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Analysis of Different Management Strategies for Annual Ryegrass (*Lolium rigidum*) Based on a Population Dynamic Model

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Weed species present high competitive capacity, rapid adaptability and herbicide resistance, hindering their effective control across worldwide cropping regions. Since field-conducted experiments are very time-consuming and usually expensive, mathematical population-based models are valuable tools to test and develop long-term weed management programs. Within this context, the objective of this paper is to formalize analytically the possible seed bank dynamics of the *Lolium rigidum*, subjected to different control strategies. The first focus is on studying in detail the effects of integrating constant actions, promoting more environmentally and economically sustainable scenarios. From the same perspective, an alternative to applying time-variant programs is introduced. The proposed control guarantees that the weed population is sufficiently small or, alternatively, is kept below a given economic threshold level in a ten-year planning horizon. Furthermore, an optimization criterion is adopted for distributing necessary efficiency into diverse integrated options. Numerical simulations are included to illustrate the analytical findings.

Keywords: Mathematical model; weed management; control; stability; bifurcation theory.

1. Introduction

Lolium rigidum (or, commonly called, annual ryegrass) is one of the most conspicuous weed species around the world [Heap, 1997]. Its highly competitive nature, fecundity capacity, genetic variability and hybrization potential (within the *Lolium* and *Festuca* genera) favor its rapid geographical expansion as well as the increasingly problematic infestation of winter cereal fields [Castellanos-Frias *et al.*, 2016; Goggin *et al.*, 2012]. The ryegrass management becomes even more complicated due to the occurrence of multiple resistance episodes caused by the repeated applications of herbicides with the same mode of action (such as glyphosate or ACCase inhibitors) [Heap, 2017; Loureiro *et al.*, 2010; Owen & Powles, 2010]. Besides the hundreds of millions of dollars invested annually for controlling *L. rigidum* [Panell *et al.*, 2004], the environmental impact of herbicides makes non-chemical management necessary. Some of the practical alternatives are catching and removal of weed seeds during crop harvest, delay of the crop seeding, selection of more competitive crops, etc.

Field-conducted experiments to study the long-term effects of applying different integrated weed management (IWM) strategies present many difficulties since they are time-consuming and usually expensive. Mathematical models providing information about the dynamics of weed species have proven to be useful not only to predict its emergence [Chantre *et al.*, 2012; Haj Seyed Hadi & González-Andujar, 2009; Panell *et al.*, 2004] and spatial distribution [Basak *et al.*, 2013; González-Díaz *et al.*, 2015; Somerville *et al.*, 2017; Wang *et al.*, 2008] but also to test different control strategies [González-Andujar & Fernández-Quintanilla, 2004; Izquierdo *et al.*, 2003; Parsons *et al.*, 2009]. In particular, population-based models of annual weed species can predict the actual seed bank density based on its value in the previous year and the progress during different life-cycle stages (e.g. seedlings, mature individuals, reproduction, etc.). Relatively simple but nonlinear models, including factors such as the intra- and the inter-specific competitions due to limited available resources (i.e. water, nutrients and light), biotic (predators) and abiotic (climatic factors) agents as well as management decisions, can be found in the literature [Holst *et al.*, 2007; Sakai, 2001]. Except for the population density-dependence, the rest of the phenomena are, in general, represented as proportional rates of reduction in seedlings or seed production.

The comprehensive work developed in [González-Andujar & Fernández-Quintanilla, 2004] gives valuable knowledge on the dynamics of *L. rigidum*. The proposed population-based model takes into account fecundity density-dependence, seedling emergence and survival, seed losses by predators before entering the seed bank as well as anthropic actions such as crop competition, crop delayed seeding, catching and removal of weed seeds at crop harvest, and herbicide applications. Despite its simplicity, the model describes the behavior of the population appropriately according to the reported data, which were collected from various field studies [Fernández-Quintanilla *et al.*, 2000]. Simulations show that, although the use of herbicides seems to be very effective, there always exist residual weed populations if the management only consists of this chemical control action. Long-term effects of some integrated agronomic practices are numerically described in [González-Andujar & Fernández-Quintanilla, 2004]. In all cases, control tactics are kept constant throughout the ten-year planning horizon.

The objective of this paper is to study analytically the performance of different strategies for managing the population dynamics of *L. rigidum* in a crop field. In contrast to [González-Andujar & Fernández-Quintanilla, 2004], possible dynamical scenarios are formalized explicitly based on the variability of demographic parameters and the efficiencies of the control actions. For that purpose, general expressions of the two possible equilibria and sufficient conditions for their local stability are deduced. The first development focuses on characterizing in detail the effects of constant (individual or integrated) management strategies. The occurrence of a transcritical bifurcation is observed, which explains that the *L. rigidum* eradication is impossible to achieve in the fields by applying an individual control action. This nonlinear phenomenon also shows that such a situation could be well handled if an IWM strategy is planned carefully. Furthermore, the obtained results permit to specify adequate IWM programs to maintain the ryegrass at very low densities by implementing control tactics at average efficiencies.

The application of time-variant management programs is also examined. In this case, traditional linear control theory is inadequate since the model is uncontrollable around the origin [Vaccaro, 1995]. Different bifurcation control techniques based on the use of time delays, polynomial functions, washout filters, etc., could be used to stabilize the trivial equilibrium [D'Amico *et al.*, 2009; D'Amico & Calandrini, 2015; Franco & Liz, 2013; Huang *et al.*, 2017; Wang *et al.*, 2014]. The proposed strategy arises from the idea of dynamic cancellation, prioritizing fundamentally its simple and intuitive agronomic implementation. Basically, the efficiency required for controlling *L. rigidum* is calculated every year according to the actual seed bank density. In the first control option, the seed bank tends to zero asymptomatically in an infinite time horizon. This traduces into a dynamical response where density achieves a sufficiently small level in a ten-year planning horizon. In the second one, the control is adapted to keep the population below a given economic threshold level. An optimization criterion is finally introduced to distribute the necessary total efficiency into diverse integrated options. The combination of different strategies aims to promote a more sustainable agroecosystem less dependent on herbicide applications.

The remaining part of the paper is organized as follows. Section 2 describes the mathematical model with its basic properties and the behavior of the ryegrass in the absence of control. Analytical results concerning constant control actions are given in Sec. 3. A detailed analysis of the proposed time-variant control is presented in Sec. 4. Finally, conclusions are drawn in Sec. 5.

2. Mathematical Model and Its Properties

2.1. *The life-cycle and control actions of L. rigidum*

Lolium rigidum is an annual plant whose life-cycle can be divided into various stages, as outlined in Fig. 1. Each stage can be modeled independently, including specific characteristics and control

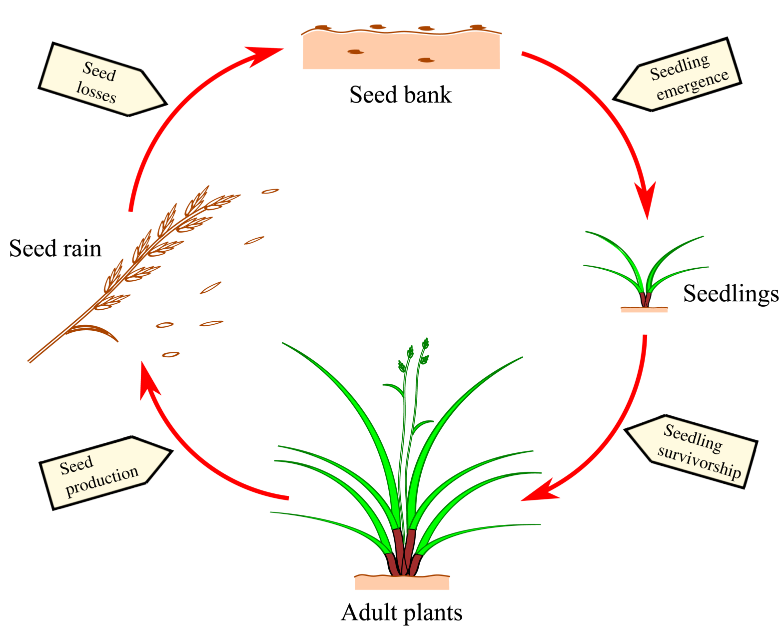


Fig. 1. Scheme of life-cycle of an annual plant.

Table 1. Life-cycle-based model for *L. rigidum* developed in [González-Andujar & Fernández-Quintanilla, 2004].

