AUTOMORPHISMS AND ISOMORPHISM OF QUANTUM GENERALIZED WEYL ALGEBRAS

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Abstract. We classify up to isomorphism the quantum generalized Weyl algebras and determine their automorphism groups in all cases in a uniform way, including where the parameter q is a root of unity, thereby completing the results obtained by [Bavula, V. V.; Jordan, D. A. Isomorphism problems and groups of automorphisms for generalized Weyl algebras. Trans. Amer. Math. Soc. 353 (2001), no. 2, 769–794] and [Richard, L.; Solotar, A. Isomorphisms between quantum generalized Weyl algebras. J. Algebra Appl. 5 (2006), no. 3, 271–285]

Introduction

If k is a field, $q \in k \setminus \{0,1\}$, D is one of $k[h]$ or $k[h^{\pm 1}]$ and $a \in D \setminus 0$, the *quantum generalized Weyl algebra* $A = \mathcal{A}(D,q,a)$ is the k-algebra freely generated by letters y, x, h (and its inverse h^{-1} when $D = k[h^{\pm 1}]$) subject to the relations

$$hy = qyh, \qquad xh = qhx, \qquad yx = a(h), \qquad xy = a(qh).$$

This construction, a special case of a general one introduced by V.V. Bavula in [3], provides an interesting class of algebras containing the quantum plane, the quantum Weyl algebra, certain well-known quotients of the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ related to the primitive quotients of the classical enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$, studied by J. Alev and F. Dumas in [2], some invariant subalgebras of these under finite group actions, the so-called ambiskew polynomial rings, and several other examples. They have notably appeared also under the name of non-commutative deformations of Kleinian singularities of type A in work of T.J. Hodges [9] and are, in fact, somewhat ubiquitous.

It is the purpose of this paper to present a solution to the problem —initially posed by Hodges in [9] in general— of determining which pairs of quantum generalized Weyl algebras are isomorphic, and to describe the automorphism groups of these algebras. Our first result solves the isomorphism problem:

Theorem A. *The two algebras $A_1 = \mathcal{A}(D,q_1,a_1)$ and $A_2 = \mathcal{A}(D,q_2,a_2)$, with a_1 and a_2 non-units, are isomorphic if and only if $q_2 \in \{q_1, q_1^{-1}\}$ and there exist a unit $\alpha \in D$, a non-zero scalar $\beta \in k$ and $\varepsilon \in \{\pm 1\}$ such that $a_2(h) = \alpha a_1(\beta h^{\varepsilon})$. If $D = k[h]$ then necessarily $\varepsilon = 1$ and $\alpha \in k^\times$.*

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Let us single out two interesting special cases of this theorem. If $q \in k \setminus \{0,1\}$, the *quantum plane* is the algebra $M_q = k\langle x, y : yx = qxy\rangle$ and the *quantum Weyl algebra* is the algebra $A_q^1(k) = k\langle x, y : yx - qxy = 1\rangle$. As $M_q \cong \mathcal{A}(k[h], q, h)$ and $A_q^1(k) \cong \mathcal{A}(k[h], q, h-1)$, Theorem A tells us that for all q_1, $q_2 \in k \setminus \{0,1\}$ we have $M_{q_1} \cong M_{q_2}$ iff $A^1_{q_1}(k) \cong A^2_{q_2}(k)$ iff $q_2 \in \{q_1, q_1^{-1}\}$. This characterization of isomorphisms between quantum planes and between quantum Weyl algebras had been established when the parameters are *not* roots of unity by Alev and Dumas in [1].

Theorem B. *Let $A = \mathcal{A}(k[h], q, a)$ be a quantum generalized Weyl algebra with a not a unit, let $N = \deg a$ and write $a = \sum_{i=0}^{N} a_i h^i$, and let $g = \gcd\{i-j : a_i a_j \neq 0\}$ and $C_g \subseteq k^\times$ be the subgroup of gth roots of unity; if a is a monomial, we make the convention that $g = 0$ and $C_g = k^\times$. If $(\gamma, \mu) \in C_g \times k^\times$, there is an automorphism $\eta_{\gamma,\mu} : A \to A$ such that $\eta_{\gamma,\mu}(y) = \mu y$, $\eta_{\gamma,\mu}(h) = \gamma h$ and $\eta_{\gamma,\mu}(x) = \mu^{-1}\gamma^N x$. The set $G = \{\eta_{\gamma,\mu} : (\gamma,\mu) \in C_g \times k^\times\}$ is a subgroup of $\operatorname{Aut}(A)$ isomorphic to $C_g \times k^\times$.*

(i) *If $q \neq -1$, we in fact have $\operatorname{Aut}(A) = G$, and*
(ii) *if $q = -1$, there is a right split short exact sequence of groups*

$$1 \longrightarrow G \hookrightarrow \operatorname{Aut}(A) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

The cyclic group $\mathbb{Z}/2\mathbb{Z}$ appearing here is generated by the image of the involutory automorphism $\Omega : A \to A$ such that $\Omega(y) = x$, $\Omega(h) = -h$ and $\Omega(x) = y$.

