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Intrinsic complexity estimates in polynomial optimization☆

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ABSTRACT

It is known that point searching in basic semialgebraic sets and the search for globally minimal points in polynomial optimization tasks can be carried out using $(s\,d)^{O(n)}$ arithmetic operations, where n and s are the numbers of variables and constraints and d is the maximal degree of the polynomials involved.

Subject to certain conditions, we associate to each of these problems an intrinsic system degree which becomes in worst case of order $(n\,d)^{O(n)}$ and which measures the intrinsic complexity of the task under consideration.

We design non-uniform deterministic or uniform probabilistic algorithms of intrinsic, quasi-polynomial complexity which solve these problems.

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1. Introduction

We develop uniform bounded error probabilistic and non-uniform deterministic algorithms of intrinsic, quasi-polynomial complexity for the point searching problem in basic semialgebraic sets and for the search of isolated local and global minimal points in polynomial optimization. The semialgebraic sets and optimization problems have to satisfy certain well motivated geometric restrictions which allow to associate with them an intrinsic *system degree* (see Section 3.2) that controls the complexity of our algorithms and constitutes the core of their intrinsic character. The algorithms we are going to design will become then polynomial in the length of the extrinsic description of the problem under consideration and its system degree (we take only arithmetic operations and comparisons in $\mathbb{Q}$ into account at unit costs). The idea is that the system degree constitutes a geometric invariant which measures the intrinsic "complexity" of the *concrete* problem under consideration (not of all problems like a worst case complexity). In worst case the sequential time complexity will be of order $\binom{s}{p}(n\,d)^{O(n)}$ (respectively $(n\,d)^{O(n)}$), where n is the number of variables and d the maximal degree of the polynomials occurring in the problem description, s their number and $1 \leq p \leq n$ the maximal codimension of the real varieties given by the active constraints. We shall suppose that these polynomials are represented as outputs of an essentially division-free arithmetic circuit in $\mathbb{Q}[X_1, \ldots, X_n]$ of size L (here, we mean by essentially division-free that only divisions by rational numbers are allowed). The (sequential) complexity of our algorithms is then of order $L\binom{s}{p}n^{O(p)}d^{O(1)}\delta^3$ (respectively $L(n\,d)^{O(1)}\delta^3$), where δ is the intrinsic system degree which in worst case becomes of order $(n\,d)^{O(n)}$. We call this type of complexity bounds *intrinsic* and *quasi-polynomial*.

For the problem of deciding the consistency of a given set of inequality constraints and of finding, in case the answer is positive, a real algebraic sample point for each connected component of the corresponding semialgebraic set, sequential time bounds of simply exponential order, e.g. $(sd)^{O(n)}$, are exhibited in Grigor'ev and Vorobjov [25], Canny [11], Renegar [38,39], Heintz, Roy and Solernó [29], Basu, Pollack and Roy [6] and the book [7]. Such bounds can also be derived from efficient quantifier elimination algorithms over the reals [29,39,6,7]. Since two alternating blocks of quantifiers become involved, one would expect at first glance that only a $(sd)^{O(n^2)}$ time complexity bound could be deduced from efficient real quantifier elimination for polynomial optimization problems. But, at least for global optimization, one can do much better with an $s^{2n+1}d^{O(n)}$ sequential time bound (see [7, Algorithm 14.46]). For particular global polynomial optimization problems the constant hidden in this bound can be made precise and the algorithms become implementable (see Greuet and Safey El Din [24,23], Safey El Din [42,41], Greuet [21] and Jeronimo and Perrucci [30]). Accurate estimations for the minima are contained in [31]. The main difference with our approach is that these papers contain *extrinsic* worst case complexity bounds whereas our bounds are *intrinsic*.

Nevertheless, this article does not focus on the improvement of known worst case complexity bounds in optimization theory. Our aim is to exhibit classes of point searching problems in semialgebraic sets and polynomial optimization problems where it makes sense to speak about intrinsic complexity of solution algorithms. This is the reason why we put the accent on geometrical aspects of these problems. The algorithms become then borrowed from [20] (see also [18,19,28,16,10]) and, in particular, from [3,4] (or alternatively from [43]).

1.1. Notions and notations

We shall freely use standard notions, results and notations from algebraic and semialgebraic geometry, commutative algebra and algebraic complexity theory which can be found e.g. in the books [35,44,34,9].

Let $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the fields of the rational, real and complex numbers, respectively, let $X_1, \ldots, X_n$ be indeterminates over $\mathbb{C}$ and let $F_1, \ldots, F_p$, $1 \leq p \leq n$, be polynomials of $\mathbb{R}[X_1, \ldots, X_n]$ defining a closed, $\mathbb{R}$-definable subvariety S of the n-dimensional complex affine space $\mathbb{C}^n$.

We denote by $S_{\mathbb{R}} := S \cap \mathbb{R}^n$ the real trace of the complex variety S. We shall use also the following notations:

$$\{F_1 = 0, \ldots, F_p = 0\} := S \quad \text{and} \quad \{F_1 = 0, \ldots, F_p = 0\}_{\mathbb{R}} := S_{\mathbb{R}}.$$

For a given polynomial $Q \in \mathbb{R}[X_1, \ldots, X_n]$ we denote by S_Q and $(S_{\mathbb{R}})_Q$ the sets of points of S and $S_{\mathbb{R}}$ at which Q does not vanish and call S_Q the localization of S at $\{Q = 0\}$.

We call a *regular* sequence $F_1, \ldots, F_p$ *reduced* if for any index $1 \le k \le p$ the ideal $(F_1, \ldots, F_k)$ is radical. A point x of $\mathbb{C}^n$ is called $(F_1, \ldots, F_p)$-*regular* if the Jacobian $J(F_1, \ldots, F_p) := \left[\frac{\partial F_j}{\partial X_k}\right]_{\substack{1\le j\le p\\ 1\le k\le n}}$ has maximal rank p at x. Observe, that for each *reduced* regular sequence $F_1, \ldots, F_p$ defining the variety S, the locus of $(F_1, \ldots, F_p)$-regular points of S is the same. In this case we call an $(F_1, \ldots, F_p)$-regular point of S simply *regular* (or *smooth*) or we say that S is regular (or smooth) at x. The variety S is called $(F_1, \ldots, F_p)$-regular or smooth if S is $(F_1, \ldots, F_p)$-regular at any of its points.