Seedling emergence	$Z_t = e(1-c_1)SB_t$	Z_t: Number of seedlings/m^2 in year t, SB_t: Density of the seed bank (seeds/m^2) in year t, e: Proportional emergence of seed bank ($0 \leq e \leq 1$), c_1: Reduction by delaying crop seeding ($0 \leq c_1 < 1$),
Seedling survivorship	$M_t = sZ_t$	M_t: Number of mature plants/m^2 in year t, s: Rate of seedling survival ($0 \leq s \leq 1$),
Fecundity	$F_t = \dfrac{f}{1+aM_t}$	F_t: Seeds per mature plant in year t, f: Number of seeds produced by an isolated individual, a: Area required to produce f,
Seed production	$S_t = F_t(1-c_2)(1-c_3)M_t$	S_t: Seed production/m^2 in year t, c_2: Reduction by herbicide application ($0 \leq c_2 < 1$), c_3: Reduction by competitive crops ($0 \leq c_3 < 1$),
Seed losses	$R_t = (1-l_1)(1-c_4)S_t$	R_t: Seed production/m^2 entering the seed bank in year t, l_1: Reduction by biotic factors ($0 \leq l_1 \leq 1$), c_4: Reduction by seed catching and removal at harvest, ($0 \leq c_4 < 1$)
Seed bank	$SB_{t+1} = (1-e)(1-m)SB_t + R_t$	m: Proportional mortality ($0 \leq m \leq 1$).

actions. The mathematical expressions proposed in [González-Andujar & Fernández-Quintanilla, 2004] are summarized in Table 1. Basically, to quantify the seed bank density for the next year, it is necessary to take into account the seedling emergence Z_t, the seedling survival M_t, the seed production S_t of mature plants, considering density-dependence in the reproduction, and the seed losses before entering the seed bank. So, the soil will store the resulting quantity R_t together with the proportion of SB_t that do not germinate but are still viable. This model structure is similar to the one given, for example, in [González-Andujar & Fernández-Quintanilla, 1991] for the weed species *Avena sterilis*.

Control actions are represented by proportional rates of reductions during different stages of the life-cycle [González-Andujar & Fernández-Quintanilla, 2004]. To date, the control of *L. rigidum* is based fundamentally on the use of herbicides (c_2). However, this tactic is threatened by the development of herbicide resistance and the negative environmental impact on the agroecosystem biota. Other agronomic practices suggested for the management of ryegrass, which help reduce the resistance levels, include delayed seeding (c_1), competitive crops (c_3), and catching and removal at harvest (c_4). The first tactic implies that crop seeding is delayed until the onset of *L. rigidum* emergence so as to facilitate weed seedling suppression by using mechanical or nonselective herbicides. The other two are nonchemical control options, which consist in the selection of crops with different weed suppression capacities and the weed seed catching and destruction at harvest (by using, for instance, a Harrington seed destructor). In all cases, maximum efficiency levels could go around 90–95%.

2.2. *The difference equation and its general characteristics*

Replacing successively the mathematical expressions of the variables presented in Table 1 into the seed bank equation, it is possible to find an explicit map that predicts the size of SB in the year $t+1$ as a function of its state value in t. This map is given by

$$SB_{t+1} = \gamma SB_t + \alpha(1-c_1)(1-c_2)(1-c_3) \times (1-c_4)\frac{SB_t}{1+\beta(1-c_1)SB_t}, \tag{1}$$

where constants $\alpha = fse(1-l_1)$, $\beta = ase$ and $\gamma = (1-e)(1-m)$ are composed of parameters and biologic factors unrelated to anthropic control and the c_i with $i = 1$–4 recast the possible control actions, i.e. delay seeding, herbicide, crop competition, and catching and removal at harvest, respectively.

Solving (1) under the steady-state condition of $SB_{t+1} = SB_t = \overline{SB}$, two equilibrium points are identified. The trivial one is $\overline{SB}_1 = 0$, which represents the weed eradication, and the nontrivial one is

$$\overline{SB}_2 = \frac{\gamma - 1 + \alpha(1-c_1)(1-c_2)(1-c_3)(1-c_4)}{(1-\gamma)\beta(1-c_1)}, \tag{2}$$

which gives the infestation level of the seed bank. The local stability of the equilibrium points is determined using the Jacobian function of (1), i.e.

$$J(\overline{SB}) = \left.\frac{dSB_{t+1}}{dSB_t}\right|_{SB_t=\overline{SB}} = \gamma + \frac{\alpha(1-c_1)(1-c_2)(1-c_3)(1-c_4)}{[1+\beta(1-c_1)\overline{SB}]^2}.$$

The eigenvalue associated with $\overline{SB}_1$ is

$$\lambda_1 = J(\overline{SB}_1) = \gamma + \alpha(1-c_1)(1-c_2)(1-c_3)(1-c_4), \tag{3}$$

while that related to $\overline{SB}_2$ is

$$\lambda_2 = J(\overline{SB}_2) = \gamma + \frac{(1-\gamma)^2}{\alpha(1-c_1)(1-c_2)(1-c_3)(1-c_4)}. \tag{4}$$

2.3. *L. rigidum parameter values*

Lolium rigidum parameter values extracted from [González-Andujar & Fernández-Quintanilla, 2004] are summarized in Table 2. They represent, on average, the field observations carried out across different sites in Spain over several years [Fernández-Quintanilla *et al.*, 2000]. In practice, most of these parameters can vary from year to year due to climatic factors, or between localities and countries around the world. According to the analysis given in [González-Andujar & Fernández-Quintanilla, 2004], the model manifests very low sensitivity to the emergence rate e, the seedling survival s, and the area a required for an individual to produce f seeds. Therefore, average values were used for such parameters. Conversely, results present high sensitivity to the fecundity capacity f as well as to the mortality m in the seed bank and the seed losses l_1 by biotic factors. For completeness, the possible ranges of these three parameters are also included in Table 2. Developments of the following section show that these parameter variations do not produce topological changes in the dynamical behavior of the model.

2.4. *Dynamics in the absence of control*

In the absence of control ($c_1 = c_2 = c_3 = c_4 = 0$), considering the parameter values in Table 2, density $\overline{SB}_1$ is always unstable ($2,252 \le \lambda_1 = \gamma + \alpha \le 547.34$) while $\overline{SB}_2$ is stable. Thus, *L. rigidum* achieves the maximum level of

$$\overline{SB}_{2\max} = \frac{\gamma - 1 + \alpha}{(1-\gamma)\beta} \tag{5}$$

in the long-term with a local growth rate given by the eigenvalue $\lambda_2 = \gamma + (1-\gamma)^2/\alpha$. The dynamical characteristics concerning this equilibrium as a function of the reproductive capacity f (with $m = 0.84$ and $l_1 = 0.19$) are represented by the solid lines in Figs. 2(a) and 2(b). Shaded zones indicate the possible variations according to the mortality and biotic factor ranges (Table 2). As can be observed, there exists a linear relation between the final seed bank size and f. Thus, for example, the annual ryegrass achieves an infestation level of $\overline{SB}_{2\max} = 2357\,\text{seeds/m}^2$ under normal conditions [square symbol in Fig. 2(a)], while it could only reach $250\,\text{seeds/m}^2$ if f would decrease more than ten times (circle symbol). The associate eigenvalue λ_2 manifests a kind of inverse variation with parameters. However, it can be seen that its magnitude keeps below 0.2 for almost all the possible values.

Table 2. Parameter values used to represent *L. rigidum* dynamical behavior.

Parameter	Brief Description	Average Value	Maximum	Minimum
e	Emergence rate	0.64	—	—
s	Survival rate	0.76	—	—
f	Fecundity capacity	935	1250	7
a	Area	0.34	—	—
l_1	Biotic factor reduction	0.19	0.35	0.1
m	Mortality	0.84	0.89	0.6

