worthy to notice that there are four possible behaviours for the mapping $g(x) := \frac{x^m}{1+x^n}$, namely: strictly increasing and bounded ($n = m > 0$), single-humped ($n > m > 0$), strictly decreasing ($m = 0$) and strictly increasing and unbounded ($0 < n < m$). The latter case does not have biological relevance, although it is of mathematical interest in order to obtain a complete picture.

Periodic effects for this type of population dynamics have been intensively analysed [1, 2, 4, 6, 19, 26, 27, 34]. In all these works time-dependent parameters and delays have a fixed period T, and sufficient conditions to ensure the existence of positive T-periodic solutions have been obtained. However, assuming the same period T for the parameters, delays and solutions may result, in some cases, somewhat artificial from the biological point of view. From the mathematical point of view, the advantage of T-periodic assumptions is that standard methods of the nonlinear analysis such as Mawhin's continuation theorem, Schauder and Kranoselskii's fixed point theorems can be used (see e.g. [1, 7, 17, 31]).

A more realistic way to avoid the periodicity conditions consists in considering almost periodic effects. This is interesting for several reasons: on the one hand, these more general effects include periodicity and allow more realistic assumptions: for example, time-dependent parameters with different periods. On the other hand, since almost periodicity is more general, a central mathematical issue relies on the fact that the involved operators are no longer compact. Due to this fact, the aforementioned methods cannot be extended in a direct way for the almost periodic problem (see [25, 28]) and other methods must be employed.

Based on these facts we establish a fixed point theorem in abstract cones without compactness conditions. It is worthy to mention that the fixed point theorems formulated by Wang et al. in [29] and [30] are direct consequences of our fixed point theorem, see Corollaries 3.2 and 3.3 below. The same can be said about the theorem established by Ding et al. in [12], as we show in Corollary 3.4. Moreover, the fixed point presented in [10] employs a stronger monotonicity assumption than our result and, in consequence, the existence proofs in the present paper improve some previous ones (see e.g. [11]).

In [5, 32–34] sufficient criteria were established for the existence of positive almost periodic solutions of (2) with $m = 0$ (monotone decreasing nonlinearity). In [33], a fixed point theorem was employed to prove the existence and uniqueness of almost periodic solutions under conditions that can be regarded as particular applications of Theorem 2.1 and Theorem 2.4 case (a) below. In [34], using the contraction mapping principle, the authors obtained sufficient criteria for existence in a bounded region under the assumption $n > 0$. However, as pointed out in [33], Theorem 3.1 in [34] has a mistake, which invalidates the case $n \leq 1$. In [30], the authors proved a fixed point theorem that allows to deduce the existence and uniqueness of positive almost periodic solutions of (2) with $M, m = 1$ and $n > m$ (single-humped nonlinearity) in a bounded region.

More recently, using similar methods to those in [5], criteria for existence and uniqueness were established in [20] when $n \geq m$ for $0 \leq m \leq 1$ (sum of single-humped functions when $n > m$, or monotone increasing and bounded nonlinearity when $n = m$). This case was also considered in [9] by employing a fixed point theorem in a cone. The results obtained for the several cases treated in [11] using the previously mentioned fixed point theorem established by Ding et al. in [12] can be regarded as particular applications of Theorems 2.1-2.2 and Theorem 2.4 below. However, when the nonlinearity is more general, Ding's fixed point theorem cannot be applied. Such are the cases of Theorems 2.3 and 2.5 below.

As pointed out in [3], equations with oscillating coefficients appear in linearizations of population dynamics models with seasonal fluctuations, where during some seasons the death or harvesting rates may be greater than the birth rate. Based on this fact, some results were considered in [18] for an oscillating loss rate coefficient $b(t)$ in (2) for $m \leq n$.

Besides existence, another relevant matter is to determine whether or not the obtained solutions are stable. In particular, exponential stability is specially important for two reasons: on the one hand, the rate of convergence is quantified and, on the other hand, it is robust to perturbations.

For example, in [32] sufficient conditions for the global attractiveness of positive almost periodic solutions of (2) with $m = 0$ were established as an answer to a question raised by Gyori and Ladas [15, p.322], although global exponential stability was not discussed. In [11, 33], Gronwall's inequality was employed to establish global exponential stability under restrictions on the delay. In [5, 34], authors studied the stability for the case $m = 0$ and in [20] for $0 \leq m \leq 1$. Recently, using similar methods to those in [20], authors studied the case $0 \leq m \leq n$ when $b(t)$ is oscillatory. However, to the best of our knowledge, the global exponential stability has not been sufficiently studied when $m \neq 0, 1$.

Motivated by the preceding discussion, we shall consider the following more general nonlinear non-autonomous model with several delays

$$x'(t) = \sum_{k=1}^{M} \lambda_k r_k(t) \frac{x^{m_k}(t - \tau_k(t))}{1 + x^{n_k}(t - \tau_k(t))} - b(t)x(t), \tag{3}$$

where $r_k, \tau_k : \mathbb{R} \to [0, +\infty)$ are almost periodic functions, λ_k and n_k are positive constants and $0 \leq m_k \leq 1$. We remark that the different choices of the exponents m_k and n_k may lead to different behaviours of the terms in the nonlinearity.

We shall introduce sufficient conditions to guarantee the existence and uniqueness of positive almost periodic solutions of (3) with an almost periodic oscillating coefficient in the circulation loss rate $b(t)$. To this end, we prove a new fixed point theorem in abstract cones. In addition, for the particular case when $\inf_{t\in\mathbb{R}} b(t)$ and $m_j > n_j$ for some j, we prove the global exponential stability of such solutions. It is worth noticing that our criteria do not impose restrictions for the delay. By means of a Halanay-type inequality [16, Chapter 4], we shall establish a simple global exponential stability lemma, which is quite different from the methods employed in the previous works.

The paper is organized as follows. Sufficient criteria for the existence, uniqueness and global exponential stability of such almost periodic solutions are presented in Section 2. In Section 3, we introduce some definitions, lemmas and theorems that shall be employed in Section 4 to prove the main results. Finally, in Section 5 we give examples to demonstrate the validity of our results obtained in Section 2 and we explain why results in previous references cannot be applicable. Thus, we show that our results improve and generalize previously known results.

Throughout the paper, it will be assumed that $b(t)$ is an almost periodic function with

$$M[b] = \lim_{t\to+\infty} \frac{1}{T} \int_t^{t+T} b(s)ds > 0.$$

For a bounded continuous function f, the supremum and the infimum of f shall be denoted respectively f^* and f_*, namely

$$f^* = \sup_{t\in\mathbb{R}} f(t), \qquad f_* = \inf_{t\in\mathbb{R}} f(t).$$

Moreover, we assume that

$$\upsilon := \max_{1\leq k\leq M} \left\{ \sup_{t\in\mathbb{R}} \tau_k(t) \right\} > 0 \quad \text{and} \quad (r_j)_* > 0 \quad \text{for some } j. \tag{4}$$

In addition, in our existence results we assume that there exist positive constants F^i and F^s, such that

$$F^i e^{-\int_s^t \tilde{b}(u)du} \leq e^{-\int_s^t b(u)du} \leq F^s e^{-\int_s^t \tilde{b}(u)du}, \tag{5}$$

where $\tilde{b} : \mathbb{R} \to (0, +\infty)$ is a bounded and continuous function with positive infimum.

Remark 1.1 *It is worth noticing that, when* $\inf_{t\in\mathbb{R}} b(t) > 0$, *inequalities* (5) *are fulfilled with* $F^i = F^s = 1$ *and* $\tilde{b}(t) \equiv b(t)$. *Thus, the techniques employed in our existence results are applicable when the circulation rate* $b(t)$ *is persistent.*

Due to the biological interpretation of the model, we shall consider as an admissible initial condition for equation (3) only continuous positive functions, namely

$$x(t_0 - t) = \varphi(t), \qquad \varphi \in C([0, \upsilon], (0, +\infty)). \tag{6}$$

A solution of the initial value problem (3) satisfying (6) shall be denoted by $x(t; t_0, \varphi)$.

2 Main results

In this section, we state our results on existence, uniqueness and global exponential stability of positive almost periodic solutions of (3).

Existence and uniqueness

The main part of our existence and uniqueness analysis shall be based on the study of the behaviour of the term production.

For simplicity of notation, let us define the constants

$$V := \min_{k: n_k > m_k > 0} \left\{ \left(\frac{m_k}{n_k - m_k} \right)^{\frac{1}{n_k}} \right\}, \tag{7}$$

$$S := \min_{\{n_k > 1 : m_k = 0\}} \left\{ \left(\frac{1}{n_k - 1} \right)^{\frac{1}{n_k}} \right\} \tag{8}$$

and

$$T := \min\{V, S\}. \tag{9}$$

Theorem 2.1 *Assume that $n_k \leq m_k$ for all k such that $m_k > 0$ and $n_k \leq 1$ for all k such that $m_k = 0$. Furthermore, assume that one of the following conditions is fulfilled:*

(a) $0 \leq m_j < 1$ *for some* j.

(b) $m_k = 1$ *for all* k *and* $(H) : \int_{-\infty}^{t} e^{-\int_s^t b(u)du} \sum_{k=1}^{M} \lambda_k r_k(s) ds > 1$.

Then (3) has exactly one almost periodic solution with positive infimum.

Theorem 2.2 *Assume that $n_k \geq m_k$ for all k such that $m_k > 0$ and $n_k \leq 1$ for all k such that $m_k = 0$. Moreover, suppose there exists i such that $n_i > m_i > 0$. Let*

$$\int_{-\infty}^{t} e^{-\int_s^t b(u)du} \sum_{k=1}^{M} \lambda_k r_k(s) ds \leq V. \tag{10}$$

Furthermore, assume that one of the following conditions is fulfilled:

(a) $0 \leq m_j < 1$ *for some* j.

(b) $m_k = 1$ *for all* k *and* $(H) : \int_{-\infty}^{t} e^{-\int_s^t b(u)du} \sum_{k=1}^{M} \lambda_k r_k(s) ds > 1$.

Then (3) has exactly one almost periodic solution with positive infimum.

