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Anti-Kählerian Geometry on Lie Groups

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Abstract Let G be a Lie group of even dimension and let (g, J) be a left invariant anti-Kähler structure on G. In this article we study anti-Kähler structures considering the distinguished cases where the complex structure J is abelian or bi-invariant. We find that if G admits a left invariant anti-Kähler structure (g, J) where J is abelian then the Lie algebra of G is unimodular and (G, g) is a flat pseudo-Riemannian manifold. For the second case, we see that for any left invariant metric g for which J is an anti-isometry we obtain that the triple (G, g, J) is an anti-Kähler manifold. Besides, given a left invariant anti-Hermitian structure on G we associate a covariant 3-tensor θ on its Lie algebra and prove that such structure is anti-Kähler if and only if θ is a skew-symmetric and pure tensor. From this tensor we classify the real 4-dimensional Lie algebras for which the corresponding Lie group has a left invariant anti-Kähler structure and study the moduli spaces of such structures (up to group isomorphisms that preserve the anti-Kähler structures).

Keywords Anti-Hermitian geometry · Norden metrics · B-manifolds · Anti-Kähler manifold · Lie groups · Abelian complex structure · Bi-invariant complex structure

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1 Introduction

Anti-Hermitian geometry can be considered as a counterpart of Hermitian geometry: an almost anti-Hermitian manifold is a triple (M, g, J), where (M, g) is a pseudo-Riemannian manifold and J is an almost complex structure on M such that J is symmetric for g. The idea of an almost anti-Hermitian manifold goes back at least as far as [10, §2], where such a manifold is called a *generalized B-manifold*. In the literature, other names are also used for this class of manifolds: *Norden Manifolds* [6] or *almost complex manifolds with a Norden metric* [9], in honour to the Russian mathematician Aleksandr P. Norden.

Since 1985, anti-Hermitian geometry has been very extensively studied, and continues to be a subject of intense interest in complex geometry and mathematical physics ([4, 5]). Many contributions to the field have been made by the Bulgarian geometry school (for instance, see [9, 10, 13–15, 20]).

In [10] it was given a division of almost anti-Hermitian geometry in 8 types of geometries by means of representation theory of $O(n, n) \cap GL(n, \mathbb{C}) \cong O(n, \mathbb{C})$. By following such result, the fundamental class of almost anti-Hermitian manifolds is the family of *anti-Kähler manifolds* (also known as *B-manifold* [10, §2], *Kählerian manifold with a Norden metric* [9] or *Kähler-Norden manifold* [18]), which are anti-Hermitian manifolds with a parallel complex structure.

Other strong motivation to study anti-Kähler manifolds comes from the work of the physicists-mathematicians Andrzej Borowiec, Mauro Francaviglia, Marco Ferraris and Igor Volovich in [4, 5], where it is proved that there is a one-to-one correspondence between (Einstein) holomorphic-Riemannian manifolds and (Einstein) anti-Kähler manifolds [5, Theorem 2.2, Theorem 5.1]. The curvature properties of anti-Kähler manifolds have been studied by Karina Olszak (formerly known as Karina Słuka) in [17–19] and by Arif Salimov and Murat İşcan in [12]. Spinor geometry and geometric analysis on anti-Kähler manifolds have been considered by Nedim Değirmenci and Şenay Karapazar in [7, 8]. Very recently, Antonella Nannicini has studied the generalized geometry of anti-Hermitian manifolds in [16] where she builds complex Lie algebroids over anti-Kähler manifolds.

This work is intended as an attempt to study anti-Kähler geometry on Lie groups and to motivate new properties of anti-Kähler manifolds. In this paper, we focus on anti-Kähler structures on Lie groups in the left invariant setting. In the complex geometry of Lie groups, we have two distinguished classes of left invariant complex structures, namely, abelian and bi-invariant complex structures. We study anti-Kähler structures with complex structures in each class.

In Section 3 we deal with the case when J is a bi-invariant complex structure on a Lie group G. In such case, we see that if J is an anti-isometry of a left invariant metric g, the triple (G, g, J) is an anti-Kähler manifold. Besides, we prove a sort of converse of [20, Proposition 3.3] which states that a semisimple Lie group G admitting a bi-invariant complex structure J satisfies that (G, g, J) is an

anti-Kähler-Einstein manifold with non-vanishing cosmological constant, where g is the bi-invariant metric on G induced by the Killing form of the Lie algebra of G.

In Section 4 we study left invariant anti-Kähler structures on Lie groups whose complex structure is abelian. In this context we find that if a Lie group G admits a left invariant anti-Kähler structure (g, J), where J is an abelian complex structure, then the Lie algebra of G is unimodular and (G, g) is a flat pseudo-Riemannian manifold.

In Section 5, given a left invariant anti-Hermitian structure on a Lie group G, we associate a covariant 3-tensor θ on its Lie algebra $\mathfrak{g}$ and prove that such structure is anti-Kähler if and only if θ is a skew-symmetric and pure tensor. The tensor θ allows us to define a subclass of *anti-Kähler Lie groups* formed by those one whose 3-tensor θ vanishes. Any four dimensional Lie group with left invariant anti-Kähler structure belongs to such family, and so, in Section 6, we are concerned with the classification of such Lie groups and prove that there are exactly two non-abelian Lie algebras in dimension 4 (up to isomorphism) admitting anti-Kähler structures. The end of Section 6 is devoted to study how many left invariant anti-Kähler structures a four dimensional simply connected Lie group can admit (up to group isomorphisms that preserve the anti-Kähler structure).

2 Preliminaries

We start this section by giving the definition of an almost anti-Hermitian manifold.

Definition 2.1 (Almost anti-Hermitian Manifold) An *almost anti-Hermitian manifold* is a triple (M, g, J), where M is a differentiable manifold of real dimension $2n$, J is an almost complex structure on M and g is an *anti-Hermitian* metric on (M, J), that is

$$g(JX, JY) = -g(X, Y), \ \forall X, Y \in \mathfrak{X}(M), \tag{2.1}$$

or equivalently, J is symmetric with respect to g.

If additionally J is integrable, then the triple (M, g, J) is called an *anti-Hermitian manifold* or *complex Norden manifold*.

Remark 2.2 If (M, g, J) is an almost anti-Hermitian manifold, it is straightforward to check that the signature of g is (n, n), i.e. g is a neutral metric.

Remark 2.3 The linear (algebra) model of an almost anti-Hermitian manifold is given by a vector space V of real dimension $2n$ with a linear complex structure J on V ($J^2 = -\mathrm{Id}$) and an inner product $\langle \cdot, \cdot \rangle$ on V such that J is an anti-isometry of $(V, \langle \cdot, \cdot \rangle)$; which, from now on, we shall call it an *anti-Hermitian vector space*. The action of J on V induces a complex vector space structure on V defined by $(a + \sqrt{-1}b) \cdot v := av + bJv$, for all $v \in V$. Let $\langle\langle \cdot, \cdot \rangle\rangle : V \times V \to \mathbb{C}$ given by:

$$\langle\langle v, w \rangle\rangle = \langle v, w \rangle - \sqrt{-1}\langle Jv, w \rangle. \tag{2.2}$$

It is straightforward to show that $\langle\langle \cdot, \cdot \rangle\rangle$ is a $\mathbb{C}$-symmetric inner product on the complex space $(V, \mathbb{C})$ and even more $\langle \cdot, \cdot \rangle$ is the real part of $\langle\langle \cdot, \cdot \rangle\rangle$.

By considering an orthonormal basis for the complex inner product space $(V, \langle\langle\cdot,\cdot\rangle\rangle)$ (see [11, The Basis Theorem 2.46.]), say $\{X_1, \ldots, X_n\}$, it follows that $\{X_1, JX_1, \ldots, X_n, JX_n\}$ is an orthonormal basis for the real inner product space $(V, \langle\cdot,\cdot\rangle)$ (see [10, Theorem 1.2], where such basis is called *orthonormal J-basis*). And conversely, the real part (and the imaginary part) of a $\mathbb{C}$-symmetric inner product on a complex vector space V, together with the linear complex structure J given by $Jv := \sqrt{-1}\cdot v$, define an anti-Hermitian vector space structure on V.

Remark 2.4 Let $(V, \langle\cdot,\cdot\rangle, J)$ be an anti-Hermitian vector space and let

$$G := \{T : V \to V : TJT^{-1} = J \text{ and } \langle Tv, Tw\rangle = \langle v, w\rangle,\ \forall v, w \in V\},$$

i.e. G is the intersection of the group of isometries of $(V, \langle\cdot,\cdot\rangle)$ with the group that preserves the linear complex structure J. Because of Remark 2.3, we have $G \cong \mathrm{O}(n,n)\cap\mathrm{GL}(n,\mathbb{C})$. Now, let us consider the complex inner product space $(V, \langle\langle\cdot,\cdot\rangle\rangle)$ associated to $(V, \langle\cdot,\cdot\rangle, J)$, just as in Remark 2.3, and let

$$\widehat{G} := \{T : V \to V : \langle\langle Tv, Tw\rangle\rangle = \langle\langle v, w\rangle\rangle,\ \forall v, w \in V\},$$

the isometry group of $(V, \langle\langle\cdot,\cdot\rangle\rangle)$. It follows from [11, The Basis Theorem 2.46.] that $\widehat{G} \cong \mathrm{O}(n,\mathbb{C})$. An easy computation shows that the groups G and $\widehat{G}$ are equal (and consequently, $\mathrm{O}(n,n)\cap\mathrm{GL}(n,\mathbb{C}) \cong \mathrm{O}(n,\mathbb{C})$).

Definition 2.5 (Anti-Kähler manifold) An *Anti-Kähler manifold* is an almost anti-Hermitian manifold (M, g, J) such that J is parallel with respect to the Levi-Civita connection of the pseudo-Riemannian manifold (M, g).

