Eigenfrequencies of generally restrained Timoshenko beams

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Abstract: This article deals with the free transverse vibration of a Timoshenko beam with intermediate elastic constraints and ends elastically restrained against rotation and translation. A combination of the Ritz method and the Lagrange multiplier method and also the standard Ritz method are used to examine the free vibration characteristics of the mentioned beam. Trial functions denoting the transverse deflections and the normal rotations of the cross-section of the beam are expressed in polynomial forms.

In order to obtain an indication of the accuracy of the developed mathematical models, some cases available in the literature have been considered. New results are presented for different end conditions and intermediate elastic restraints.

Keywords: vibrations, Timoshenko beams, elastically restrained, Lagrange multiplier, Ritz

1 INTRODUCTION

Timoshenko proposed a beam theory that adds the effects of shear distortion and the rotatory inertia to the Euler–Bernoulli model [**1**, **2**]. Later, there has been a considerable interest in developing techniques for the solutions of equations according to the Timoshenko theory. Several authors have obtained the frequencies equations for various end conditions. One of the first and more complete studies was carried out by Traill-Nash and Collar [**3**]. Dolph [**4**] analysed a hinged–hinged beam with external forces and determined the orthogonality conditions.

Huang [**5**] derived the exact solutions of eigenfrequency and modes for a one-span Timoshenko beam under various classical end conditions. Carnegie and Thomas [**6**] studied the effect of shear deformation and rotatory inertia on the frequencies of flexural vibration of uniform and tapered cantilever beams. Anderson [**7**] presented the general solution of mode superposition form for the flexural response of a uniform beam according to the Timoshenko theory. Bhashyam and Prathap [**8**] confirmed the existence of a second spectrum of frequency for the hinged–hinged case by using a finite-element model.

The problem of elastic end restraints has also received considerable attention. Abbas [**9**] treated the problem of free vibration of Timoshenko beams with elastically supported ends by using a finite-element model, which can satisfy all the geometric and natural boundary conditions. Farghaly [**10**] investigated the natural frequencies and the critical buckling load coefficients for a multi-span Timoshenko beam elastically supported. The free vibrations of Timoshenko beams having classical boundary conditions satisfied by Lagrange multipliers were analysed by Kocatürk and Simsek [**11**]. These authors [**12**] also studied the free vibration of elastically supported Timoshenko beams by using the Lagrange equations with the trial functions expressed in the power series form. Zhou [**13**] analysed the free vibration of multi-span Timoshenko beams by the Rayleigh–Ritz method using static Timoshenko beam functions. Lee and Schultz [**14**] presented a study of the free vibration of Timoshenko beams and axisymmetric Mindlin plates by the pseudospectral method. Grossi and Aranda [**15**] applied the Ritz method in the variational formulation of Timoshenko beams with elastically restrained ends.

The free vibration of multi-span Timoshenko beams with an arbitrary number of flexible intermediate constraints was analysed by Lin and Chang [**16**], by using a hybrid analytical numerical solution. The full

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development and analysis of four theories, including the Timoshenko model, for the transversely vibrating uniform beam were presented by Han *et al.* [**17**].

A review of the literature further reveals that there is only a limited amount of information for the vibration of Timoshenko beams with intermediate elastic restraints.

The aim of the present article is to investigate the natural frequencies and mode shapes of a Timoshenko beam with intermediate elastic constraints and generally restrained ends. Several cases are solved by a combination of the Ritz method and the Lagrange multiplier method in conjunction with sets of simple polynomials as trial functions. In order to obtain an indication of the accuracy of the developed mathematical model, some cases available in the literature have been considered and comparisons of numerical results are included. The algorithms developed can be applied to a wide range of elastic restraint conditions. The generally restrained beam analysed includes the classical end conditions: clamped, simply supported, sliding, and free as simply particular cases.

A great number of problems were solved and, since this number of cases is prohibitively large, results are presented for only a few cases.

2 THEORY AND FORMULATIONS

A uniform Timoshenko beam of length l with elastically restrained ends and constrained at an intermediate point c with variable position is considered as in Fig. 1.

According to the Timoshenko beam theory, two independent variables, transverse deflection w and normal rotational angle ϕ due to bending, are used to describe the deformation of the beam. The elastic strain energy because of the beam and the elastic restraints at any instant t is given by

$$U = \frac{1}{2}\int_0^l \left\{ EI\left[\frac{\partial \phi(\bar{x},t)}{\partial \bar{x}}\right]^2 + kGA\left[\frac{\partial w(\bar{x},t)}{\partial \bar{x}} - \phi(\bar{x},t)\right]^2\right\} \times \mathrm{d}\bar{x} + \frac{1}{2}[t_1 w^2(0,t) + r_1\phi^2(0,t) + t_c w^2(c,t) + r_c\phi^2(c,t) + t_2 w^2(l,t) + r_2\phi^2(l,t)] \quad (1)$$

where E is the Young's modulus, G is the transverse shear modulus, I is the moment of inertia, A is the area of the cross-section, and k is the shear correction factor. The rotational restraints are characterized by the spring constants r_1, r_2, and r_c, and the translational restraints by the spring constants t_1, t_2, and t_c.

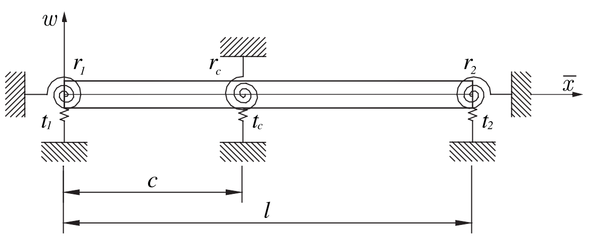

Fig. 1 Vibrating system under study

The kinetic energy of the beam at any instant t is given by

$$T = \frac{1}{2}\int_0^l \left\{ \rho A\left[\frac{\partial w(\bar{x},t)}{\partial t}\right]^2 + \rho I\left[\frac{\partial \phi(\bar{x},t)}{\partial t}\right]^2\right\} \mathrm{d}\bar{x} \quad (2)$$

where ρ is the mass per unit volume.