Theorem 2.3 *Assume that $n_k \leq 1$ for all k such that $m_k = 0$. Moreover, suppose there exist i and l such that $n_i > m_i > 0$ and $n_l < m_l$. Let*

$$\int_{-\infty}^{t} e^{-\int_s^t b(u)du} \left(\sum_{k:n_k<m_k} \lambda_k r_k(s) \frac{V^{m_k}}{1+V^{n_k}} + \sum_{\{k:m_k=0\}\cup\{k:n_k\geq m_k>0\}} \lambda_k r_k(s) \right) ds \leq V. \tag{11}$$

Furthermore, assume that one of the following conditions is fulfilled:

(a) $0 \leq m_j < 1$ *for some* j.

(b) $m_k = 1$ *for all* k *and* $(H) : \int_{-\infty}^{t} e^{-\int_s^t b(u)du} \sum_{k=1}^{M} \lambda_k r_k(s) ds > 1$.

Then (3) *has at least one almost periodic solution with positive infimum.*

Theorem 2.4 *Assume that $n_k \geq m_k$ for all k such that $m_k > 0$. Moreover, supposse that $n_q > 1$ for some q such that $m_q = 0$ and*

$$\int_{-\infty}^{t} e^{-\int_s^t b(u)du} \sum_{k=1}^{M} \lambda_k r_k(s) ds \leq T. \tag{12}$$

Then (3) *has exactly one almost periodic solution with positive infimum.*

Theorem 2.5 *Assume that $n_i < m_i$ for some i and $n_q > 1$ for some q such that $m_q = 0$. Moreover, supposse that*

$$\int_{-\infty}^{t} e^{-\int_s^t b(u)du} \left(\sum_{k:n_k<m_k} \lambda_k r_k(s) \frac{T^{m_k}}{1+T^{n_k}} + \sum_{\{k:m_k=0\}\cup\{k:n_k\geq m_k>0\}} \lambda_k r_k(s) \right) ds \leq T. \tag{13}$$

Then (3) *has at least one positive almost periodic solution with positive infimum.*

Remark 2.1 *In view of Remark 1.1, we clearly have*

$$\int_{-\infty}^{t} e^{-\int_s^t b(u)du} \sum_{k=1}^{M} \lambda_k r_k(s) ds \leq F^s \sum_{k=1}^{M} \frac{\lambda_k r_k^*}{\tilde{b}_*} < +\infty.$$

It follows that assumption (10) in Theorem 2.2 is satisfied under the following condition, which is easier to verify:

$$\sum_{k=1}^{M} \frac{\lambda_k r_k^*}{\tilde{b}_*} \leq \frac{V}{F^s}.$$

Similarly, condition (11) *in Theorem 2.3 can be replaced by the stronger assumption*

$$\sum_{k:n_k<m_k} \lambda_k r_k^* \frac{V^{m_k}}{\tilde{b}_*(1+V^{n_k})} + \sum_{\{k:m_k=0\}\cup\{k:n_k\geq m_k>0\}} \lambda_k r_k^* \frac{1}{\tilde{b}_*} \leq \frac{V}{F^s},$$

and conditions (12) *and* (13) *by*

$$\sum_{k=1}^{M} \frac{\lambda_k r_k^*}{\tilde{b}_*} \leq \frac{T}{F^s} \quad \text{and} \quad \sum_{k:n_k<m_k} \lambda_k r_k^* \frac{T^{m_k}}{\tilde{b}_*(1+T^{n_k})} + \sum_{\{k:m_k=0\}\cup\{k:n_k\geq m_k>0\}} \lambda_k r_k^* \frac{1}{\tilde{b}_*} \leq \frac{T}{F^s},$$

respectively. Also, condition (H) *can be replaced by* $\sum_{k=1}^{M} \lambda_k (r_k)_* > b^*$.

Remark 2.2 ***(Uniqueness of periodic solution)*** *Sufficient criteria for the existence of positive T-periodic solutions of (3) were established in [1] by using topological degree methods. It is worth mentioning that the referred work deals only with existence and multiplicity, and conditions for uniqueness of solutions are not given.*

As remarked above, some properties of T-periodic functions do not hold for the more general case of almost periodic functions and, consequently, the results on existence of positive T-periodic solutions in [1] cannot be directly applied to (3). Despite of that, it is still possible to compare Theorem 3.2 in [1] with Theorems 2.1, 2.2 and 2.3 assuming that b, r_k and τ_k are positive T-periodic functions and $m_k > 0$ for all k. The methods used in the present paper provide also uniqueness of solutions, although more restrictive conditions are needed. For example, the results in [1] do not impose conditions on the (positive) constants m_k, but the uniqueness result requires that $m_k \leq 1$ for all k. Moreover, if $m_k = 1$ for all k, then the existence result [1, Thm. 3.2] assumes that $\sum_{k=1}^{M} \lambda_k r_k(t) > b(t)$, while the existence and uniqueness result provided by this paper employs the stronger condition $\sum_{k=1}^{M} \lambda_k (r_k)_* > b^*$.

Global exponential stability

Let $x(t; t_0, \varphi)$ be a solution of (3) with initial condition(6), and $\tilde{x}(t)$ an almost periodic solution with positive infimum of (3), and define

$$A := \{k : n_k > m_k(3 + 2\sqrt{2})\}.$$

Remark 2.3 *As we will see in Section 3 Lemma 3.5, under appropriate conditions it is possible to find positive constants η and t_φ such that*

$$x(t; t_0, \varphi) > \eta, \text{ for all } t \geq t_\varphi.$$

Theorem 2.6 *Let $0 \leq m_k \leq 1$, $n_k > 0$ for all $k = 1, \ldots, M$ and $m_j > n_j$ for some j. Let η and $t_{\varphi,\tilde{x}}$ be positive constants such that $\tilde{x}(t), x(t; t_0, \varphi) > \eta$, for all $t \geq t_{\varphi,\tilde{x}}$.*

Set

$$p(t) = \sum_{k \in A} \eta^{m_k - 1} \lambda_k r_k(t) \frac{(n_k - m_k)^2}{4 n_k} + \sum_{k \notin A} \eta^{m_k - 1} \lambda_k r_k(t) m_k,$$

and suppose that

$$\inf_{t \geq t_{\varphi,\tilde{x}}} \{b(t) - p(t)\} > 0.$$

*Then $\tilde{x}(t)$ is **globally exponentially stable**. i.e., there exist positive constants ρ, $K_{\varphi,\tilde{x}}$ and $t_{\varphi,\tilde{x}}$ such that*

$$|x(t; t_0, \varphi) - \tilde{x}(t)| < K_{\varphi,\tilde{x}} e^{-\rho t} \text{ for all } t \geq t_{\varphi,\tilde{x}}.$$

Remark 2.4 ***(Uniqueness)*** *Theorem 2.3 and Theorem 2.5 only ensure the existence of at least one positive almost periodic solutions of (3). However, under extra assumptions, the global exponential stability of such solutions is ensured. Thus, we can conclude the existence of a unique almost periodic solution of (3) with positive infimum (see Corollary 3.1 below.)*

Remark 2.5 *If we allow $b(t)$ to oscillate, then global exponential stability can be obtained when $m_k \leq n_k$ for all k. The proof is similar to that given by Jiang in [18] for and $m_k = m, n_k = n$ for all k and we omit it. The method cannot be extended to the case $m_k > n_k$ for some k, which is left as an interesting open problem to be studied.*

3 Preliminaries

In this section, we provide preliminary results which will be used in the proofs of our main results. For the reader's convenience, we also include a formal definition of almost periodicity.

Definition 3.1 *(Corduneanu [8]) Let X be a Banach space. A function $f : \mathbb{R} \to X$ is called* ***almost periodic*** *if for any $\epsilon > 0$ there exists a number $l(\epsilon) > 0$ such that any interval on $\mathbb{R}$ of length $l(\epsilon)$ contains at least one point ξ with the property that*

$$||f(t+\xi) - f(t)|| < \epsilon \qquad \textit{for all}\, t \in \mathbb{R}.$$

It is proved that the previous definition is equivalent to the following one due to Bochner, expressed in terms of sequential convergence of families translates.

Definition 3.2 *(Fink [13]) A $f : \mathbb{R} \to \mathbb{C}$ is almost periodic if from every sequence $\{\alpha'_n\}$ one can extract a subsequence $\{\alpha_n\}$ such that*

$$\lim_{n\to\infty} f(t+\alpha_n)$$

exists uniformly on the real line.

Definition 3.3 *Let X be a real Banach space. A nonempty closed set $C \subset X$ is called a* ***cone*** *if the following conditions are fulfilled:*

$$(a)\; C + C \subset C \qquad (b)\; C \cap -C = \{0\} \qquad (c)\; C \textit{ is convex,}$$

where 0 denotes the zero element of X.

Every cone C induces a partial order $\leq$ in X given by

$$x \leq y \textit{ if and only if } y - x \in C.$$

If $x \leq y$ and $x \neq y$, we write $x < y$. A set $\{z \in X/x \leq z \leq y\}$ is called an ***order interval*** *and shall be denoted as $[x, y]$. The interior of C shall be denoted by C°. A cone C satisfying $C^\circ \neq \emptyset$ is called a* ***solid cone****. A cone C is called* ***normal*** *if there exists a constant $N > 0$ such that*

$$0 \leq x \leq y \textit{ implies that } ||x|| \leq N||y||.$$

The smaller constant N satisfying the inequality is called the normal constant of C.

The Banach space of almost periodic real functions defined on $\mathbb{R}$, equipped with the usual uniform norm, shall be denoted as $AP(\mathbb{R})$. Also, we denote

$$P := \{x \in AP(\mathbb{R}) : x(t) \geq 0, \forall t \in \mathbb{R}\},$$

the normal solid cone of nonnegative functions. It is readily verified that

$$P^\circ = \{x \in P : \exists\epsilon > 0 \text{ such that } x(t) \geq \epsilon, \text{ for all } t \in \mathbb{R}\}.$$

Lemma 3.1 *Let $f, g \in AP(\mathbb{C})$. Suppose that $\lim_{t\to\infty} f(t) = 0$, then $f \equiv 0$.*

Proof: Consider the sequence $\alpha'_n = n \in \mathbb{N}$, by Definition 3.2, there exists an increasing subsequence $\{\alpha_n\}$ such that $f(t+\alpha_n)$ converges uniformly on the real line. Moreover, $f(t+\alpha_n)$ converges uniformly to 0, the limit given by the pointwise convergence.

Thus, defining $f_{\alpha_n}(t) := f(t+\alpha_n)$, we get

$$||f||_{\infty,\mathbb{R}} = ||f_{\alpha_n}||_{\infty,\mathbb{R}} \to 0$$

as $n \to +\infty$. We conclude that $f \equiv 0$. ■

The following Corollary is a direct consequence of Lemma 3.1.