To state the analog of this theorem for the case where $D = k[h^{\pm 1}]$, we need a definition. We say that a Laurent polynomial $f \in k[h^{\pm 1}]$ is *symmetric* if there exist $l \in \mathbb{N}$, $\gamma \in k$ and $\delta \in k$ such that $\delta f(h) = h^l f(\gamma h^{-1})$.

Theorem C. *Let $A = \mathcal{A}(k[h^{\pm 1}], q, a)$ be a quantum generalized Weyl algebra, with $a = \sum_{i \in I} a_i h^i$ a non-unit in $k[h^{\pm 1}]$, and let $g = \gcd\{i-j : a_i a_j \neq 0\}$ and $C_g \subseteq k^\times$ be the subgroup of gth roots of unity; fix $i_0 \in I$. If $(\gamma, \mu) \in C_g \times k^\times$, there is an automorphism $\eta_{\gamma,\mu} : A \to A$ such that $\eta_{\gamma,\mu}(y) = \mu y$, $\eta_{\gamma,\mu}(h) = \gamma h$ and $\eta_{\gamma,\mu}(x) = \mu^{-1}\gamma^{i_0} x$. The set $G = \{\eta_{\gamma,\mu} : (\gamma,\mu) \in C_g \times k^\times\}$ is a subgroup of $\operatorname{Aut}(A)$ isomorphic to $C_g \times k^\times$. Consider the subgroup K of $\operatorname{Aut} A$ of all automorphisms η such that $\eta(h)$ is a scalar multiple of h.*

(i) *If a is symmetric then $\operatorname{Aut} A \cong K \ltimes \mathbb{Z}/2\mathbb{Z}$ and, if not, $\operatorname{Aut}(A) = K$.*
(ii) *If $q = -1$ then $K \cong G \ltimes \mathbb{Z}/2\mathbb{Z}$ and otherwise $K = G$.*

To avoid complicating statements and proofs, we have chosen to postpone to the end of the paper the results in the line of these three theorems for the case in which the polynomial a is invertible in the ring D.

The results corresponding to these theorems for the case of classical generalized Weyl algebras —in which "there is no q"— have been given by Bavula and Jordan in [4] and the quantum case as above but with q not a root of unity has been solved for $D = k[h]$ by L. Richard and A. Solotar in [12] and for $D = k[h^{\pm 1}]$ by Bavula and Jordan also in [4].

Our approach makes no hypothesis on the scalar parameter, and it is interesting to remark one key point which makes the difference. In [1], Alev and Dumas attached to a k-algebra Λ the subgroup $G(\Lambda) = (\Lambda^\times)' \cap k^\times \subseteq k^\times$ —where $(\Lambda^\times)'$ is the derived subgroup of the group of units of Λ— and showed that if $k_q(x, y)$ denotes the quantum Weyl field we have $G(k_q(x, y)) = \langle q\rangle$, the cyclic subgroup generated by q. Richard and Solotar prove that the fraction field of a quantum generalized Weyl algebra $A = \mathcal{A}(q, a)$ is isomorphic to $k_q(x, y)$ and, since in their situation q is not a root of unity, notice that one can recover q from A, up to inversion, as one of the two generators of $G(\operatorname{Frac} A)$. If instead q has finite order in $k^\times$, the subgroup $\langle q\rangle$ has many generators and their approach cannot get started. We replace below their consideration of $G(\operatorname{Frac} A)$ by a detailed study of certain derivations of A and their eigenvalues, and this avoids that difficulty: in a very loose sense, this is like "taking the logarithm" of $G(\operatorname{Frac} A)$. Similar difficulties with parameters of finite order appear when trying to classify other classes of algebras, like that of down-up algebras introduced by G. Benkart and T. Roby in [5], and one can hope that similar ideas may possibly overcome these too.

In [13], together with A. Solotar, we computed the Hochschild cohomology of quantum generalized Weyl algebras defined over $k[h]$. The results of the present paper arose in the process of studying the algebraic structure of the cohomology —the cup product and the Gerstenhaber bracket.

We finish by emphasizing that the theorems stated above, as well as all the related work we referred to, exclude the case where $q = 1$, which is precisely that in which the algebras are commutative. When $D = k[h]$, the problem of determining the automorphisms is that of finding the automorphism group of the affine surface $\operatorname{Spec} k[x, y, h]/(xy - a(h))$. L. Makar-Limanov gave in [10] explicit generators for these groups and recently J. Blanc and A. Dubouloz showed in [6] that they have an amalgamated product structure similar to that of $\operatorname{Aut}(k[x, y])$ described by the classical theorems of L. Makar-Limanov, H.W.E. Jung and W. van der Kulk, and that the surfaces are classified under isomorphism exactly as in Theorem A. While Makar-Limanov deals systematically with locally nilpotent derivations, as we do, the methods with which these commutative results are obtained are quite different from ours —the work [6], for example, is a paper on algebraic geometry.