Notice that the polynomials $F_1, \ldots, F_p$ form locally a reduced regular sequence at any $(F_1, \ldots, F_p)$-regular point of S.

Suppose for the moment that V is a closed subvariety of $\mathbb{C}^n$. For V irreducible we define its degree $\deg V$ as the maximal number of points we can obtain by cutting V with finitely many affine hyperplanes of $\mathbb{C}^n$ such that the intersection is finite. Observe that this maximum is reached when we intersect V with dimension of V many *generic* affine hyperplanes of $\mathbb{C}^n$. In case that V is not irreducible let $V = C_1 \cup \cdots \cup C_s$ be the decomposition of V into irreducible components. We define the degree of V as $\deg V := \sum_{1\le j\le s} \deg C_j$.

With this definition we can state the so-called *Bézout Inequality*:

Let V and W be closed subvarieties of $\mathbb{C}^n$. Then we have

$$\deg(V \cap W) \le \deg V \cdot \deg W.$$

If V is a hypersurface of $\mathbb{C}^n$ then its degree equals the degree of its minimal equation. The degree of a point of $\mathbb{C}^n$ is just one. For more details we refer to [27,17,45].

Let $1 \le i \le n - p$ and let $a := \left[a_{k,l}\right]_{\substack{1\le k\le n-p-i+1\\ 0\le l\le n}}$ be a real $((n-p-i+1) \times (n+1))$-matrix with $(a_{1,0}, \ldots, a_{n-p-i+1,0}) \ne 0$ and suppose that $\left[a_{k,l}\right]_{\substack{1\le k\le n-p-i+1\\ 1\le l\le n}}$ has maximal rank $n - p - i + 1$.

The ith dual polar variety of S associated with the matrix a is defined as closure of the locus of the $(F_1, \ldots, F_p)$-regular points of S where all $(n-i+1)$-minors of the polynomial $((n-i+1) \times n)$-matrix

$$\begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_n} \\ a_{1,1} - a_{1,0}X_1 & \cdots & a_{1,n} - a_{1,0}X_n \\ \vdots & \vdots & \vdots \\ a_{n-p-i+1,1} - a_{n-p-i+1,0}X_1 & \cdots & a_{n-p-i+1,n} - a_{n-p-i+1,0}X_n \end{bmatrix}$$

vanish.

Strictly speaking this notion of dual polar variety depends rather on the scheme given by the ideal generated by the polynomials $F_1, \ldots, F_p$ than on the variety S itself. We shall not stick on the distinction between schemes and varieties, because it will be irrelevant in the sequel.

Observe that this definition of dual polar varieties may be extended to the case that there is given a Zariski open subset O of $\mathbb{C}^n$ and that S is now the locally closed subvariety of $\mathbb{C}^n$ given by

$$S := \{F_1 = 0, \ldots, F_p = 0\} \cap O.$$

In [3,4] we have introduced the notion of dual polar variety of S and motivated by geometric arguments the calculatory definition above of these objects. Moreover, we have shown that, for a real $((n-p-i+1) \times (n+1))$-matrix $a = [a_{k,l}]_{\substack{1\le k\le n-p-i+1\\ 0\le l\le n}}$ with $[a_{k,l}]_{\substack{1\le k\le n-p-i+1\\ 1\le l\le n}}$ *generic*, the ith dual polar variety is either empty or of pure codimension i in S. Further, we have shown that this polar variety is normal and Cohen–Macaulay (but not necessarily smooth) at any of their $(F_1, \ldots, F_p)$-regular points (see [5, Corollary 2 and Section 3.1]). This motivates the consideration of the so-called *generic* dual polar varieties associated with real $((n-p-i+1)\times(n+1))$-matrices a which are generic in the above sense, as invariants of the variety S.

For our use of the word "generic" we refer to [5, Definition 1].

In case that S is closed and that any point of $S_{\mathbb{R}}$ is $(F_1,\dots,F_p)$-regular, the ith dual polar variety associated with a contains at least one point of each connected component of $S_{\mathbb{R}}$ and is therefore not empty (see [3] and [4, Proposition 2]).

If S is only locally closed and a is generic, then any $(F_1,\dots,F_p)$-regular point of $S_{\mathbb{R}}$, which is a local minimizer of the distances of $(\frac{a_{1,1}}{a_{1,0}},\dots,\frac{a_{1,n}}{a_{1,0}})$ to the points of $S_{\mathbb{R}}$, belongs to the ith dual polar variety of S associated with a (this fact is an immediate consequence of the proof [3] and [4, Proposition 2]).

When speaking about generic dual polar varieties we shall always suppose that there is given a generic real or rational $(n-p)\times(n+1)$ matrix and that for $1\le i\le n-p$ the ith dual polar variety is associated with the first $n-p-i+1$ rows of this matrix. Hence our generic dual polar varieties will be arranged in descending chains.

1.2. Algorithmic tools

In the sequel we shall make use of the Kronecker algorithm in the form of [20, Theorems 1 and 2] and of the following result which constitutes a reformulation and a slight sharpening of [3, Theorem 11] and [4, Theorem 13]. This sharpening is obtained by applying [1], Lemma 10 in the spirit of [1, Section 5.1] to the original proofs.

Let $Q, F_1,\dots,F_p\in\mathbb{Q}[X_1,\dots,X_n]$, $1\le p\le n$, be polynomials with $Q\neq 0$ and $\deg F_j\le d$, $1\le j\le p$. Assume that the polynomials $Q, F_1,\dots,F_p$ are given as outputs of an essentially division-free circuit β in $\mathbb{Q}[X_1,\dots,X_n]$ of size L.