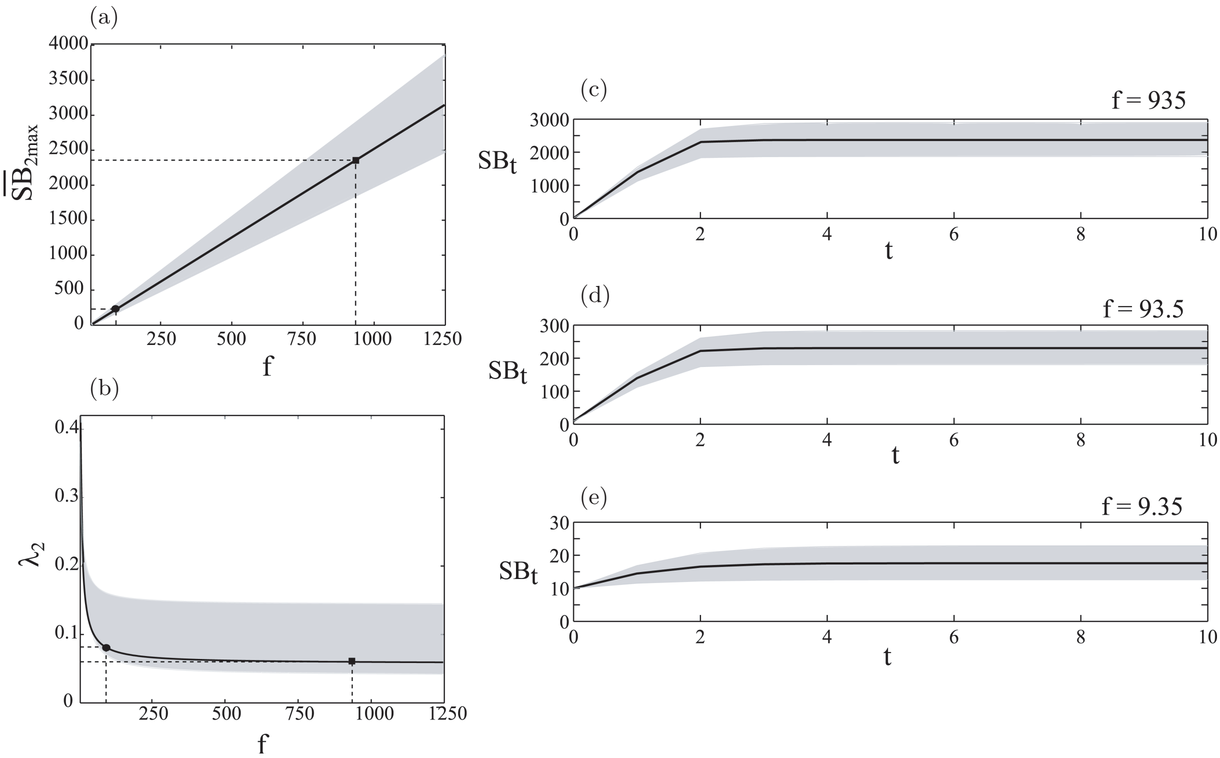


Fig. 2. *Lolium rigidum* population dynamics in the absence of control actions. (a) Steady-state seed bank density as a function of fecundity capacity, (b) eigenvalue λ_2 representing the growth rate of the population, (c) seed bank dynamics for $f = 935$ seeds per plant [square symbol in (a)], (d) seed bank dynamics for $f = 93.5$ seeds per plant [circle symbol in (a)] and (e) seed bank dynamics for $f = 9.35$ seeds per plant. Shaded regions represent maximum and minimum variations according to l_1 and m ranges.

This implies that SB_t shows predominantly a fast rise dynamics, reaching its steady-state value after two to four years [Figs. 2(c) and 2(d)].

From an agronomic perspective, it would be highly desirable to find, for example, that the population density level were always below 20 seeds/m^2, at least within a ten-year planning horizon [Fig. 2(e)]. In that ideal scenario, it would be unnecessary to adopt any anthropic control. Based on the parameter ranges in Table 2, this situation is within the possibilities but it rarely occurs in practice. Usually, infestation levels are sufficiently high to provoke substantial losses in crops. Developments of this work are mainly focused on considering the average parameter values in Table 2, since they are sufficiently representative for those field scenarios [Fig. 2(c)], where it can become critical to control the weed before reaching its higher levels.

3. Dynamics and Results Responding to Constant Control

Actions for controlling a weed species are traditionally planned to be constant over the whole time horizon. Thus, they can be seen as parameters of the dynamical system modeled by (1). Provided that $c_i \neq 0$, for $i = 1, 2, 3$ or 4, the nontrivial point $\overline{SB}_2$ becomes lower than level $\overline{SB}_{2\max}$, which is given by (5). The eigenvalues associated with both equilibria is always kept greater than zero. So, it can be affirmed that the model does not exhibit nonlinear phenomena such as period-doubling bifurcations [Wiggins, 2003; Sakai, 2001]. However, there exists a possibility that λ_1 and λ_2 are equal to 1 simultaneously (for which $\overline{SB}_1 = \overline{SB}_2 = 0$). As proved below, this critical condition corresponds to the existence of a transcritical bifurcation. Thus, it is possible that $\lambda_1 > 1$ and $0 < \lambda_2 < 1$ for a certain combination of weed control options maintaining the original condition of $\overline{SB}_1$ unstable while $\overline{SB}_2$ holds stable. But, it could also be possible that $0 < \lambda_1 < 1$ and $\lambda_2 > 1$ if the control is changed, leading to the situation where $\overline{SB}_1$ is stable and $\overline{SB}_2$ is unstable. This would be the ideal control scenario since external actions are able to stabilize $\overline{SB}_1 = 0$, causing the definitive species eradication.

Theorem 1. *Assume $c_i = c \neq 0$ for $i = 1, 2, 3$ or 4. Then, model (1) undergoes a transcritical bifurcation around the origin, whenever $c = c_o = 1 - (1-\gamma)/\alpha'$, where $\alpha' = \alpha\prod_{k=1}^{4}(1-c_k)$ with $k \neq i$.*

Proof. Consider model (1) as the difference equation $SB_{t+1} = g(SB_t, c)$, where $g(\cdot)$ is a smooth nonlinear function and $c_i = c$ (with $i = 1, 2, 3$ or 4) is the bifurcation parameter. The equilibrium points are not hyperbolic at the origin since their associate eigenvalues (3) and (4) satisfy $\lambda_1 = \lambda_2 = 1$ for $c = c_o = 1-(1-\gamma)/\alpha'$ with $\alpha' = \alpha\prod_{k=1}^{4}(1-c_k)$ ($k \neq i$). Based on [Wiggins, 2003], this critical condition corresponds to the occurrence of a transcritical bifurcation, provided that

$$\text{(i)}\ \ D_1 = \left.\frac{dg(SB_t, c)}{dc}\right|_{SB_t=0, c=c_o} = 0,$$

$$\text{(ii)}\ \ D_2 = \left.\frac{d^2g(SB_t, c)}{dSB_t^2}\right|_{SB_t=0, c=c_o} \neq 0,$$

$$\text{(iii)}\ \ D_3 = \left(\left.\frac{d^2g(SB_t, c)}{dSB_t dc}\right|_{SB_t=0, c=c_o}\right)^2 - \left.\frac{d^2g(SB_t, c)}{dSB_t^2}\right|_{SB_t=0, c=c_o} \left.\frac{d^2g(SB_t, c)}{dc^2}\right|_{SB_t=0, c=c_o} \neq 0.$$

According to the role of c in function $g(\cdot)$, the calculation of these expressions leads to two situations. When bifurcation parameter c corresponds to any c_i with $i \neq 1$, map (1) reduces to

$$\begin{aligned} SB_{t+1} &= g(SB_t, c) \\ &= \gamma SB_t + \alpha'(1-c)\frac{SB_t}{1+(1-c_1)\beta SB_t}. \end{aligned}$$

So, by calculating the first- and second-order derivatives with respect to SB_t and c, one obtains $D_1 = 0$, $D_2 = -2(1-c_1)\beta(1-\gamma) < 0$ and $D_3 = \alpha'^2$. On the other hand, when parameter c corresponds to c_1, map (1) transforms into

$$\begin{aligned} SB_{t+1} &= g(SB_t, c) \\ &= \gamma SB_t + \alpha'(1-c)\frac{SB_t}{1+\beta(1-c)SB_t}, \end{aligned}$$

so that, by recalculating the derivatives, it results in $D_1 = 0$, $D_2 = -2\beta(1-\gamma)^2/\alpha' < 0$ and $D_3 = \alpha'^2$. In both cases, conditions (i)–(iii) are verified, demonstrating that the model presents a transcritical bifurcation regardless of which control action is chosen as the parameter. ■

Different control scenarios with special agronomic interest are formalized in the following. Theorem 1 explains why *L. rigidum* eradication is impossible to achieve in practice if individual control tactics are implemented. Conversely, this is possible if IWM strategies are planned carefully. In that context, it will be shown that there exist very good control alternatives to maintain very low densities of *L. rigidum* at average field efficiency control levels.

3.1. *Individual control*

As can be inferred from (1), control actions c_2, c_3 and c_4 affect the system dynamics in the same way, provided that they are implemented individually. However, c_1 associated with the delayed crop seeding option acts differently. Thus, two generic cases can be defined.

3.1.1. *Case I. Individual control without delayed crop seeding*

Assuming that $c_i = c$ for $i = 2, 3$ or 4 and $c_k = 0$ for all $k \neq i$ (with $k = 1, 2, 3$ or 4), model (1) maintains the trivial equilibrium and the nontrivial (2) becomes

$$\overline{SB}_2 = -\frac{1}{\beta} + \frac{\alpha}{(1-\gamma)\beta}(1-c).$$

As can be seen, the relation between $\overline{SB}_2$ and c is linear. The higher the c value, the smaller the seed bank. The eigenvalues associated with the equilibria reduce to $\lambda_1 = \gamma + \alpha(1-c)$ and $\lambda_2 = \gamma + (1-\gamma)^2/[\alpha(1-c)]$. Thus, $\overline{SB}_2$ is stable and $\overline{SB}_1$ is unstable if $c < c_0 = 1-(1-\gamma)/\alpha$, while the opposite occurs if $c \geq c_0$. The following corollary of Theorem 1 can be easily established.