Let (M, g, J) be an almost anti-Hermitian manifold. From now on, let us denote by ∇ the Levi-Civita connection of (M, g) and we denote by $(\nabla_X J)$ the covariant derivative of J in the direction of the vector field X. We recall that $(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y$.

Remark 2.6 Note that an anti-Kähler manifold (M, g, J) satisfies that J is integrable; it follows from the well known relation between the Nijenhuis tensor N and the covariant derivative of J with respect to the Levi-Civita connection of (M, g)

$$\begin{aligned} N(X,Y) &:= [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] \\ &= (\nabla_{JX}J)Y - J(\nabla_X J)Y - (\nabla_{JY}J)X + J(\nabla_Y J)X, \end{aligned} \tag{2.3}$$

for all X, Y in $\mathfrak{X}(M)$.

The following lemma can be considered as an analogue to a well-known result in Hermitian geometry and it is a direct consequence of the identity $X(g(JY, Z)) = X(g(Y, JZ))$ for all $X, Y, Z \in \mathfrak{X}(M)$ (since J is symmetric with respect to g).

Lemma 2.7 [10, Lemma 2.1.] *Let (M, g, J) be an almost anti-Hermitian manifold. Then $(\nabla_X J)$ is a symmetric operator with respect to the metric g, i.e.*

$$g((\nabla_X J)Y, Z) = g(Y, (\nabla_X J)Z),\ \forall X, Y, Z \in \mathfrak{X}(M). \tag{2.4}$$

Proposition 2.8 [10, Theorem 2.4.c] *Let (M, g, J) be an almost anti-Hermitian manifold. Then, (M, g, J) is an anti-Kähler manifold if and only if*

$$(\nabla_{JX}J)Y = \varepsilon J(\nabla_X J)Y,\ \forall X, Y \in \mathfrak{X}(M) \tag{2.5}$$

where ε is a real constant.

Proof Define $\alpha(X, Y, Z) = g((\nabla_X J)Y, Z)$. By Lemma 2.7, α is a tensor which is symmetric in the last two variables:

$$\alpha(X, Y, Z) = \alpha(X, Z, Y) \tag{2.6}$$

Under the hypothesis we have

$$\begin{aligned}\alpha(JX, Y, Z) &= g((\nabla_{JX}J)Y, Z)\\ &= \varepsilon g(J(\nabla_X J)Y, Z)\\ &= \varepsilon g((\nabla_X J)Y, JZ),\ \text{since } J \text{ is symmetric for } g\\ &= \varepsilon\alpha(X, Y, JZ).\end{aligned} \tag{2.7}$$

On the other hand,

$$\begin{aligned}\alpha(JX, Y, Z) &= \varepsilon g(J(\nabla_X J)Y, Z)\\ &= -\varepsilon g((\nabla_X J)JY, Z),\ \text{since } J \text{ anti-commute with } (\nabla_X J)\\ &= -\varepsilon\alpha(X, JY, Z).\end{aligned} \tag{2.8}$$

Combining these three relations, we have

$$\begin{aligned}\alpha(JX, Y, Z) &\overset{(2.6)}{=} \alpha(JX, Z, Y)\\ &\overset{(2.8)}{=} -\varepsilon\alpha(X, JZ, Y)\\ &\overset{(2.6)}{=} -\varepsilon\alpha(X, Y, JZ)\\ &\overset{(2.7)}{=} -\alpha(JX, Y, Z)\end{aligned} \tag{2.9}$$

Therefore $\nabla J = 0$. □

Definition 2.9 (Twin metric) Let (M, g, J) be an almost anti-Hermitian manifold. The tensor defined by the formula $\widetilde{g}(X, Y) := g(JX, Y)$, $\forall X, Y \in \mathfrak{X}(M)$ is symmetric because of (2.1); we have even more that $(M, \widetilde{g}, J)$ is an almost anti-Hermitian manifold. The metric $\widetilde{g}$ is called *associated metric* ([9]), *twin metric* or *dual metric* ([4]).

Remark 2.10 Let (M, g, J) be an anti-Kähler manifold and $\widetilde{g}$ its twin metric. It is proved in [12, Theorem 5] that the Levi-Civita connection of the twin metric coincides with the Levi-Civita connection of g. In particular $(M, \widetilde{g}, J)$ is also an anti-Kähler manifold. Using this fact, it is proved in [12, Theorem 6] that the Riemannian curvature tensor of (M, g, J) is *pure*, i.e. for smooth vector fields X, Y, Z, W,

$$R(JX, Y, Z, W) = R(X, JY, Z, W) = R(X, Y, JZ, W) = R(X, Y, Z, JW). \tag{2.10}$$

It had been proved previously in [19, Equation (16)].

2.1 Left invariant geometric structures on Lie groups

We now proceed to consider Lie groups endowed with left invariant geometric structures. Let G be a Lie group and let us denote by $\mathfrak{g}$ its Lie algebra, which is the finite dimensional real vector space consisting of all smooth vector fields invariant under left translations L_p, $p \in G$. If g is a left invariant pseudo-Riemannian metric on G, i.e. the left translations are isometries of (G, g), then g is completely determined by the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ induced by g:

$$\langle X, Y \rangle = g(X, Y), \ \forall X, Y \in \mathfrak{g}.$$

Conversely, every inner product on T_eG (here, e is the identity element of G), or equivalently an inner product on $\mathfrak{g}$, defines a left invariant pseudo-Riemannian metric on G. The Levi-Civita connection of (G, g) is a left invariant affine connection, that is, if $X, Y \in \mathfrak{g}$ then $\nabla_X Y \in \mathfrak{g}$. Besides, since $g(U, V)$ is a constant function on G for all U, V in $\mathfrak{g}$, we have that ∇_X satisfies

$$0 = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{2.11}$$

and, from the Koszul formula,

$$g(\nabla_X Y, Z) = \tfrac{1}{2} \{ g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y) \}, \tag{2.12}$$

for X, Y, Z in $\mathfrak{g}$.

An almost complex structure J on a Lie group G is said to be *left invariant* if $(\mathrm{d}\, L_p) \circ J = J \circ (\mathrm{d}\, L_p)$ for all $p \in G$; equivalently for all $X \in \mathfrak{g}$, $J \circ X \in \mathfrak{g}$. Therefore, J is completely determined by the linear complex structure $J_e : T_eG \to T_eG$. Conversely, every linear complex transformation on T_eG determines a left invariant almost complex structure J on G, which is integrable if the Nijenhuis tensor N given in (2.3) vanishes on $\mathfrak{g}$.

Definition 2.11 (Abelian complex structure) A left invariant almost complex structure J on a Lie group G is called *abelian* when it satisfies

$$[JX, JY] = [X, Y], \ \forall X, Y \in \mathfrak{g}. \tag{2.13}$$

Remark 2.12 Note that an abelian complex structure J on a Lie group G is in fact integrable, hence (G, J) is a complex manifold, but (G, J) is not a complex Lie group (unless $\mathfrak{g}$ is an abelian Lie algebra).

The notion of abelian complex structure has an important role in the complex geometry of Lie groups. Such notion was introduced by Isabel Dotti, Roberto Miatello and Laura Barberis in [3] and since then, it has been extensively studied.

Definition 2.13 (Bi-invariant complex structure) A left invariant almost complex structure J on a Lie group is called *bi-invariant* if it satisfies

$$[JX, Y] = J[X, Y] (= [X, JY]), \ \forall X, Y \in \mathfrak{g}. \tag{2.14}$$

Remark 2.14 Note that a bi-invariant complex structure J on a Lie group G is in fact integrable, and even more, (G, J) is a complex Lie group.

We are interested in studying anti-Kähler structures in the left invariant setting. From now on, we say that (g, J) is a *left invariant almost anti-Hermitian structure* on a Lie group G if (G, g, J) is an almost anti-Hermitian manifold where g and J are left invariant geometric structures on G. In addition, if (G, g, J) is an anti-Kähler manifold, we say that (g, J) is a *left invariant anti-Kähler structure* on G. The following proposition provides sufficient conditions for a left invariant almost anti-Hermitian structure on a Lie group to be a left invariant anti-Kähler structure.

Proposition 2.15 *Let (g, J) be a left invariant almost anti-Hermitian structure on a Lie group G. If any of the following conditions are satisfied:*

$$\nabla_{JX}Y = -J\nabla_X Y, \quad \forall X, Y \in \mathfrak{g}, \tag{2.15}$$

$$\nabla_{JX}Y = J\nabla_X Y, \quad \forall X, Y \in \mathfrak{g}, \tag{2.16}$$

then (G, g, J) is an anti-Kähler manifold, and even more, J is an abelian complex structure if condition (2.15) holds and J is a bi-invariant complex structure if condition (2.16) is satisfied.

Proof We begin by noting that any of the conditions implies that

$$(\nabla_{JX}J)Y = \varepsilon J(\nabla_X J)Y, \quad \forall X, Y \in \mathfrak{g};$$

with $\varepsilon = -1$ when the condition (2.15) is satisfied, and $\varepsilon = 1$ in the other case.

Since $B(X, Y) = (\nabla_{JX}J)Y - \varepsilon J(\nabla_X J)Y$ is a $(2, 1)$-tensor field on G and B vanishes for all X, Y in $\mathfrak{g}$, we have that B vanishes identically. From Proposition 2.8, it follows that (G, g, J) is an anti-Kähler manifold.