When the beam executes free vibrations, transverse deflection and normal rotation can be written as

$$w(\bar{x},t) = \bar{W}(\bar{x})\sin(\omega t), \quad \phi(\bar{x},t) = \bar{\Phi}(\bar{x})\sin(\omega t)$$

where ω is the radian frequency.

By introducing the following non-dimensional parameters

$$x = \frac{\bar{x}}{l}, \quad W = \frac{\bar{W}}{l}, \quad \Phi = \bar{\Phi} \quad (3)$$

the Lagrangian functional L_0 of the problem can be written as

$$\begin{aligned} L_0 &= U - T \\ &= \frac{1}{2}\int_0^1 \left[\left(\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right)^2 + \gamma\left(\frac{l}{r}\right)^2\left(\frac{\mathrm{d}W}{\mathrm{d}x} - \Phi\right)^2\right]\mathrm{d}x \\ &\quad + \frac{1}{2}[T_1W^2(0) + R_1\Phi^2(0) + T_cW^2(c_1) \\ &\quad + R_c\Phi^2(c_1) + T_2W^2(1) + R_2\Phi^2(1)] \\ &\quad - \frac{1}{2}\Omega^2\int_0^1\left[\left(\frac{r}{l}\right)^2\Phi^2 + W^2\right]\mathrm{d}x \end{aligned} \quad (4)$$

where

$$\gamma = \frac{kG}{E}, \quad r = \sqrt{\frac{I}{A}}, \quad \Omega = \omega l^2\sqrt{\frac{\rho A}{EI}}, \quad c_1 = \frac{c}{l}$$

$$T_i = \frac{t_i l^3}{EI}, \quad R_i = \frac{r_i l}{EI}, \quad i = 1, 2, \quad R_c = \frac{r_c l}{EI}, \quad T_c = \frac{t_c l^3}{EI}$$

2.1 Combination of the Ritz method and the Lagrange multiplier method

Assuming that $W(x)$ and $\Phi(x)$ can be written in the form

$$W(x) = \begin{cases} W_1(x), \forall x \in [0, c_1] \\ W_2(x), \forall x \in (c_1, 1] \end{cases}$$

and

$$\Phi(x) = \begin{cases} \Phi_1(x), \forall x \in [0, c_1] \\ \Phi_2(x), \forall x \in (c_1, 1] \end{cases} \quad (5)$$

and considering the compatibility requirements on the intermediate elastically restrained point, the relationships between two adjacent spans can be expressed as

$$\Phi_1(c_1) - \Phi_2(c_1) = 0 \tag{6a}$$

$$W_1(c_1) - W_2(c_1) = 0 \tag{6b}$$

Now the problem can be posed as one of extremizing the given functional in equation (4) subjected to the following constraints

$$G_1(\Phi_1, \Phi_2) = \Phi_1(c_1) - \Phi_2(c_1) \tag{7a}$$

$$G_2(W_1, W_2) = W_1(c_1) - W_2(c_1) \tag{7b}$$

These constraints may be incorporated into the energy functional given by equation (4) by using the Lagrange multiplier method [**18**] as

$$L_{\mathrm{L}} = L_0 + \lambda_1 G_1 + \lambda_2 G_2 \tag{8}$$

where L_{L} is the new Lagrangian functional and $\lambda_i,\ i = 1, 2$ are the Lagrange multipliers.

The normal rotation and the transverse deflection, at the kth span, can be represented by the sets of polynomials $\{p_i^{(k)}(x)\}$ and $\{q_j^{(k)}(x)\}$, respectively, as

$$\Phi_k = \sum_{i=1}^{M} a_i^{(k)} p_i^{(k)}(x), \quad k = 1, 2 \tag{9}$$

$$W_k = \sum_{j=1}^{N} b_j^{(k)} q_j^{(k)}(x), \quad k = 1, 2 \tag{10}$$

where both $a_i^{(k)}$ and $b_j^{(k)}$ are unknown coefficients to be determined. It is sufficient that the trial functions satisfy the geometric boundary conditions of the beam since, as the number of functions approaches infinity, the natural boundary conditions will be exactly satisfied [**19**]. In consequence, the first members of the sets $p_i^{(k)}(x)$ and $q_j^{(k)}(x)$ are obtained as the simplest polynomial that satisfies all the geometric boundary conditions of the kth span. Assume that

$$p_i^{(k)}(x) = \sum_{i=1}^{5} \bar{a}_i^{(k)} x^{i-1}, \quad k = 1, 2 \tag{11}$$

where the arbitrary constants $\bar{a}_i^{(k)}$ are determined by substituting equation (11) into the corresponding boundary conditions. In the case of beams involving free ends simpler starting members of order zero, one, or two are used. The corresponding polynomials of higher order are obtained as

$$p_i^{(k)} = p_1^{(k)} x^{i-1}, \quad i = 2, \ldots, M, \ \ k = 1, 2 \tag{12}$$

The polynomials set $\{q_j^{(k)}(x)\}$ are also generated using the same procedure, i.e.