1. Preliminaries

We fix a field k of characteristic zero and identify $\mathbb{Q}$ with its prime field. If $q \in k$, D is one of $k[h]$ or $k[h^{\pm 1}]$ and $a \in D$, the *quantum generalized Weyl algebra* $A = \mathcal{A}(D, q, a)$ is the k-algebra freely generated by letters y, x, h (and h^{-1} when $D = k[h^{\pm 1}]$) subject to the relations;

$$hy = qyh, \qquad xh = qhx, \qquad yx = a(h), \qquad xy = a(qh).$$

The set $\{y^i h^j x^k : ik = 0\}$ is a k-basis of A; we call its elements *standard monomials*. The algebra is a domain iff $q \neq 0$ and $a \neq 0$: we will always assume this is the case. We will moreover suppose throughout that $q \neq 1$, thereby excluding all the commutative examples and no other.

We write $a = \sum_{i=M}^{N} a_i h^i$ with $a_M a_N \neq 0$. Notice that if a is a unit, that is, if $M = N = 0$ when $D = k[h]$ or $M = N$ when $D = k[h^{\pm 1}]$, then A is isomorphic to the Ore extension $D[x^{\pm 1}, \sigma]$. As the results and methods needed to deal with this case are different, we will do this separately at the end of this paper.

The algebra A is $\mathbb{Z}$-graded in a unique way so that the degrees of y, h and x are 1, 0, and -1, respectively; we refer to the degree $|a|$ of an homogeneous element $a \in A$ in this grading as its *weight*, and extend this convention to related contexts. For $r \in \mathbb{Z}$, we let $A^{(r)}$ be the homogeneous component of A of degree r; we have $A^{(0)} = k[h]$ and, for each $r \in \mathbb{N}$, $A^{(r)} = y^r k[h]$ and $A^{(-r)} = k[h]x^r$.

Let $V = \bigoplus_{i\in\mathbb{Z}} V_i$ be a graded vector space and let $d : V \to V$ a not necessarily homogeneous linear endomorphism. We say that d is *locally finite* if for each $v \in V$ the cyclic subspace $\langle v\rangle_d$ of V generated by d and v is finite-dimensional, and that d is *locally nilpotent* if for each $v \in V$ we have $d^i(v) = 0$ for $i \gg 0$. It is enough to check these conditions on homogeneous elements of V.

Lemma 1.1. *Suppose $d = d_1 + \cdots + d_l$ with $d_1, \ldots, d_l : V \to V$* homogeneous *endomorphisms of V of degrees $\alpha_1, \ldots, \alpha_l$ such that $\alpha_1 < \cdots < \alpha_l$. If d is locally-finite then d_1 and d_l are locally-finite.* □

An homogeneous endomorphism of V of non-zero degree is locally finite iff it is locally nilpotent. It follows that if in the lemma we have $\alpha_l \neq 0$ then in fact d_l is locally nilpotent, and similarly for d_1.

Let now A be a graded algebra. If $d : A \to A$ is a homogeneous derivation of positive degree which is locally nilpotent, there is a function $\deg_d : A \setminus 0 \to \mathbb{N}$ such that for each $u \in A \setminus 0$ we have $\deg_d(u) = \max\{r \in \mathbb{N}_0 : d^r(u) \neq 0\}$. It is straightforward to check that $\deg_d$ is such that for all $u, v \in A$ we have

$$\deg_d(u + v) \le \max\{\deg_d(u), \deg_d(v)\},$$
$$\deg_d(uv) = \deg_d(u) + \deg_d(v).$$

It follows from this that the subalgebra $\ker d$ is *factorially closed*: if $u, v \in A \setminus 0$ then $d(uv) = 0 \implies d(u) = d(v) = 0$. In particular, d vanishes on the units of A.

In contexts where this makes sense, we will write $x \doteq y$ to mean that y is a non-zero scalar multiple of y.

2. Derivations

Let $A = \mathcal{A}(D, q, a)$ be a quantum generalized Weyl algebra. If $u_1, u_2, u_3 \in A$, we write $u_1\frac{\partial}{\partial y} + u_2\frac{\partial}{\partial h} + u_3\frac{\partial}{\partial x}$ the unique derivation $A \to A$ whose values at y, h and x are u_1, u_2 and u_3, respectively, *assuming there is one*.

Lemma 2.1. *The algebra A has no non-zero locally nilpotent homogeneous derivations.*

Proof. Let $d : A \to A$ be a locally nilpotent homogeneous derivation. Suppose first that $D = k[h^{\pm1}]$. As we observed above, d vanishes on h and h^{-1} because they are units, so $d(yx) = d(a) = 0$ and therefore $d(y) = d(x) = 0$: we see that $d = 0$.