We now proceed with the proof of the last part of the proposition. If condition (2.15) is satisfied, let X, Y in $\mathfrak{g}$:

$$\begin{aligned} [JX, JY] &= \nabla_{JX}JY - \nabla_{JY}JX \\ &\overset{(2.15)}{=} -J\nabla_X JY + J\nabla_Y JX \\ &\overset{(\nabla J)\equiv 0}{=} \nabla_X Y - \nabla_Y X \\ &= [X, Y]. \end{aligned}$$

By a similar argument, it is easy to check that condition (2.16) implies that J is a bi-invariant complex structure. □

In the following two propositions we state and prove the converse of Proposition 2.15.

Proposition 2.16 *Let (g, J) be a left invariant anti-Kähler structure on a Lie group G such that J is an abelian complex structure. Then (G, g, J) satisfies the condition (2.15), i.e.*

$$\nabla_{JX}Y = -J\nabla_X Y, \quad \forall X, Y \in \mathfrak{g}.$$

Proof Let us consider the real inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ induced by the metric g. Define $\alpha(X, Y, Z) = \langle \nabla_{JX} Y + J\nabla_X Y, Z \rangle, \forall X, Y, Z$ in $\mathfrak{g}$.

From property (2.11) and the fact that J is parallel and symmetric for $\langle \cdot, \cdot \rangle$, it follows that α is skew-symmetric in the last two variables

$$\alpha(X, Y, Z) = -\alpha(X, Z, Y). \tag{2.17}$$

Since J is an abelian complex structure, it is immediate to see that

$$\nabla_{JX} Y + J\nabla_X Y = \nabla_{JY} X + J\nabla_Y X$$

and therefore α is symmetric in the first two variables:

$$\alpha(X, Y, Z) = \alpha(Y, X, Z). \tag{2.18}$$

Combining (2.17) with (2.18) yields that α is symmetric in the first and last variables:

$$\begin{aligned} \alpha(X, Y, Z) &\overset{(2.17)}{=} -\alpha(X, Z, Y) \\ &\overset{(2.18)}{=} -\alpha(Z, X, Y) \\ &\overset{(2.17)}{=} -(-\alpha(Z, Y, X)). \end{aligned} \tag{2.19}$$

Finally, the symmetry given by (2.19) and (2.18) implies that α is a symmetric tensor on $\mathfrak{g}$, since the group S_3 is generated by the transpositions $(1\,2)$ and $(1\,3)$. In particular, α is symmetric in the last two variables:

$$\begin{aligned} \alpha(X, Y, Z) &\overset{(2.17)}{=} \alpha(Y, X, Z) \\ &\overset{(2.19)}{=} \alpha(Z, X, Y) \\ &\overset{(2.17)}{=} \alpha(X, Z, Y). \end{aligned} \tag{2.20}$$

Comparing (2.20) and (2.17) we obtain that α vanishes identically, which establishes the proof. □

Proposition 2.17 *Let (g, J) be a left invariant anti-Kähler structure on a Lie group G such that J is a bi-invariant complex structure. Then (G, g, J) satisfies condition (2.16), i.e.*

$$\nabla_{JX} Y = J\nabla_X Y, \quad \forall X, Y \in \mathfrak{g}.$$

Proof The proof is straightforward and follows directly from the fact that the Levi-Civita connection is a torsion-free connection: Since $[JX, Y] = J[X, Y], \forall X, Y \in \mathfrak{g}$, we have $\nabla_{JX} Y - \nabla_Y JX = J(\nabla_X Y - \nabla_Y X)$. Since $\nabla J \equiv 0$, we have $\nabla_Y JX = J\nabla_Y X$, and so $\nabla_{JX} Y = J\nabla_X Y$. □

3 Anti-Kähler geometry on complex Lie groups

It has already been proved in [5] that

Proposition 3.1 [5, Proposition 4.1] *Every complex parallelizable manifold M admits an anti-Kähler structure.*

This proposition is a very contrasting result with the Kähler case, where the only compact complex parallelizable manifold admitting Kähler structures are *complex tori*; a well known result due to Hsien-Chung Wang ([21, Corollary 2]).

Another proof of the existence of an anti-Kähler structure on complex Lie group is given by Marta Teofilova in [20, Proposition 3.1]. The following proposition may be considered as a generalization of the above mentioned result.

Proposition 3.2 *Let (g, J) be a left invariant almost anti-Hermitian structure on a Lie group G where J is a bi-invariant complex structure on G. Then (G, g, J) is an anti-Kähler manifold.*

Proof Since $(\nabla J)(\cdot,\cdot)$ is a tensor field on G, it is sufficient to verify that $\nabla_X JY = J\nabla_X Y$ for all X, Y in $\mathfrak{g}$. For an arbitrary $Z \in \mathfrak{g}$ we have

$$\begin{aligned}
2g(\nabla_X JY, Z) &\overset{(2.12)}{=} g([X, JY], Z) - g([JY, Z], X) + g([Z, X], JY)\\
&\overset{(2.14)}{=} g(J[X, Y], Z) - g([Y, JZ], X) + g([Z, X], JY)\\
&\overset{(2.1)}{=} g([X, Y], JZ) - g([Y, JZ], X) + g(J[Z, X], Y)\\
&\overset{(2.14)}{=} g([X, Y], JZ) - g([Y, JZ], X) + g([JZ, X], Y)\\
&\overset{(2.12)}{=} 2g(\nabla_X Y, JZ)\\
&\overset{(2.1)}{=} 2g(J\nabla_X Y, Z),
\end{aligned}$$

and the proposition follows. □

Remark 3.3 Combining Remark 2.3 and Proposition 3.2 with well-known results of representation of algebras and *wild problems* (also known as *hopeless problems*), we have that the classification of anti-Kähler manifolds could be a wild problem.

To finish this section, we want to study a result due to Marta Teofilova concerning almost anti-Hermitian structures on complex semisimple Lie groups (see [20, Proposition 3.3]). We improve slightly this result, by using the previous proposition and well-known results on the Killing form of semisimple Lie algebras, namely that the Killing form of a semisimple Lie algebra $\mathfrak{g}$ is an inner product (Cartan's criterion) inducing a bi-invariant Einstein metric on a Lie group G with Lie algebra $\mathfrak{g}$. We denote by R and Rc the Riemannian curvature tensor and the Ricci tensor of a pseudo-Riemannian manifold, respectively.

Proposition 3.4 *Let G be a semisimple Lie group admitting a bi-invariant complex structure J. If g is the bi-invariant metric on G induced by the Killing form of $\mathfrak{g}$, then (G, g, J) is an anti-Kähler-Einstein manifold with non-vanishing cosmological constant.*

About the converse of above proposition, we can prove:

Proposition 3.5 *Let G be a Lie group admitting a left invariant anti-Kähler-Einstein structure (g, J) with non-vanishing cosmological constant and g a bi-invariant metric. Then, G is a semisimple Lie group and J is a bi-invariant complex structure on G.*

Proof Since g is a bi-invariant metric on G, we have $\mathrm{Rc}(X, Y) = \frac{1}{4}B(X, Y)$ for all X, Y in $\mathfrak{g}$, where B is the Killing form of $\mathfrak{g}$. And so, from the hypothesis that g is an Einstein metric with non-vanishing cosmological constant, it follows that $\mathfrak{g}$ is a semisimple Lie algebra (Cartan's criterion).

Next, we show that J is a bi-invariant complex structure on G. Again, since g is a bi-invariant metric on G, we have $R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$ for all X, Y, Z in $\mathfrak{g}$. Since (G, g, J) is an anti-Kähler manifold, it is easy to see that $R(JX, JY) = -R(X, Y)$ for all X, Y in $\mathfrak{X}(G)$, because the *symmetry by pairs* of the Riemannian curvature tensor of (G, g) and $(\nabla J) \equiv 0$. This clearly forces that the complex structure J is *anti-abelian*; i.e. $[JX, JY] = -[X, Y]$ for all X, Y in $\mathfrak{g}$, since $\mathfrak{g}$ is centerless. From the vanishing of the Nijenhuis tensor (see (2.3)) and the anti-abelian property of J, we have that J is bi-invariant complex structure. □

Remark 3.6 There exist Lie groups admitting (non-flat) Ricci-flat anti-Kähler structures where the metric and the complex structure are bi-invariant. Such a Lie group is necessarily a solvable Lie group with vanishing Killing form.

4 Anti-Kähler geometry and abelian complex structures

In this section, we study left invariant anti-Kähler structures with abelian complex structures. We begin with an elementary observation about the Levi-Civita connection under the mentioned hypothesis.

Proposition 4.1 *Let G be a Lie group admitting a left invariant anti-Kähler structure (g, J) with J an abelian complex structure on G. Then the Levi-Civita connection of (G, g) is completely determined just by the complex structure J and the Lie algebra $\mathfrak{g}$:*

$$\nabla_X Y = \tfrac{1}{2}\left([X, Y] - J[X, JY]\right), \ \forall X, Y \in \mathfrak{g} \tag{4.1}$$

Proof Since $(\nabla J) \equiv 0$ and ∇ is torsion-free, it is easy to see that $\forall X, Y$ in $\mathfrak{X}(G)$

$$[JX, Y] - J[X, Y] = \nabla_{JX} Y - J\nabla_X Y.$$

Combining this with Proposition 2.16, we have

$$\begin{aligned} \nabla_X Y &= \tfrac{1}{2}\left([X, Y] + J[JX, Y]\right) \\ &\overset{(2.13)}{=} \tfrac{1}{2}\left([X, Y] - J[X, JY]\right) \end{aligned}$$

for all X, Y in $\mathfrak{g}$. □

Corollary 4.2 *Let G be a Lie group and let (g, J) be a left invariant anti-Kähler structure on G where J is an abelian complex structure on G. If g is a bi-invariant metric on G then the Lie algebra of G is commutative.*

The following proposition gives an important obstruction to a Lie group admitting a left invariant anti-Kähler structure (g, J) with J an abelian complex structure on G.