Theorem 1. *Let δ be the maximal degree of the Zariski closures of the $(F_1,\dots,F_p)$-regular loci of $\{F_1=0,\dots,F_j=0\}_Q$, $1\le j\le p$, and of all generic dual polar varieties of $S_Q=\{F_1=0,\dots,F_p=0\}_Q$.*

There exists a uniform bounded error probabilistic algorithm over $\mathbb{Q}$ which computes from the input β in time $L(n\,d)^{O(1)}\delta^2\le(n\,d)^{O(n)}$ a representation by univariate polynomials of degree at most δ of a suitable, over $\mathbb{Q}$-defined, $(n-p)$th generic dual polar variety of $S_{\mathbb{R}}$.

For any $n,\ d,\ p,\ L,\ \delta\in\mathbb{N}$ with $1\le p\le n$ this algorithm may be realized by an algebraic computation tree over $\mathbb{Q}$ of depth $L\,(n\,d)^{O(1)}\,\delta^2\le(n\,d)^{O(n)}$ that depends on certain parameters which are chosen randomly.

In view of the comments made at the end of Section 1.1, we may apply the algorithm of Theorem 1 in two ways to the problem of finding real algebraic sample points of $S_{\mathbb{R}}$.

The first way is to suppose $Q=1$ and that any real point of the closed variety $S=\{F_1=0,\dots,F_p=0\}$ is $(F_1,\dots,F_p)$-regular. Then the algorithm returns a real algebraic sample point for each connected component of $S_{\mathbb{R}}$. In Section 2 we shall proceed in this manner.

The second way works for locally closed varieties as well (i.e., in case $Q\notin\mathbb{Q}$) and consists in the search for $(F_1,\dots,F_p)$-regular real points of S_Q which are local minimizers of the distances of a suitable chosen point of $\mathbb{R}^n$ to the elements of the real trace of S_Q. In Section 3.2.2 below we shall proceed in this manner in order to prove Theorem 12.

2. Inequalities

Let $F_1,\dots,F_s\in\mathbb{R}[X_1,\dots,X_n]$ and $1\le p\le\min\{s,n\}$.

Condition A. *Let $1\le j_1<\cdots<j_k\le s$, $1\le k\le p$. Then any point of the semialgebraic set $\{F_{j_1}=0,\dots,F_{j_k}=0\}_{\mathbb{R}}$ is $(F_{j_1},\dots,F_{j_k})$-regular. Moreover, any $p+1$ polynomials of $F_1,\dots,F_s$ have no common real zero.*

Until the end of this section we shall tacitly assume that the polynomials $F_1,\dots,F_s$ satisfy Condition A.

For $\varepsilon_1,\dots,\varepsilon_s\in\{-1,1\}$ let

$$\{\operatorname{sign}F_1=\varepsilon_1,\dots,\operatorname{sign}F_s=\varepsilon_s\}:=\{x\in\mathbb{R}^n\mid\operatorname{sign}F_1(x)=\varepsilon_1,\dots,\operatorname{sign}F_s(x)=\varepsilon_s\}.$$

From now on we shall suppose without loss of generality $\varepsilon_1 = \cdots = \varepsilon_s = 1$ and write

$$\{F_1 > 0, \ldots, F_s > 0\} \quad \text{instead of } \{\text{sign}\, F_1(x) = 1, \ldots, \text{sign}\, F_s(x) = 1\}.$$

For the next two statements, let us fix a maximal index set $1 \le j_1 < \cdots < j_k \le s,\ 1 \le k \le p$ with

$$\{F_{j_1} = 0, \ldots, F_{j_k} = 0\}_{\mathbb{R}} \cap \overline{\{F_1 > 0, \ldots, F_s > 0\}} \neq \emptyset.$$

Lemma 2.

$$\begin{aligned} &\{F_{j_1} = 0, \ldots, F_{j_k} = 0\}_{\mathbb{R}} \cap \overline{\{F_1 > 0, \ldots, F_s > 0\}} \\ &\quad = \{F_{j_1} = 0, \ldots, F_{j_k} = 0\}_{\mathbb{R}} \cap \{F_j > 0,\ 1 \le j \le s,\ j \neq j_1, \ldots, j \neq j_k\}. \end{aligned}$$

Proof. We show first the inclusion of the left hand side of the set equation in the right hand side. For this purpose, let x be an arbitrary point of $\{F_{j_1} = 0, \ldots, F_{j_k} = 0\}_{\mathbb{R}} \cap \overline{\{F_1 > 0, \ldots, F_s > 0\}}$. Suppose that there exists an index $1 \le j \le s, j \neq j_1, \ldots, j \neq j_k$ with $F_j(x) = 0$. Then x belongs to $\{F_{j_1} = 0, \ldots, F_{j_k} = 0, F_j = 0\}_{\mathbb{R}} \cap \overline{\{F_1 > 0, \ldots, F_s > 0\}}$ which by the maximal choice of $1 \le j_1 < \cdots < j_k \le s$ is empty. Therefore, we have $F_j(x) \neq 0$ for any index $1 \le j \le s, j \neq j_1, \ldots, j \neq j_k$. Since x belongs to $\overline{\{F_1 > 0, \ldots, F_s > 0\}}$, we have $F_j(x) > 0$.

We are now going to show the inverse inclusion. Consider an arbitrary point $x \in \{F_{j_1} = 0, \ldots, F_{j_k} = 0\}_{\mathbb{R}} \cap \{F_j > 0,\ 1 \le j \le s,\ j \neq j_1, \ldots, j \neq j_k\}$ and let U be an arbitrary neighborhood of x in $\mathbb{R}^n$. Without loss of generality we may assume that U is contained in $\{F_j > 0,\ 1 \le j \le s,\ j \neq j_1, \ldots, j \neq j_k\}$. Since by Condition A the point x is contained in the $(F_{j_1}, \ldots, F_{j_k})$-regular set $\{F_{j_1} = 0, \ldots, F_{j_k} = 0\}_{\mathbb{R}}$, the polynomial map from $\mathbb{R}^n$ to $\mathbb{R}^k$ given by $(F_{j_1}, \ldots, F_{j_k})$ is a submersion at x and therefore there exists a point $y \in U$ with $F_{j_1}(y) > 0, \ldots, F_{j_k}(y) > 0$. Because U was an arbitrary neighborhood of x, we conclude that x belongs to $\{F_{j_1} = 0, \ldots, F_{j_k} = 0\}_{\mathbb{R}} \cap \overline{\{F_1 > 0, \ldots, F_s > 0\}}$. $\square$