Let now D be $k[h]$ and r be the weight of d. We will assume that $r > 0$; if we had $r < 0$ the same reasoning would apply, and the situation is even simpler if $r = 0$. There are homogeneous elements of positive weight in $\ker d$, so there exist $s \in \mathbb{N}$ and $u \in k[h]$ such that $d(y^s u) = 0$. Since $\ker d$ is factorially closed, this implies that in fact $d(y) = 0$. On the other hand, there is a polynomial $p \in k[h]$ such that $d(h) = y^r p$, and from the relation $hy = qyh$ we see that $y^r py = qy^{r+1}p$, so that $\sigma(p) = qp$: it follows from this that we can write $p = p_1 h$ for some $p_1 \in k[h]$. If $k \ge 0$, then $d(A_k h) \subseteq A_{k+r}h$: indeed, if $f \in k[h]$ we have

$$d(y^k fh) = y^k d(f)h + y^k f d(h) = y^k d(f)h + y^k f y^r p_1 h \in A_{k+r}h$$

because $d(f) \in A_r$. This tells us that $d^i(h) \in A_{ir}h$ for all $i \ge 0$. If $i_0 = \deg_d(h)$, then $0 \neq d^{i_0}(h) \in A_{i_0 r}h \cap \ker d$ and, since $\ker d$ is factorially closed, $d(h) = 0$. An immediate consequence of this is that $yd(x) = d(yx) = d(a) = 0$, so also $d(x) = 0$, and we see that $d = 0$, as we wanted. □

That D is $k[h]$ or $k[h^{\pm 1}]$ is important in this lemma, for these two algebras have very few locally nilpotent derivations. Let us exhibit an example with $D = k[h_1, h_2]$ where its conclusion does not hold. We take $q \in k \setminus \{0, 1\}$ a root of unity of order $e > 1$, $\sigma : D \to D$ the automorphism such that $\sigma(h_i) = qh_i$ for $i \in \{1, 2\}$, an arbitrary $a \in D$ and consider the algebra $A = \mathcal{A}(k[h_1, h_2], \sigma, a)$. There is a unique derivation $\tilde{d} : D \to D$ such that $\tilde{d}(h_1) = h_2$ and $\tilde{d}(h_2) = 0$, and it is locally nilpotent, and using this it is easy to check that for each $r > 0$ there is a locally nilpotent derivation $d_r : A \to A$ with $d_r(y) = 0$, $d_r(h_1) = y^{re}h_2$, $d_r(h_2) = 0$ and $d_r(x) = y^{re-1}\tilde{d}(a)$. Since d_r is clearly homogeneous, we see that the conclusion of the lemma does not apply to A.

Corollary 2.2. *The locally finite derivations of A are homogeneous of weight zero.*

Proof. Let $d : A \to A$ be a locally finite derivation. Since A is finitely generated, there are non-zero homogeneous derivations $d_1, \dots, d_l : A \to A$ of strictly increasing weights such that $d = d_1 + \cdots + d_l$. The weight of d_l cannot be positive, for then d_l would be locally nilpotent —because d is locally finite— and the lemma would imply that $d_l = 0$; similarly, the weight of d_1 cannot be negative. It follows that d itself is homogeneous of weight zero. □

Proposition 2.3. *Let $d : A \to A$ be a locally finite derivation, and consider the derivation $\xi = y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}$.*

(i) *If a is not a monomial then d is a scalar multiple of ξ.*

(ii) *If a is a monomial then d is a linear combination of ξ and $\tau = h\frac{\partial}{\partial h} + Nx\frac{\partial}{\partial x}$.*

All locally finite derivations are diagonalizable with the standard monomials as eigenvectors and, in particular, they commute.

We will refer to $\xi : A \to A$ in what follows as the *Eulerian derivation* of A. It is easy to check that its eigenvalues are exactly the integers, and that for each $r \in \mathbb{Z}$ the eigenspace of ξ corresponding to r is precisely $A^{(r)}$, the homogeneous component of A of weight r.

Proof. According to Corollary 2.2 the derivation d is of weight zero, so there are polynomials p_1, p_2, $p_3 \in D$ such that $d = yp_1\frac{\partial}{\partial y} + p_2\frac{\partial}{\partial h} + p_3x\frac{\partial}{\partial x}$. In particular d restricts to a locally finite derivation $D \to D$, and therefore this restriction has to be of the form $(\alpha h + \beta)\frac{\partial}{\partial h}$, with α, $\beta \in k$. Looking at the coefficients of y in both sides of the equality $d(hy) = qd(yh)$, we see that in fact $\beta = 0$.

There is a sequence $(g_i)_{i \geq 0}$ in D such that $g_0 = 1$, $d^i(y) = yg_i$ and $g_{i+1} = p_1g_i + \alpha g_i'h$ for all $i \geq 0$. If $D = k[h]$ we have $\deg g_i = i \deg p_1$ and the local finiteness of d implies that $p_1 \in k$; if $D = k[h^{\pm 1}]$ we reach the same conclusion by considering the degree of the first or last monomials of the g_i.