Corollary 3.1 *Let $f, g \in AP(\mathbb{R})$. Let $\epsilon > 0$ and assume there exists $t_0(\epsilon) > 0$ such that $|f(t) - g(t)| < \epsilon$ for all $t \geq t_0$. Then $f \equiv g$ for all $t \in \mathbb{R}$.*

Definition 3.4 *(Guo and Lakshmikantham [14]) Let $(X, \leq)$ be an ordered Banach space and let $E \subset X$. An operator $\Phi : E \times E \to X$ is called a* ***mixed monotone operator*** *if $\Phi(x,y)$ is nondecreasing in x and nonincreasing in y. An element $\tilde{x} \in E$ is called a* ***fixed point*** *of Φ if $\Phi(\tilde{x}, \tilde{x}) = \tilde{x}$.*

The following fixed point theorem shall play an important role in Section 4.

Theorem 3.1 *Let P be a normal cone in a real Banach space X, and $\Phi : P^\circ \times P^\circ \to P^\circ$. Assume that*

(I) there exist $u_0, v_0 \in P^\circ$, $u_0 \leq v_0$, $u_0 \leq \Phi(u_0, v_0)$ and $v_0 \geq \Phi(v_0, u_0)$;

(II) Φ is a mixed monotone operator on $[u_0, v_0] \times [u_0, v_0]$;

(III) there exists a function $\phi : (0,1) \to (0, +\infty)$ such that $\phi(\gamma) > \gamma$ for all $\gamma \in (0,1)$, and for any $x, y \in [u_0, v_0]$

$$\Phi(\gamma x, \gamma^{-1} y) \geq \phi(\gamma)\Phi(x,y), \quad \textit{for all } \gamma \in (0,1).$$

Then Φ has exactly one fixed point $\tilde{x}$ in $[u_0, v_0]$. Moreover, for any initial $x_0, y_0 \in [u_0, v_0]$, the iterative sequences

$$x_n = \Phi(x_{n-1}, y_{n-1}), \quad y_n = \Phi(y_{n-1}, x_{n-1}), \;\; n \in \mathbb{N}, \tag{14}$$

satisfy

$$||x_n - \tilde{x}||, ||y_n - \tilde{x}|| \to 0 \;\; (n \to +\infty).$$

Proof: For $n \in \mathbb{N}$, define $u_n := \Phi(u_{n-1}, v_{n-1})$ and $v_n := \Phi(v_{n-1}, u_{n-1})$. Since Φ is a mixed monotone operator, by (I) we deduce

$$u_0 \leq u_1 = \Phi(u_0, v_0) \leq \Phi(v_0, u_0) = v_1 \leq v_0,$$

and inductively we obtain

$$u_0 \leq u_1 \leq \ldots \leq u_n \leq \ldots \leq v_n \leq \ldots \leq v_1 \leq v_0. \tag{15}$$

Since P° is an open set and $u_n \in P^\circ$, there exists a constant $\delta > 0$ such that $u_n - \lambda v_n \in P^\circ$ for any $\lambda \in (0, \delta)$. Thus, the constant $\lambda_n := \sup\{\lambda : u_n \geq \lambda v_n\}$ is well defined and positive. It is clear that

$$u_n \geq \lambda_n v_n \tag{16}$$

and the inequality $u_n \leq v_n$ implies $\lambda_n \leq 1$. Moreover, since $u_{n+1} \geq u_n \geq \lambda_n v_n \geq \lambda_n v_{n+1}$, it is seen that $\lambda_{n+1} \geq \lambda_n$. We claim that $\overline{\lambda} := \lim_{n \to +\infty} \lambda_n = 1$. Indeed, if this is not true then $\overline{\lambda} \in (0,1)$ and there are two cases:

Case 1. There exists $\overline{n}$ such that $\lambda_{\overline{n}} = \overline{\lambda}$. Then $\lambda_n = \overline{\lambda}$, $u_n \geq \overline{\lambda} v_n$ for all $n > \overline{n}$ which, together with $(II), (III)$ and (15), yields

$$u_{n+1} = \Phi(u_n, v_n) \geq \Phi(\overline{\lambda} v_n, \overline{\lambda}^{-1} u_n) \geq \phi(\overline{\lambda})\Phi(v_n, u_n) = \phi(\overline{\lambda}) v_{n+1}.$$

Thus $\lambda_{n+1} \geq \phi(\overline{\lambda}) > \overline{\lambda}$, which contradicts the fact that $\lambda_{n+1} = \overline{\lambda}$.

Case 2. $\lambda_n < \overline{\lambda}$, for all n. Then

$$u_{n+1} = \Phi(u_n, v_n) \geq \Phi\left(\lambda_n v_n, \lambda_n^{-1} u_n\right) = \Phi\left(\frac{\lambda_n}{\overline{\lambda}} \overline{\lambda} v_n, \frac{\overline{\lambda}}{\lambda_n} \overline{\lambda}^{-1} u_n\right)$$

$$\geq \phi\left(\frac{\lambda_n}{\overline{\lambda}}\right) \Phi\left(\overline{\lambda} v_n, \overline{\lambda}^{-1} u_n\right) > \frac{\lambda_n}{\overline{\lambda}} \phi\left(\overline{\lambda}\right) \Phi(v_n, u_n) \geq \frac{\lambda_n}{\overline{\lambda}} \phi\left(\overline{\lambda}\right) v_{n+1}.$$

Thus $\lambda_{n+1} \geq \frac{\lambda_n}{\overline{\lambda}} \phi\left(\overline{\lambda}\right)$. Letting $n \to \infty$, we deduce that $\overline{\lambda} \geq \phi\left(\overline{\lambda}\right) > \overline{\lambda}$, a contradiction.

Hence $\overline{\lambda} = 1$ and from (15)-(16) it follows that, for any k,

$$0 \leq u_{n+k} - u_n \leq v_n - u_n \leq v_n - \lambda_n v_n = (1 - \lambda_n) v_n \leq (1 - \lambda_n) v_0. \tag{17}$$

By the normality of P and (17),

$$||u_{n+k} - u_n|| \leq N(1 - \lambda_n)||v_0|| \to 0 \quad \text{as } n \to \infty.$$

Thus, $\{u_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. This implies that there exists $\tilde{x} \in [u_0, v_0]$ such that $u_n \to \tilde{x}$. Similarly,

$$0 \leq v_n - u_n \leq v_n - \lambda_n v_n = (1 - \lambda_n) v_n \leq (1 - \lambda_n) v_0.$$

Again, by the normality of P

$$||v_n - u_n|| \leq N(1 - \lambda_n)||v_0|| \to 0 \quad \text{as } n \to \infty,$$

and consequently $v_n \to \tilde{x}$. Hence, since Φ is a mixed monotone operator on $[u_0, v_0]$, it follows that

$$u_{n+1} = \Phi(u_n, v_n) \leq \Phi(\tilde{x}, \tilde{x}) \leq \Phi(v_n, u_n) = v_{n+1.}$$

We conclude that $\tilde{x} = \Phi(\tilde{x}, \tilde{x})$.

Suppose now that $\overline{w} \in [u_0, v_0]$ is another fixed point of Φ. Let $\alpha := \sup\{\tilde{\alpha} \in (0,1) : \tilde{\alpha}\overline{w} \leq \tilde{x} \leq \frac{1}{\tilde{\alpha}}\overline{w}\}$. Then, $\alpha\overline{w} \leq \tilde{x} \leq \alpha^{-1}\overline{w}$ and $\alpha \in (0, 1]$. Suppose that $\alpha \in (0, 1)$, then $\phi(\alpha) > \alpha$,

$$\tilde{x} = \Phi(\tilde{x}, \tilde{x}) \leq \Phi(\frac{1}{\alpha}\overline{w}, \alpha\overline{w}) \leq \phi(\alpha)^{-1}\Phi(\overline{w}, \overline{w}) = \phi(\alpha)^{-1}\overline{w},$$

and

$$\tilde{x} = \Phi(\tilde{x}, \tilde{x}) \geq \Phi(\alpha\overline{w}, \frac{1}{\alpha}\overline{w}) \geq \phi(\alpha)\Phi(\overline{w}, \overline{w}) = \phi(\alpha)\overline{w}.$$

Thus, by the definition of α we have $\phi(\alpha) \leq \alpha$, which is a contradiction. We conclude that $\alpha = 1$ and therefore $\overline{w} = \tilde{x}$.

Finally, let be (x_0, y_0) any initial condition in $[u_0, v_0] \times [u_0, v_0]$ and (x_n, y_n) the iterative sequences given by (14). Since Φ is a mixed monotone operator, we have

$$u_1 = \Phi(u_0, v_0) \leq x_1 = \Phi(x_0, y_0) \leq \Phi(v_0, u_0) = v_1$$

and

$$u_1 = \Phi(u_0, v_0) \leq y_1 = \Phi(y_0, x_0) \leq \Phi(v_0, u_0) = v_1,$$

and inductively we obtain $x_n, y_n \in [u_n, v_n]$. Thus, it is clear that

$$||x_n - \tilde{x}||, ||y_n - \tilde{x}|| \to 0 \text{ as } n \to +\infty.$$

The proof is complete. ■

Remark 3.1 *It is worth noticing that the function ϕ in the previous theorem is not necessarily continuous.*

Corollary 3.2 *Let P be a normal cone in a real Banach space X, and $\Phi : P^\circ \to P^\circ$. Assume that*

(I) there exist $u_0, v_0 \in P^\circ$, $u_0 \leq v_0$, $u_0 \leq \Phi(u_0)$ and $v_0 \geq \Phi(v_0)$;

(II) Φ is a nondecreasing operator on $[u_0, v_0]$;

(III) there exists a function $\phi : (0,1) \to (0, +\infty)$ such that $\phi(\gamma) > \gamma$ for all $\gamma \in (0,1)$, and for any $x \in [u_0, v_0]$

$$\Phi(\gamma x) \geq \phi(\gamma)\Phi(x), \quad \text{for all } \gamma \in (0,1).$$

Then Φ has exactly one fixed point $\tilde{x}$ in $[u_0, v_0]$.