Corollary 3. *Let C be a connected component of $\{F_{j_1} = 0, \ldots, F_{j_k} = 0\}_{\mathbb{R}}$ with $C \cap \overline{\{F_1 > 0, \ldots, F_s > 0\}} \neq \emptyset$. Then*

$$C \subset \{F_j > 0,\ 1 \le j \le s,\ j \neq j_1, \ldots, j \neq j_k\}.$$

Proof. Lemma 2 implies that the set

$$\{F_{j_1} = 0, \ldots, F_{j_k} = 0\}_{\mathbb{R}} \cap \overline{\{F_1 > 0, \ldots, F_s > 0\}}$$

is open and closed in $\{F_{j_1} = 0, \ldots, F_{j_k} = 0\}_{\mathbb{R}}$. Therefore this set is the union of all the connected components of $\{F_{j_1} = 0, \ldots, F_{j_k} = 0\}_{\mathbb{R}}$ which have a nonempty intersection with $\overline{\{F_1 > 0, \ldots, F_s > 0\}}$. This implies Corollary 3. $\square$

2.1. *Converting non-strict inequalities into strict ones*

Let $1 \le k \le p$, $1 \le j_1 < \cdots < j_k \le s$ and let $x = (x_1, \ldots, x_n)$ be a point of $\{F_{j_1} = 0, \ldots, F_{j_k} = 0\}_{\mathbb{R}} \cap \{F_1 > 0, \ldots, F_s > 0\}$. Thus x satisfies the system of non-strict inequalities

$$F_{j_1}(x) \ge 0, \ldots, F_{j_k}(x) \ge 0, \qquad F_j(x) > 0, \quad 1 \le j \le s,\ j \neq j_1, \ldots, j \neq j_k.$$

Starting from x we wish to construct a point $y \in \mathbb{R}^n$ which satisfies the strict inequalities

$$F_j(y) > 0, \quad 1 \le j \le s.$$

From Condition A we conclude that the Jacobian $J(F_{j_1}, \ldots, F_{j_k})$ has full rank k at x. Therefore we may efficiently find a vector $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ such that the entries of $J(F_{j_1}, \ldots, F_{j_k})(x)\mu^T$ are all positive (here, μ^T denotes the transposed vector of μ).

Let Y be a new indeterminate and for $1 \leq j \leq s$ let $G_j := F_j(\mu_1 Y + x_1, \ldots, \mu_n Y + x_n)$. Observe, that the univariate polynomial G_j satisfies the equation $\frac{dG_j}{dY}(0) = \sum_{1 \leq i \leq n} \frac{\partial F_j}{\partial X_i}(x)\mu_i$. In particular, the entries of

$$\left(\frac{dG_{j_1}}{dY}(0), \ldots, \frac{dG_{j_k}}{dY}(0)\right) = J(F_{j_1}, \ldots, F_{j_k})(x)\mu^T$$

are all positive. Let $c > 0$ be the smallest positive zero of $\prod_{1 \leq j \leq s} G_j$ (if there exists none, c may be any positive real number). Then one verifies immediately that $z := x + \frac{c}{2}\mu$ satisfies for any index $1 \leq j \leq s$ the condition $F_j(z) = G_j(\frac{c}{2}) > 0$.

2.2. Finding sample points for all consistent sign conditions

Let $(\varepsilon_1, \ldots, \varepsilon_s) \in \{-1, 0, 1\}^s$. The polynomial inequality system $\operatorname{sign} F_1 = \varepsilon_1, \ldots, \operatorname{sign} F_s = \varepsilon_s$ is called a *sign condition* on $F_1, \ldots, F_s$ which we say to be *consistent* if there exists a point $x \in \mathbb{R}^n$ satisfying it. In case $(\varepsilon_1, \ldots, \varepsilon_s) \in \{-1, 1\}^s$ we call the sign condition *strict*, otherwise *nonstrict*. A real algebraic point of $\mathbb{R}^n$ which is supposed to be encoded "à la Thom" [12] and to satisfy the sign condition is called a *sample point* of it.

Let $F_1, \ldots, F_s$ be given as outputs of an essentially division-free arithmetic circuit β in $\mathbb{Q}[X_1, \ldots, X_n]$ (hence, $F_1, \ldots, F_s$ belong to $\mathbb{Q}[X_1, \ldots, X_n]$). Let $d \geq 2$ be an upper bound of $\deg F_1, \ldots, \deg F_s$. For $1 \leq k \leq p$ and $1 \leq j_1 < \cdots < j_k \leq s$ let $\delta_{j_1,\ldots,j_k}$ be the maximal degree of $\{F_{j_1} = 0, \ldots, F_{j_k} = 0\}$ and all generic dual polar varieties of this variety. Let finally

$$\delta := \max\{\delta_{j_1,\ldots,j_k} \mid 1 \leq j_1 < \cdots < j_k \leq s,\ 1 \leq k \leq p\}.$$

We call δ the *degree* of the sample point finding problem for all consistent sign conditions of $F_1, \ldots, F_s$. From the Bézout Inequality we deduce $\delta \leq (nd)^{O(n)}$. Using Theorem 1 we construct for each $1 \leq k \leq p$ and $1 \leq j_1 < \cdots < j_k \leq s$ real algebraic sample points for each connected component of $\{F_{j_1} = 0, \ldots, F_{j_k} = 0\}_{\mathbb{R}}$. Then we evaluate the signs of all F_j, $1 \leq j \leq s$ $j \neq j_1, \ldots, j \neq j_k$ on these sample points, which we think encoded "à la Thom" by univariate polynomials over $\mathbb{Q}$ of degree at most δ. For this purpose we apply [40, Proposition 4.9], (compare also [13,37]) at a computational cost of $O(\delta^3)$.