Proposition 4.3 *If G is a $2n$-dimensional Lie group admitting a left invariant anti-Kähler structure (g, J) with J an abelian complex structure, then $\mathfrak{g}$ is a unimodular Lie algebra, i.e. for all X in $\mathfrak{g}$*

$$\mathrm{Tr}(\mathrm{ad}_X) = 0.$$

Proof Combining the above proposition and condition (2.11), we have for all X, Y in $\mathfrak{g}$

$$\begin{aligned} 0 &= 2g(\nabla_X Y, Y) \\ &= g([X, Y] - J[X, JY], Y) \\ &= g([X, Y], Y) - g([X, JY], JY). \end{aligned}$$

Now we consider the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ induced by the left invariant metric g on G and let $\{Y_1, \ldots, Y_{2n}\}$ be an orthonormal basis of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. By using this basis, we have for all X in $\mathfrak{g}$

$$\mathrm{Tr}(\mathrm{ad}_X) = \sum_{i=1}^{2n} \langle Y_i, Y_i \rangle \langle [X, Y_i], Y_i \rangle.$$

On the other hand, by using the orthonormal basis $\{JY_1, \ldots, JY_{2n}\}$ of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ we have

$$\begin{aligned} \mathrm{Tr}(\mathrm{ad}_X) &= \sum_{i=1}^{2n} \langle JY_i, JY_i \rangle \langle [X, JY_i], JY_i \rangle \\ &\overset{(2.1)}{=} \sum_{i=1}^{2n} -\langle Y_i, Y_i \rangle \langle [X, JY_i], JY_i \rangle. \end{aligned}$$

By adding the last two expressions, we have for all X in $\mathfrak{g}$

$$2\,\mathrm{Tr}(\mathrm{ad}_X) = \sum_{i=1}^{2n} \langle Y_i, Y_i \rangle (\langle [X, Y_i], Y_i \rangle - \langle [X, JY_i], JY_i \rangle),$$

but, it follows from the first expression that each summand in the last formula for $\mathrm{Tr}(\mathrm{ad}_X)$ is equal to zero, and so $\mathrm{Tr}(\mathrm{ad}_X) = 0$. □

Under the hypothesis of the previous proposition, we have as a corollary that the abelian complex structure J is symmetric for the Killing form.

Corollary 4.4 *If G is a Lie group admitting a left invariant anti-Kähler structure (g, J) with J an abelian complex structure, then*

$$B(JX, JY) = -B(X, Y),\ \forall X, Y \in \mathfrak{g},$$

where B is the Killing form of $\mathfrak{g}$.

Proof First, it follows from the above proposition that $\mathrm{Tr}(\mathrm{ad}_{J[X,Y]}) = 0$ for all X, Y in $\mathfrak{g}$. Since $\mathrm{ad}_{J[X,Y]} = -\mathrm{ad}_{[X,Y]} J = \mathrm{ad}_Y \mathrm{ad}_X J - \mathrm{ad}_X \mathrm{ad}_Y J$, we have

$$\begin{aligned} B(JX, Y) &= B(Y, JX) \\ &= \mathrm{Tr}(\mathrm{ad}_Y \mathrm{ad}_{JX}) \\ &= -\mathrm{Tr}(\mathrm{ad}_Y \mathrm{ad}_X J) \\ &= -\mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y J) \\ &= \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_{JY}) \\ &= B(X, JY),\ \forall X, Y \in \mathfrak{g}, \end{aligned}$$

and the corollary follows. □

We state and prove the next proposition which we will use to prove the main theorem of this section (see Theorem 4.6).

Proposition 4.5 *If G is a Lie group admitting a left invariant anti-Kähler structure (g, J) with J an abelian complex structure, then for all X, Y, Z in $\mathfrak{g}$*

$$\nabla_X \nabla_Y Z = \nabla_Y \nabla_X Z.$$

Proof Let X, Y, Z in $\mathfrak{g}$. From Proposition 4.1, we have

$$\begin{aligned} \nabla_X \nabla_Y Z &= \frac{1}{2} \nabla_X ([Y, Z] - J[Y, JZ]) \\ &= \frac{1}{4} ([X, [Y, Z] - J[Y, JZ]] - J[X, J([Y, Z] - J[Y, JZ])] \\ &= \frac{1}{4} ([X, [Y, Z]] - [X, J[Y, JZ]] - J[X, J[Y, Z]] - J[X, [Y, JZ]]), \end{aligned}$$

and so $4(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z)$ is equal to

$$\begin{aligned} &= [X, [Y, Z]] - [X, J[Y, JZ]] - J[X, J[Y, Z]] - J[X, [Y, JZ]] \\ &\quad -[Y, [X, Z]] + [Y, J[X, JZ]] + J[Y, J[X, Z]] + J[Y, [X, JZ]] \\ &\overset{(2.13)}{=} [X, [Y, Z]] - [JX, [JY, Z]] + J[JX, [JY, JZ]] - J[X, [Y, JZ]] \\ &\quad -[Y, [X, Z]] + [JY, [JX, Z]] - J[JY, [JX, JZ]] + J[Y, [X, JZ]] \\ &\overset{\text{Jacobi}}{=} [[X, Y], Z] + [[JY, JX], Z] + J[[JX, JY], JZ] + J[[Y, X], JZ] \\ &\overset{(2.13)}{=} [[X, Y], Z] + [[Y, X], Z] + J[[X, Y], JZ] + J[[Y, X], JZ] \\ &= 0, \end{aligned}$$

as desired. □

Theorem 4.6 *Let (g, J) be a left invariant anti-Kähler structure on a Lie group G such that J is an abelian complex structure. Then (G, g) is a flat pseudo-Riemannian Lie group.*

Proof It is sufficient to prove that $R(X, Y)Z = 0$ for all X, Y, Z in $\mathfrak{g}$. By the above proposition, we have $R(X, Y)Z = \nabla_{[X,Y]}Z$ and therefore

$$R(JX, JY)Z = \nabla_{[JX,JY]}Z = R(X, Y)Z.$$

While on the other hand, by virtue of the *symmetry by pairs* of the Riemannian curvature tensor of (G, g) and $(\nabla J) \equiv 0$,

$$R(JX, JY)Z = -R(X, Y)Z$$

and the proof is completed. □

As a consequence of the above theorem and Proposition 4.1 we have the following obstruction: if a Lie group admits a left invariant anti-Kähler structure with abelian complex structure, then its Lie algebra $\mathfrak{g}$ together with the abelian complex structure must satisfy a distinguished 3-degree polynomial identity.

Corollary 4.7 *Let (g, J) be a left invariant anti-Kähler structure on a Lie group G such that J is an abelian complex structure, then for all X, Y, Z in $\mathfrak{g}$*

$$[J[X, Y], Z] = J[[X, Y], Z].$$

We want to give an example of a left invariant anti-Kähler structure (g, J) on a Lie group with J an abelian complex structure. The 6-dimensional Lie algebras that can be endowed with abelian complex structure were classified in [1] (with a corrigendum in [2]). We focus on the nilpotent Lie algebra $\mathfrak{n}_7$ of [1, Theorem 3.4] with its abelian complex structure J_{-1}.

Example 4.8 Consider a nilpotent Lie group N with Lie algebra

$$\mathfrak{n}_7 := \begin{cases} [X_1, X_2] = X_4,\ [X_1, X_3] = X_5,\ [X_1, X_4] = X_6, \\ [X_2, X_3] = X_6,\ [X_2, X_4] = -X_5. \end{cases}$$

Such Lie group N admits an abelian complex structure determined by the linear complex structure $J := J_{-1}$ on $\mathfrak{n}_7$ defined by

$$JX_1 = X_2,\ JX_3 = -X_4,\ JX_5 = -X_6.$$

Consider the left invariant metric g on N such that

$$\{X_1 + X_5,\ X_1 - X_5,\ X_2 + X_6,\ X_2 - X_6,\ X_3,\ X_4\}$$

is an orthonormal frame field, where $\{X_1 + X_5, X_2 + X_6, X_3\}$ are spacelike vector fields and $\{X_1 - X_5, X_2 - X_6, X_4\}$ are timelike vector fields. In the frame field $\{X_1, \ldots, X_6\}$, all possible non-zero functions of the form $g(X_i, X_j)$ are

$$g(X_1, X_5) \equiv \frac{1}{2},\ g(X_2, X_6) \equiv \frac{1}{2},\ g(X_3, X_3) \equiv 1,\ g(X_4, X_4) \equiv -1.$$

It is a simple matter to check that

$$\nabla_{X_1} X_1 = -\frac{1}{2} X_3,\ \nabla_{X_1} X_2 = \frac{1}{2} X_4,\ \nabla_{X_1} X_3 = X_5,\ \nabla_{X_1} X_4 = X_6,$$
$$\nabla_{X_2} X_1 = -\frac{1}{2} X_4,\ \nabla_{X_2} X_2 = -\frac{1}{2} X_3,\ \nabla_{X_2} X_3 = X_6,\ \nabla_{X_2} X_4 = -X_5,$$

and therefore we can prove that (N, g, J) is an anti-Kähler manifold.