Moreover, for any initial $x_0 \in [u_0, v_0]$, the iterative sequence

$$x_n = \Phi(x_{n-1}), \;\; n \in \mathbb{N}, \tag{18}$$

satisfies

$$||x_n - \tilde{x}|| \to 0 \;\; (n \to +\infty).$$

Corollary 3.3 *Let P be a normal cone in a real Banach space X, and $\Phi : P^\circ \to P^\circ$. Assume that*

(I) there exist $u_0, v_0 \in P^\circ$, $u_0 \leq v_0$, $u_0 \leq \Phi(v_0)$ and $v_0 \geq \Phi(u_0)$;

(II) Φ is a nonincreasing operator on $[u_0, v_0]$;

(III) there exists a function $\phi : (0,1) \to (0, +\infty)$ such that $\phi(\gamma) > \gamma$ for all $\gamma \in (0,1)$, and for any $x \in [u_0, v_0]$

$$\Phi(\gamma^{-1} x) \geq \phi(\gamma)\Phi(x), \quad \text{for all } \gamma \in (0,1).$$

Then Φ has exactly one fixed point $\tilde{x}$ in $[u_0, v_0]$.

Moreover, for any initial $x_0 \in [u_0, v_0]$, the iterative sequence

$$x_n = \Phi(x_{n-1}), \;\; n \in \mathbb{N}, \tag{19}$$

satisfies

$$||x_n - \tilde{x}|| \to 0 \;\; (n \to +\infty).$$

Corollary 3.4 *Let P be a normal cone in a real Banach space X, and $\Phi : P^\circ \times P^\circ \to P^\circ$ a mixed monotone operator. Assume that there exists a function $\phi : (0,1) \times P^\circ \times P^\circ \to (0,+\infty)$ such that for each $\gamma \in (0,1)$ and $x, y \in P^\circ$, $\phi(\gamma, x, y) > \gamma$, $\phi(\gamma, \cdot, y)$ is nondecreasing in P°, $\phi(\gamma, x, \cdot)$ is nonincreasing in P° and*

$$\Phi(\gamma x, \gamma^{-1} y) \geq \phi(\gamma, x, y)\Phi(x, y).$$

Assume, in addition, there exists $z \in P^\circ$ such that $\Phi(z,z) \geq z$. Then Φ has a unique fixed point $\tilde{x} \in P^\circ$.

Moreover, for any initial $(x_0, y_0) \in P^\circ \times P^\circ$, the iterative sequences

$$x_n = \Phi(x_{n-1}, y_{n-1}), \;\; y_n = \Phi(y_{n-1}, x_{n-1}), \;\; n \in \mathbb{N},$$

satisfy

$$||x_n - \tilde{x}||, \;\; ||y_n - \tilde{x}|| \to 0 \;\; (n \to +\infty).$$

Proof: In the first place, we shall prove the existence of a fixed point of Φ. If $z \in P^\circ$ is such that $\Phi(z,z) = z$, then there is nothing to prove; otherwise, we can choose $\alpha \in (0,1)$ such that

$$\Phi(z,z) \leq \frac{1}{\alpha} z.$$

Since, $\phi(\alpha, z, z) > \alpha$, there exists $N_0(\alpha) > 0$ such that for all $N \geq N_0(\alpha)$, $\left(\frac{\phi(\alpha,z,z)}{\alpha}\right)^N \geq \frac{1}{\alpha}$, that is,

$$\phi^N(\alpha, z, z) \geq \alpha^{N-1}.$$

Let $u_0 = \alpha^N z$, $v_0 = \alpha^{-N} z$ and for $n \in \mathbb{N}$ define $u_n := \Phi(u_{n-1}, v_{n-1})$ and $v_n := \Phi(v_{n-1}, u_{n-1})$. Thus,

$$\Phi(u_0, v_0) = \Phi(\alpha^N z, \alpha^{-N} z) \geq \alpha \Phi(\alpha^{N-1} z, \alpha^{-(N-1)} z) \geq \cdots \geq \alpha^N z = u_0,$$

and

$$\begin{aligned}
\Phi(v_0, u_0) = \Phi(\alpha^{-N} z, \alpha^N z) &\leq \frac{\Phi(\alpha^{-(N-1)} z, \alpha^{(N-1)} z)}{\phi(\alpha, \alpha^{-N} z, \alpha^N z)} \\
&\leq \frac{\Phi(\alpha^{-(N-1)} z, \alpha^{(N-1)} z)}{\phi(\alpha, z, z)} \\
&\leq \frac{\Phi(z,z)}{\phi^N(\alpha, z, z)} \leq \alpha^{-N} z = v_0.
\end{aligned}$$

In addition, Φ is a mixed monotone operator on $[u_0, v_0] \times [u_0, v_0]$. Moreover, for each $\gamma \in (0,1)$ and $x, y \in [u_0, v_0]$, we have

$$\begin{aligned}
\Phi(\gamma x, \gamma{-}1 y) &\geq \phi(\gamma, x, y)\Phi(\gamma, x, y) \\
&\geq \phi(\gamma, u_0, v_0)\Phi(\gamma, x, y) \\
&= \phi(\gamma, \alpha^N z, \alpha^{-N} z)\Phi(\gamma, x, y) \\
&= \tilde{\phi}(\gamma)\Phi(\gamma, x, y),
\end{aligned}$$

where $\tilde{\phi} : (0,1) \to (0,+\infty)$ is defined by $\tilde{\phi}(\gamma) := \phi(\gamma, \alpha^N z, \alpha^{-N} z)$. Thus, by Theorem 3.1, Φ has a unique fixed point $\tilde{x}$ in $[u_0, v_0]$.

Suppose that $\tilde{y} \in P^\circ$ is a fixed point of Φ. Since P° is an open set, there exists a constant $\beta \in (0, \alpha)$ and $M \in \mathbb{N}$, $M > N$ such that

$$\tilde{u}_0 := \beta^M z \leq \tilde{y} \leq \beta^{-M} z := \tilde{v}_0.$$

Again, by Theorem 3.1 we can prove that Φ has a unique fixed point on $[\tilde{u}_0, \tilde{v}_0]$. Thus, the inclusion $[u_0, v_0] \subset [\tilde{u}_0, \tilde{v}_0]$ implies that $\tilde{x} = \tilde{y}$.

The proof is complete.

■

Remark 3.2 *It is worth noticing that Corollaries 3.2, 3.3 and 3.4 are the same results as those fixed point theorems established by Wang et al. in [30], [29] and by Ding et al. in [12] respectively. Hence, Theorem 3.1 generalizes their results.*

Our stability result shall be based on the following result, which is a generalization of [35, Lemma 3] for the case with time-dependent parameters. Moreover, we shall give explicit bounds for the convergence rate.

Lemma 3.2 *Let $x(t)$ be a continuous nonnegative function on $t \geq t_0 - \upsilon$ satisfying the following inequality*

$$D^+x(t) \leq -k_1(t)x(t) + k_2(t)\overline{x}(t) \text{ for } t \geq t_0 \tag{20}$$

where $k_1(t)$ and $k_2(t)$ are nonnegative, continuous and bounded functions and $\overline{x}(t) = \sup_{t-\upsilon \leq s \leq t} x(s)$. Suppose

$$\alpha = \inf_{t \geq t_0} \{k_1(t) - k_2(t)\} > 0.$$

Then there exists a positive constant $\tilde{\rho} > 0$ such that

$$x(t) \leq \overline{x}(t_0)e^{-\tilde{\rho}(t-t_0)}$$

holds for all $t \geq t_0$. Moreover, the decay rate $\tilde{\rho}$ is such that

$$0 < \inf_{t \in \mathbb{R}} \left\{ \frac{(k_1(t) - k_2(t))k_1(t)}{k_1(t) - k_2(t) + k_2(t)e^{\upsilon k_1(t)}} \right\} < \tilde{\rho} < k_1^*.$$

<u>*Proof*</u>: Define the function f by

$$f(t, \rho) := -k_1(t) + k_2(t)e^{\rho\upsilon} + \rho.$$

For each fixed t, f is a strictly increasing function; in addition, $f(t, 0) = -k_1(t) + k_2(t) < 0$ and $f(t, k_1(t)) = k_2(t)e^{k_1(t)\upsilon} > 0$. Thus, for each t there exists a unique $\rho_t \in (0, k_1^*)$ which satisfies $f(t, \rho_t) = 0$. Moreover, because $f(t, \cdot)$ is a convex function we deduce that

$$\rho_t > \frac{(k_1(t) - k_2(t))k_1(t)}{k_1(t) - k_2(t) + k_2(t)e^{\upsilon k_1(t)}}. \tag{21}$$

Now let

$$\tilde{\rho} := \inf\{\rho_t : t \in \mathbb{R}\} \tag{22}$$

and

$$y(t) := \overline{x}(t_0)e^{-\tilde{\rho}(t-t_0)}, \ \ t \geq t_0 - \upsilon.$$

Let $c > 1$ be an arbitrary constant, then

$$x(t) < cy(t), \ \ t_0 - \upsilon \leq t \leq t_0.$$

We claim that

$$x(t) < cy(t) \text{ for } t > t_0. \tag{23}$$

Indeed, suppose that (23) does not hold, then there exists $t_1 > t_0$ for which

$$x(t) \leq cy(t) \text{ for } \quad t_0 - \upsilon \leq t \leq t_1 \quad \text{and} \quad x(t_1 + \tilde{\delta}) > cy(t_1) \text{ for all } \tilde{\delta} \in (0, \delta). \tag{24}$$

According to (20) and (24), it follows that

$$\begin{aligned}
D^+x(t_1) &\leq -k_1(t_1)x(t_1) + k_2(t_1)\overline{x}(t_1) \\
&\leq -k_1(t_1)cy(t_1) + k_2(t_1)cy(t_1 - \upsilon) \\
&= c[-k_1(t_1) + k_2(t_1)e^{\tilde{\rho}\upsilon}]\overline{x}(t_0)e^{-\tilde{\rho}(t_1 - t_0)} \\
&< c(-\tilde{\rho}\overline{x}(t_0)e^{-\tilde{\rho}(t_1 - t_0)}) = cy'(t_1),
\end{aligned} \tag{25}$$

then (24) contradicts (25). Hence (23) holds for any $t > t_0$. By letting $c \to 1$ we obtain

$$x(t) \leq \overline{x}(t_0)e^{-\tilde{\rho}(t - t_0)}.$$

Finally, (21)-(22) yield

$$\inf_{t \in \mathbb{R}} \left\{ \frac{(k_1(t) - k_2(t))k_1(t)}{k_1(t) - k_2(t) + k_2(t)e^{\upsilon k_1(t)}} \right\} < \tilde{\rho} < k_1^*,$$

and the proof is complete.