Applying d to both sides of the equality $yx = a$, we see that $a\sigma^{-1}(p_1 + p_3) = \alpha a'h$, which is possible only if $p_3 \in k$. If we now solve this equation for the three scalars p_1, α and p_3 we obtain the claims (i) and (ii) of the statement. The last claim, finally, can be proved directly by inspection. □

Since the dimension of the vector space of locally finite derivations of an algebra is invariant under isomorphisms, the above Proposition 2.3 has the following consequence:

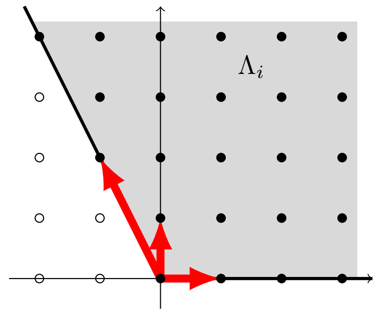

FIGURE 1. The semigroup Λ_i.

Corollary 2.4. *If $A_1 = \mathcal{A}(D, q_1, a_1)$ and $A_2 = \mathcal{A}(D, q_2, a_2)$ are two isomorphic quantum generalized Weyl algebras, then either both a_1 and a_2 are monomials or neither of them are. Moreover if one of them is a unit the other one also.*

Proof. The first claim is an immediate consequence of the proposition. On the other hand, it is easy to see that a_1 is a unit if and only if $A_1^\times/k^\times$ is a non-trivial group; in that case, it is isomorphic to $\mathbb{Z}$ when $D = k[h]$ and to $\mathbb{Z}^2$ when $D = k[h^{\pm 1}]$. As this quotient is invariant under isomorphisms of k-algebras, the second claim follows. □

We are now in position to establish the key fact that will allow us to describe the isomorphisms and automorphisms of our algebras in the next section:

Proposition 2.5. *Let $A_1 = \mathcal{A}(D, q_1, a_1)$ and $A_2 = \mathcal{A}(D, q_2, a_2)$ two quantum generalized Weyl algebras with a_1 and a_2 not units, and let ξ_1 and ξ_2 be their respective Eulerian derivations. If $\eta : A_1 \to A_2$ is an isomorphism, then $\eta \circ \xi_1 \circ \eta^{-1}$ is a scalar multiple of ξ_2.*

Proof. Let us write $\xi_2' = \eta \circ \xi_1 \circ \eta^{-1}$, which is a locally finite derivation of A_2. If a_2 is not a monomial, then the first part of Proposition 2.3 immediately implies that ξ_2' must be a scalar multiple of ξ_2. We need only consider, then, the case where $a_2 = h^{N_2}$ is a monomial and therefore, by our assumption that a_2 is not a unit, that $N_2 > 0$ and $D = k[h]$. We have, then, a derivation $\tau_2' = \eta \circ \tau_1 \circ \eta^{-1}$ with the notation of Proposition 2.3. The second part of that proposition implies that there is a matrix $M = \left(\begin{smallmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{smallmatrix}\right) \in \mathrm{GL}_2(k)$ such that

$$\begin{pmatrix} \xi_2' \\ \tau_2' \end{pmatrix} = M \begin{pmatrix} \xi_2 \\ \tau_2 \end{pmatrix}. \tag{2.1}$$

If $i \in \{1, 2\}$, the derivations ξ_i and τ_i are simultaneously diagonalizable with integer eigenvalues, so there is a direct sum decomposition $A_i = \bigoplus_{\lambda \in \mathbb{Z}^2} A_i^\lambda$ with $A_i^\lambda = \{u \in A_i : \xi_i(u) = \lambda_1 u, \tau_i(u) = \lambda_2 u\}$ for all $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$, which is a $\mathbb{Z}^2$-grading. The set $\Lambda_i = \{\lambda \in \mathbb{Z}^2 : A_i^\lambda \neq 0\}$ is a submonoid of $\mathbb{Z}^2$, and it is generated as such by $(1, 0)$, $(0, 1)$ and $(-1, N_i)$ because y, h and x are, respectively, of those degrees. Morover, the vectors $(1, 0)$ and $(-1, N_i)$ are the unique indecomposable elements of Λ_i which are not interior to the convex hull of Λ_i; see Figure 1.

If $\lambda \in \mathbb{Z}^2$ and $u \in A_2^\lambda$, we see from (2.1) that $\eta^{-1}(u) \in A_1^{M\lambda}$: it follows that η^{-1} restricts to an isomorphism $A_2^\lambda \to A_1^{M\lambda}$ and, therefore, the linear map $\lambda \in \mathbb{Z}^2 \mapsto M\lambda \in \mathbb{Z}^2$ induced by M restricts to an isomorphism $\phi : \Lambda_2 \to \Lambda_1$. As $(1, 0)$

and $(0,1)$ are in Λ_2 and their images under this map are in $\mathbb{Z}^2$, we see that M has integral coefficients; considering the inverse map η^{-1} we see that the same applies to M^{-1}, so that $M \in \mathrm{GL}_2(\mathbb{Z})$. Since the restriction ϕ must preserve indecomposable non-interior points, it maps the vectors $(1,0)$ and $(-1,N_2)$ to $(1,0)$ and $(-1,N_1)$ in some order. Exploring the two possibilities shows that $N_1 = N_2$ and that M is either the identity matrix or $\left(\begin{smallmatrix}-1 & 0\\ N_1 & 1\end{smallmatrix}\right)$. In any case, we see that $\xi_2' \doteq \xi_2$. □