5 Some 3-forms associated with left invariant anti-Kähler structures on Lie groups

Let G be a Lie group admitting a left invariant anti-Kähler structure (g, J). We begin by defining a family of bilinear maps on $\mathfrak{g}$ in the following way: let $\{a_1, \ldots, a_4\}$ be real constants and $D : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ the bilinear map on $\mathfrak{g}$ given by

$$D(X, Y) = a_1 \nabla_X Y + a_2 \nabla_{JX} Y + a_3 J \nabla_X Y + a_4 J \nabla_{JX} Y.$$

Consider the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ induced by the metric and let δ be the covariant 3-tensor on $\mathfrak{g}$ defined by

$$\delta(X, Y, Z) = \langle D(X, Y), Z \rangle,\ \forall X, Y, Z \in \mathfrak{g}.$$

Note that δ is skew-symmetric in the last two arguments, because of property (2.11) and the anti-Kähler hypothesis. Therefore, the skew-symmetric part of δ is a multiple of

$$\theta(X, Y, Z) = \langle D(X, Y), Z \rangle + \langle D(Y, Z), X \rangle + \langle D(Z, X), Y \rangle. \tag{5.1}$$

We want to highlight an important member in this family of skew-symmetric tensors which is defined from the particular bilinear map

$$D(X, Y) = \nabla_{JX} Y + J \nabla_X Y. \tag{5.2}$$

In this case, we have that δ is a *pure* tensor on $\mathfrak{g}$ since $D(JX, Y) = D(X, JY)$, $D(JX, Y) = JD(X, Y)$ and $\langle J\cdot, \cdot \rangle = \langle \cdot, J\cdot \rangle$. So θ is a pure skew-symmetric 3-tensor on $\mathfrak{g}$.

Now, we consider on $\mathfrak{g}$ the complex vector space structure induced by J $(a + \sqrt{-1}\, b) \cdot X := aX + JbX$ and call $\widehat{\theta}(X, Y, Z) = \theta(X, Y, Z) - \sqrt{-1}\, \theta(JX, Y, Z)$. In this way, we have on $(\mathfrak{g}, \mathbb{C})$ a complex skew-symmetric 3-tensor and θ is its real part. Furthermore, θ has the following very nice expression: for all X, Y, Z in $\mathfrak{g}$

$$\theta(X, Y, Z) = \langle [JX, Y], Z \rangle + \langle [JY, Z], X \rangle + \langle [JZ, X], Y \rangle. \tag{5.3}$$

Note that the above θ vanishes identically, for instance, when J is an abelian complex structure (see Proposition 2.16) or when the dimension of G is 4. If (g, J) is a left invariant anti-Kähler structure on a Lie group G with g and J bi-invariant geometric structures, then θ is a multiple of $\langle [JX, Y], Z \rangle$.

Here is an important property of the 3-tensor θ. Left invariant anti-Kähler structures on Lie groups are determined by the skew-symmetry and pureness of its 3-tensor θ:

Theorem 5.1 *Let (g, J) be a left invariant almost anti-Hermitian structure on a Lie group G and let θ be the associated 3-tensor as in (5.1) from (5.2). Then, (G, g, J) is an anti-Kähler manifold if and only if θ is skew-symmetric and pure on $\mathfrak{g}$; equivalently,*

$$\theta(X, Y, Z) = -\theta(X, Z, Y) \tag{5.4}$$

and

$$\theta(JX, Y, Z) = \theta(X, JY, Z) \tag{5.5}$$

for all X, Y, Z in $\mathfrak{g}$.

Proof As we need to show that $\nabla J \equiv 0$, let us consider $\alpha(X, Y, Z) = \langle(\nabla_X J)Y, Z\rangle$ for all X, Y, Z in $\mathfrak{g}$ and we will show that α vanishes identically on $\mathfrak{g}$.

From Lemma 2.7, we have that α is symmetric in the last two arguments:

$$\alpha(X, Y, Z) = \alpha(X, Z, Y),\ \forall X, Y, Z \in \mathfrak{g}.$$

We also have the following equality:

$$\alpha(X, Y, JZ) = -\alpha(X, JY, Z),\ \forall X, Y, Z \in \mathfrak{g}. \tag{5.6}$$

From (5.4), it follows $\theta(X, Y, Z) + \theta(X, Z, Y) = 0$ for all X, Y, Z in $\mathfrak{g}$. Now $\theta(X, Z, Y)$ is also equal to

$$\langle Z, -\nabla_{JX}Y - \nabla_X JY\rangle + \langle Y, -\nabla_{JZ}X - \nabla_Z JX\rangle + \langle X, -\nabla_{JY}Z - \nabla_Y JZ\rangle,$$

hence

$$\begin{aligned}
0 &= \theta(X, Y, Z) + \theta(X, Z, Y)\\
&= \langle\nabla_{JX}Y + J\nabla_X Y, Z\rangle + \langle\nabla_{JY}Z + J\nabla_Y Z, X\rangle + \langle\nabla_{JZ}X + J\nabla_Z X, Y\rangle\\
&\quad -\langle\nabla_{JX}Y + \nabla_X JY, Z\rangle - \langle\nabla_{JY}Z + \nabla_Y JZ, X\rangle - \langle\nabla_{JZ}X + \nabla_Z JX, Y\rangle\\
&= -\langle\nabla_X JY - J\nabla_X Y, Z\rangle - \langle\nabla_Y JZ - J\nabla_Y Z, X\rangle - \langle\nabla_Z JX - J\nabla_Z X, Y\rangle\\
&= -\alpha(X, Y, Z) - \alpha(Y, Z, X) - \alpha(Z, X, Y).
\end{aligned}$$

That is, for all X, Y, Z in $\mathfrak{g}$

$$\alpha(X, Y, Z) + \alpha(Y, Z, X) + \alpha(Z, X, Y) = 0. \tag{5.7}$$

From (5.5), we have $\theta(X, JY, Z) - \theta(JX, Y, Z) = 0$. Since $\theta(JX, Y, Z)$ is also equal to

$$\langle -\nabla_X Y + J\nabla_{JX}Y, Z\rangle + \langle J\nabla_{JY}Z - \nabla_Y Z, X\rangle + \langle\nabla_{JZ}JX + J\nabla_Z JX, Y\rangle,$$

it follows

$$\begin{aligned}
0 &= \theta(X, JY, Z) - \theta(JX, Y, Z)\\
&= \langle\nabla_{JX}JY + J\nabla_X JY, Z\rangle + \langle -\nabla_Y Z + J\nabla_{JY}Z, X\rangle + \langle J\nabla_{JZ}X - \nabla_Z X, Y\rangle\\
&\quad +\langle\nabla_X Y - J\nabla_{JX}Y, Z\rangle + \langle -J\nabla_{JY}Z + \nabla_Y Z, X\rangle - \langle\nabla_{JZ}JX + J\nabla_Z JX, Y\rangle\\
&= \langle\nabla_{JX}JY - J\nabla_{JX}Y, Z\rangle + \langle\nabla_X Y + J\nabla_X JY, Z\rangle\\
&\quad +\langle J\nabla_{JZ}X - \nabla_{JZ}JX, Y\rangle - \langle\nabla_Z X + J\nabla_Z JX, Y\rangle\\
&= \alpha(JX, Y, Z) - \alpha(X, JY, Z) - \alpha(JZ, X, Y) + \alpha(Z, JX, Y).
\end{aligned}$$

Or even better, by adding $\alpha(Y, Z, JX)$ to both sides of the preceding identity and by using equality (5.7), we have

$$\alpha(Y, Z, JX) = -\alpha(X, JY, Z) - \alpha(JZ, X, Y).$$

Now, by (5.6) we have that the above equality can be written as

$$-\alpha(Y, JZ, X) = \alpha(X, Y, JZ) - \alpha(JZ, X, Y),$$

and since, by (5.7), $\alpha(X, Y, JZ) = -\alpha(Y, JZ, X) - \alpha(JZ, X, Y)$, finally we have that

$$-\alpha(Y, JZ, X) = -\alpha(Y, JZ, X) - \alpha(JZ, X, Y) - \alpha(JZ, X, Y)$$

So, $\alpha(JZ, X, Y) = 0$ for all X, Y, Z in $\mathfrak{g}$, which establishes the proof. □

Corollary 5.2 *Let (g, J) be a left invariant almost anti-Hermitian structure on a Lie group G such that the associated 3-tensor θ vanishes identically on $\mathfrak{g}$. Then (G, g, J) is an anti-Kähler manifold.*

Let us mention an important consequence of the theorem, which is the key to obtain the main result of the following section:

Corollary 5.3 *Let (g, J) be a left invariant almost anti-Hermitian structure on a 4-dimensional Lie group G. Then, (G, g, J) is an anti-Kähler manifold if and only if the associated 3-tensor θ as in (5.1) from (5.2) vanishes identically on $\mathfrak{g}$.*

6 Left invariant anti-Kähler structures on four dimensional Lie groups

We begin this section studying the 4-dimensional Lie groups which admit a left invariant anti-Kähler structure.

Theorem 6.1 *Let G be a 4-dimensional Lie group. Then, G admits a left invariant anti-Kähler structure if and only if its Lie algebra $\mathfrak{g}$ is abelian or is isomorphic to*

$$\begin{aligned}\mathfrak{r}_{-1,-1} &= \{[e_1, e_2] = e_2, [e_1, e_3] = -e_3, [e_1, e_4] = -e_4,\\ &\quad or\\ \mathfrak{aff}(\mathbb{C})_{\mathbb{R}} &= \{[e_1, e_3] = e_3, [e_1, e_4] = e_4, [e_2, e_3] = e_4, [e_2, e_4] = -e_3.\end{aligned}$$

Proof Let G be a real Lie group of dimension four which admits a left invariant anti-Kähler structure (g, J). As before, let us denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathfrak{g}$ induced by the left invariant metric g on G.