■

The following Lemma gives us an integral formula for the almost periodic solutions of (3).

Lemma 3.3 *Let $x(t) := x(t; t_0, \varphi)$ be a solution of the initial value problem given by(3) and (6). Then for $t_1 \geq t_0$*

$$x(t) = x(t_1)e^{-\int_{t_1}^t b(u)du} \int_{t_1}^t e^{-\int_s^t b(u)du} \sum_{k=1}^{M} \lambda_k r_k(s) \frac{x^{m_k}(s - \tau_k(s))}{1 + x^{n_k}(s - \tau_k(s))} ds, \quad \textit{for all } t \geq t_1.$$

Moreover, if $x(t)$ is defined on the whole real line, then

$$x(t) = \int_{-\infty}^t e^{-\int_s^t b(u)du} \sum_{k=1}^{M} \lambda_k r_k(s) \frac{x^{m_k}(s - \tau_k(s))}{1 + x^{n_k}(s - \tau_k(s))} ds, \quad \textit{for all } t \in \mathbb{R}.$$

Proof: From (3) we have:

$$\begin{aligned}
\left(x(t)e^{\int_{t_0}^t b(u)du}\right)' &= x'(t)e^{\int_{t_0}^t b(u)du} + x(t)e^{\int_{t_0}^t b(u)du}b(u) \\
&= e^{\int_{t_0}^t b(u)du} \sum_{k=1}^{M} \lambda_k r_k(t) \frac{x^{m_k}(t - \tau_k(t))}{1 + x^{n_k}(t - \tau_k(t))}
\end{aligned}$$

and integrating from t_1 to t we obtain

$$x(t)e^{\int_{t_0}^t b(u)du} = x(t_1)e^{\int_{t_0}^{t_1} b(t)du} + \int_{t_1}^t e^{\int_{t_0}^s b(u)du} \sum_{k=1}^{M} \lambda_k r_k(s) \frac{x^{m_k}(s - \tau_k(s))}{1 + x^{n_k}(s - \tau_k(s))} ds$$

and thus,

$$x(t)=x(t_1)e^{-\int_{t_1}^t b(u)du}+\int_{t_1}^t e^{-\int_s^t b(u)du}\sum_{k=1}^M \lambda_k r_k(s)\frac{x^{m_k}(s-\tau_k(s))}{1+x^{n_k}(s-\tau_k(s))}ds.$$

In addition, if $x(t)$ is defined on the whole real line, taking the limit on the right-hand side of the equality we deduce that

$$x(t)=\int_{-\infty}^t e^{-\int_s^t b(u)du}\sum_{k=1}^M \lambda_k r_k(s)\frac{x^{m_k}(s-\tau_k(s))}{1+x^{n_k}(s-\tau_k(s))}ds,$$

and the proof is now complete. ■

Observe that Theorem 2.6 requires the existence of constants η and $t_{\varphi,\tilde{x}}$ such that $\tilde{x}(t), x(t;t_0,\varphi)>\eta$. This fact shall be guaranteed by the following Lemmas.

The following assumptions will be needed throughout the rest of the section:

$$\sum_{k=1}^M \lambda_k (r_k)_* > b^* \qquad \text{and} \qquad \inf_{t\in\mathbb{R}} b(t)>0. \tag{26}$$

Remark 3.3 *Let $x(t)$ be a positive solution of (3). Then*

$$x'(t)\geq -b(t)x(t),$$

and hence $\frac{x(t_1)}{x(t_2)}\leq e^{\int_{t_1}^{t_2} b(t)dt}$ for any $t_1\leq t_2$. In particular, this implies that

$$x(t-\tau_k(t))\leq Lx(t), \tag{27}$$

where $L:=\max_{t\in\mathbb{R}} e^{\int_{t-\upsilon}^t b(s)ds}$.

Lemma 3.4 *If $x(t):=x(t;t_0,\varphi)$ is a solution of the initial value problem given by (3) and (6), then $x(t)$ is positive and bounded.*

<u>*Proof*</u>: Suppose firstly there exists $\tilde{t}$ such that $x(\tilde{t})=0$ and $x(t)>0$ for all $t\in[t_0,\tilde{t})$, then

$$\lim_{t\to\tilde{t}^-} x'(\tilde{t})=\sum_{k=1}^M \frac{x^{m_k}(\tilde{t}-\tau_k(\tilde{t}))}{1+x^{n_k}(\tilde{t}-\tau_k(\tilde{t}))}>0,$$

a contradiction. Next, suppose that $x(t)$ is unbounded, then there exists a sequence $t_j\to+\infty$ such that $\lim_{t_j\to+\infty} x(t_j-\upsilon)=+\infty$. From Remark 3.3, it follows that $x(t_j-\upsilon)\leq Lx(t_j)$ and $x(t_j-\upsilon)\leq Lx(t_j-\tau_k(t_j))$ for $k=1,\ldots,k$, which implies

$$x(t_j), x(t_j-\tau_k(t_j))\to+\infty \text{ as } t_j\to+\infty. \tag{28}$$

Due to (3) and (27) we get

$$\begin{aligned}
x'(t_j) &= \left[\sum_{k=1}^M \lambda_k r_k(t_j)\frac{x^{m_k}(t_j-\tau_k(t_j))}{x(t_j)\left(1+x^{n_k}(t_j-\tau_k(t_j))\right)}-b(t_j)\right]x(t_j)\\
&\leq \left[\sum_{k=1}^M \lambda_k r_k(t_j)L\frac{x^{m_k-1}(t_j-\tau_k(t_j))}{1+x^{n_k}(t_j-\tau_k(t_j))}-b(t_j)\right]x(t_j).
\end{aligned}$$

Thus, from (28) we deduce the existence of a positive constant J such that $x'(t_j) < -J < 0$ for all j large enough. In addition,

$$x(t_j) = x(t_0) + \int_{t_0}^{t_j} x'(s)ds \leq x(t_0) - J(t_j - t_0), \text{ for } j \text{ large enough.}$$

This yields

$$x(t_j) \to -\infty \text{ as } j \to +\infty,$$

a contradiction.

■

Remark 3.4 *If (26) holds, then we may fix a positive constant $\eta > 0$ such that*

$$\frac{\alpha^{m_j-1}}{1+\alpha^{n_j}} > \frac{b^*}{\sum_{k=1}^{M} \lambda_k (r_k)_*}, \quad \text{for all} \quad \alpha \in (0, \eta] \tag{29}$$

for all $j = 1, \ldots, M$. Furthermore, if $n_k > m_k > 0$ for some k, then we can observe that the constant η previously defined can be chosen in such a way that $0 < \eta < V$ with V defined in (7) and, consequently, we may also fix $\tilde{\eta} > \eta$ such that

$$\sum_{k:n_k>m_k>0} \frac{\eta^{m_k}}{1+\eta^{n_k}} = \sum_{k:n_k>m_k>0} \frac{\tilde{\eta}^{m_k}}{1+\tilde{\eta}^{n_k}}, \text{ if } n_j > m_j > 0 \text{ for some } j, \tag{30}$$

and $\tilde{\eta} = +\infty$ otherwise.

Lemma 3.5 *Let $n_j < m_j$ for some j and let η and $\tilde{\eta}$ be defined as in Remark 3.4. Suppose there exists a positive constant $W \in (\eta, \tilde{\eta}]$ such that*

$$\sup_{t \geq t_0} \left\{ \sum_{k:m_k \leq n_k} \lambda_k r_k(t) + \left(\sum_{k:m_k > n_k} \lambda_k r_k(t) \frac{(LW)^{m_k - n_k}}{W} - b(t) \right) W \right\} < 0. \tag{31}$$

Then there exists $\bar{t}_\varphi > t_0$ such that

$$x(t; t_0, \varphi) < W \text{ for all } t \geq \bar{t}_\varphi.$$

Proof: In the first place, suppose that $x(t_1) < W$ for some $t_1 > t_0$. We claim that $x(t) < W$ for all $t > t_1$. Indeed, otherwise there exists $\bar{t} \in (t_1, +\infty)$ such that

$$x(\bar{t}) = W \text{ and } x(t) < W \text{ for all } t \in [t_1, \bar{t}),$$

which together with (27),(31) and from the fact that

$$\sup_{u>0} \left\{ \frac{u^{m_k}}{1+u^{n_k}} \right\} \leq 1, \text{ for } n_k \geq m_k \geq 0 \tag{32}$$

implies

$$
\begin{aligned}
0 < x'(\bar{t}) &= \sum_{k=1}^{M} \lambda_k r_k(\bar{t}) \frac{x^{m_k}(t-\tau_k(\bar{t}))}{1+x^{n_k}(\bar{t}-\tau_k(\bar{t}))} - b(\bar{t})x(\bar{t}) \\
&\leq \sum_{k:m_k\leq n_k} \lambda_k r_k(\bar{t}) + \sum_{k:m_k>n_k} \lambda_k r_k(\bar{t}) x^{m_k-n_k}(\bar{t}-\tau_k(\bar{t})) - b(\bar{t})x(\bar{t}) \\
&= \sum_{k:m_k\leq n_k} \lambda_k r_k(\bar{t}) + \left(\sum_{k:m_k>n_k} \lambda_k r_k(\bar{t}) L^{m_k-n_k} W^{m_k-n_k-1} - b(\bar{t}) \right) W \\
&\leq \sup_{t>t_0} \left\{ \sum_{k:m_k\leq n_k} \lambda_k r_k(t) + \left(\sum_{k:m_k>n_k} \lambda_k r_k(t) L^{m_k-n_k} W^{m_k-n_k-1} - b(t) \right) W \right\} := \zeta < 0,
\end{aligned}
$$

a contradiction.

Suppose that $x(t_0) \geq W$, again in view of (27),(31) and (32) we have

$$
\begin{aligned}
x'(t_0) &\leq \sum_{k:m_k\leq n_k} \lambda_k r_k(t_0) + \left(\sum_{k:m_k>n_k} \lambda_k r_k(t_0) L^{m_k-n_k} x^{m_k-n_k-1}(t_0) - b(t_0) \right) x(t_0) \\
&\leq \sum_{k:m_k\leq n_k} \lambda_k r_k(t_0) + \left(\sum_{k:m_k>n_k} \lambda_k r_k(t_0) L^{m_k-n_k} W^{m_k-n_k-1} - b(t_0) \right) x(t_0) \\
&:= \zeta < 0.
\end{aligned}
$$

Furthermore, by continuity, there exists $\beta \geq t_0$ such that

$$
x'(t) \leq \zeta < 0 \qquad \text{for all } t \in [t_0, \beta]. \tag{33}
$$

Thus, β can be chosen in such a way that $x(\beta) = W$, so there exists $t_1 > \beta$ such that $x(t_1) < W$ and the proof follows.