By Corollary 3 we obtain in this way sample points for all non-strict consistent sign conditions on $F_1, \ldots, F_s$. As far as only sample points for the strict sign conditions on $F_1, \ldots, F_s$ are required, we limit our attention to sample points of the connected component of $\{F_{j_1} = 0, \ldots, F_{j_k} = 0\}_{\mathbb{R}}$ where the signs of all F_j, $1 \leq j \leq s$ $j \neq j_1, \ldots, j \neq j_k$ are all strict. Let x be such a sample point with $\operatorname{sign} F_j(x) = \varepsilon_j$, and $\varepsilon_j \in \{-1, 1\}$ for $1 \leq j \leq s$ with $j \neq j_1, \ldots, j \neq j_k$.

Then, following Section 2.1, for any $\varepsilon_{j_1}, \ldots, \varepsilon_{j_k} \in \{-1, 1\}$ we may convert x into a real algebraic sample point of the strict sign conditions $\operatorname{sign} F_1 = \varepsilon_1, \ldots, \operatorname{sign} F_s = \varepsilon_s$. The whole procedure can be realized in time $L \binom{s}{p} n^{O(p)} d^{O(1)} \delta^3$ (here arithmetic operations and comparisons in $\mathbb{Q}$ are taken into account at unit costs). We have therefore shown the following statement which constitutes a simplified variant of [33, Theorem 5].

Theorem 4. *Let $n, d, p, s, L, \delta \in \mathbb{N}$ with $1 \leq p \leq n$ be arbitrary and let $F_1, \ldots, F_s \in \mathbb{Q}[X_1, \ldots, X_n]$ be polynomials of degree at most d satisfying Condition A and having sample point finding degree at most δ. Suppose that $F_1, \ldots, F_s$ are given as outputs of an essentially division-free circuit β in $\mathbb{Q}[X_1, \ldots, X_n]$ of size L.*

There exists a uniform bounded error probabilistic algorithm $\mathcal{A}$ over $\mathbb{Q}$ which computes from the input β in time $L \binom{s}{p} n^{O(p)} d^{O(1)} \delta^3 \leq \binom{s}{p} (nd)^{O(n)}$ real algebraic sample points for each consistent sign condition on $F_1, \ldots, F_s$.

For any $n, d, p, s, L, \delta \in \mathbb{N}$ with $1 \leq p \leq n$ the probabilistic algorithm $\mathcal{A}$ may be realized by an algebraic computation tree over $\mathbb{Q}$ of depth $L \binom{s}{p} n^{O(p)} d^{O(1)} \delta^3 \leq \binom{s}{p} (n\, d)^{O(n)}$ that depends on certain parameters which are chosen randomly.

Condition A requires that for every $1 \le k \le p$ and $1 \le j_1 < \cdots < j_k \le s$ any point of the semialgebraic set $\{F_{j_1} = 0, \ldots, F_{j_k} = 0\}_{\mathbb{R}}$ is $(F_{j_1}, \ldots, F_{j_k})$-regular. This requirement may be relaxed using the algorithmic tools developed in [1]. More restrictive is the requirement that any $p+1$ polynomials of $F_1, \ldots, F_s$ have no common real zero. If we drop this requirement, we have to modify the notion of the degree δ of the sample point finding problem for all consistent sign conditions of $F_1, \ldots, F_s$. We obtain then a complexity bound of order $L\binom{s}{n} n^{O(n)} d^{O(1)} \delta^3 \le \binom{s}{n} (nd)^{O(n)}$. The exponential behavior of the "combinatorial" complexity $\binom{s}{n} n^{O(n)}$ cannot be avoided, since for $F_1, \ldots, F_s$ being generic polynomials of degree one at least $\binom{s}{n} 2^n$ distinct sign conditions become satisfied.

3. Optimization

We associate with a polynomial optimization problem with smooth equality constraints certain natural geometric conditions and an intrinsic invariant that controls the complexity of the algorithm which we are going to develop in order to solve this problem. Our approach has some features in common with that of [22].

3.1. Geometric considerations

Let be given polynomials $G, F_1, \ldots, F_p \in \mathbb{R}[X_1, \ldots, X_n]$, $1 \le p \le n$, and let $V := \{F_1 = 0, \ldots, F_p = 0\}$. We suppose for the rest of this paper that $V_{\mathbb{R}}$ is not empty and that any point of $V_{\mathbb{R}}$ is $(F_1, \ldots, F_p)$-regular.

From now on, let $\mathcal{J}$ denote an ordered sequence of indices $j_1, \ldots, j_p$ with $1 \le j_1 < \cdots < j_p \le n$.

For such an *index sequence* $\mathcal{J}$ denote by $\Delta_{\mathcal{J}}$ the p-minor of the Jacobian $J(F_1, \ldots, F_p)$ given by the columns numbered by the elements of $\mathcal{J}$ and by

$$M_1^{\mathcal{J}}, \ldots, M_{n-p}^{\mathcal{J}}$$

the $(p+1)$-minors of $((p+1) \times n)$-matrix

$$\begin{bmatrix} J(F_1, \ldots, F_p) \\ \dfrac{\partial G}{\partial X_1} \cdots \dfrac{\partial G}{\partial X_n} \end{bmatrix}$$

given by the columns numbered by the elements of $\mathcal{J}$ to which we add, one by one, the columns numbered by the indices belonging to the set $\{1, \ldots, n\} \setminus \mathcal{J}$.