From Remark 2.3 there exists an orthonormal basis of the Lie algebra $\mathfrak{g}$ of G of the form $\mathcal{B} = \{X, JX, Y, JY\}$, where X and Y are spacelike, and so, we have

$$[g]_{\mathcal{B}} = \text{diag}\,(1, -1, 1, -1),$$

and

$$[J]_{\mathcal{B}} = \text{diag}\,(j, j),$$

where $j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

From Corollary 5.3 we have that the 3-form θ vanishes on $\mathfrak{g}$. In particular, we have $\theta(U, V, JV) = 0$ for all U, V in $\mathfrak{g}$; equivalently

$$\langle [V, U], V\rangle = -\langle [V, JU], JV\rangle.$$

So the following equalities for the elements of $\mathcal{B}$ hold:

$$\begin{array}{llll} \langle [X,Y],X\rangle &= -\langle [X,JY],JX\rangle, & \langle [X,Y],JX\rangle &= \langle [X,JY],X\rangle, \\ \langle [X,Y],Y\rangle &= -\langle [JX,Y],JY\rangle, & \langle [X,Y],JY\rangle &= \langle [JX,Y],Y\rangle, \\ \langle [X,JY],Y\rangle &= -\langle [JX,JY],JY\rangle, & \langle [X,JY],JY\rangle &= \langle [JX,JY],Y\rangle, \\ \langle [JX,Y],X\rangle &= -\langle [JX,JY],JX\rangle, & \langle [JX,Y],JX\rangle &= \langle [JX,JY],X\rangle, \\ \langle [X,JX],X\rangle &= 0, & \langle [X,JX],JX\rangle &= 0, \\ \langle [Y,JY],Y\rangle &= 0, & \langle [Y,JY],JY\rangle &= 0. \end{array} \tag{6.1}$$

Hence, by (6.1) the Lie bracket of elements of $\mathcal{B}$ satisfies the following equations

$$\begin{aligned} [X, JX] &= a\,Y + b\,JY, \\ [X, Y] &= t_1\,X + t_2\,JX + t_3\,Y + t_4\,JY, \\ [X, JY] &= -t_2\,X + t_1\,JX + t_5\,Y + t_6\,JY, \\ [JX, Y] &= t_7\,X + t_8\,JX - t_4\,Y + t_3\,JY, \\ [JX, JY] &= -t_8\,X + t_7\,JX - t_6\,Y + t_5\,JY, \\ [Y, JY] &= c\,X + d\,JX, \end{aligned} \tag{6.2}$$

where a, b, c, d and t_i, for $i = 1, \cdots, 8$, are real numbers.

Besides, by using again the 3-form θ, since $\theta(U, U, V) = 0$ for all $U, V \in \mathfrak{g}$, we obtain

$$\begin{array}{llll} a &= \quad t_2 + t_7, & b &= t_8 - t_1, \\ c &= -(t_4 + t_5), & d &= t_3 - t_6. \end{array} \tag{6.3}$$

Now, computing all the Jacobi equations involving the elements of $\mathcal{B}$ and considering the equalities given in (6.2) we obtain the following equations

$$\begin{aligned} -(t_8 + t_1)\,[X, JX] - b\,[Y, JY] + t_3\Delta(X, Y) + t_4\Delta(X, JY) &= 0, \\ (t_2 - t_7)\,[X, JX] + a\,[Y, JY] + t_5\Delta(X, Y) + t_6\Delta(X, JY) &= 0, \\ -d\,[X, JX] + (t_3 + t_6)\,[Y, JY] - t_1\Delta(X, Y) + t_2\Delta(X, JY) &= 0, \\ -c\,[X, JX] + (t_4 - t_5)\,[Y, JY] + t_7\Delta(X, Y) - t_8\Delta(X, JY) &= 0, \end{aligned} \tag{6.4}$$

where $\Delta(U, V) = [JU, V] - [U, JV]$, for all $U, V \in \mathfrak{g}$.

If $\{[X, JX], [Y, JY], \Delta(X, Y), \Delta(X, JY)\}$ is a linearly independent set, then we have that all the constants are zero and so $\mathfrak{g}$ is the four dimensional abelian Lie algebra.

From here on, we assume that this set is linearly dependent. By (6.2) and (6.3), we see that

$$\begin{array}{rl} \Delta(X, Y) &= aX + bJX + cY + dJY, \\ \Delta(X, JY) &= -bX + aJX + dY - cJY. \end{array} \tag{6.5}$$

Next, we study the set of the 4-tuples $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ satisfying

$$\lambda_1[X, JX] + \lambda_2[Y, JY] + \lambda_3\Delta(X, Y) + \lambda_4\Delta(X, JY) = 0.$$

In terms of $\mathcal{B}$, using (6.2) and (6.5), this set coincides with the set of solutions of the homogeneous linear system $Ax = 0$, where

$$A = \begin{bmatrix} 0 & c & a & -b \\ 0 & d & b & a \\ a & 0 & c & d \\ b & 0 & d & -c \end{bmatrix}.$$

We have two cases to analize:

Case 1. If at least one of the coefficients of A is non zero, we have that the Lie algebra is

$$\mu_{a,b,\varepsilon} = \begin{cases} [X, JX] &= a\,Y + b\,JY, \\ [X, Y] &= a\,JX - \varepsilon b\,JY, \\ [X, JY] &= -a\,X + \varepsilon a\,JY, \\ [JX, Y] &= b\,JX + \varepsilon b\,Y, \\ [JX, JY] &= -b\,X - \varepsilon a\,Y, \\ [Y, JY] &= \varepsilon b\,X - \varepsilon a\,JX \end{cases} \tag{6.6}$$

with $\varepsilon^2 = 1$ and $a, b \in \mathbb{R}$ $(a, b \neq 0)$.

To prove this assertion, first we call

$$v_1 = \begin{bmatrix} -t_1 - t_8 \\ t_1 - t_8 \\ t_3 \\ t_4 \end{bmatrix}, \; v_2 = \begin{bmatrix} t_2 - t_7 \\ t_2 + t_7 \\ t_5 \\ t_6 \end{bmatrix}, \; v_3 = \begin{bmatrix} t_6 - t_3 \\ t_3 + t_6 \\ -t_1 \\ t_2 \end{bmatrix} \text{ and } v_4 = \begin{bmatrix} t_4 + t_5 \\ t_4 - t_5 \\ t_7 \\ -t_8 \end{bmatrix}.$$

By the assumption that at least one of the coefficients of A is different from zero, the rank of A is three. By (6.4), v_1, v_2, v_3 and v_4 are in the kernel of A, which has dimension one. So, these vectors are parallel and not simultaneously zero. We will suppose that v_1 is nonzero (the remaining cases give the same conditions on the coefficients a, b, c, d and t_i). So, there exist real numbers α, β and ε such that

$$v_2 = \alpha v_1, \;\; v_3 = \beta v_1, \;\; v_4 = \varepsilon v_1.$$

In particular, using the equality $v_3 = \beta\, v_1$, by the equations we have

$$\begin{cases} t_6 - t_3 &= \beta\,(-t_1 - t_8), \\ t_3 + t_6 &= \beta\,(t_1 - t_8), \\ -t_1 &= \beta\,t_3 \end{cases}$$

and so, we obtain that $t_3(1 + \beta^2) = 0$. Therefore, $t_3 = 0$ and in consequence $t_1 = t_5 = t_7 = 0$, since $-t_1 = \beta\,t_3$, $t_5 = \alpha\,t_3$ and $t_7 = \varepsilon\,t_3$. Now, by (6.3) and the above computations we have

$$\begin{aligned} a &= t_2, \quad b = t_8 \\ c &= -t_4 \quad d = -t_6, \end{aligned}$$

and using that $v_4 = \varepsilon\, v_1$ we obtain the following two equalities

$$\begin{cases} c &= \varepsilon b, \\ b &= \varepsilon c. \end{cases}$$

So, $(\varepsilon^2 - 1)c = 0$. That is, $c = 0$ or $\varepsilon^2 = 1$. But, if $c = 0$ we have that all the coefficients are zero (since we have the equalities $d = \alpha\, c$, $a = -\beta\, c$ and $b = \varepsilon\, c$), which contradicts the hypothesis. Hence, $\varepsilon^2 = 1$ and this gives that $c = \varepsilon b$ and $d = -\varepsilon a$ (the last one holds since $a = -\beta b = -\varepsilon\beta c$ and $\beta c = d$).

Finally, in this case we have that the family of Lie algebras $\mu_{a,b,\varepsilon}$ that was obtained is isomorphic to the Lie algebra $\mathfrak{r}_{-1,-1}$. If we consider the change of basis given by the matrix

$$\phi = \begin{bmatrix} -\varepsilon a & -\varepsilon b & b & -a \\ \varepsilon b & -\varepsilon a & a & b \\ 0 & -\varepsilon & -1 & 0 \\ \varepsilon & 0 & 0 & -1 \end{bmatrix}$$

whose inverse is given by

$$\phi^{-1} = \frac{1}{2(a^2+b^2)} \begin{bmatrix} -\varepsilon a & \varepsilon b & 0 & \varepsilon\left(a^2+b^2\right) \\ -\varepsilon b & -\varepsilon a & -\varepsilon\left(a^2+b^2\right) & 0 \\ b & a & -(a^2+b^2) & 0 \\ -a & b & 0 & -(a^2+b^2) \end{bmatrix},$$

we have that ϕ is an isomorphism between the Lie algebra $\mu_{a,b,\varepsilon}$ and the Lie algebra $\mathfrak{r}_{-1,-1}$.