■

Lemma 3.6 *Under the assumptions of Lemma 3.5, there exists a positive constant $t_\varphi \geq \bar{t}_\varphi$ such that*

$$
x(t; t_0, \varphi) > \eta \quad \textit{for all } t \geq t_\varphi.
$$

Proof: The proof is analogous to those given in [20, 33] (see [20, Lemma 2.2] and [33, Lemma 5]) and we omit it. ■

Lemma 3.7 *Let $m \geq 0$, $n > 0$ be constants. The function $g_{m,n}(u) = \frac{m+(m-n)u}{(1+u)^2}$ satisfies:*

$$
\left| g_{m,n}(u) \right| \leq \begin{cases} \frac{(n-m)^2}{4n} & \textit{if } n > m(3+2\sqrt{2}) \\ m & \textit{otherwise,} \end{cases} \tag{34}
$$

for all $u \geq 0$

Proof: In the case of $m \geq n$, the function $g_{m,n}$ is positive, nonincreasing and $g_{m,n}(0) = m$. In the case $m < n$, it is easy to verify that $g_{m,n}$ is nonincreasing on $\left[0, \frac{n+m}{n-m}\right)$ and increasing on $\left(\frac{n+m}{n-m}, +\infty\right)$. Moreover, $g_{m,n}(\frac{n+m}{n-m}) = -\frac{(n-m)^2}{4n}$ and $\lim_{u\to+\infty} g(u) = 0$. Finally, is easy to see that $\frac{(n-m)^2}{4n} > m$ if and only if $n > m(3+2\sqrt{2})$. This analysis completes the proof. ■

4 Proofs of the main results

In this section, we shall give a detailed proof of some of the main results of Section 2. The remaining proofs follow analogously and are consequently omitted.

Proof of Theorem 2.1: First we consider the case $0 \le m_j < 1$ for some j. Let us verify that the assumptions of Theorem 3.1 are satisfied. Let $P \subset AP(\mathbb{R})$ be the cone of nonnegative functions defined in Section 3 and set the operator

$$\Phi(x,y)(t) := \int_{-\infty}^{t} e^{-\int_s^t b(u)du} \left(\sum_{k:m_k>0} \lambda_k r_k(s) \frac{x^{m_k}(s-\tau_k(s))}{1+x^{n_k}(s-\tau_k(s))} + \sum_{k:m_k=0} \lambda_k r_k(s) \frac{1}{1+y^{n_k}(s-\tau_k(s))} \right) ds \quad (35)$$

for all $t \in \mathbb{R}$.

It is clear that $h_k(y) := \frac{1}{1+y^{n_k}}$ is a nondecreasing function. In addition, from the fact that $n_k \le m_k$, it is readily seen that $g_k(x) := \frac{x^{m_k}}{1+x^{n_k}}$ is a nondecreasing function. Due to the monotonicity of these functions, the nonlinear operator Φ is mixed monotone in $P^\circ \times P^\circ$. Moreover, by properties of almost periodic functions it follows that $\Phi(x,y) \in AP(\mathbb{R})$. Moreover, $\Phi(P^\circ \times P^\circ) \subset P^\circ$. Indeed, for each $(x,y) \in P^\circ \times P^\circ$ there exist $\kappa_1, \kappa_2 > 0$ such that $\kappa_1 \le x(t), y(t) \le \kappa_2$ for all $t \in \mathbb{R}$. Thus,

$$\begin{aligned} \Phi(x,y)(t) &\ge \int_{-\infty}^{t} e^{-\int_s^t b(u)du} \left(\sum_{k:m_k>0} \lambda_k r_k(s) \left(\min_{\kappa_1 \le w \le \kappa_2} \frac{w^{m_k}}{1+w^{n_k}} \right) + \sum_{k:m_k=0} \lambda_k r_k(s) \frac{1}{1+\kappa_2^{n_k}} \right) ds \\ &\ge \sum_{k:m_k>0} \frac{\lambda_k (r_k)_*}{b^*} \left(\min_{\kappa_1 \le w \le \kappa_2} \frac{w^{m_k}}{1+w^{n_k}} \right) + \sum_{k:m_k=0} \frac{\lambda_k (r_k)_*}{b^*} \frac{1}{1+\kappa_2^{n_k}} := \tilde{\epsilon} > 0. \end{aligned}$$

Next, for K large enough we have

$$\sum_{k:m_k>0} F^s \frac{\lambda_k r_k^*}{\tilde{b}_*} \frac{K^{m_k-1}}{1+K^{n_k}} + \sum_{k:m_k=0} F^s \frac{\lambda_k r_k^*}{\tilde{b}_*} \frac{1}{K} \le 1. \quad (36)$$

Let us fix the constant function $v_0 := K > 1$, where K satisfies (36). In addition, we choose a constant $\epsilon \in (0, K)$ such that

$$\frac{\lambda_j (r_j)_*}{b^*} \frac{\epsilon^{m_j-1}}{1+\epsilon^{n_j}} \ge 1, \qquad \text{if } 0 < m_j < 1 \quad (37)$$

or such that

$$\frac{\lambda_j (r_j)_*}{b^*} \frac{\epsilon^{-1}}{1+K^{n_j}} \ge 1, \qquad \text{if } m_j = 0. \quad (38)$$

Define $u_0 := \epsilon$ and, by virtue of (36) and (5) we obtain

$$\begin{aligned} \Phi(v_0, u_0) &\le \int_{-\infty}^{t} F^s e^{-\int_s^t \tilde{b}(u)du} \left(\sum_{k:m_k>0} \lambda_k r_k(s) \frac{K^{m_k}}{1+K^{n_k}} + \sum_{k:m_k=0} \lambda_k r_k(s) \frac{1}{1+\epsilon^{n_k}} \right) ds \\ &\le \sum_{k:m_k>0} F^s \frac{\lambda_k r_k^*}{\tilde{b}_*} \frac{K^{m_k}}{1+K^{n_k}} + \sum_{k:m_k=0} F^s \frac{\lambda_k r_k^*}{\tilde{b}_*} \\ &\le K = v_0, \end{aligned}$$

and from (37)-(38) it follows that

$$\Phi(u_0, v_0) \geq \frac{\lambda_j (r_j)_*}{b^*} \frac{\epsilon^{m_j}}{1+\epsilon^{n_j}} \geq \epsilon := u_0, \qquad \text{if } m_j < 1 \tag{39}$$

and,

$$\Phi(u_0, v_0) \geq \frac{\lambda_j (r_j)_*}{b^*} \frac{1}{1+K^{n_j}} \geq \epsilon := u_0, \qquad \text{if } m_j = 0. \tag{40}$$

We conclude that $\Phi(u_0, v_0) \geq u_0$.

In addition, for each $\gamma \in (0,1)$ and $x, y \in [u_0, v_0]$, we obtain

$$\begin{aligned}\Phi(\gamma x, \gamma^{-1} y)(t) \geq \int_{-\infty}^{t} e^{-\int_s^t b(u)du} \Bigg(& \sum_{k:m_k>0} \lambda_k r_k(s) \frac{x^{m_k}(s-\tau_k(s))}{1+x^{n_k}(s-\tau_k(s))} \frac{\gamma^{m_k}(1+u_0^{n_k})}{1+\gamma^{n_k} u_0^{n_k}} \\ & + \sum_{k:m_k=0} \lambda_k r_k(s) \frac{1}{1+y^{n_k}(s-\tau_k(s))} \frac{1+v_0^{n_k}}{1+\gamma^{-n_k} v_0^{n_k}} \Bigg) ds \\ & \geq \phi(\gamma)\Phi(x,y)(t),\end{aligned}$$

where $\phi : (0,1) \to (0, +\infty)$ is the mapping defined by

$$\phi(\gamma) = \min\left\{ \min_{k:m_k>0} \left\{ \frac{\gamma^{m_k}(1+\epsilon^{n_k})}{1+\gamma^{n_k}\epsilon^{n_k}} \right\}, \min_{\{k:m_k=0\}} \left\{ \frac{1+K^{n_k}}{1+\gamma^{-n_k} K^{n_k}} \right\} \right\}. \tag{41}$$

Thus,

$$\Phi(\gamma x, \gamma^{-1} y)(t) \geq \phi(\gamma)\Phi(x,y)(t), \text{ for each } \gamma \in (0,1) \text{ and } x, y \in [u_0, v_0].$$

Furthermore, it is easy to see that $\phi(\gamma) > \gamma$ for all $\gamma \in (0,1)$. Hence, by Theorem 3.1, Φ has a unique fixed point $\tilde{x} \in [u_0, v_0]$. Thus, by Lemma 3.3, the operator $\tilde{x}$ is the unique solution of (3) such that $\epsilon \leq \tilde{x}(t) \leq K$.

It remains to analyze the case $m_k = 1$ for all k. As before, we choose $v_0 = K$ large enough satisfying (36). By virtue of (H) there is a positive constant $\tilde{\epsilon} \in (0, K)$ small enough such that

$$\int_{-\infty}^{t} e^{-\int_s^t b(u)du} \sum_{k=1}^{M} \lambda_k r_k(s) \frac{\tilde{\epsilon}}{1+\tilde{\epsilon}^{n_k}} \geq \tilde{\epsilon}. \tag{42}$$

Define, $u_0 := \tilde{\epsilon}$ and consider the nondecreasing operator

$$\tilde{\Phi}(x)(t) := \int_{-\infty}^{t} e^{-\int_s^t b(u)du} \sum_{k=1}^{M} \lambda_k r_k(s) \frac{x(s-\tau_k(s))}{1+x^{n_k}(s-\tau_k(s))} ds \tag{43}$$

and the function

$$\tilde{\phi}(\gamma) := \min_{k=1,\ldots,M} \left\{ \frac{\gamma(1+\tilde{\epsilon}^{n_k})}{1+\gamma^{n_k}\tilde{\epsilon}^{n_k}} \right\}. \tag{44}$$

It is seen that $\tilde{\Phi}$ and $\tilde{\phi}$ satisfy all assumptions of Corollary 3.2. Then $\tilde{\Phi}$ has a unique fixed point $\tilde{x} \in [u_0, v_0]$ and, by Lemma 3.3 $\tilde{x}$ is the unique solution of (3) such that $\epsilon \leq \tilde{x}(t) \leq K$.