Let x be a local minimal point of G on $V_{\mathbb{R}}$. Then the Karush–Kuhn–Tucker conditions imply that $\mathrm{rk} \begin{bmatrix} J(F_1, \ldots, F_p) \\ \frac{\partial G}{\partial X_1} \cdots \frac{\partial G}{\partial X_n} \end{bmatrix} (x) \le p$ holds. We consider the subset W of V satisfying this rank condition, i.e.,

$$W := \left\{ x \in V \mid \mathrm{rk} \begin{bmatrix} J(F_1, \ldots, F_p) \\ \dfrac{\partial G}{\partial X_1} \cdots \dfrac{\partial G}{\partial X_n} \end{bmatrix} (x) \le p \right\}.$$

Let $\mathcal{J}$ be fixed and let us consider the localization $W_{\Delta_{\mathcal{J}}}$ of W outside of the hypersurface $\{\Delta_{\mathcal{J}} = 0\}$. In [2] we developed a succinct local description of determinantal varieties. The main tool for this description was a general Exchange lemma which depicts an exchange relation between certain minors of a given matrix. Applying this Exchange lemma we conclude

$$W_{\Delta_{\mathcal{J}}} = V_{\Delta_{\mathcal{J}}} \cap \{M_1^{\mathcal{J}} = 0, \ldots, M_{n-p}^{\mathcal{J}} = 0\}.$$

For the rest of this paper we shall assume that the polynomials G and $F_1, \ldots, F_p$ satisfy the following condition.

Let D_k denote the union of all irreducible components of W of dimension strictly larger than $n-p-k$. Observe that D_{n-p+1} is also well defined.

Condition B. *Let $\mathcal{J}$ be an arbitrary index sequence, $1 \le k \le n-p$ and denote by $C_1^{\mathcal{J}}, \ldots, C_{s_{\mathcal{J}}}^{\mathcal{J}}$ the irreducible components of $(V_{\Delta_{\mathcal{J}}} \cap \{M_1^{\mathcal{J}} = 0, \ldots, M_k^{\mathcal{J}} = 0\}) \setminus D_k$. Then any point of*

$$\left((V_{\mathbb{R}})_{\Delta_{\mathcal{J}}} \cap \{M_1^{\mathcal{J}} = 0, \ldots, M_k^{\mathcal{J}} = 0\}_{\mathbb{R}}\right) \setminus (D_k)_{\mathbb{R}}$$

is $(F_1, \ldots, F_p, M_1^{\mathcal{J}}, \ldots, M_k^{\mathcal{J}})$-regular. Moreover for any $1 \le j \le s_{\mathcal{J}}$ the semialgebraic set $(C_j^{\mathcal{J}})_{\mathbb{R}}$ is non-empty and each irreducible component of W is of dimension strictly smaller than $n-p$ and contains a real point.

In particular, Condition B implies that the real trace of $V_{\Delta_{\mathcal{J}}} \cap \{M_1^{\mathcal{J}} = 0, \ldots, M_k^{\mathcal{J}} = 0\} \setminus D_k$ is smooth at any of its points. This entails that for $1 \le i < j \le s_{\mathcal{J}}$ the semialgebraic sets $(C_i^{\mathcal{J}})_{\mathbb{R}}$ and $(C_j^{\mathcal{J}})_{\mathbb{R}}$ have an empty intersection.

Condition B allows to establish a bridge between semialgebraic and algebraic geometry (see below the proofs of Lemmas 5 and 6).

Lemma 5. *Let notations be as in Condition B, which we suppose to be satisfied, and let $\mathcal{J}$ be an arbitrary index sequence. Then for any $1 \le j \le s_{\mathcal{J}}$, we have $\dim C_j^{\mathcal{J}} = n-p-k$.*

Proof. From Condition B we infer that there exists an open semialgebraic subset U of $\mathbb{R}^n$, disjoint from $(\{D_k \cup C_1^{\mathcal{J}} \cup \ldots \cup C_{j-1}^{\mathcal{J}} \cup C_{j+1}^{\mathcal{J}} \cup \ldots \cup C_{s_{\mathcal{J}}}^{\mathcal{J}}\})_{\mathbb{R}}$, with $U \cap (C_j^{\mathcal{J}})_{\mathbb{R}} \neq \emptyset$. This implies

$$U \cap (V_{\mathbb{R}})_{\Delta_{\mathcal{J}}} \cap \{M_1^{\mathcal{J}} = 0, \ldots, M_k^{\mathcal{J}} = 0\}_{\mathbb{R}} = U \cap (C_j^{\mathcal{J}})_{\mathbb{R}}.$$

From $U \cap (D_k)_{\mathbb{R}} = \emptyset$ and Condition B we deduce now that any point of the non-empty semialgebraic set $U \cap (C_j^{\mathcal{J}})_{\mathbb{R}}$ is $(F_1, \ldots, F_p, M_1^{\mathcal{J}}, \ldots, M_k^{\mathcal{J}})$-regular. In particular, $C_j^{\mathcal{J}}$ is included in $\{F_1 = 0, \ldots, F_p = 0, M_1^{\mathcal{J}} = 0, \ldots, M_k^{\mathcal{J}} = 0\}$ and contains a $(F_1, \ldots, F_p, M_1^{\mathcal{J}}, \ldots, M_k^{\mathcal{J}})$-regular point. This implies $\dim C_j^{\mathcal{J}} \le n-p-k$. From the definition of $C_j^{\mathcal{J}}$ we infer $\dim C_j^{\mathcal{J}} \ge n-p-k$. Therefore we have $\dim C_j^{\mathcal{J}} = n-p-k$. $\square$

Lemma 6. *Suppose that Condition B is satisfied and let C be an irreducible component of W. Then G takes a constant real value on C.*

Proof. Since by assumption $V_{\mathbb{R}} = \{F_1 = 0, \ldots, F_p = 0\}_{\mathbb{R}}$ is smooth and since $C_{\mathbb{R}}$ is nonempty by Condition B, there exists an index sequence $\mathcal{J}$ with $(C_{\mathbb{R}})_{\Delta_{\mathcal{J}}} \neq \emptyset$. Furthermore, there exists an index $1 \le k \le n-p$ with $\dim C = n-p-k$.