$$q_j^{(k)} = q_1^{(k)} x^{j-1}, \quad j = 2, \ldots, N, \ \ k = 1, 2 \tag{13}$$

Substituting equations (9) and (10) into equation (8), and minimizing with respect to the unknown coefficients $a_i^{(k)}, b_j^{(k)}$, and the Lagrangian multipliers λ_i, leads to the following conditions

$$\frac{\partial L_{\mathrm{L}}}{\partial a_i^{(k)}} = 0, \quad i = 1, 2, \ldots, M, \ \ k = 1, 2 \tag{14}$$

$$\frac{\partial L_{\mathrm{L}}}{\partial b_j^{(k)}} = 0, \quad j = 1, 2, \ldots, N, \ \ k = 1, 2 \tag{15}$$

$$\frac{\partial L_{\mathrm{L}}}{\partial \lambda_i} = 0, \quad i = 1, 2 \tag{16}$$

By using equations (14) to (16), a set of linear algebraic equations is obtained, which can be expressed in the following matrix form

$$([\mathbf{K}] - \Omega^2[\mathbf{M}])\{\bar{c}\} = \{0\} \tag{17}$$

where

$$[\mathbf{K}] = \begin{bmatrix} [K_{aa}^{(1)}] & [K_{ab}^{(1)}] & [0] & [0] & [L_{a\lambda_1}^{(1)}] & [0] \\ & [K_{bb}^{(1)}] & [0] & [0] & [0] & [L_{b\lambda_2}^{(1)}] \\ & & [K_{aa}^{(2)}] & [K_{ab}^{(2)}] & [L_{a\lambda_1}^{(2)}] & [0] \\ & & & [K_{bb}^{(2)}] & [0] & [L_{b\lambda_2}^{(2)}] \\ & \text{symm} & & & [0] & [0] \\ & & & & & [0] \end{bmatrix} \tag{18}$$

$$[\mathbf{M}] = \begin{bmatrix} [M_{aa}^{(1)}] & [0] & [0] & [0] & [0] & [0] \\ & [M_{bb}^{(1)}] & [0] & [0] & [0] & [0] \\ & & [M_{aa}^{(2)}] & [0] & [0] & [0] \\ & & & [M_{bb}^{(2)}] & [0] & [0] \\ & \text{symm} & & & [0] & [0] \\ & & & & & [0] \end{bmatrix} \tag{19}$$

$$\{\bar{c}\} = \{\{a^{(1)}\}, \{b^{(1)}\}, \{a^{(2)}\}, \{b^{(2)}\}, \{\lambda\}\}^{\mathrm{T}} \tag{20}$$

with

$$\begin{aligned} &\{a^{(k)}\} = \{a_1^{(k)}, a_2^{(k)}, \ldots, a_M^{(k)}\} \\ &\{b^{(k)}\} = \{b_1^{(k)}, b_2^{(k)}, \ldots, b_N^{(k)}\}, \quad k = 1, 2 \\ &\{\lambda\} = \{\lambda_1, \lambda_2\} \end{aligned} \tag{21}$$

The expressions for the various elements of the stiffness matrix $[\mathbf{K}]$ and the mass matrix $[\mathbf{M}]$ are given by

$$K_{aaim}^{(1)} = \int_0^{c_1} \left[\frac{\mathrm{d}p_i^{(1)}(x)}{\mathrm{d}x} \frac{\mathrm{d}p_m^{(1)}(x)}{\mathrm{d}x} + \gamma \left(\frac{l}{r}\right)^2 p_i^{(1)}(x) p_m^{(1)}(x) \right] \mathrm{d}x + R_1 p_i^{(1)}(0) p_m^{(1)}(0) + R_c p_i^{(1)}(c_l) p_m^{(1)}(c_1)$$

$$K_{abij}^{(1)} = -\int_0^{c_1} \gamma \left(\frac{l}{r}\right)^2 p_i^{(1)} \frac{\mathrm{d}q_j^{(1)}}{\mathrm{d}x} \, \mathrm{d}x$$

$$K_{bbjn}^{(1)} = \int_0^{c_1} \gamma \left(\frac{l}{r}\right)^2 \frac{\mathrm{d}q_j^{(1)}(x)}{\mathrm{d}x} \frac{\mathrm{d}q_n^{(1)}(x)}{\mathrm{d}x} \, \mathrm{d}x + T_1 q_j^{(1)}(0) q_n^{(1)}(0) + T_c q_j^{(1)}(c_1) q_n^{(1)}(c_1)$$

$$L_{a\lambda_1 i1}^{(1)} = p_i^{(1)}(c_1)$$

$$L_{b\lambda_2 j1}^{(1)} = q_j^{(1)}(c_l)$$

$$K_{aaim}^{(2)} = \int_{c_l}^{1} \left[\frac{\mathrm{d}p_i^{(2)}(x)}{\mathrm{d}x} \frac{\mathrm{d}p_m^{(2)}(x)}{\mathrm{d}x} + \gamma \left(\frac{l}{r}\right)^2 p_i^{(2)}(x) p_m^{(2)}(x) \right] \mathrm{d}x + R_2 p_i^{(2)}(1) p_m^{(2)}(1)$$

$$K_{abij}^{(2)} = -\int_{c_l}^{1} \gamma \left(\frac{l}{r}\right)^2 p_i^{(2)} \frac{\mathrm{d}q_j^{(2)}}{\mathrm{d}x} \,\mathrm{d}x$$

$$K_{bbjn}^{(2)} = \int_{c_l}^{1} \gamma \left(\frac{l}{r}\right)^2 \frac{\mathrm{d}q_j^{(2)}(x)}{\mathrm{d}x} \frac{\mathrm{d}q_n^{(2)}(x)}{\mathrm{d}x} \,\mathrm{d}x + T_2 q_j^{(2)}(1) q_n^{(2)}(1)$$

$$L_{a\lambda_1 i1}^{(2)} = -p_i^{(2)}(c_l)$$

$$L_{b\lambda_2 j1}^{(2)} = -q_j^{(2)}(c_l)$$

$$M_{aaim}^{(1)} = \int_0^{c_l} \left(\frac{r}{l}\right)^2 p_i^{(1)}(x) p_m^{(1)}(x) \,\mathrm{d}x$$

$$M_{bbjn}^{(1)} = \int_0^{c_l} q_j^{(1)}(x) q_n^{(1)}(x) \,\mathrm{d}x$$

$$M_{aaim}^{(2)} = \int_{c_l}^{1} \left(\frac{r}{l}\right)^2 p_i^{(2)}(x) p_m^{(2)}(x) \,\mathrm{d}x$$

$$M_{bbjn}^{(2)} = \int_{c_l}^{1} q_j^{(2)}(x) q_n^{(2)}(x) \,\mathrm{d}x$$

with $i, m = 1, 2, \ldots, M$ and $j, n = 1, 2, \ldots, N$.