Corollary 3.1. *Model (1), with $c_i = c$ for $i = 2, 3$ or 4 and $c_k = 0$ for all $k \neq i$ (with $k = 1, 2, 3$*

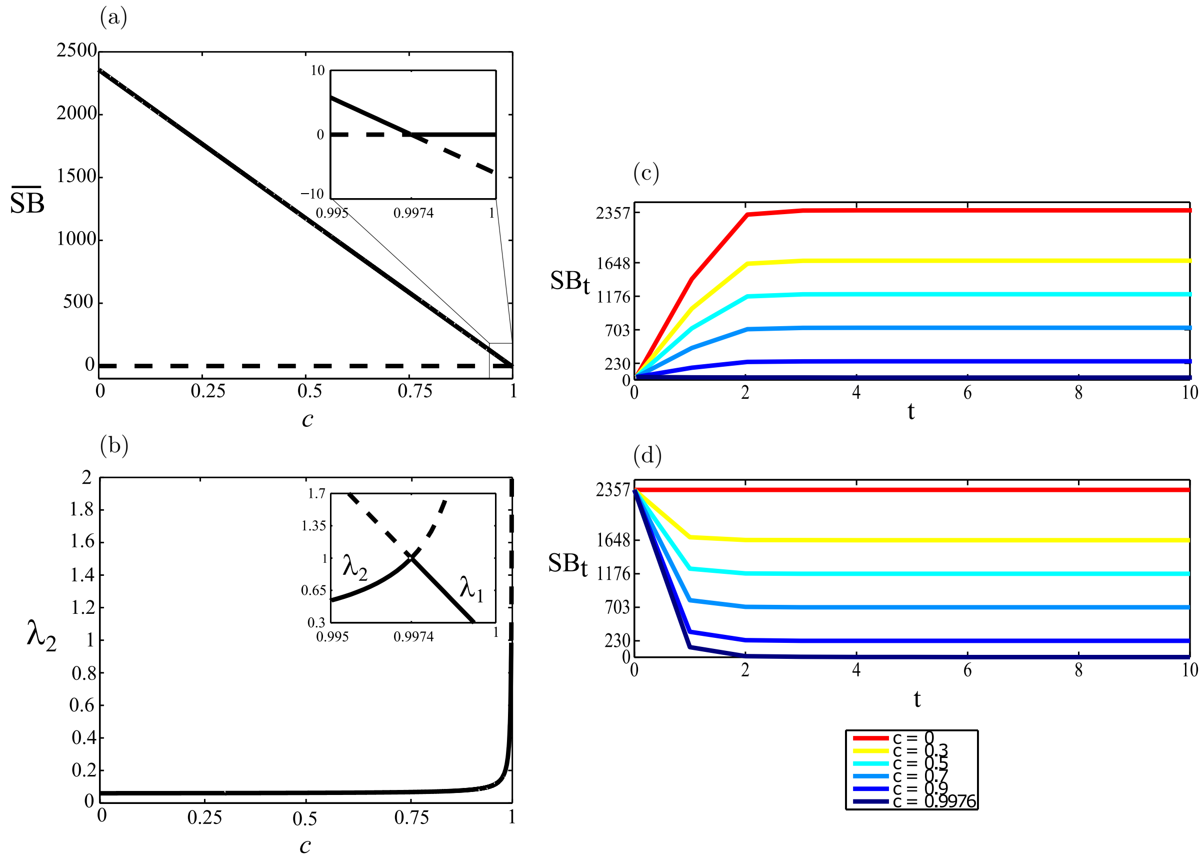


Fig. 3. *Lolium rigidum* population dynamics with a constant and individual control (without delay crop seeding). (a) Equilibria of the seed bank as a function of c, (b) associated eigenvalues. Solid line: stable point; dash line: unstable point and (c) and (d) population dynamics for different c values considering initial conditions $SB_0 = 10$ and $SB_0 = 2357$ seeds/m^2, respectively.

or 4), *undergoes a transcritical bifurcation around the origin when* $c = c_o = 1 - (1-\gamma)/\alpha$.

Figure 3(a) presents the size and stability of the equilibrium points $\overline{SB}_1$ and $\overline{SB}_2$ as a function of c, considering average parameter values. The control value c_0 for which equilibria shift their stabilities is practically equal to 1 ($c_0 = 0.9974$), as depicted in Fig. 3(b). Such control efficacy is almost impossible to be reached in practice, however. Notice that even the fecundity capacity f could decrease, for example ten times, leading naturally the seed bank to safer levels, the amount of control needed to eradicate the species should be excessively high ($c_0 = 0.9744$).

Remark 3.1. In practice, the implementation of an individual tactic such as c_2, c_3 or c_4 would only result in size reduction of the population.

Figures 3(c) and 3(d) show the dynamics of map (1) simulated over a ten-year planning horizon using different c values. In Fig. 3(c), the seed bank initially contains 10 seeds/m^2, while in Fig. 3(d) it initiates with the maximum density $\overline{SB}_{2\,\max} = 2357$ seeds/m^2. In both cases, the size of the infestation achieves levels below 250 seeds/m^2, whenever the efficacy of the control is greater than 90%.

3.1.2. *Case II. Individual control via delayed crop seeding*

If only $c_1 \neq 0$, the nontrivial equilibrium point of (1) is transformed to

$$\overline{SB}_2 = \frac{\alpha}{(1-\gamma)\beta} - \frac{1}{\beta(1-c_1)},$$

showing that the density level and c_1 are hyperbolically related. This implies that it is necessary to consider very high c_1 values to achieve a considerable reduction of the seed bank. As illustrated in Fig. 4(a), density can manifest appreciable changes for delayed seeding efficiencies superior to 90%. From a stability viewpoint, c_1 acts like any of the other individual tactics [Fig. 4(b)]. In fact, $\overline{SB}_2$ is stable and $\overline{SB}_1$ is unstable if $c_1 < c_0 = 1-(1-\gamma)/\alpha$

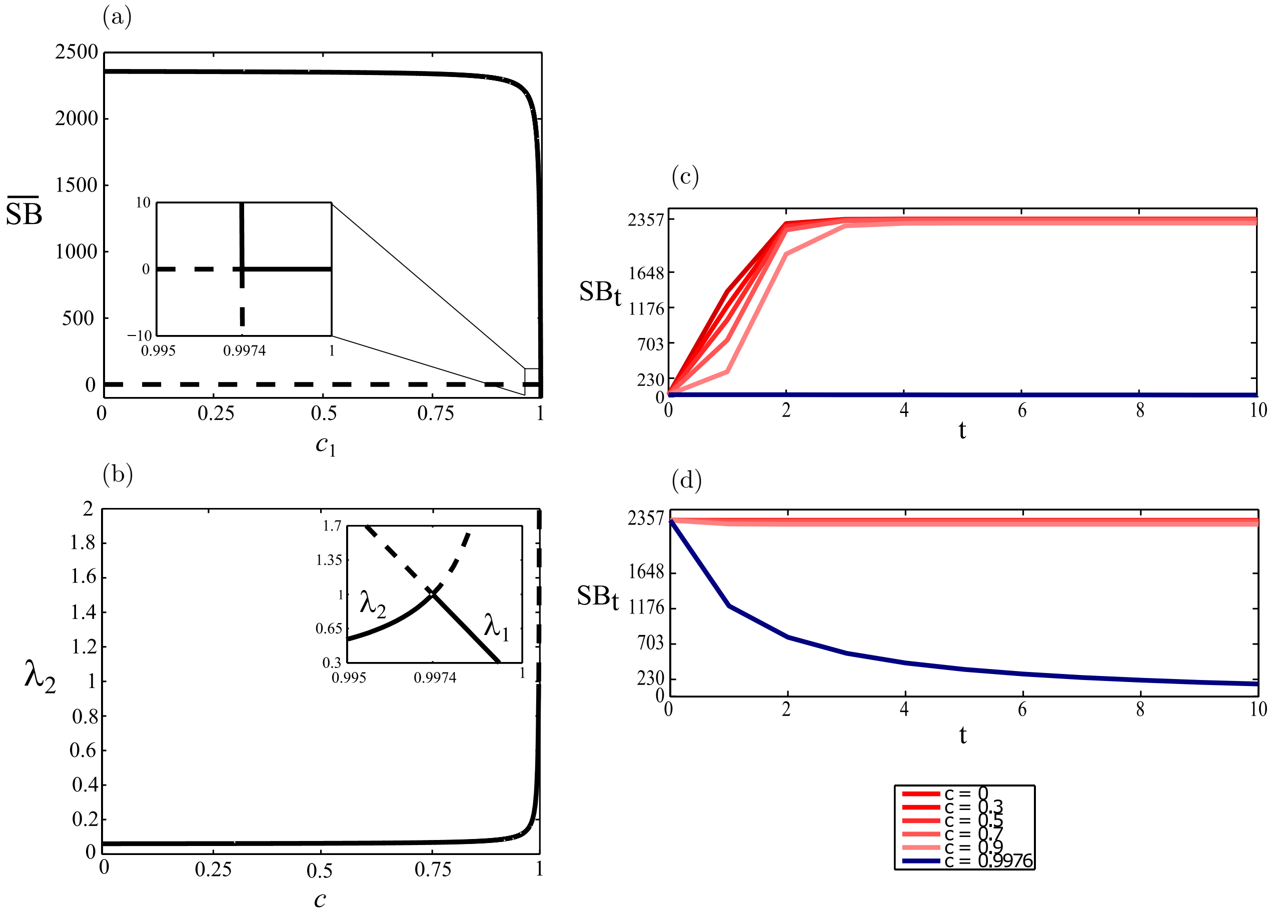


Fig. 4. *Lolium rigidum* population dynamics implementing delay seeding under nominal parameters. (a) Equilibria of the seed bank as a function of c, (b) associated eigenvalues. Solid line: stable point; dash line: unstable point and (c) and (d) temporal evolutions for different c values considering initial conditions $SB_0 = 10$ and $SB_0 = 2357$ seeds/m^2, respectively.

and the opposite occurs if $c_1 \geq c_0$. So, a new corollary of Theorem 1 can be presented.