Case 2. On the other hand, if all the coefficients of A are zero, we have that $\mathfrak{g}$ is the underlying real Lie algebra of the 2-dimensional complex Lie algebra $\mathfrak{aff}(\mathbb{C})$. Indeed, $a = b = c = d = 0$ implies that $t_5 = -t_4$, $t_6 = t_3$, $t_7 = -t_2$ and $t_8 = t_1$, and hence

$$\mu_{t_1,t_2,t_3,t_4} = \begin{cases} [X, JX] = 0, \\ [X, Y] = t_1 X + t_2 JX + t_3 Y + t_4 JY, \\ [X, JY] = -t_2 X + t_1 JX - t_4 Y + t_3 JY, \\ [JX, Y] = -t_2 X + t_1 JX - t_4 Y + t_3 JY, \\ [JX, JY] = -t_1 X - t_2 JX - t_3 Y - t_4 JY, \\ [Y, JY] = 0. \end{cases} \tag{6.7}$$

It is easy to check that $J\,[U, V] = [JU, V]$, for all $U, V \in \mathcal{B}$, and so, J is a bi-invariant complex structure on the Lie algebra μ_{t_1,t_2,t_3,t_4}.

Finally, the change of basis given by the matrix

$$\phi = \begin{bmatrix} t_3 & -t_4 & -t_1 & t_2 \\ t_4 & t_3 & -t_2 & -t_1 \\ -t_1 & -t_2 & -t_3 & -t_4 \\ t_2 & -t_1 & t_4 & -t_3 \end{bmatrix}$$

whose inverse is

$$\phi^{-1} = \frac{1}{{t_1}^2 + {t_2}^2 + {t_3}^2 + {t_4}^2}\, \phi^t$$

gives an isomorphism between μ_{t_1,t_2,t_3,t_4} and $\mathfrak{aff}(\mathbb{C})_{\mathbb{R}}$. □

Now, we will investigate a more delicate problem. We are interested in determining how many left invariant anti-Kähler structures the Lie groups given in the

Theorem 6.1 have. Rather than discussing this in full generality, let us assume that the Lie groups are simply connected and introduce the notion of *equivalence* of anti-Kähler structures.

Definition 6.2 Let (g_1, J_1) and (g_2, J_2) be left invariant anti-Kähler structures on simply connected Lie groups G_1 and G_2, respectively. We say that (g_1, J_1) is equivalent to (g_2, J_2) if there exists an isomorphism $\Psi : G_1 \to G_2$ such that it is an isometry between the pseudo-Riemannian manifolds (G_1, g_1) and (G_2, g_2), and $(\mathrm{d}\,\Psi) \circ J_1 = J_2 \circ (\mathrm{d}\,\Psi)$.

Remark 6.3 Note that two anti-Kähler structures (g_1, J_1) and (g_2, J_2) are equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that ψ is a (linear) isometry between $(\mathfrak{g}_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathfrak{g}_2, \langle\cdot,\cdot\rangle_2)$, and $\psi \circ J_1 = J_2 \circ \psi$. Here, $\langle\cdot,\cdot\rangle_i$ is the inner product on $\mathfrak{g}_i$ induced by the left invariant metric g_i.

Remark 6.4 The proof of Theorem 6.1 gives more information, namely that each Lie algebra in (6.6) and (6.7) represents a left invariant anti-Kähler structure on the respective Lie group. Our equivalence problem reduces to know when $\mu_{a,b,\varepsilon}$ and $\mu_{c,d,\varepsilon}$ are isomorphic via a linear map φ that preserves the inner product $\langle\cdot,\cdot\rangle$ and the complex structure given in the beginning of the section (that is, $\varphi \in O(2,2) \cap GL(2,\mathbb{C})$). The same reasoning for two Lie algebras in the family given in (6.7).

Theorem 6.5 *Let G be a simply connected Lie group of dimension 4 admitting a left invariant anti-Kähler structure. If the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{r}_{-1,-1}$, then G admits only one left invariant anti-Kähler structure, up to equivalence. If $\mathfrak{g}$ is isomorphic to $\mathfrak{aff}(\mathbb{C})_{\mathbb{R}}$, then G admits only a two-parameter family of non-equivalent anti-Kähler structures.*

Proof **Case 1.** First, we fix the structure $\mu_{1,0,+1}$ obtained in (6.6). We have that $\mu_{1,0,+1}$ is equivalent to $\mu_{a,b,\varepsilon}$, for all $a, b \in \mathbb{R}$ $(a, b \neq 0)$ and $\varepsilon = \pm 1$. Indeed, we take the change of basis given by the matrix

$$\varphi := \frac{1}{2(a^2+b^2)} \begin{bmatrix} \varepsilon ra & \varepsilon sb & \varepsilon rb & -\varepsilon sa \\ -\varepsilon sb & \varepsilon ra & \varepsilon sa & \varepsilon rb \\ -rb & sa & ra & sb \\ -sa & -rb & -sb & ra \end{bmatrix}$$

whose inverse is

$$\varphi^{-1} = \frac{1}{2(a^2+b^2)} \begin{bmatrix} \varepsilon ra & \varepsilon sb & -rb & sa \\ -\varepsilon sb & \varepsilon ra & -sa & -rb \\ \varepsilon rb & -\varepsilon sa & ra & sb \\ \varepsilon sa & \varepsilon rb & -sb & ra \end{bmatrix}$$

where $r = a^2 + b^2 + 1$ and $s = a^2 + b^2 - 1$.

A straightforward computation shows that φ preserves the inner product $\langle\cdot,\cdot\rangle$ and the complex structure J, and that φ is an isomorphism between $\mu_{1,0,+1}$ and $\mu_{a,b,\varepsilon}$.

Case 2. We recall that, by Remark 2.3, if we have an anti-Hermitian vector space $(V, \langle\cdot,\cdot\rangle, J)$ of real dimension 4 we have an associated 2-dimensional complex vector space $(V, \mathbb{C})$ and a $\mathbb{C}$-symmetric inner product $\langle\langle\cdot,\cdot\rangle\rangle$. And conversely, given a complex vector space endowed with a $\mathbb{C}$-symmetric inner product we have its associated anti-Hermitian vector space.

Now, we consider the structure of real Lie algebra μ_{t_1,t_2,t_3,t_4} defined in (6.7) on the anti-Hermitian vector space $(V, \langle\cdot,\cdot\rangle, J)$, with $V = \mathrm{Span}_{\mathbb{R}}\{X, JX, Y, JY\}$. Since J is a bi-invariant complex structure, μ_{t_1,t_2,t_3,t_4} induces a complex Lie bracket μ_{z_1,z_2} on the 2-dimensional complex vector space $(V, \mathbb{C})$ given by:

$$\mu_{z_1,z_2}(X, Y) = z_1X + z_2Y,$$

where $z_1 = t_1 + \sqrt{-1}t_2$ and $z_2 = t_3 + \sqrt{-1}t_4$. Conversely, if $(V, \mathbb{C})$ is endowed with a Lie bracket μ_{z_1,z_2}, where z_1, z_2 are complex numbers $(z_1, z_2 \neq 0)$, we have a member of (6.7), μ_{t_1,t_2,t_3,t_4}, where $t_1 = \mathfrak{Re}(z_1)$, $t_2 = \mathfrak{Im}(z_1)$, $t_3 = \mathfrak{Re}(z_2)$ and $t_4 = \mathfrak{Im}(z_2)$.

From Remark 2.4, we have that two Lie algebra structures μ_{t_1,t_2,t_3,t_4} and μ_{s_1,s_2,s_3,s_4} on $(V, \langle\cdot,\cdot\rangle, J)$ are isomorphic via a linear map φ that preserves the inner product $\langle\cdot,\cdot\rangle$ and the complex structure J if and only if the corresponding structures of complex Lie algebras μ_{z_1,z_2} and μ_{w_1,w_2} on $(V, \langle\langle\cdot,\cdot\rangle\rangle)$ are isomorphic via a linear map $\widehat{\varphi}$ that preserves $\langle\langle\cdot,\cdot\rangle\rangle$, where $z_1 = t_1 + \sqrt{-1}t_2$, $z_2 = t_3 + \sqrt{-1}t_4$, $w_1 = s_1 + \sqrt{-1}s_2$ and $w_2 = s_3 + \sqrt{-1}s_4$.

Thus, in this case, we can study the equivalence of anti-Khäler structures from the equivalence of structures of Lie algebras μ_{z_1,z_2} on $(V, \langle\langle\cdot,\cdot\rangle\rangle)$ via isomorphisms that preserve the $\mathbb{C}$-symmetric inner product $\langle\langle\cdot,\cdot\rangle\rangle$.

We fix $z_1, z_2 \in \mathbb{C}$ and take $\varphi \in O(V, \langle\langle\cdot,\cdot\rangle\rangle) \cong O(2, \mathbb{C})$. One can verify that

$$\varphi \cdot \mu_{z_1,z_2} = \mu_{\epsilon w_1,\epsilon w_2}, \tag{6.8}$$

with $w_1X + w_2Y = \varphi(z_1X + z_2Y)$ and $\epsilon = \det\varphi = \pm 1$. Thus, if μ_{z_1,z_2} and μ_{w_1,w_2} are equivalent (isomorphic) then $z_1^2 + z_2^2 = w_1^2 + w_2^2$, since $\varphi \in O(V, \langle\langle\cdot,\cdot\rangle\rangle)$.

Now, we suppose that z_1, z_2, w_1, w_2 are complex numbers such that $z_1^2 + z_2^2 = w_1^2 + w_2^2$. We want to prove that its corresponding structures of Lie algebras μ_{z_1,z_2} and μ_{w_1,w_2} are in the same $O(V, \langle\langle\cdot,\cdot\rangle\rangle)$-orbit. We have two possibilities:

a. If $z_1^2 + z_2^2 \neq 0$, we consider the following orthogonal bases of $(V, \langle\langle\cdot,\cdot\rangle\rangle)$

$$\mathcal{B} = \{z_1X + z_2Y, -z_2X + z_1Y\} \quad \text{and} \quad \mathcal{B}' = \{w_1X + w_2Y, -w_2X + w_1Y\}$$

and we take the linear operator φ of V which sends $z_1X + z_2Y$ onto $w_1X + w_2Y$ and $-z_2X + z_1Y$ onto $-w_2X + w_1Y$. A straightforward computation shows that φ preserves square norms, so by polarization we obtain that $\varphi \in O(V, \langle\langle\cdot,\cdot\rangle\rangle)$ and furthermore, by (6.8), μ_{z_1,z_2} and μ_{w_1,w_2} are equivalent.

b. If $z_1^2 + z_2^2 = 0$, this implies that

$$(z_1, z_2) = \begin{cases} z(1, \sqrt{-1}) \text{ or} \\ z(1, -\sqrt{-1}), \end{cases}$$

for some $z \in \mathbb{C}$, $(z \neq 0)$.