The eigenvalues Ω^2 are found from the condition that the determinant of the system of equations given by equation (17) must vanish.

The starting polynomials used for several combinations of classical boundary conditions are described in Appendix 2, where the symbols F, C, and S denote free, clamped, and simply supported ends, respectively.

Table 1 Convergence study of the first six values of the frequency parameter $\Omega = \omega l^2 \sqrt{(\rho A/EI)}$ of a Timoshenko beam with an intermediate support ($T_c \to \infty$) located at $c_l = 0.4$ for $\sqrt{12}(r/l) = 0.1$

Boundary conditions	$N = M$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
Ritz with the Lagrange multipliers method							
S–S	6	31.3371	66.9615	104.3698	186.7780	205.4217	348.4253
	7	31.3371	66.9553	103.9240	186.4887	204.5803	300.6391
	8	31.3371	66.9551	103.9222	185.3377	203.2434	299.8346
	9	31.3371	66.9551	103.9195	185.3363	203.2218	293.0323
	10	31.3371	66.9551	103.9195	185.3183	203.1968	293.0078
	11	31.3371	66.9551	103.9195	185.3183	203.1966	292.7685
	12	31.3371	66.9551	103.9195	185.3182	203.1964	292.7682
Standard Ritz method							
	10	31.4665	67.9108	106.9151	185.8660	236.2251	334.1349
	12	31.4282	67.6183	105.7391	185.3308	215.9283	304.4182
	14	31.4088	67.4754	105.3104	185.3184	211.2248	298.2840
	16	31.3982	67.3982	105.0949	185.3183	209.7171	296.7557
	18	31.3915	67.3500	104.9635	185.3183	208.9424	296.1967
	20	31.3864	67.3130	104.8637	185.3183	208.3807	295.8568
	25	31.3751	67.2309	104.6442	185.3183	207.1680	295.1484
	30	31.3682	67.1815	104.5131	185.3183	206.4515	294.7315
	35	31.3639	67.1504	104.4308	185.3183	206.0023	294.4684
	40	31.3609	67.1284	104.3730	185.3183	205.6866	294.2822
	Reference [**13**]*	31.3370	66.9554	103.9200	185.3186	203.2250	292.8410
Ritz with the Lagrange multipliers method							
C–C	6	44.8971	89.3768	120.3082	204.3584	221.9158	313.2971
	7	44.8970	89.3751	120.3068	202.1147	220.3916	313.0603
	8	44.8970	89.3750	120.2982	202.0986	220.3812	304.0233
	9	44.8970	89.3750	120.2982	202.0523	220.3466	304.0080
	10	44.8970	89.3750	120.2982	202.0522	220.3465	303.6564
	11	44.8970	89.3750	120.2982	202.0519	220.3462	303.6561
	12	44.8970	89.3750	120.2982	202.0519	220.3462	303.6512
Standard Ritz method							
	10	45.1259	90.2482	123.2656	202.3702	236.9757	319.6643
	12	45.0729	90.0472	122.4846	202.2452	229.9998	310.7379
	14	45.0433	89.9360	122.0961	202.2128	227.8754	308.4639
	16	45.0256	89.8695	121.8714	202.1966	226.8682	307.6734
	18	45.0133	89.8232	121.7179	202.1854	226.2146	307.2536
	20	45.0029	89.7843	121.5898	202.1758	225.6790	306.9265
	25	44.9793	89.6945	121.2983	202.1526	224.4701	306.1952
	30	44.9646	89.6385	121.1193	202.1372	223.7345	305.7518
	35	44.9554	89.6031	121.0069	202.1271	223.2729	305.4724
	40	44.9501	89.5828	120.9429	202.1211	223.0090	305.3113
	Reference [**13**]*	44.8968	89.3762	120.3006	202.0673	220.4041	303.7835

*Rayleigh–Ritz method using static Timoshenko beam functions.

2.2 Standard Ritz procedure

In order to apply the standard Ritz procedure, the expressions for the transverse deflection W and normal rotation Φ are assumed

$$\Phi = \sum_{i=1}^{M} a_i p_i(x) \tag{22a}$$

$$W = \sum_{j=1}^{N} b_j q_j(x) \tag{22b}$$

where

$$p_i(x) = p_1(x)x^{i-1}, \quad i = 1, 2, \ldots, M \tag{23}$$

$$q_j(x) = q_1(x)x^{j-1}, \quad j = 1, 2, \ldots, N \tag{24}$$

First members $p_1(x)$ and $q_1(x)$ are obtained as the ones stated in section 2.1. In this case the governing eigenvalue equation is given by

$$\left(\begin{bmatrix} [K_{aa}] & [K_{ab}] \\ [K_{ab}] & [K_{bb}] \end{bmatrix} - \Omega^2 \begin{bmatrix} [M_{aa}] & [0] \\ [0] & [M_{bb}] \end{bmatrix}\right) \begin{pmatrix} \{\bar{a}\} \\ \{\bar{b}\} \end{pmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{25}$$

where

$$K_{aaim} = \int_0^1 \left[\frac{\mathrm{d}p_i(x)}{\mathrm{d}x}\frac{\mathrm{d}p_m(x)}{\mathrm{d}x} + \gamma\left(\frac{l}{r}\right)^2 p_i(x)p_m(x)\right]\mathrm{d}x$$
$$+ R_1 p_i(0)p_m(0) + R_c p_i(c_l)p_m(c_l)$$
$$+ R_2 p_i(1)p_m(1)$$