Corollary 3.2. *Model* (*1*) *with* $c_2 = c_3 = c_4 = 0$ *undergoes a transcritical bifurcation around the origin when* $c_1 = c_o = 1 - (1-\gamma)/\alpha$.

Figures 4(c) and 4(d) show the population dynamics of *L. rigidum* for average parameter values with two possible initial conditions. Unlike the individual control case in Sec. 3.1.1, the growth of the seed bank becomes slower as c_1 approaches the critical value c_0. Even though the eigenvalues manifest the same dependence with c or c_1, the neighborhoods around the equilibria, for which they represent the growth rate of the map, can change. For sufficiently high c_1 values, nonlinearities become more significant and that neighborhood reduces considerably.

According to experimental results [Borger *et al.*, 2013; Monaghan, 1980], the delayed crop seeding strategy would generate better responses than those predicted by the model. Indeed, such cultural control practice is strongly related to weed-crop interactions. Intuitively, it is expected that the reduction of weed seedlings facilitates crop establishment, thus generating greater competition on the surviving ryegrass individuals. Model (1) is not able to represent this indirect effect as c_1 and c_3 are considered independent. In order to achieve more realistic results, the implementation of c_1 should be combined, at least, with an increase of the c_3 value (representing more competitive crops). Related simulations can be found in the following case as a part of the integrated control strategies.

3.2. *Integrated control*

The combinations of different control strategies are analyzed in this section, since they can promote a more sustainable agroecosystem less dependent on herbicide applications.

3.2.1. *Case III. Integrated control without delayed crop seeding*

As the three controls c_2, c_3 and c_4 play the same role in model (1), their values could be alternated to yield the same results. The different infestation levels obtained by combining strategies c_2, c_3 and c_4 (with $c_1 = 0$) are resumed in Fig. 5. In all cases, the region delimited by the critical condition of Theorem 1 (characterized by $\overline{SB}_2 = 0$) and lines $c_2 = c_3 = 1$ corresponds to the control values for which $\overline{SB}_2$ is unstable and $\overline{SB}_1$ is stable. This zone tends to be larger as c_4 increases. As α' diminishes with the contribution of the control actions, the critical value c_0 also decreases. However, the involved efficiencies are still high to achieve the eradication of the ryegrass in practice. On the other hand, the steady-state levels reached by equilibrium (2) tend to diminish significantly as c_4 increases. These simulations show that it is possible to define appropriate IWM programs to maintain, at least, very low weed densities at average efficiency levels.

Remark 3.2. Fixing c_4 and $\overline{SB}_2$ at desired values, the admissible c_2 and c_3 ranges are defined by means of the vertex $(\bar{c}, \bar{c})$ of an hyperbola (Fig. 5), i.e.

$$\bar{c} = 1 - \sqrt{\frac{(1-\gamma)(1+\beta\overline{SB}_2)}{\alpha(1-c_4)}}$$

and the maximum control value

$$c_{\max} = 1 - \frac{(1-\gamma)(1+\beta\overline{SB}_2)}{\alpha(1-c_4)},$$

to be adopted when one of them is equal to zero. Thus, if c_2 belongs to $[0, \bar{c}]$, then c_3 will belong inevitably to $[\bar{c}, c_{\max}]$ via the function

$$c_3 = 1 - \frac{(1-\gamma)(1+\beta\overline{SB}_2)}{\alpha(1-c_2)(1-c_4)},$$

or vice versa.

Thus, for example, if it is expected that $c_4 = 0$ and density $\overline{SB}_2$ reaches $250\,\text{seeds/m}^2$, control actions c_2 and c_3 will be limited by the intervals $[0.67, 0.891]$ and $[0, 0.67]$, respectively [Fig. 5(a)]. But, these bounds diminish if the efficiency of c_4 is increased. In fact, it results in $c_2 \in [0.399, 0.639]$ and $c_3 \in [0, 0.399]$ if $c_4 = 0.7$ [Fig. 5(d)].

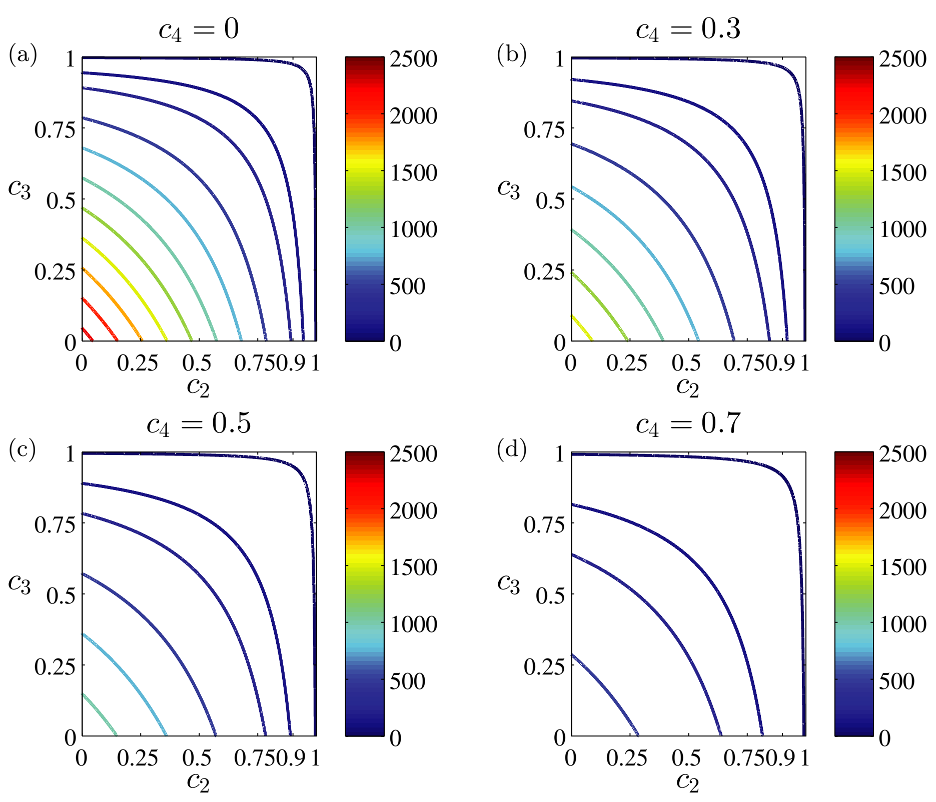


Fig. 5. Levels of the seed bank $\overline{SB}_2$ combining the three control strategies c_2, c_3 and c_4 with $c_1 = 0$. Region delimited by the blue curve ($\overline{SB}_2 = 0$) and lines $c_2 = c_3 = 1$ correspond to the control values for which $\overline{SB}_2$ is unstable and $\overline{SB}_1$ is stable.

3.2.2. *Case IV. Integrated control with delayed crop seeding*

This last case corresponds to those integrated strategies where the control action c_1 is necessarily different from 0. Results do not change significantly with respect to Fig. 5 if the delayed crop seeding is included in the IWM program but keeping it at low efficiency levels. In coincidence with the analysis in Sec. 3.1.2, effects are appreciable when c_1 is larger than 0.9, as presented in Fig. 6. Compared to Fig. 5, differences are minor for c_2, c_3 and c_4 combinations resulting in $\overline{SB}_2 > 1000\,\text{seeds/m}^2$. The most visible benefits occur for $\overline{SB}_2 < 1000\,\text{seeds/m}^2$. Very low infestation levels can be reached by implementing standard control rates. Furthermore, the region where $\overline{SB}_1 = 0$ becomes stable is clearly enlarged. This implies, for example, that it would be plausible to have a seed bank tending to zero by means of standard control actions with $c_1 = 0.9$, $c_2 = 0.6$, $c_3 = 0.8$ and $c_4 = 0.7$.