To conclude, observe that, in both cases, the constant function v_0 can be chosen arbitrarily large, as well as u_0 can be chosen arbitrarily small. Thus, if $z(t)$ is another almost periodic solution with a positive infimum of (3), then we may assume that $u_0 \leq z(t) \leq v_0$. Hence, $\tilde{x} = z$ and the proof is complete.

■

Next we shall prove Theorem 2.2. Observe that the assumptions allow not only bounded monotone nonlinear terms ($n_k = m_k > 0$ or $m_k = 0$) but also nonlinear single-humped terms ($n_k > m_k > 0$), which are neither monotone increasing nor decreasing. Thus the fixed point Theorem 3.1 cannot be applied for an arbitrary large interval as in the aforementioned cases.

Proof of Theorem 2.2: The proof is divided into 2 steps.
Step 1. Let $x(t)$ an almost periodic solution of (3). In view of Lemma 3.3, (10) and (32), we have

$$\begin{aligned} x(t) &= \int_{-\infty}^{t} e^{-\int_s^t b(u)du} \sum_{k=1}^{M} \lambda_k r_k(s) \frac{\tilde{x}^{m_k}(s-\tau_k(s))}{1+\tilde{x}^{n_k}(s-\tau_k(s))} \\ &\le \int_{-\infty}^{t} e^{-\int_s^t b(u)du} \sum_{k=1}^{M} \lambda_k r_k(s) ds \\ &\le V, \qquad \text{for all } t \in \mathbb{R}. \end{aligned}$$

Step 2. First, we consider the case $0 \le m_j < 1$ for some j. Let $v_0 := V$ and define $u_0 := \epsilon$, with $\epsilon \in (0, V)$ such that

$$\frac{\lambda_j (r_j)_*}{b^*} \frac{\epsilon^{m_j - 1}}{1+\epsilon^{n_j}} \ge 1, \qquad \text{if } 0 < m_j < 1 \tag{45}$$

or such that

$$\frac{\lambda_j (r_j)_*}{b^*} \frac{\epsilon^{-1}}{(1+V^{n_j})} \ge 1, \qquad \text{if } m_j = 0. \tag{46}$$

Let the operator $\Phi : P^\circ \times P^\circ \times \to P^\circ$ be defined as in (35). Due to the monotonicity of the functions $\frac{u^{m_k}}{1+u^{n_k}}$ on $(0, V)$ and $\frac{1}{1+u^{n_k}}$ on $(0, +\infty)$, Φ is a mixed monotone operator on $[u_0, v_0] \times [u_0, v_0]$.

Moreover, from (37),(38) and (10),(32) we get

$$\Phi(u_0, v_0) \ge u_0 \quad \text{and} \quad \Phi(v_0, u_0) \le v_0.$$

Let the mapping $\varphi : (0,1) \to (0, +\infty)$ be given by

$$\varphi(\gamma) := \min \left\{ \min_{k:m_k>0} \left\{ \frac{\gamma^{m_k}(1+\epsilon^{m_k})}{1+\gamma^{n_k}\epsilon^{n_k}} \right\}, \min_{k:m_k=0} \left\{ \frac{1+V^{n_k}}{1+\gamma^{-n_k}V^{n_k}} \right\} \right\}. \tag{47}$$

Then, it is not difficult to show that $\varphi(\gamma) > \gamma$ for each $\gamma \in (0,1)$. In addition, it is readily verified that for each $\gamma \in (0,1)$ and $x, y \in [u_0, v_0]$

$$\Phi(\gamma x, \gamma^{-1} y)(t) \ge \varphi(\gamma) \Phi(x, y).$$

Hence, by Theorem 3.1, Φ has a unique fixed point $\tilde{x} \in [u_0, v_0]$. Thus, by Lemma 3.3, $\tilde{x}$ is the unique solution of (3) such that $\epsilon \le \tilde{x}(t) \le V$.

For the case $m_k = 1$ for all k, let $\tilde{\Phi}$ be the operator defined in (43) and the function $\tilde{\phi}$ defined in (44).

Analogously to the preceding proofs, one can show that all the assumptions of Corollary 3.2 are fulfilled. Then $\tilde{\Phi}$ has a unique fixed point $\tilde{x} \in [u_0, v_0]$ and, by Lemma 3.3 $\tilde{x}$ is the unique solution of (3) such that $\tilde{\epsilon} \le \tilde{x}(t) \le K$.

To conclude, observe that, in both cases, the positive constant function u_0 can be chosen arbitrarily small. Thus, if $z(t) \in (0, V]$ is another almost periodic solution with a positive infimum of (3), then we may assume that $u_0 \le z(t) \le V$. Hence, $\tilde{x} = z$.

Furthermore, by *Step 1* all almost periodic solutions are uniformly bounded by the constant V. We conclude that $\tilde{x}$ is the unique almost periodic solution with positive infimum and the proof is complete.

■

In the previous proof, the existence of a uniform bound for almost periodic solutions of (3) allowed us to obtain the unique almost periodic solution with positive infimum of problem (3) in the presence of nonlinear single-humped terms ($n_k > m_k > 0$). Observe that the assumptions in Theorem 2.3 admit also unbounded terms ($n_k < m_k$). Thus, uniform bounds cannot be obtained as before and the preceding argument cannot be applied.

Proof of Theorem 2.3: First we consider the case $0 \leq m_j < 1$ for some j. Let $\Phi : P^\circ \times P^\circ \to P^\circ$ be the operator defined in (35). Set $v_0 = V$, with V the constant defined in (7) and $u_0 = \epsilon < V$ defined as in (45)-(46). Then, $\Phi(u_0, v_0) \geq u_0$ and, in view of (11), we obtain $\Phi(v_0, u_0) \leq v_0$. Again, it is easy to verify that Φ is a mixed monotone operator $[u_0, v_0] \times [u_0, v_0]$ and $\Phi(P^\circ \times P^\circ) \subset P^\circ$.

Setting the function ϕ defined as in (41), with similar arguments as used in the previous theorems it is easy to see that all remaining assumptions of Theorem 3.1 are fullfilled. Hence, Φ has a unique fixed point $\tilde{x} \in [u_0, v_0]$. Thus, by Lemma 3.3 $\tilde{x}(t)$ is the unique almost periodic solution of (3) such that $\epsilon \leq \tilde{x}(t) \leq V$.

In the case of $m_k = 1$ for all k, we consider the operator $\tilde{\Phi}$ and the function ϕ defined in (35) and (44) respectively. The remaining proof for this case is similar to that of Theorem 2.2 and we omit it.

To conclude, observe that, in both cases, the positive constant function u_0 can be chosen arbitrarily small. Thus, if $z(t) \in (0, V]$ is another almost periodic solution with a positive infimum of (3), then we may assume that $u_0 \leq z(t) \leq V$. Hence, $\tilde{x} = z$.

■

The arguments in the proof of Theorem 2.4 are similar to those given in the proof of Theorem 2.2. However, the assumption $n_q > 1$ for some q such that $m_q = 0$ implies that condition $\phi(\gamma) > \gamma$, with ϕ defined above, is not fulfilled for all $\gamma \in (0, 1)$ and $x, y \in [u_0, v_0]$. Thus, more restrictive assumptions are needed.

Proof of Theorem 2.4: The proof is divided into two steps.
Step 1. Let us define the functions h_k as follows. If k is such that $n_k \leq 1$ and $m_k = 0$ then

$$h_k(y) = \frac{1}{1 + y^{n_k}},$$

and if k is such that $n_k > 1$ and $m_k = 0$, then we set

$$h_k(y) = \begin{cases} \frac{1}{1+y^{n_k}} & \text{if} \quad y \leq T \\ \frac{1}{1+S^{n_k}} & \text{if} \quad y > T \end{cases},$$

where T is the constant defined in (9).

Let us consider the following associated equation

$$x'(t) = \sum_{k:m_k>0} \lambda_k r_k(t) \frac{x^{m_k}(t - \tau_k(t))}{1 + x^{n_k}(t - \tau_k(t))} + \sum_{k:m_k=0} \lambda_k r_k(t) h_k(x(s - \tau_k(s))) - b(t)x(t). \tag{48}$$

Now, similarly to Theorem 2.2 *Step 1*, in view of (12) one can show that all positive almost periodic solutions $x(t)$ of equations (3) and (48) satisfy $x(t) \leq T$ for all $t \in \mathbb{R}$. Moreover, note that this statement implies that equations (3) and (48) have the same positive almost periodic solutions.
Step 2. Set the constant functions $v_0 = T$ and $u_0 = \epsilon$, with $\epsilon \in (0, T)$ as in (45), if $0 < m_j < 1$, or such that

$$\frac{\lambda_j (r_j)_*}{b^*} \frac{\epsilon^{-1}}{(1 + T^{n_j})} \geq 1, \qquad \text{if } m_j = 0. \tag{49}$$

Define operator Θ by

$$\Theta(x,y)(t)=\int_{-\infty}^{t}e^{-\int_s^t b(u)du}\left(\sum_{k:m_k>0}\lambda_k r_k(s)\frac{x^{m_k}(t-\tau_k(t))}{1+x^{n_k}(t-\tau_k(t))}+\sum_{k:m_k=0}\lambda_k r_k(s)h_k(y(s-\tau_k(s)))\right)ds. \tag{50}$$

It is readily seen that Θ is mixed monotone in $[u_0,v_0]\times[u_0,v_0]$ and $\Theta(P^\circ\times P^\circ)\subset P^\circ$. In addition, in view of (12), (45) and (49), we obtain $\Theta(v_0,u_0)\leq v_0$ and $\Theta(u_0,v_0)\geq u_0$.

Let the mapping $\theta:(0,1)\to(0,+\infty)$ be given by

$$\theta(\gamma)=\min\left\{\varphi(\gamma),\min_{\{n_k>1:m_k=0\}}\left\{\frac{1+\gamma^{n_k}T^{n_k}}{1+T^{n_k}}\right\}\right\}, \tag{51}$$

where $\varphi(\gamma)$ is defined as in (47).