By Lemma 5, we may assume without loss of generality that $C_{\Delta_{\mathcal{J}}} \setminus D_k = C_1^{\mathcal{J}}$ holds. Let x be an arbitrary point of $(C_1^{\mathcal{J}})_{\mathbb{R}}$. As we have seen in the proof of Lemma 5, there exists an open semialgebraic neighborhood U of x in $\mathbb{R}^n$ with

$$U \cap (V_{\mathbb{R}})_{\Delta_{\mathcal{J}}} \cap \{M_1^{\mathcal{J}} = 0, \ldots, M_k^{\mathcal{J}} = 0\}_{\mathbb{R}} = U \cap (C_1^{\mathcal{J}})_{\mathbb{R}}$$

and $U \cap (D_k)_{\mathbb{R}} = \emptyset$. Condition B implies now that $U \cap (C_1^{\mathcal{J}})_{\mathbb{R}}$ is a smooth semialgebraic manifold which we may suppose to be connected by continuously differentiable paths. Let y be an arbitrary point of $U \cap (C_1^{\mathcal{J}})_{\mathbb{R}}$ and let τ be a continuously differentiable path in $U \cap (C_1^{\mathcal{J}})_{\mathbb{R}}$ that connects x with y. We may suppose that τ can be extended to a suitable open neighborhood of $[0, 1]$ in $\mathbb{R}$ and that $\tau(0) = x$ and $\tau(1) = y$ holds. Observe that $\tau([0, 1])$ is contained in $V_{\mathbb{R}}$. The path τ depends on a parameter T defined in the given neighborhood of $[0, 1]$. Let $\tau(T) := (\tau_1(T), \ldots, \tau_n(T))$. Since $\tau([0, 1])$ is contained in $V_{\mathbb{R}}$ the vector $(\frac{d\tau_1}{dT}(t), \ldots, \frac{d\tau_n}{dT}(t))$ belongs to the kernel of $J((F_1, \ldots, F_p))(\tau(t))$ for any $t \in [0, 1]$. On the other hand, $C_1^{\mathcal{J}} \subset C \subset W$ implies that $(\frac{\partial G}{\partial X_1}(\tau(t)), \ldots, \frac{\partial G}{\partial X_n}(\tau(t)))$ is linearly dependent on the full rank matrix $J(F_1, \ldots, F_p)(\tau(t))$. Therefore we have $\frac{d(G \circ \tau)}{dT}(t) = \sum_{i=1}^{n} \frac{\partial G}{\partial X_i}(\tau(t)) \frac{d\tau_i}{dT}(t) = 0$ for any $t \in [0, 1]$. Hence, $G \circ \tau$ is constant on $[0, 1]$. Consequently, we have $G(x) = G(\tau(0)) = G(\tau(1)) = G(y)$. From the arbitrary choice of x and y in $U \cap (C_1^{\mathcal{J}})_{\mathbb{R}}$ we infer that G takes on $U \cap (C_1^{\mathcal{J}})_{\mathbb{R}}$ a constant value.

Thus the restriction of G to the semialgebraic set $(C_1^{\mathcal{J}})_{\mathbb{R}}$ is locally constant and takes therefore only finitely many values in $\mathbb{R}$.

By Condition B there exists an $(F_1,\dots,F_p,M_1^{\mathcal{J}},\dots,M_k^{\mathcal{J}})$-regular point $x=(x_1,\dots,x_n)$ of $(C_1^{\mathcal{J}})_{\mathbb{R}}$. Hence, there exists an open semialgebraic neighborhood U' of x in $\mathbb{R}^n$ and $n-p-k$ parameters $\xi_1,\dots,\xi_{n-p-k}$ of $U'\cap C_1^{\mathcal{J}}$ such that the restriction of G to $U'\cap(C_1^{\mathcal{J}})_{\mathbb{R}}$ can be developed into a convergent power series $P(\xi_1-x_1,\dots,\xi_{n-p-k}-x_{n-p-k})$ around $(x_1,\dots,x_{n-p-k})$. Since G is locally constant on $U'\cap(C_1^{\mathcal{J}})_{\mathbb{R}}$ we conclude that $P(\xi_1-x_1,\dots,\xi_{n-p-k}-x_{n-p-k})$ equals its constant term, say $b\in\mathbb{R}$.

On the other hand, there exists an open neighborhood O of x in $\mathbb{C}^n$ such that the restriction of G to $O\cap C_1^{\mathcal{J}}$ can be developed into a convergent power series in $\xi_1-x_1,\dots,\xi_{n-p-k}-x_{n-p-k}$. This power series must necessarily be $P(\xi_1-x_1,\dots,\xi_{n-p-k}-x_{n-p-k})$. Thus G takes on $O\cap C_1^{\mathcal{J}}$ only the real value b. Suppose that G takes on $C_1^{\mathcal{J}}$ a value different from b. Then $(C_1^{\mathcal{J}})_{G-b}$ is nonempty and therefore (by [35, Chapter I, Section 10, Corollary 1]) dense in the Euclidean topology of $C_1^{\mathcal{J}}$. In particular, there exists a point $y\in O\cap C_1^{\mathcal{J}}$ with $G(y)\neq b$. This contradiction implies that G takes on $C_1^{\mathcal{J}}$ the constant value b. Lemma 6 follows now from the fact that $C_1^{\mathcal{J}}$ is dense in C. $\square$

A formally different result of the same spirit as Lemma 6 is [15, Lemma 3.3]. Its proof can be transformed into an alternative argument for Lemma 6.

Let $1\leq k\leq n-p$. By Lemma 6 the polynomial G takes on D_k only finitely many values which are all real algebraic. We denote $B_k:=G(D_k)$ the set of these values.

To any index sequence $\mathcal{J}$ we may associate a Hessian matrix $H_{\mathcal{J}}$ of G on $V_{\Delta_{\mathcal{J}}}$ whose entries belong to $\mathbb{R}[X_1,\dots,X_n]_{\Delta_{\mathcal{J}}}$. The following condition reflects the intuitive meaning of the Hessian.