Remark 3.3. From the analysis of Cases III and IV, it can be inferred that several control combinations could lead the seed bank to the same steady-state density.

Different options are proposed in Table 3, considering three possible infestation scenarios. The second target, for example, is near the economic threshold of $23\,\text{seeds/m}^2$ (or, equivalently, $15\,\text{seedlings/m}^2$) established by Spain. Control values are chosen with the aim of reducing the application of herbicides. This can be reached by defining an adequate program of both cultural and mechanical tactics. As previously mentioned, the delayed seeding option could be implemented by weed seedlings suppression using cover crops or mechanical control instead of chemical fallows. Thus, c_1 could contribute to reduce the negative impact of herbicide-based control programs. The simulated responses of map (1) over a ten-year planning horizon under the proposed programs are presented in Fig. 7. As could be expected, differences in the settling time appear when c_1 is included in the weed management program. In all cases, the transient response becomes slower when $c_1 = 0.9$. For programs P3–P4 in Fig. 7(a) and P7–P8 in Fig. 7(b), this effect is beneficial since the bank reaches the targets of $250\,\text{seeds/m}^2$ or $25\,\text{seeds/m}^2$ in the long run more gradually. If the initial condition is above the admissible final value, as in Fig. 7(c), the

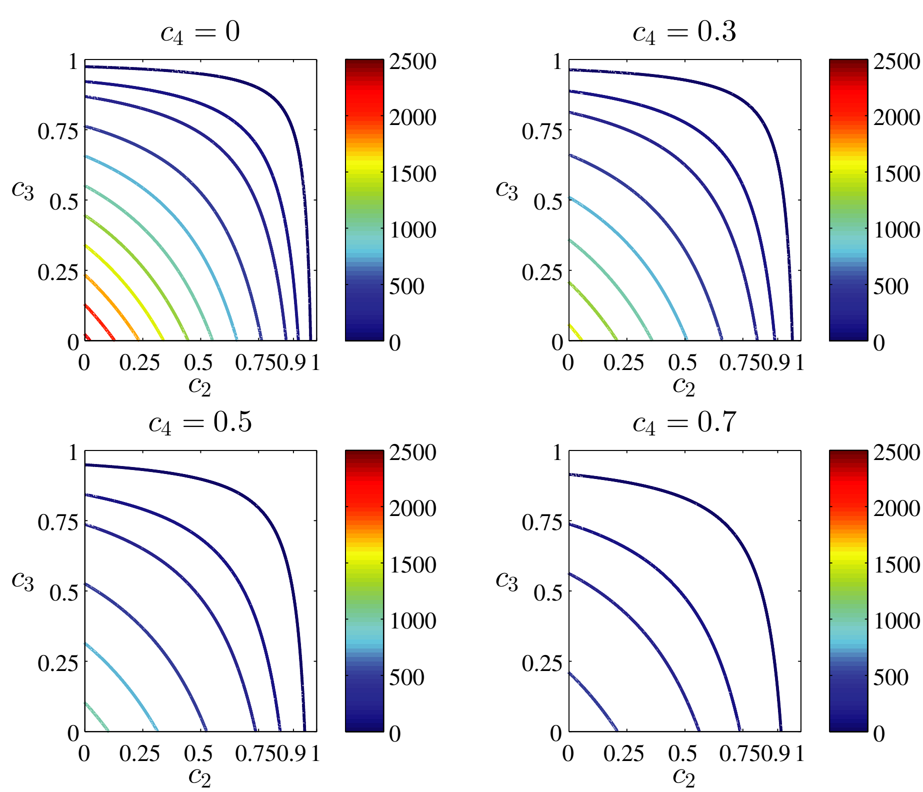


Fig. 6. Levels of the bank $\overline{SB}_2$, considering $c_1 = 0.9$ and combining the control actions c_2, c_3 and c_4. Region delimited by the blue curve ($\overline{SB}_2 = 0$) and line $c_2 = c_3 = 1$ corresponds to the control values for which $\overline{SB}_2$ is unstable and $\overline{SB}_1$ is stable.

Table 3. Different IWM control options for reducing *L. rigidum* seed bank density. Target stands for the final seed bank density.

Target	Program	Delayed Seeding c_1	Herbicide c_2	Crop Competition c_3	Catching and Removal c_4
250 seeds/m^2	P1	—	0.458	0.8	—
	P2	—	0.278	0.5	0.7
	P3	0.9	0.124	0.5	0.7
	P4	0.9	0.061	0.8	0.3
25 seeds/m^2	P5	—	0.934	0.8	—
	P6	—	0.671	0.8	0.8
	P7	0.9	0.397	0.8	0.7
	P8	0.9	0.096	0.8	0.8
5 seeds/m^2	P9	—	0.953	0.9	—
	P10	—	0.88	0.8	0.8
	P11	0.9	0.538	0.8	0.7
	P12	0.9	0.307	0.8	0.8

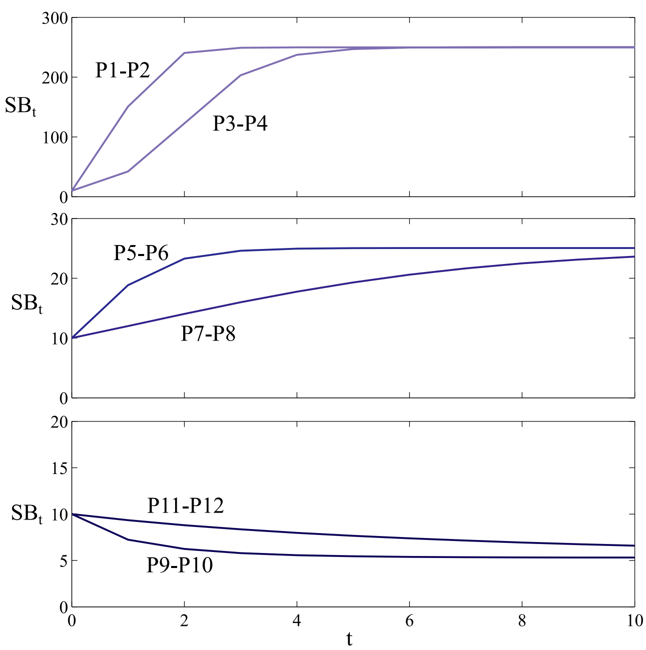


Fig. 7. *Lolium rigidum* population dynamics reaching three different targets according to the integrated control programs described in Table 3. P1 to P4: $\overline{SB}_2 = 250$ seeds/m^2; P5 to P8: $\overline{SB}_2 = 25$ seeds/m^2; P9 to P12: $\overline{SB}_2 = 5$ seeds/m^2.

delay of P11–P12 in reaching the steady-state of 5 seeds/m^2 could be inadequate. On the other hand, the application of c_1 causes the efficiency of herbicides (c_2) to be almost halved.

4. Dynamics and Results Applying Time-Variant Control

In this section, necessary control actions are calculated each year according to the seed bank dynamics. The aim is to reduce the application of herbicides or even the efficiency of the cultural tactics, but achieving the same steady-state results as those obtained by constant actions.

Model (1) is rewritten here as

$$SB_{t+1} = \gamma SB_t + \alpha \frac{SB_t}{1 + \beta SB_t} U_t, \tag{6}$$

where the three actions c_2, c_3 and c_4 transform into the unique variant control U_t. Moreover, constants α and β are redefined as $\alpha = fse(1 - l_1)(1 - c_1)$, $\beta = ase(1 - c_1)$, respectively. As shown below, once the U_t function is found, this could be implemented by means of individual or integrated tactics.

Remark 4.1. The temporal variation of c_1 has no practical meaning since the efficiency of this control has to be maintained at high levels (superior to 90%) in order to have impact on the weed population.

In principle, it might seem that the expression of the control law U_t could be obtained by using classical linear control theory [Phillips & Nagle, 1995; Vaccaro, 1995]. With the objective of weed eradication in mind, this traditional approach cannot be applied since the map is uncontrollable around the origin. As can be noticed, U_t multiplies the state SB_t so that the linearization of (6) around $\overline{SB}_1 = 0$ results in the elimination of the control action. Based on the idea of a simple and practical agronomic application, the control design is carried out by means of the cancellation strategy, i.e. an appropriate law U_t is obtained by canceling the undesired dynamics of map (6) and, at the same time, forcing it to have an specific desirable behavior.

4.1. *Control without uncertainties*

Addressing the issue of *L. rigidum* eradication, the following result is useful.

Theorem 2. *Let* $U_t = \tau(1+\beta SB_t)/\alpha$, *where* τ *is the constant gain of the controller. Then*, (6) *possesses an unique equilibrium at the origin. The point is stable and* $0 \le U_t \le 1$ *whenever* $\tau < 1-\gamma$.