Let us first prove that, given $z \in \mathbb{C} \setminus \{0\}$, there exists $\varphi_z \in O(V, \langle\langle \cdot, \cdot \rangle\rangle)$ such that $\varphi_z \cdot \mu_{1,\sqrt{-1}} = \mu_{z,z\sqrt{-1}}$. In fact, we consider the bases of the complex vector space $(V, \mathbb{C})$ given by

$$\mathcal{B} = \{X + \sqrt{-1}Y, X - \sqrt{-1}Y\} \quad \text{and} \quad \mathcal{B}' = \{zX + \sqrt{-1}zY, \tfrac{1}{z}X - \tfrac{1}{z}\sqrt{-1}Y\},$$

and φ_z the linear operator defined by $\varphi_z(X + \sqrt{-1}Y) = zX + z\sqrt{-1}Y$ and $\varphi_z(X - \sqrt{-1}Y) = \frac{1}{z}X - \frac{1}{z}\sqrt{-1}Y$. It is easy to check that φ_z is a linear operator preserving the $\mathbb{C}$-symmetric inner product $\langle\langle \cdot, \cdot \rangle\rangle$.

Now, given $w \in \mathbb{C} \setminus \{0\}$ we have that $\mu_{1,-\sqrt{-1}}$ and $\mu_{w,-w\sqrt{-1}}$ are equivalent via $\varphi_{\frac{1}{w}}$.

Besides, $\mu_{1,\sqrt{-1}}$ and $\mu_{1,-\sqrt{-1}}$ are equivalent via $\varphi_o \in O(V, \langle\langle \cdot, \cdot \rangle\rangle)$, the linear operator whose matrix with respect to the basis $\mathcal{B} = \{X + \sqrt{-1}Y, X - \sqrt{-1}Y\}$ is

$$[\varphi_o]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Finally, if $z_1^2 + z_2^2 = w_1^2 + w_2^2 = 0$, we have that

$$(z_1, z_2) = \begin{cases} z(1, \sqrt{-1}) \text{ or} \\ z(1, -\sqrt{-1}), \end{cases} \quad \text{and} \quad (w_1, w_2) = \begin{cases} w(1, \sqrt{-1}) \text{ or} \\ w(1, -\sqrt{-1}), \end{cases}$$

for certain $z, w \in \mathbb{C} \setminus \{0\}$. Making the suitable compositions of the above linear operators, we have that if $z_1^2 + z_2^2 = w_1^2 + w_2^2 = 0$, then μ_{z_1,z_2} and μ_{w_1,w_2} are isomorphic via $\widehat{\varphi} \in O(V, \langle\langle \cdot, \cdot \rangle\rangle)$. □

6.1 The curvature of the left invariant anti-Kähler structures in dimension 4.

We consider the anti-Kähler structures in the family obtained in (6.6). An easy computation shows that

$$\nabla_X X = -aJY, \ \nabla_X Y = aJX, \ \nabla_Y Y = -\varepsilon bJX.$$

By using that ∇ is torsion free and $\nabla J \equiv 0$, we can compute the remaining values $\nabla_U V$ with $U, V \in \mathcal{B} := \{X, JX, Y, JY\}$ and show that

$$\nabla_U \nabla_V W = \nabla_V \nabla_U W, \ \forall\, U, V, W \in \mathcal{B}.$$

Furthermore, it is fairly easy to show that $\nabla_{[U,V]} W = 0$ for all U, V, W in $\mathcal{B}$. Then, anti-Kähler structures in the family are flat.

For the anti-Kähler structures obtained in (6.7), we have

$$\nabla_X X = -t_1 Y - t_2 JY, \ \nabla_X Y = t_1 X + t_2 JX, \ \nabla_Y Y = t_3 X + t_4 JX,$$

and

$$\nabla_U \nabla_V W = \nabla_V \nabla_U W, \ \forall\, U, V, W \in \mathcal{B}.$$

Furthermore, we have

$$R(X, Y) = \begin{bmatrix} 0 & H \\ -H & 0 \end{bmatrix}$$

where

$$H = \begin{bmatrix} \mathfrak{Re}(\zeta) & -\mathfrak{Im}(\zeta) \\ \mathfrak{Im}(\zeta) & \mathfrak{Re}(\zeta) \end{bmatrix},$$

here $\zeta = \langle\langle z_1 X + z_2 Y, z_1 X + z_2 Y \rangle\rangle = z_1^2 + z_2^2$, $z_1 := t_1 + t_2\sqrt{-1}$ and $z_2 := t_3 + t_4\sqrt{-1}$. And so, the anti-Kähler structure is flat if and only if $\zeta = 0$. Regarding the Ricci curvature tensor, we have that the Ricci operator Ric, $\mathrm{Rc}(\cdot,\cdot) = g(\mathrm{Ric}\,\cdot,\cdot)$, is given by the matrix

$$\mathrm{Ric} = -2\begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \tag{6.9}$$

with respect to the basis $\mathcal{B}$. Hence, the anti-Kähler structure is Einstein if and only if $\mathfrak{Im}(\zeta) = 0$, and in that case, the cosmological constant is -2ζ. Moreover, the anti-Kähler structure is Ricci flat if and only if it is flat.

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References

1. Andrada, A., Barberis, M.L., Dotti, I.: Classification of abelian complex structures on 6-dimensional Lie algebras. J. Lond. Math. Soc. **83**(1), 232–255 (2011)
2. Andrada, A., Barberis, M.L., Dotti, I.: Corrigendum: Classification of abelian complex structures on six-dimensional Lie algebras. J. Lond. Math. Soc. **87**(2), 319–320 (2013)
3. Barberis, M.L., Dotti, I., Miatello, R.: On certain locally homogeneous Clifford manifolds. Ann. Glob. Anal. Geom. **13**(3), 289–301 (1995)
4. Borowiec, A., Ferraris, M., Francaviglia, M., Volovich, I.: Almost-complex and almost-product Einstein manifolds from a variational principle. J. Math. Phys. **40**(7), 3446–3464 (1999)
5. Borowiec, A., Francaviglia, M., Volovich, I.: Anti-Kählerian manifolds. Differential Geometry and its Applications **12**(3), 281–289 (2000)
6. Castro, R., Hervella, L.M., García-Rio, E.: Some examples of almost complex manifolds with Norden metric. Rivista di Matematica della Università di Parma Serie 4 **15**, 133–141 (1989)
7. Değirmenci, N., Karapazar, Ş.: Spinors on Kähler-Norden manifolds. J. Nonlinear Math. Phys. **17**(1), 27–34 (2010)
8. Değirmenci, N., Karapazar, Ş.: Schrödinger-lichnerowicz like formula on Kähler-Norden manifolds. Int. J. Geom. Meth. Mod. Phys. **9**(1), 1250010 (14 pages) (2012)
9. Ganchev, G.T., Borisov, A.W.: Note on the almost complex manifolds with Norden Metric. Comptes rendus de l'Académie Bulgare des Sciences **39**(5), 31–34 (1986)
10. Gribachev, K.I., Mekerov, D.G., Djelepov, G.D.: Generalized B-manifold. Comptes rendus de l'Académie Bulgare des Sciences **38**(3), 299–302 (1985)
11. Harvey, F.R.: Spinors and calibrations perspectives in mathematics, vol. 9, 1st edn. Academic Press, Boston (1990)
12. İşcan, M., Salimov, A.A.: On Kähler-Norden manifolds. Proc. Math. Sci. **119**(1), 71–80 (2009)
13. Manev, M.: Classes of real isotropic hypersurfaces of a Kähler manifold with B-metric. Comptes rendus de l'Académie Bulgare des Sciences **55**(4), 27–32 (2002)
14. Mekerov, D.: Connection with parallel totally skew-symmetric torsion on almost complex manifolds with Norden metric. Comptes rendus de l'Académie Bulgare des Sciences **62**(12), 1501–1508 (2009)
15. Mekerov, D.: On the geometry of the connection with totally skew-symmetric torsion on almost complex manifolds with Norden metric. Comptes rendus de l'Académie Bulgare des Sciences **63**(1), 19–28 (2010)

16. Nannicini, A.: Generalized geometry of Norden manifolds. J. Geom. Phys. **99**, 244–255 (2016)
17. Olszak, K.: On the Bochner conformal curvature of Kähler-Norden manifolds. Central European Journal of Mathematics **3**(2), 309–317 (2005)
18. Słuka, K.: On Kähler manifolds with Norden metrics. Analele Ştiinţifice ale Universităţii "Alexandru Ioan cuza" din Iaşi. Matematică. Tomul XLVII f. **1**, 105–122 (2001)
19. Słuka, K.: On the curvature of Kähler-Norden manifolds. J. Geom. Phys. **54**(2), 131–145 (2005)
20. Teofilova, M.: Lie groups as Kähler manifolds with Killing Norden metric. Comptes rendus de l'Académie Bulgare des Sciences **65**(6), 733–742 (2012)
21. Wang, H.C.: Complex parallelisable manifolds. Proc. Am. Math. Soc. **5**(5), 771–776 (1954)