We remark that the result of this proposition is false when a_1 (and then a_2) is a unit. For example, there is an automorphism $\eta : A(D,q,1) \to A(D,q,1)$ such that $\eta(x) = x$, $\eta(h) = hx$, $\eta(y) = y$, and it does not preserve the Eulerian derivation. This fact is what forces us to consider this case separately.

3. Automorphisms and isomorphisms

We have shown that isomorphisms of quantum generalized Weyl algebras preserve, up to scalars, their Eulerian derivations. This fact evinces a non-trivial rigidity of these algebras which strongly restricts the form of isomorphisms between them:

Proposition 3.1. *If $\eta : A_1(D,q_1,a_1) \to A_2(D,q_2,a_2)$ is an isomorphism of quantum generalized Weyl algebras with a_1 and a_2 not units, then there exist $\gamma \in k$, $\varepsilon \in \{\pm 1\}$ and μ, $\nu \in D^\times$ such that $\eta(h) = \gamma h^\varepsilon$ and*

(♣) *either $\eta(y) = y\mu$ and $\eta(x) = \nu x$*

(♠) *or $\eta(y) = \mu x$ and $\eta(x) = y\nu$.*

If $D = k[h]$ then necessarily $\varepsilon = 1$ and μ, $\nu \in k^\times$.

Proof. According to Proposition 2.5, there exists a non-zero scalar $\lambda \in k$ such that

$$\eta \circ \xi_1 = \lambda \, \xi_2 \circ \eta \tag{3.1}$$

and, therefore, for each $r \in \mathbb{Z}$ the subspace $\eta(A_1^{(r)})$ is the eigenspace of ξ_2 corresponding to the eigenvalue r/λ; in particular, $D = \ker \xi_2 = \eta(\ker \xi_1) = \eta(D)$ and η restricts to an algebra isomorphism $D \to D$. As ξ_2 has integer eigenvalues, we must have $\lambda \in \{\pm 1\}$.

Let us suppose that $\lambda = 1$; the other possibility can be handled similarly and will lead to the second possibility (♠) in the statement. There exists an $f \in D$ such that $y = \eta(yf) = \eta(y)\eta(f)$: since $\eta(f) \in D$, this implies that $\eta(y)$ generates $A_2^{(1)}$ as a right D-module. This module is free of rank one and y and $\eta(y)$ are two generators: it follows that there is an unit $\mu \in D$ such that $\eta(y) = y\mu$. The same argument applied to x shows that there is also an unit $\nu \in D$ such that $\eta(x) = \nu x$.

Consider the case $D = k[h]$. As the restriction $\eta : D \to D$ is an isomorphism, we have $\eta(h) = \gamma h + \delta$ for some $\gamma \in k \setminus 0$ and $\delta \in k$. Since $hy = q_1 yh$ in A_1, we have $(\gamma h + \delta)\mu y = q_1 \mu y (\gamma h + \delta)$ in A_2. We conclude that $\delta = 0$ and the proposition follows. In the case $D = k[h^{\pm 1}]$ we must have $\eta(h) = \gamma h^\varepsilon$ for some $\varepsilon \in \{\pm 1\}$ since h generates $D^\times / k^\times \cong \mathbb{Z}$. □

From the locally nilpotent derivation d of the algebra $A = \mathcal{A}(k[h_1,h_2],\sigma,a)$ considered in the example given in Section 2, we obtain, by exponentiation, a 1-parameter family of automorphisms $\eta_t : A \to A$ such that

$$\eta_t(y) = y, \qquad \eta_t(h_1) = h_1 + t y^{re} h_2,$$
$$\eta_t(x) = y^{-1} a(h_1 + t y^{re} h_2, h_2) \qquad \eta_t(h_2) = h_2,$$

which is neither homogeneous not linear. This shows that the conclusion of the proposition above does not hold when $D = k[h_1, h_2]$ and, in fact, as this construction can be carried out starting from any locally nilpotent derivation of D —in this case the group of automorphisms is much larger.

At this point, we have everything we need to prove the theorems from the introduction.

Proof of Theorem A. The sufficiency of the condition can be checked by a straightforward verification, which we omit, so we only prove the necessity.