Table 2 Values of the fundamental frequency parameter $\sqrt{\Omega_1} = \sqrt[4]{(\rho A/EI)\omega_1^2}\,l$ of a cantilever Timoshenko beam with an intermediate point elastically restrained against rotation and translation located at $c_l = 0.6$ for $\sqrt{12}(r/l) = 0.0001$

		T_c				
R_c		0	10	100	1000	10 000
0	RLMM	1.875 104	2.130 286	2.936 571	3.572 321	3.671 740
	SRM	1.875 104	2.130 287	2.936 592	3.572 374	3.671 796
	Reference [**20**]*	1.875 104	2.130 285	2.936 571	3.572 320	3.671 740
10	RLMM	2.608 757	2.746 179	3.377 897	4.033 222	4.136 166
	SRM	2.622 658	2.758 980	3.389 049	4.046 734	4.150 045
	Reference [**20**]*	2.608 757	2.746 178	3.377 896	4.033 222	4.136 165
100	RLMM	2.949 918	3.067 338	3.679 377	4.446 696	4.569 468
	SRM	2.982 249	3.098 362	3.710 585	4.496 739	4.623 319
	Reference [**20**]*	2.949 918	3.067 338	3.679 377	4.446 696	4.569 468
1000	RLMM	3.004 578	3.119 844	3.732 400	4.532 750	4.662 299
	SRM	3.040 564	3.154 515	3.767 811	4.593 127	4.728 104
	Reference [**20**]*	3.004 577	3.119 843	3.732 400	4.532 749	4.662 298
10 000	RLMM	3.010 371	3.125 421	3.738 080	4.542 271	4.672 637
	SRM	3.046 754	3.160 488	3.773 944	4.603 837	4.739 838
	Reference [**20**]*	3.010 371	3.125 421	3.738 079	4.542 270	4 672 637

*Exact solution based on the Euler-Bernoulli beam theory.
RLMM, the Ritz with Lagrange multipliers method; SRM, the standard Ritz method

Table 3 First three values of the frequency parameter $\sqrt{\Omega} = \sqrt[4]{(\rho A/EI)\omega^2}\,l$ of a Timoshenko beam with ends elastically restrained against rotation and translation ($R_1 = R, R_2 = 0, T_1 = 1 \times 10^8, T_2 = T, T_c = R_c = 0$) for $\sqrt{12}(r/l) = 0.005$

	T	∞		1000		100		10		1	
R	Mode number	RLMM	Reference [**12**]*	RLMM	Reference [**12**]*	RLMM	Reference [**12**]*	RLMM	Reference [**12**]*	RLMM	Reference [**12**]*
∞	1	3.926 40	3.926 41	3.897 60	3.897 61	3.640 41	3.640 42	2.638 90	2.638 90	2.010 00	2.009 99
	2	7.067 60	7.067 61	6.875 45	6.875 46	5.615 66	5.615 67	4.793 44	4.793 45	4.703 46	4.703 47
	3	10.207 40	10.207 44	9.550 73	9.550 74	8.082 80	8.082 81	7.874 34	7.874 35	7.855 51	7.855 52
1000	1	3.922 50	3.922 50	3.893 81	3.893 82	3.637 59	3.637 60	2.637 58	2.637 59	2.008 32	2.008 34
	2	7.060 59	7.060 60	6.869 24	6.869 25	5.612 18	5.612 18	4.788 98	4.788 99	4.698 82	4.698 82
	3	10.197 30	10.197 35	9.543 29	9.543 30	8.075 53	8.075 54	7.866 61	7.866 62	7.847 74	7.847 75
100	1	3.889 00	3.889 00	3.861 28	3.861 28	3.613 23	3.613 23	2.626 12	2.626 14	1.993 93	1.993 93
	2	7.002 29	7.002 30	6.817 36	6.817 37	5.582 80	5.582 80	4.751 12	4.751 12	4.659 30	4.659 31
	3	10.115 90	10.115 94	9.482 58	9.482 59	8.015 72	8.015 73	7.802 73	7.802 74	7.783 53	7.783 54
10	1	3.664 51	3.664 51	3.642 14	3.642 14	3.441 12	3.441 13	2.538 82	2.538 81	1.879 27	1.879 28
	2	6.686 71	6.686 71	6.530 84	6.530 84	5.409 70	5.409 71	4.515 71	4.515 71	4.410 59	4.410 59
	3	9.749 44	9.749 45	9.482 58	9.195 24	7.714 32	7.714 33	7.473 94	7.473 95	7.452 43	7.452 44
1	1	3.273 21	3.273 21	3.256 57	3.256 57	3.108 40	3.108 40	2.326 45	2.326 46	1.535 80	1.535 80
	2	6.355 42	6.355 42	6.220 83	6.220 84	5.198 48	5.198 49	4.184 44	4.184 45	4.045 97	4.045 98
	3	9.473 01	9.473 02	8.963 47	8.963 47	7.440 88	7.440 89	7.161 08	7.161 09	7.136 08	7.136 08

*Lagrange equations using trial functions expressed in the power series form.

$$K_{abij} = -\int_0^1 \gamma \left(\frac{l}{r}\right)^2 p_i \frac{\mathrm{d}q_j}{\mathrm{d}x}\,\mathrm{d}x$$

$$K_{bbjn} = \int_0^1 \gamma \left(\frac{l}{r}\right)^2 \frac{\mathrm{d}q_j(x)}{\mathrm{d}x}\frac{\mathrm{d}q_n(x)}{\mathrm{d}x}\,\mathrm{d}x + T_1 q_j(0) q_n(0) + T_c q_j(c_l) q_n(c_l) + T_2 q_j(1) q_n(1)$$

$$M_{aaim} = \int_0^1 \left(\frac{r}{l}\right)^2 p_i(x) p_m(x)\,\mathrm{d}x$$

$$M_{bbjn} = \int_0^1 q_j(x) q_n(x)\,\mathrm{d}x$$

with $i, m = 1, 2, \ldots, M$ and $j, n = 1, 2, \ldots, N$.