Proof. Replacing the proposed law U_t into (6), the resulting map is given by $SB_{t+1} = (\gamma+\tau)SB_t$. Solving the difference equation under steady-state condition, it is easy to see that the unique equilibrium is $\overline{SB} = 0$. The associated eigenvalue is $\lambda = \gamma+\tau > 0$, so the trivial point is stable whenever $\tau < 1-\gamma$. Moreover, since $0 \le SB_t \le \overline{SB}_{2\max}$, the control law satisfies that $0 \le U_t \le 1$ if $0 < \tau \le 1-\gamma$. ■

Remark 4.2. If it is desired that SB_t approaches the origin with an error less than 1% of the initial condition in a ten-year planning horizon, it is necessary that the growth rate of the controlled map satisfies $\lambda < e^{-4.62/10}$. Considering that γ is sufficiently small, this restriction defines a range for the possible gain τ, which should be $0 < \tau < 0.6$.

4.2. *Control considering uncertainties*

Due to multiple factors, the estimation of α and β as well as the measurement of density SB_t can naturally present uncertainties. Since deviations are indeterminable quantities, it is convenient to find a condition for the τ election that, even being more conservative, ensures the objective of weed eradication.

Corollary 4.1. *Let* $U_t = \tau(1+\hat{\beta}\widehat{SB}_t)/\hat{\alpha}$, *with* $\hat{\alpha} = \delta_\alpha\alpha$ *and* $\hat{\beta}\widehat{SB}_t = \delta_\beta\beta SB_t$. *Then, there exist* δ_α *and* δ_β *ranges on which the origin is a stable equilibrium of* (6) *and* $0 \le U_t \le 1$ *whenever* $0 < \tau < 0.6$.

Proof. Substituting U_t into (6), we get

$$SB_{t+1} = \gamma SB_t + \frac{\tau}{\delta_\alpha}\frac{1+\delta_\beta\beta SB_t}{1+\beta SB_t}SB_t. \tag{7}$$

This difference equation possesses the desired equilibrium at $\overline{SB}_1 = 0$. Since its associated eigenvalue is $\lambda_1 = \gamma + \tau/\delta_\alpha$, this point is stable if $\tau < (1-\gamma)\delta_\alpha$. But, due to the existence of uncertainties, (7) presents another equilibrium, at

$$\overline{SB}_2 = \frac{\tau-(1-\gamma)\delta_\alpha}{\beta[(1-\gamma)\delta_\alpha - \delta_\beta\tau]}.$$

The numerator of $\overline{SB}_2$ is less than zero when the origin is stable. So, to ensure $\overline{SB}_2$ is bounded and less than zero (becoming a nonexistent equilibrium point in practice), it should be $\delta_\beta < (1-\gamma)\delta_\alpha/\tau$.

On the other hand, it is necessary to check if the proposed control law is bounded, i.e. $0 \le U_t \le 1$ for $0 \le \widehat{SB}_t \le \overline{SB}_{2\max}$. Provided that α is a sufficiently large positive number [even comparing to $(1-\gamma)(1-\delta_\beta)$], this condition is also verified under the stability restriction of the origin, since

$$0 \le \tau \le (1-\gamma)\delta_\alpha\frac{\alpha}{(1-\gamma)(1-\delta_\beta)+\alpha} \approx (1-\gamma)\delta_\alpha.$$

Therefore, since $0 < \tau < 0.6$, it can be affirmed that the origin is stable and $0 \le U_t \le 1$ for $\delta_\alpha > 0.6/(1-\gamma) = 0.637$ and $\delta_\beta < 0.6/\tau$. ■

The δ_α restriction can be easily fulfilled by calculating $\hat{\alpha}$ with fecundity capacities around the maximum value. As the parameter β normally does not have large variations, the δ_β margin mainly represents the admissible seed bank uncertainties. This restriction could be critical for control gains near its maximum limit. However, it is probable that measurement errors tend to be defective as only seeds located at superficial soil depths (0–5 cm) are normally quantified.

The population dynamics of *L. rigidum* under control U_t of Corollary 4.1 is shown in Fig. 8. For simplicity, U_t is implemented as an individual tactic, $c_t = 1-U_t$. As expected, the density bank goes to a quantity near zero in the ten-year planning horizon, considering different systematic uncertainties. As the τ value increases, the growth rate for which SB_t reaches the steady-state level diminishes. This phenomenon of making the seed bank change more gradually compared to the case of constant control implies that the involved efficiencies are less demanding year by year.

Remark 4.3. In general, it is preferable that the seed bank reaches certain density SP while the control efficiency does not exceed a specific economic threshold level. The expression of U_t in

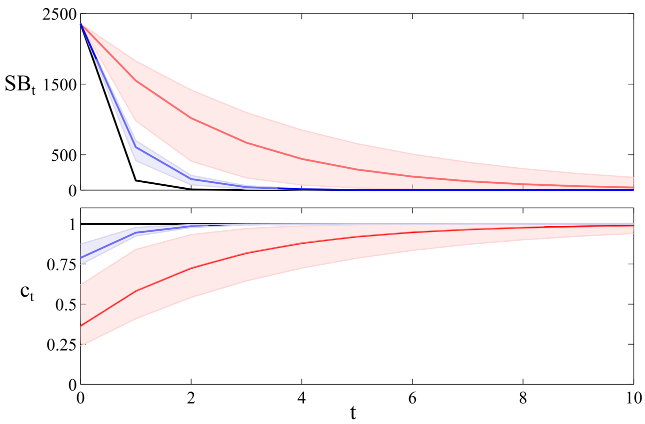


Fig. 8. Performance of the time-variant control for $\tau = 0.2$ (blue) and $\tau = 0.6$ (red). Solid lines: simulations without considering uncertainties. Shaded regions: deviations due to systematic uncertainties in the α estimation ($\delta_\alpha = 1250/935$) and in the SB_t measurements ($0.8 \le \delta_\beta \le 1.6$). For comparison, constant control with a 100% efficiency is included (black line).

Corollary 4.1 can be modified in order to achieve this objective, i.e.

$$U_t = \max\left\{\tau \frac{1 + \hat{\beta}\widehat{SB}_t}{\hat{\alpha}}, \overline{U}\right\}, \tag{8}$$

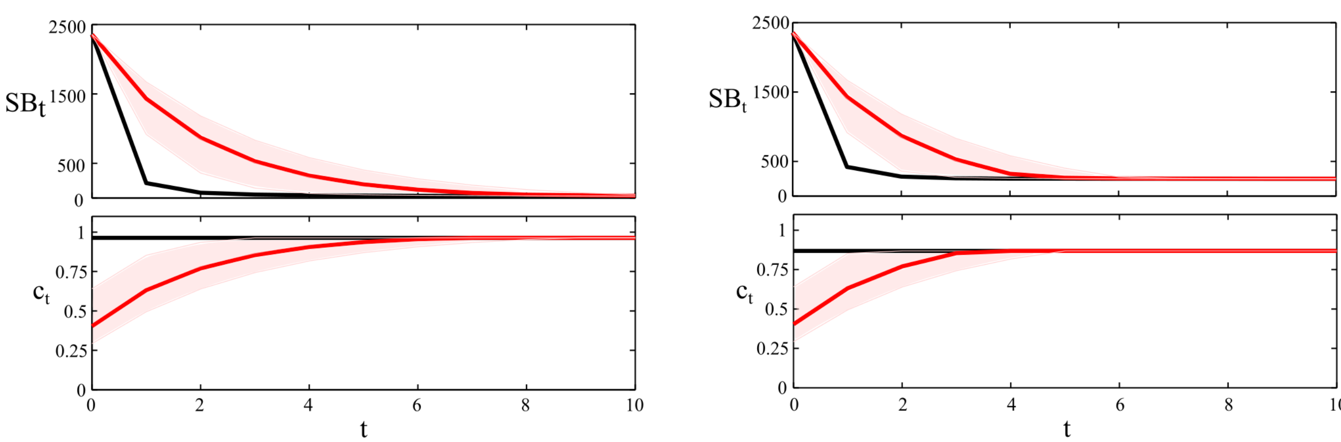
where $\overline{U} = (1-\gamma)(1+\beta SP)/\alpha$ is the fixed quantity that keeps the chosen equilibrium (and coincides with the value of the constant control).

Figures 9(a) and 9(b) show the numerical results, with $c_t = 1-U_t$ and a target of $25\,\text{seeds/m}^2$ and $250\,\text{seeds/m}^2$, respectively. Fixing the τ value, the magnitude of saving control with respect to the constant case decreases if the admissible infestation is less restrictive. The simulations of two more realistic scenarios with a target of $250\,\text{seeds/m}^2$ are presented in Fig. 10. In the diagrams, the fecundity f develops random variations (solid line) over the years representing natural fluctuations due to climatic factors. However, the estimation of this parameter (used to calculate U_t) is kept at the fixed value of 935 seeds per plant (red circle). The quantification of the seed bank also manifests irregular errors which are inherent to the way the seeds are measured or dispersed in the field (red circles in the SB_t plot). Both constant and time-variant controls achieve similar results when densities are below $500\,\text{seeds/m}^2$. In the first four to six years, the efficiency of c_t ($c_t = 1 - U_t$) is always less than the constant value, implying a great amount of saving in the management program.