Let $\eta : A_1 \to A_2$ be an isomorphism. From Proposition 3.1 we know there is $\gamma \in k$, $\varepsilon \in \{\pm 1\}$ and μ, $\nu \in D^\times$ such that $\eta(h) = \gamma h^\varepsilon$ and (♣) either $\eta(y) = y\mu$ and $\eta(x) = \nu x$ (♠) or $\eta(y) = \mu x$ and $\eta(x) = y\nu$; if $D = k[h]$ then moreover $\varepsilon = 1$ and μ, $\nu \in k^\times$. If we are in the first case, we have

$$\sigma^{-1}(\mu\nu)a_2(h) = y\mu\nu x = \eta(yx) = \eta(a_1(h)) = a_1(\gamma h^\varepsilon)$$

and

$$\gamma q_2^\varepsilon y h^\varepsilon \mu = \gamma h^\varepsilon y \mu = \eta(hy) = q_1 \eta(yh) = \gamma q_1 y \mu h^\varepsilon.$$

As $\sigma^{-1}(\mu\nu) \in D^\times$, the necessity of the conditions is clear. □

Proof of Theorem B. The verification that the set G is indeed a subgroup of $\mathrm{Aut}(A)$ is routine, so we only check (i) and (ii). Let $\eta : A \to A$ be an automorphism. According to Proposition 3.1, there are γ, μ, $\nu \in D^\times = k \setminus 0$ such that $\eta(h) = \gamma h$ and either (♣) $\eta(y) = y\mu$ and $\eta(x) = \nu x$, or (♠) $\eta(y) = \mu x$ and $\eta(x) = y\nu$. If (♣) holds, applying η to both sides of the equality $yx = a(h)$ shows that

$$a_i \neq 0 \implies \gamma^i = \mu\nu, \tag{3.2}$$

so that $\gamma^{i-j} = 1$ whenever $a_i a_j \neq 0$ and, in consequence, $\gamma \in C_g$. Additionally, (3.2) tells us that $\nu = \mu^{-1}\gamma^N$ and then we see that $\eta = \eta_{\gamma,\mu} \in G$.

If instead (♠) holds, applying η to the equality $hy = qyh$ shows that $q^2 = 1$ so that in fact $q = -1$. This means that when $q = -1$ the alternative (♠) does not occur, and $\mathrm{Aut}(A) = G$. On the other hand, if $q = -1$ there is indeed an automorphism Ω as described in the statement, and $\eta \circ \Omega \in G$ because this composition falls in the case (♣) with which we have already dealt. The subgroup G together with Ω thus generate $\mathrm{Aut}(A)$ in this situation and all the claims in (ii) now follow at once. □

We need two lemmas for the proof of Theorem C; the notation is as in the statement of that theorem.

Lemma 3.2. *The Laurent polynomial a is symmetric if and only if there exists an automorphism $\Omega_{\mathrm{sym}} : A \to A$ such that $\Omega_{\mathrm{sym}}(h) \doteq h^{-1}$.*

Proof. If a is symmetric then there exist $l \in \mathbb{N}$ and γ, $\delta \in k$ such that $\delta a(h) = h^l a(\gamma h^{-1})$. The automorphism Ω_{sym} is defined by

$$\Omega_{\mathrm{sym}}(y) = x, \qquad \Omega_{\mathrm{sym}}(h) = q^{-1}\gamma h^{-1}, \qquad \Omega_{\mathrm{sym}}(x) = \delta q^{-l} y h^{-l}.$$

Conversely, if exists such an automorphism Ω_{sym} then

$$\Omega_{\mathrm{sym}}(y)\Omega_{\mathrm{sym}}(x) = \Omega_{\mathrm{sym}}(yx) = \Omega_{\mathrm{sym}}(a) = a(\gamma h^{-1}),$$

and it is easy to see, applying Proposition 3.1, that the left hand side of this equation is equal to $\delta a(h)h^{-l}$ for some $\delta \in k$ and $l \in \mathbb{Z}$. □

Lemma 3.3. *The parameter q is equal to -1 if and only if there exists an automorphism $\Omega_{-1} : A \to A$ such that $\Omega_{-1}(y) \doteq x$, $\Omega_{-1}(x) \doteq y$ and $\Omega_{-1}(h) \doteq h$.*

Proof. If $q = -1$ then Ω_{-1} is defined by

$$\Omega_{-1}(y) = x, \qquad \Omega_{-1}(h) = qh, \qquad \Omega_{-1}(x) = y.$$

If there exists an automorphism as in the statement then

$$hx \doteq \Omega_{-1}(h)\Omega_{-1}(y) = \Omega_{-1}(hy) = q\Omega_{-1}(yh) = q\Omega_{-1}(y)\Omega_{-1}(h) \doteq qxh,$$

so that $q = q^{-1}$. $\square$

Proof of Theorem C. Let $\eta \in \mathrm{Aut}(A)$. From Proposition 3.1, we know that $\eta(h) \doteq h^{\pm 1}$. If $\eta(h) \doteq h^{-1}$ then the first lemma above shows that a is symmetric and, moreover, that $\eta \circ \Omega_{\mathrm{sym}} \in K$. If a is not symmetric then we must have $\eta(h) \doteq h$ and therefore $\eta \in K$. This proves part (i) of the theorem.