The eigenvalues Ω^2 are found from the condition that the determinant of the system of equations given by equation (25) must vanish.

Table 4 Values of the fundamental frequency parameter $\Omega_1 = \omega_1 l^2 \sqrt{(\rho A/EI)}$ of a Timoshenko beam with ends elastically restrained against rotation ($T_1 \to \infty$, $T_2 \to \infty$, $k = 0.85$, $\mu = 0.3$, $(r/l) = 0.08$)

	R_2	10		1		0	
R_1	Ω	Reference [**9**]*	RLMM	Reference [**9**]*	RLMM	Reference [**9**]*	RLMM
0.01	1	11.632	11.188	9.227	9.469	8.845	8.922
	2	30.702	30.255	28.699	28.861	28.465	28.511
	3	52.825	52.557	51.645	51.736	51.511	51.528
	4	76.019	75.873	75.502	75.503	75.426	75.383
	5	91.751	91.501	89.768	90.036	89.345	89.436
10	1	14.644	13.5182	12.006	11.7836	11.637	11.1803
	2	32.670	31.8575	30.907	30.6026	30.705	30.2507
	3	54.093	53.5823	52.955	52.7832	52.826	52.5538
	4	76.503	76.3022	76.081	75.9891	76.019	75.8716
	5	97.771	96.2700	92.595	92.8000	91.762	91.4854

*Result obtained using a finite-element model.

Table 5 Values of the fundamental frequency parameter $\sqrt{\Omega_1} = \sqrt[4]{(\rho A/EI)\omega_1^2}\,l$ of a Timoshenko beam with symmetric boundary conditions and with an intermediate point elastically restrained against rotation and translation located at three different positions

			$\sqrt{12}(r/l)$								
			$c_l = 0.1$			$c_l = 0.3$			$c_l = 0.5$		
	T_c	R_c	0.001	0.01	0.1	0.001	0.01	0.1	0.001	0.01	0.1
S–S	0	0	3.1416	3.1413	3.1157	3.1416	3.1413	3.1157	3.1416	3.1413	3.1157
		10	3.7858	3.7852	3.7225	3.4497	3.4493	3.4075			
		100	4.2400	4.2389	4.1302	3.7740	3.7733	3.7006			
	10	0	3.1568	3.1565	3.1311	3.2409	3.2407	3.2164	3.2913	3.2911	3.2675
		10	3.7897	3.7890	3.7267	3.5322	3.5318	3.4919			
		100	4.2404	4.2392	4.1309	3.8450	3.8442	3.7740			
	100	0	3.2756	3.2754	6.1612	3.8148	3.8145	3.7855	4.1315	4.1313	4.1063
		10	3.8216	3.8210	3.7612	4.0415	4.0411	3.9998			
		100	4.2436	4.2424	4.1363	4.3071	4.3064	4.2385			
C–C	0	0	4.7300	4.7284	4.5795	4.7300	4.7284	4.5795	4.7300	4.7284	4.5795
		10	4.9115	4.9095	4.7338	5.0303	5.0280	4.8255			
		100	5.1643	5.1619	4.9435	5.4649	5.4612	5.1420			
	10	0	4.7309	4.7293	4.5807	4.7579	4.7563	4.6103	4.7884	4.7868	4.6419
		10	4.9119	4.9099	4.7345	5.0548	5.0526	4.8535			
		100	5.1644	5.1619	4.9437	5.4859	5.4821	5.1675			
	100	0	4.7383	4.7367	4.5910	4.9733	4.9718	4.8395	5.2304	5.2290	5.1026
		10	4.9154	4.9135	4.7408	5.2458	5.2438	5.0645			
		100	5.1648	5.1624	4.9460	5.6509	5.6475	5.3611			
F–F	0	0	4.7300	4.7292	4.6485	4.7300	4.7292	4.6485	4.7300	4.7292	4.6485
		10	2.3629	2.3628	2.3522	2.6968	2.6966	2.6805	2.9448	2.9446	2.9227
		100	2.5178	2.5177	2.5048	3.0239	3.0236	2.9994	3.6131	3.6126	3.5641
	10	0	2.3101	2.3100	2.3054	1.9558	1.9557	1.9534	1.7644	1.7644	1.7634
		10	1.5891	1.5891	1.5868	1.7178	1.7178	1.7162			
		100	1.6416	1.6415	1.6391	1.7366	1.7366	1.7351			
	100	0	3.7974	3.7971	3.7692	3.3784	3.3783	3.3623	2.9265	2.9264	2.9123
		10	1.8626	1.8625	1.8564	2.2995	2.2994	2.2887			
		100	2.0086	2.0085	2.0001	2.4900	2.4898	2.4757			

3 CONVERGENCE AND COMPARISON STUDIES

Through all the analysis, beams with shear correction factor $k = (5/6)$ and Poisson's ratio $\mu = 0.3$ are considered.