By a direct computation it is readily verified that for each k such that $n_k>1$ and $m_k=0$,

$$\frac{h_k(\gamma^{-1}y)}{h_k(y)}\geq\frac{1+\gamma^{n_k}T^{n_k}}{1+T^{n_k}},$$

for all $\gamma\in(0,1)$ and $y\in[u_0,v_0]$, which together with (47) yields that,

$$\Theta(\gamma x,\gamma^{-1}y)\geq\theta(\gamma)\Theta(x,y).$$

Similarly to Theorem 2.2 *Step 1*, one can conclude that Θ has a unique fixed point $\tilde{x}(t)\in[u_0,v_0]$ and that this is the unique almost periodic solution with positive infimum of equation (48) such that $\tilde{x}(t)\leq T$.

Furthermore, by *Step 1*, we conclude that $\tilde{x}(t)$ is the unique almost periodic solution with positive infimum of equation (48). Thus, $\tilde{x}(t)$ is the unique almost periodic solution with positive infimum of (3) and the proof is complete. ■

In order to prove Theorem 2.6 we employ Lemma 3.2. Moreover, we assume that there exist constants $\eta,t_{\varphi,\tilde{x}}>0$ such that $\tilde{x}(t),x(t;t_0,\varphi)>\eta$ for all $t\geq t_{\varphi,\tilde{x}}$. Such bound can be obtained under the conditions of Lemma 3.5.

Proof of Theorem 2.6: Let $\tilde{x}(t)$ be a positive almost periodic solution of (3) and $x(t)=x(t;t_0,\varphi)$ the solution of the initial value problem (3) and (6). Define $y(t):=\tilde{x}(t)-x(t)$ with $t\in[t_0-\upsilon,+\infty)$, then we have

$$y'(t)=\sum_{k=1}^{M}\lambda_k r_k(t)\left[\frac{x^{m_k}(t-\tau_k(t))}{1+x^{n_k}(t-\tau_k(t))}-\frac{\tilde{x}^{m_k}(t-\tau_k(t))}{1+\tilde{x}^{n_k}(t-\tau_k(t))}\right]-b(t)y(t). \tag{52}$$

Computing the upper right Dini derivative of $|y(t)|$ and from the mean-value theorem we obtain:

$$\begin{aligned}D^+|y(t)|&\leq\sum_{k=1}^{M}\lambda_k r_k(t)\left|\frac{x^{m_k}(t-\tau_k(t))}{1+x^{n_k}(t-\tau_k(t))}-\frac{\tilde{x}^{m_k}(t-\tau_k(t))}{1+\tilde{x}^{n_k}(t-\tau_k(t))}\right|-b(t)|y(t)|\\&=\sum_{k=1}^{M}\lambda_k r_k(t)\left|\frac{\theta^{m_k-1}(t-\tau_k(t))\left[m_k+(m_k-n_k)\theta^{n_k}(t-\tau_k(t))\right]}{(1+\theta^{n_k}(t-\tau_k(t)))^2}\right|\left|x(t-\tau_k(t))-\tilde{x}(t-\tau_k(t))\right|-b(t)|y(t)|\\&<\sum_{k=1}^{M}\lambda_k r_k(t)\eta^{m_k-1}\left|g_{m_k,n_k}\left(\theta^{n_k}(t-\tau_k(t))\right)\right|\left|x(t-\tau_k(t))-\tilde{x}(t-\tau_k(t))\right|-b(t)|y(t)|,\end{aligned}$$

where $\theta(t)$ lies between $x(t)$ and $\tilde{x}(t)$. In view of Lemma 3.7 we obtain

$$D^+|y(t)| \le p(t)\overline{|y(t)|} - b(t)|y(t)|, \text{ for all } t \ge t_{\varphi,\tilde{x}}$$

where $\overline{|y(t)|} := \sup_{t-\upsilon\le s\le t}\{|y(s)|\}$.

Thus, by Lemma 3.2 there exists $\rho > 0$ such that

$$|\tilde{x}(t) - x(t)| = |y(t)| \le \overline{|y(t_{\varphi,\tilde{x}})|}e^{-\rho(t-t_{\varphi,\tilde{x}})} = K_{\varphi,\tilde{x}}e^{-\rho t}, \text{ for all } t \ge t_{\varphi,\tilde{x}}$$

and the proof is complete. ∎

5 Examples

In this section, we give examples to demonstrate the results obtained in Section 2.

Example 5.1 *Consider the following model of hematopoiesis with multiple time-varying delays:*

$$x'(t) = \frac{1}{4}\left(2 + \frac{1}{2}|\cos(\sqrt{2}t)|\right)\frac{x^{\frac{1}{4}}(t - 2e^{\cos t})}{1 + x^{\frac{1}{2}}(t - 2e^{\cos t})} \tag{53}$$

$$+\frac{1}{4}\left(2 + \frac{1}{2}|\sin(\sqrt{3}t)|\right)\frac{x^{\frac{1}{4}}\left(t - 2e^{\sin t}\right)}{1 + x^{\frac{1}{2}}\left(t - 2e^{\sin t}\right)} - (1.5 + 2\cos(400t))x(t).$$

It is seen that,

$$m = \frac{1}{4}, \ \ n = \frac{1}{2} \qquad \textit{and}$$

$$M[b] = \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} b(s)ds = 1.5 + \lim_{T\to\infty}\frac{1}{T}\frac{1}{200}[\sin(400(t+T)) - \sin(400t)] = 1.5.$$

Thus, (53) *satisfies the assumptions of Theorem 2.2. Therefore, equation* (53) *has a unique positive almost periodic solution with positive infimum.*

However, existence and stability results in [18] cannot be applied. It is due to the fact that the following assumption, employed in the mentioned work,

$$\eta^i := \inf_{t\in\mathbb{R}}\left\{-\tilde{b}(t) + F^i\sum_{k=1}^{M}\lambda_k r_k(t)\right\} > 0$$

is not satisfied. Indeed,

$$\tilde{b} = 1.5, \ \ F^i = e^{-\frac{1}{100}}, \ \ \lambda_1 = \lambda_2 = \frac{1}{4}, \ \ (r_1)^* = (r_2)^* = 2.5 \quad \textit{and}$$

$$\eta^i < -1.5 + F^i(\lambda_1 2.5 + \lambda_2 2.5) \approx -0.262.$$

Example 5.2 *Consider the following model of hematopoiesis with both bounded and unbounded nonlinear terms:*

$$x'(t) = \frac{1}{2}\left(2 + \frac{1}{2}|\cos(\sqrt{2}t)|\right)\frac{x^{\frac{1}{2}}\left(t - 2e^{\cos t}\right)}{1 + x^{\frac{1}{4}}\left(t - 2e^{\cos t}\right)} \tag{54}$$

$$+\frac{1}{2}\left(2+\frac{1}{2}|\sin(\sqrt{3}t)|\right)\frac{1}{1+x^{\frac{1}{2}}(t-2e^{\sin t})}-1.5x(t).$$

It is seen that,

$$m_1=\frac{1}{2},\ \ n_1=\frac{1}{4},\ \ m_2=0,\ \ n_2=\frac{1}{2},\ \ \upsilon=2e\quad \textit{and}\quad M[b]=1.5.$$

Thus, (54) *satisfies the assumptions of Theorem 2.1. Therefore, equation* (54) *has a unique positive almost periodic solution with positive infimum.*

Moreover, this solution is globally exponentially stable. Indeed, let $\eta = 0.5$ *and* $M = 15$*, by Remark 3.4* $\tilde{\eta} = +\infty$*. Then*

$$\sum_{k=1}^{2}\lambda_k(r_k)_* = 2 > 1.5 = b^*,\ \ L = e^{\int_0^\upsilon 1.5} = e^{3e}$$

$$\sup_{t\in\mathbb{R}}\left\{\lambda_2 r_2(t) + \left(\lambda_1 r_1(t) L^{m_1-n_1} W^{m_1-n_1-1} - b(t)\right) W\right\} \leq \lambda_2 r_2^* + \left(\lambda_1 r_1^* e^{\frac{3e}{4}} W^{-\frac{3}{4}} - 1.5\right) W \approx -2.3555 < 0,$$

$$\inf_{\alpha\in(0,\eta]}\frac{\alpha^{m_1-1}}{1+\alpha_1^n} \geq \frac{\left(\frac{1}{2}\right)^{\frac{1}{2}-1}}{1+\left(\frac{1}{2}\right)^{\frac{1}{4}}} \approx 0.76822 > 0.75 = \frac{b^*}{\sum_{k=1}^{2}\lambda_k(r_k)_*}$$

$$\inf_{\alpha\in(0,\eta]}\frac{\alpha^{m_2-1}}{1+\alpha_2^n} \geq \frac{\left(\frac{1}{2}\right)^{-1}}{1+\left(\frac{1}{2}\right)^{\frac{1}{2}}} \approx 1.1715 > 0.75 = \frac{b^*}{\sum_{k=1}^{2}\lambda_k(r_k)_*},$$

$$\inf_{t\in\mathbb{R}}\left\{b(t) - \sum_{k=1}^{2}\eta^{m_k-1}\lambda_k r_k(t) m_k\right\} \geq 1.5 - \eta^{-\frac{1}{2}}\lambda_1(r_1)^* m_1 \approx 0.6161 > 0$$

which imply that (54) *satisfies the assumptions of Lemma 2.6. Thus, the unique almost periodic solution with positive infimum is globally exponentially stable.*

6 Conclusions and open problems

It is worth to notice that the authors in [5,9,18,20,30,33] only considered the hematopoiesis model for $m_k \leq n_k$. Moreover, in these works, the term production is assumed to be the sum of functions with the same behaviour, that is $m_k = m$ and $n_k = n$ for all k. Thus, the results in [5,9,18,20,30,33] and references therein cannot be applied to prove the existence and global exponential stability of the positive almost periodic solution of (3).

By applying a new fixed point theorem, this paper provides sufficient conditions for existence and uniqueness of positive almost periodic solutions for a generalized hematopoiesis model. For the global exponential stability we apply a Halanay-type inequality. We remark that this method is quite different from those employed by other authors. The results are new and complement previously known results.

However, it is difficult to establish sufficient criteria ensuring global exponential stability of the positive almost periodic solution of (3) when the loss rate $b(t)$ is oscillatory and $m_j > n_j$ for some j. In addition, condition $0 \leq m_k \leq 1$ for all k has been adopted as fundamental for the existence and stability analysis of (3). The approach used in this paper and in [5,9,18,20,30,33] cannot be applied to equation (3). We find that these open problems might be of interest for scientists who plan to start future research in this field.

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