Assume now that $\eta \in K$. In this case, if $\eta(y) \doteq x$ we must have $q = -1$ and in this case $\eta \circ \Omega_{-1} \in G$. Conversely, if $q \neq -1$ then necessarily $\eta(y) \doteq y$, so that K is as in (ii). $\square$

The theorems stated in the introduction leave untouched the case in which the parameter a of the generalized Weyl algebras $\mathcal{A}(D, a, q)$ is a unit in D. As promised there, we now state and prove the corresponding results for this case.

Theorem D. (i) *Let $A = \mathcal{A}(k[h], a, q)$ with $a \in k^\times$. If $q \neq -1$, let H be the subgroup $\{\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) : z \in \mathbb{Z}\}$ and let H be $\{\left(\begin{smallmatrix} \pm 1 & z \\ 0 & 1 \end{smallmatrix}\right) : z \in \mathbb{Z}\}$ otherwise. There is then a right-split short exact sequence of groups*

$$0 \longrightarrow (k^\times)^2 \longrightarrow \mathrm{Aut}(A) \longrightarrow H \longrightarrow 1$$

(ii) *Let $A = \mathcal{A}(k[h^{\pm 1}], a, q)$ with $a = \alpha h^N$ and $N \in \mathbb{Z}$. If $q \neq -1$, let H be $\mathrm{SL}_2(\mathbb{Z})$ if $q \neq -1$ and $\mathrm{GL}_2(\mathbb{Z})$ otherwise. There is then a right-split short exact sequence of groups*

$$0 \longrightarrow (k^\times)^2 \longrightarrow \mathrm{Aut}(A) \longrightarrow H \longrightarrow 1$$

Proof. In both cases the algebra A is generated by x and h, because a is a unit.

(i) We proceed exactly as in the beginning of the proof of Proposition 2.5. Given an automorphism $\eta : A \to A$, this constructs a matrix $M \in \mathrm{GL}_2(\mathbb{Z})$ such that for all $\lambda \in \mathbb{Z}^2$ and $u \in A^\lambda$, we have that $\eta^{-1}(u) \in A^{M\lambda}$ and which therefore preserves the subsemigroup $\Lambda \subseteq \mathbb{Z}^2$ which, in this case, is generated by $(\pm 1, 0)$ and $(0, 1)$; we remark that in this situation the semigroup Λ does not have indecomposable elements, so that the argument given in the proof of Proposition 2.5 cannot be continued. In any case, as M preserves Λ, we must have $M = \left(\begin{smallmatrix} \varepsilon & \ell \\ 0 & 1 \end{smallmatrix}\right)$ for some $\varepsilon \in \{\pm 1\}$ and $\ell \in \mathbb{Z}$. Since A^λ is one-dimensional for all $\lambda \in \Lambda$, this implies that

$$\eta^{-1}(x) \doteq x^\varepsilon, \qquad \eta^{-1}(h) \doteq hx^\ell. \tag{3.3}$$

From the relation $xh = qhx$ we see that if $q \neq -1$ we must have $\varepsilon = 1$, so that $M \in H$. In this way we obtain a morphism of groups $\pi : \mathrm{Aut}(A) \to H$, and it is easy to see that it is surjective and right-split —one can use formulas (3.3) to construct a section. The kernel of π, isomorphic to $(k^\times)^2$, can be identified at once.

(ii) There an obvious group homomorphism $\pi : \mathrm{Aut}(A) \to \mathrm{Aut}(A^\times/k^\times)$. Since A is generated by x and h, which are units, the kernel of π is easily seen to be the group of automorphisms which multiply those generators by non-zero scalars and therefore isomorphic to $(k^\times)^2$.

It is easy to see that $A^\times/k^\times$ is an abelian group freely generated by the classes of h and x, so we can identify it with $\mathbb{Z}^2$. If $\eta : A \to A$ is an automorphism and $M = \pi(\eta) = \left(\begin{smallmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{smallmatrix}\right)$, we must have

$$\eta(h) \doteq h^{m_{1,1}} x^{m_{2,1}}, \qquad \eta(x) \doteq h^{m_{1,2}} x^{m_{2,2}}. \tag{3.4}$$

From the q-commutation relation between h and x, we see that necessarily $\det M = 1$ if $q \neq 1$; this means that, in any case, M is in the subgroup H and π can be corestricted to a morphism $\mathrm{Aut}(A) \to H$. Using formulas (3.4) we can easily construct a section for this map, thereby finishing the proof of the theorem. □

An argument completely parallel to that of this proof establishes the following final result. We omit the details.

Theorem E. *Let D be $k[h]$ or $k[h^{\pm 1}]$, let a_1, $a_2 \in D$ be two units, and let q_1, $q_2 \in k \setminus \{0, 1\}$. The algebras $\mathcal{A}(D, a_1, q_1)$ and $\mathcal{A}(D, a_2, q_2)$ are isomorphic iff $q_2 \in \{q_1, q_1^{-1}\}$.* □

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