In order to illustrate the accuracy of solutions obtained using the Ritz method with Lagrange multipliers and to demonstrate quantitatively the lack of accuracy obtained from the use of the standard Ritz procedure, several test problems are considered. Results of a convergence study of the first six values of the dimensionless frequency parameter Ω of simply–simply supported and clamped–clamped beams with an intermediate support located at $c_1 = 0.4$ are presented in Table 1. The convergence of the mentioned eigenvalues is studied by gradually increasing the number of trial functions. In this case, it has been determined that it is convenient to use the Lagrange multipliers method. It is well known that the Ritz method gives upper bounds eigenvalues. A comparison of values of the fundamental frequency with those of reference [**13**] shows that $N = M = 12$ in the Ritz with the Lagrange multipliers method is enough to reach stable convergence in all cases and to give results with the same precision. On the other hand, one can see that the use of standard Ritz method (SRM) requires more than 40 terms in the approximate functions in order to obtain the numerical results with the same accuracy. Table 2 depicts values of the fundamental frequency parameter $\sqrt{\Omega_1}$ of a cantilever Timoshenko beam with an intermediate point elastically restrained against rotation and translation located at $c_1 = 0.6$. In order to compare values of the mentioned parameter with the classical solutions based on the Euler–Bernoulli beam theory [**20**], the ratio $\sqrt{12}(r/l) = 0.0001$ is taken. The numerical results obtained by using the combination of the Ritz method with the Lagrange multiplier method with $N = M = 7$ are in good agreement with those of Grossi and Albarracín [**20**]. The SRM was used with $N = M = 30$ and small discrepancies appear as the values of R_c and T_c increase.

Table 3 depicts the first three values of the frequency parameter $\sqrt{\Omega}$ of a single-span Timoshenko beam with ends elastically restrained against rotation and translation ($R_1 = R, R_2 = 0, T_1 = 1 \cdot 10^8, T_2 = T, T_c = R_c = 0$) for $\sqrt{12}(r/l) = 0.005$. The numerical results were obtained by using the Ritz with the

Table 6 Values of the fundamental frequency parameter $\sqrt{\Omega_1} = \sqrt[4]{(\rho A/EI)\omega_1^2}\, l$ of a Timoshenko beam with unsymmetrical boundary conditions and with an intermediate point elastically restrained against rotation and translation located at three different positions

			$\sqrt{12}(r/l)$								
			$c_1 = 0.25$			$c_1 = 0.50$			$c_1 = 0.75$		
	T_c	R_c	0.001	0.01	0.1	0.001	0.01	0.1	0.001	0.01	0.1
S–F	0	0	0	0	0	0	0	0	0	0	0
		10	2.0175	2.0174	2.0075	1.9745	1.9744	1.9656	1.7016	1.7016	1.6970
		100	2.3777	2.3774	2.3574	2.2882	2.2880	2.2709	1.8252	1.8251	1.8190
	10	0	1.1524	1.1524	1.1507	1.6300	1.6300	1.6277	3.9270	2.0235	2.0215
		10	2.0440	2.0439	2.0348	2.2179	2.2178	2.2108	2.2290	2.2290	2.2257
		100	2.3827	2.3825	2.3631	2.4857	2.4856	2.4709	2.2815	2.2815	2.2773
	100	0	1.8352	1.8351	1.8232	2.5643	2.5641	2.5480	3.5396	3.5394	3.5242
		10	2.1755	2.1754	2.1661	2.9331	2.9330	2.9161	3.5469	3.5468	3.5306
		100	2.4117	2.4115	2.3951	3.1856	3.1853	3.1620	3.5479	3.5477	3.5314
C–F	0	0	1.8751	1.8750	1.8677	1.8751	1.8750	1.8677	1.8751	1.8750	1.8677
		10	2.2155	2.2154	2.2006	2.5983	2.5980	2.5674	2.4969	2.4967	2.4710
		100	2.4398	2.4396	2.4176	3.0455	3.0448	2.9800	2.6729	2.6726	2.6370
	10	0	1.8890	1.8889	1.8822	2.0238	2.0237	2.0183	2.3316	2.3316	2.3263
		10	2.2190	2.2188	2.2046	2.6665	2.6662	2.6397	2.7488	2.7486	2.7282
		100	2.4403	2.4400	2.4183	3.0955	3.0949	3.0357	2.8859	2.8856	2.8562
	100	0	1.9831	1.9831	1.9763	2.5690	2.5688	2.5538	3.6290	3.6289	3.6133
		10	2.2447	2.2446	2.2329	2.9936	2.9934	2.9731	3.8216	3.8214	3.8005
		100	2.4439	2.4437	2.4239	3.3494	3.3489	3.3062	3.8777	3.8774	3.8510
C–S	0	0	3.9266	3.9258	3.8518	3.9266	3.9258	3.8518	3.9266	3.9258	3.8518
		10	4.3230	4.3216	4.1961	3.9839	3.9830	3.8968	4.2088	4.2078	4.1198
		100	4.8177	4.8153	4.5978	4.0437	4.0426	3.9413	4.5716	4.5702	4.4405
	10	0	3.9438	3.9430	3.8710	4.0098	4.0091	3.9389	3.9863	3.9855	3.9130
		10	4.3338	4.3324	4.2095	4.0655	4.0647	3.9826	4.2608	4.2599	4.1734
		100	4.8227	4.8203	4.6059	4.1235	4.1225	4.0258	4.6170	4.6156	4.4877
	100	0	4.0757	4.0750	4.0137	4.5776	4.5770	4.5197	4.3957	4.3949	4.3218
		10	4.4182	4.4170	4.3104	4.6280	4.6273	4.5601	4.6292	4.6283	4.5420
		100	4.8624	4.8602	4.6677	4.6791	4.6782	4.5986	4.9477	4.9464	4.8209

Lagrange multipliers method. A comparison of values of the fundamental frequency with those of reference [**12**] shows a very close agreement.