4.3. *Algorithm for implementing U_t*

The control law U_t can be implemented by individual or integrated tactics. If a unique control action is selected then the value of the chosen c_2, c_3 or c_4 is calculated year to year by $c_t = 1 - U_t$, as considered in Figs. 8–10. In the case of applying more than one simultaneous actions, the possibilities of U_t assignation can be infinite. Thus, for example, a simple option could be the adoption of three tactics with the same efficiency, i.e. $c_2 = c_3 = c_4 = 1 - \sqrt[3]{U_t}$.

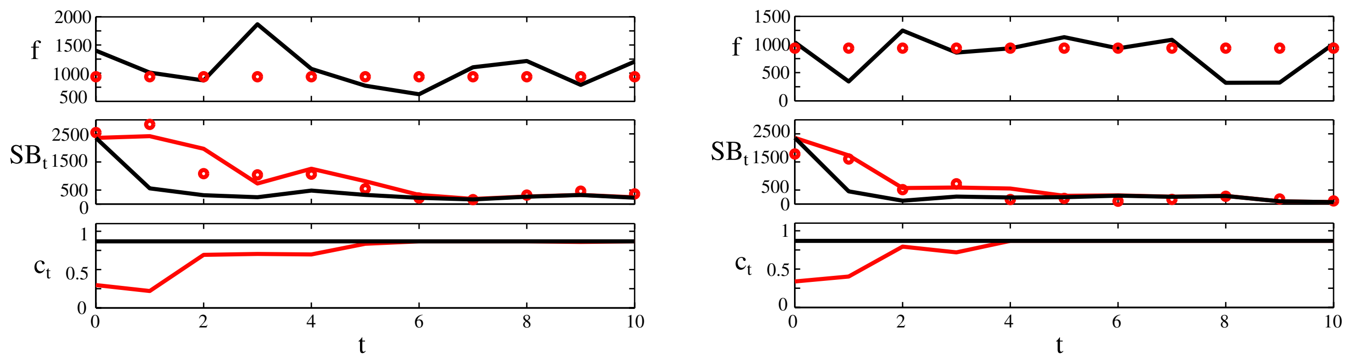
A more adequate alternative is to distribute U_t weighting the priority given to each control option



(a)

(b)

Fig. 9. Performance of the time-variant control for $\tau = 0.6$ (red) and a delay seeding of $c_1 = 0.9$. (a) Minimum seed bank of $25\,\text{seeds/m}^2$ and (b) minimum seed bank of $250\,\text{seeds/m}^2$. Solid lines: simulations without considering uncertainties. Shaded region: deviations due to uncertainties in the α estimation ($\delta_\alpha = 1250/935$) and also in the measurements of SB_t ($0.8 \le \delta_\beta \le 1.6$). For comparison, constant control efficiency is also included (black line).



(a) (b)

Fig. 10. Performance of the time-variant control ($\tau = 0.6$) when δ_α and δ_β vary year to year. It is considered a minimum seed bank of $250\,\text{seeds/m}^2$ and a delay seeding of $c_1 = 0.9$.

according to the necessity of reducing its participation in the IWM program. That rank can be established by taking into account multiple factors such as economic costs, ecological and environmental impacts, available resources, etc. The distribution is achieved by defining the performance index

$$\mathcal{J} = q_2c_2^2 + q_3c_3^2 + q_4c_4^2,$$

which has to be minimized subject to the control restriction $U_t = (1 - c_2)(1 - c_3)(1 - c_4)$. Notice that the greater the weight q_i, the higher the priority of reduction and, then, the smaller the c_i efficiency. Using the Lagrange multipliers technique, the index and restriction can be combined to generate a unique function

$$\mathcal{L} = \mathcal{J} + \Lambda[U_t - (1 - c_2)(1 - c_3)(1 - c_4)].$$

Then, the algorithm consists in finding all the real c_i and Λ values that satisfy

- $\frac{\partial \mathcal{L}}{\partial c_i} = 0$, for $i = 2, 3, 4$.
- $\frac{\partial \mathcal{L}}{\partial \Lambda} = 0$.

This set of critical solutions can be determined by using any mathematical software. Finally, an appropriate c_2, c_3 and c_4 combination (with $0 < c_i < 1$,

Table 4. Time-variant IWM controls for reducing the seed bank density from the initial state of $2357\,\text{seeds/m}^2$ to $25\,\text{seeds/m}^2$. Control law U_t is distributed according to the different priorities given to each control option in order to reduce its participation in the IWM.

	año	0	1	2	3	4	5	6	7	8	9	10
	c_1	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
	U_t	0.597	0.368	0.230	0.146	0.094	0.063	0.044	0.036	0.036	0.036	0.036
PA ($k \neq i$)	c_1	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
	c_i	0.403	0.632	0.770	0.854	0.906	0.937	0.956	0.964	0.964	0.964	0.964
	c_k	0	0	0	0	0	0	0	0	0	0	0
PB	c_1	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
	c_2	0.158	0.283	0.387	0.474	0.545	0.602	0.646	0.669	0.669	0.669	0.669
	c_3	0.158	0.283	0.387	0.474	0.545	0.602	0.646	0.669	0.669	0.669	0.669
	c_4	0.158	0.283	0.387	0.474	0.545	0.602	0.646	0.669	0.669	0.669	0.669
PC	c_1	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
	c_2	0.085	0.131	0.146	0.087	0.052	0.034	0.023	0.019	0.019	0.019	0.019
	c_3	0.192	0.349	0.481	0.199	0.112	0.070	0.047	0.038	0.038	0.038	0.038
	c_4	0.192	0.349	0.481	0.801	0.888	0.930	0.952	0.962	0.962	0.962	0.962
PD	c_1	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
	c_2	0.054	0.067	0.052	0.035	0.023	0.016	0.011	0.009	0.009	0.009	0.009
	c_3	0.116	0.145	0.112	0.074	0.048	0.032	0.022	0.018	0.018	0.018	0.018
	c_4	0.286	0.538	0.727	0.837	0.899	0.934	0.954	0.963	0.963	0.963	0.963

$i = 2, 3, 4$) will be the one that corresponds to the minimum J.

Table 4 illustrates some forms of implementation of control U_t when the initial state of the seed bank is $2357\,\text{seeds/m}^2$. Then, this density is reduced to $25\,\text{seeds/m}^2$ in ten years, with a transient response given by $\tau = 0.55$. Program PA corresponds to the use of a time-variant individual tactic, considering, for example, catching and removal of seeds at harvest (c_4) and keeping $c_1 = 0.9$, which is precisely illustrated by Fig. 9(a). Program PB represents an integrated management where, except for c_1, the rest of the control is applied with the same efficiency. Even though herbicides are used, the required efficacy of the control is moderate (70%). Such a level could be easily obtained at normal recommended label dosages (without overdosage or reiterated interventions per year). The other two proposed programs, PC and PD, are derived from the use of index J chosen with different priorities. In PC, it is considered that $q_2 = 1$, $q_3 = 0.5$ and $q_4 = 0.5$, implying that it is important to reduce the amount of herbicides. Since $q_2 = q_3$, the distribution of c_2 and c_3 could be interchanged, yielding the same index value. This program presents two admissible forms of implementation. In PD, weights are $q_2 = 1$, $q_3 = 0.5$ and $q_4 = 0.25$, so that the crop competition also presents certain relevance in the reduction. On the other hand, the practice of catching and removal has little weight on the minimization since it is important to promote its use.

5. Conclusions

Different long-term IWM programs for controlling *L. rigidum* were analyzed through this paper. Their potential effects on the steady-state value and growth rate of the seed bank were evaluated by analytical expressions obtained from nonlinear systems theory. Integrated weed control alternatives are oriented to promote a sustainable intervention of the agroecosystem, thus reducing the high current dependency on herbicide usage. The obtained results show that the integration of cultural and mechanical control options are decisive for overcoming the limitations of an individual strategy so as to successfully manage *L. rigidum*. The idea of time-variant control based on the calculation of the control efficiencies every year according to the weed dynamics was also introduced. Evaluations were carried out by considering common uncertainties and parametric variations, validating the expected performances.

Despite its simplicity, the population-based model of *L. rigidum* turns out to be very useful to provide analytical information for more precise long-term decision making. Further work should aim to increase the level of completeness of the model, including some terms to represent more precisely the weed-crop competition as well as to characterize the phenomenon of herbicide resistance at both temporal and spatial levels. These factors could be pertinent for improving the proposed control strategies, leading to a more practical implementation within IWM scenarios.

Acknowledgments

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