Table 4 depicts values of the fundamental frequency parameter $\Omega_1 = \omega_1 l^2 \sqrt{\rho A/EI}$ of a Timoshenko beam with ends elastically restrained against rotation. Following Abbas [**9**], the shear correction factor $k = 0.85$, the Poisson's ratio $\mu = 0.3$, and the ratio $(r/l) = 0.08$ are considered. It can be observed that several values are not in good agreement. The article of Abbas does not include details about the implementation of the finite-element model; nevertheless, it can be observed that one of the boundary conditions has an incorrect sign.

4 NUMERICAL EXAMPLES

In order to investigate the influence of stiffness of the intermediate elastic restraints on the free vibration characteristics of Timoshenko beams, numerical results were computed by using the combination of the Ritz method with the Lagrange multiplier method. A great number of problems were solved and, since the number of cases is extremely large, results are presented for only a few cases. All calculations have been performed taking $N = M = 12$, $k = (5/6)$, and $\mu = 0.3$.

Tables 5 and 6 depict values of the fundamental frequency parameter $\sqrt{\Omega_1}$ of a Timoshenko beam with an intermediate point elastically restrained against rotation and translation. Three different thickness ratios: $\sqrt{12}(r/l) = 0.001$, 0.01, and 0.1 are considered and the intermediate point is located at three different positions. In Table 5 three kinds of symmetric boundary conditions are considered, while in Table 6 three kinds of unsymmetrical boundary conditions are considered.

Figure 2 shows the variation of the first four values of the frequency parameters $\sqrt{\Omega}$ with respect to the intermediate rotational restraint R_c located at $c_1 = 0.5$ of a clamped-free Timoshenko beam for $\sqrt{12}(r/l) = 0.1$. From this figure it appears that the major variations of frequency parameters with the intermediate rotational restraint R_c correspond to modes 1 and 3.

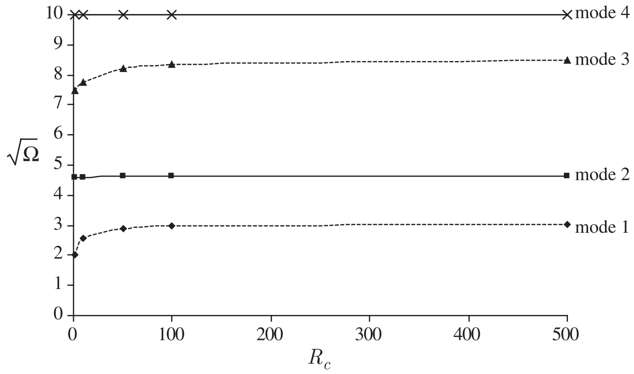

Fig. 2 Variation in the first four values of the frequency parameters $\sqrt{\Omega}$ with respect to the intermediate rotational restraint R_c located at $c_1 = 0.5$ of a C–F Timoshenko beam for $\sqrt{12}(r/l) = 0.1$

5 CONCLUSIONS

A simple, computationally efficient and accurate approach has been developed for the determination of natural frequencies of free vibration of a uniform Timoshenko beam with intermediate elastic constraints and ends elastically restrained against rotation and translation. The algorithm is very general and it is attractive regarding its versatility in handling any boundary conditions and any transition conditions, including ends and an intermediate point elastically restrained against rotation and translation. A combination of the Ritz method and the Lagrange multiplier method and also the SRM have been used. Close agreement with results presented by previous investigators is demonstrated for several examples.

It has been demonstrated that it is convenient to use the mentioned combined method, since it is more efficient than the SRM because of the higher rate of convergence.

These results obtained may provide useful information for structural designers and engineers. The algorithms developed can be easily extended to a beam with an arbitrary number of intermediate points elastically restrained.

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APPENDIX 1

Notation

A	cross-sectional area
$c_l = c/l$	geometrical parameter
E	Young's modulus
G	transverse shear modulus
I	moment of inertia
l	length of the beam
M, N	upper limits of summation
$r = \sqrt{I/A}$	radius of gyration of cross-section
r_c	rotational stiffness at the point $\bar{x} = c$
r_1, r_2	rotational stiffness at the left and right ends, respectively
$R_c, R_i, \quad i = 1, 2$	dimensionless rotational parameters
t	time
t_c	translational stiffness at the point $\bar{x} = c$
t_1, t_2	translational stiffness at the left and right ends, respectively
T	kinetic energy
$T_c, T_i, \quad i = 1, 2$	dimensionless translational parameters
U	strain energy
x	dimensionless abscissa
$\bar{x}$	abscissa
ρ	mass density
ω	radian frequency
$\Omega = \omega l^2 \sqrt{\rho A/EI}$	dimensionless natural frequency parameter

APPENDIX 2

First members of the set of polynomials $\left\{p_i^{(k)}(x)\right\}$ and $\left\{q_j^{(k)}(x)\right\}$ for all possible combinations of classical boundary conditions and with intermediate elastic restraints

Classical boundary conditions and intermediate elastic restraints at $x = c_l$	$p_1^{(1)}$	$q_1^{(1)}$	$p_1^{(2)}$	$q_1^{(2)}$
S–S	1	x	1	$x-1$
S–F	1	x	1	1
F–F	1	1	1	1
C–C	x	x	$x-1$	$x-1$
C–S	x	x	1	$x-1$
C–F	x	x	1	1
Classical boundary conditions with intermediate point support at $x = c_l$				
S–S	1	$x-c_l$	1	$(x-1)(x-c_l)$
S–F	1	$x-c_l$	1	$x-c_l$
F–F	1	$x-c_l$	1	$x-c_l$
C–C	x	$x(x-c_l)$	$x-1$	$(x-1)(x-c_l)$
C–S	x	$x(x-c_l)$	1	$x-1$
C–F	x	$x(x-c_l)$	1	1