Condition C. *Let $\mathcal{J}$ be an arbitrary index sequence. Then the rational function* $\det H_{\mathcal{J}}$ *does not vanish at any $(F_1,\dots,F_p,M_1^{\mathcal{J}},\dots,M_{n-p}^{\mathcal{J}})$-regular real point of $W_{\Delta_{\mathcal{J}}}$.*

Lemma 7. *Suppose that Conditions B and C are satisfied and let $\mathcal{J}$ be an arbitrary index sequence. Then the set of isolated local minimal points of G on $(V_{\mathbb{R}})_{\Delta_{\mathcal{J}}}$ is exactly the set of $(F_1,\dots,F_p,M_1^{\mathcal{J}},\dots,M_{n-p}^{\mathcal{J}})$-regular points of $(W_{\mathbb{R}})_{\Delta_{\mathcal{J}}}$ where $H_{\mathcal{J}}$ is positive definite.*

Proof. Let $\mathcal{J}$ be an arbitrary index sequence. From the Morse Lemma (see [14]) one deduces easily that the points of $(W_{\mathbb{R}})_{\Delta_{\mathcal{J}}}$ where $H_{\mathcal{J}}$ is positive definite, are isolated local minimizers of G on $(V_{\mathbb{R}})_{\Delta_{\mathcal{J}}}$. So, we have only to show that the isolated local minimal points of G in $(V_{\mathbb{R}})_{\Delta_{\mathcal{J}}}$ belong to $(W_{\mathbb{R}})_{\Delta_{\mathcal{J}}}$, are $(F_1,\dots,F_p,M_1^{\mathcal{J}},\dots,M_{n-p}^{\mathcal{J}})$-regular and that their Hessians are positive definite.

Let $x\in(V_{\mathbb{R}})_{\Delta_{\mathcal{J}}}$ be an isolated minimal point of G in $(V_{\mathbb{R}})_{\Delta_{\mathcal{J}}}$. Then, as we have seen, x belongs to $(W_{\mathbb{R}})_{\Delta_{\mathcal{J}}}$. Let C be an arbitrary irreducible component of $W_{\Delta_{\mathcal{J}}}$ which contains x. Let $n-p-k$ with $1\leq k\leq n-p$ be the dimension of C. Suppose that $1\leq k<n-p$ holds. Condition B implies now that there exists an open subset of $(F_1,\dots,F_p,M_1^{\mathcal{J}},\dots,M_k^{\mathcal{J}})$-regular points of $C_{\mathbb{R}}$ which is dense in $C_{\mathbb{R}}$. This implies that any neighborhood of x in $C_{\mathbb{R}}$ contains a point y different from x. Since $G(y)=G(x)$ holds by Lemma 6, the local minimal point x of G in $(V_{\mathbb{R}})_{\Delta_{\mathcal{J}}}$ cannot be isolated. Therefore, we have $k=n-p$. From Condition B we deduce now that x is $(F_1,\dots,F_p,M_1^{\mathcal{J}},\dots,M_{n-p}^{\mathcal{J}})$-regular. Hence, by Condition C, we have $\det H_{\mathcal{J}}(x)\neq 0$. The Morse Lemma implies now that $H_{\mathcal{J}}(x)$ must be positive definite for x being an isolated local minimal point of G in $(V_{\mathbb{R}})_{\Delta_{\mathcal{J}}}$. $\square$

Finally let us comment the regularity requirement contained in Condition B by two classes of examples.

Let $(a_1,\dots,a_n)\in\mathbb{R}^n$ be a generic vector and let $G:=a_1X_1+\cdots+a_nX_n$ or $G:=(a_1-X_1)^2+\cdots+(a_n-X_n)^2$. Furthermore, let $F_1,\dots,F_p$ be as at the beginning of this subsection. Mimicking the argumentation of [5], Section 3 we see that for any index sequence $\mathcal{J}$ and any index $1\leq k\leq n-p$ every point of the real trace of $V_{\Delta_{\mathcal{J}}}\cap\{M_1^{\mathcal{J}}=0,\dots,M_k^{\mathcal{J}}=0\}$ is $(F_1,\dots,F_p,M_1^{\mathcal{J}},\dots,M_k^{\mathcal{J}})$-regular. Hence the regularity requirement contained in Condition B becomes satisfied for this kind of examples.

3.1.1. Unconstrained optimization

We illustrate our argumentation in the case of unconstrained optimization. In this case we have $p := 0$ and V is the complex affine space $\mathbb{C}^n$. There is given a polynomial $G \in \mathbb{R}[X_1, \ldots, X_n]$ and the task is to characterize the isolated local and global minimal points of G in $\mathbb{R}^n$. Such a local minimal point belongs to $W := \{\frac{\partial G}{\partial X_1} = 0, \ldots, \frac{\partial G}{\partial X_n} = 0\}$. For the unconstrained optimization problem we consider the following condition.

Let D_k be the union of all irreducible components of W of dimension strictly larger than $n - k$.

Condition D. *Let $1 \le k \le n$ and let $C_1, \ldots, C_s$ be the irreducible components of $\{\frac{\partial G}{\partial X_1} = 0, \ldots, \frac{\partial G}{\partial X_k} = 0\} \setminus D_k$. Any point of $\{\frac{\partial G}{\partial X_1} = 0, \ldots, \frac{\partial G}{\partial X_k} = 0\}_{\mathbb{R}} \setminus (D_k)_{\mathbb{R}}$ is $(\frac{\partial G}{\partial X_1}, \ldots, \frac{\partial G}{\partial X_k})$-regular.*

For any $1 \le j \le s$ the semialgebraic set $(C_j)_{\mathbb{R}}$ is non empty. Moreover, any irreducible component of W contains a real point.

If Condition D is satisfied by G we can prove in the same way as in case of Lemmas 5–7 the following corresponding statements.

Lemma 8. *Let notations be as in Condition D, which we suppose to be satisfied. Then for any $1 \le j \le s$, we have $\dim C_j = n - k$.*

Lemma 9. *Suppose that Condition D is satisfied and let C be an irreducible component of W. Then G takes a constant real value on C.*

Let $1 \le k \le n$. By Lemma 9 the polynomial G takes on D_k only finitely many values which are all real algebraic. We denote by $B_k := G(D_k)$ the set of these values.

Lemma 10. *Suppose that Condition D is satisfied. Then the set of isolated local minimal points of G in $\mathbb{R}^n$ is exactly the set of points of $W_{\mathbb{R}}$ where the Hessian of G is positive definite.*

Let $G \in \mathbb{R}[X_1, \ldots, X_n]$ be a generic polynomial of degree two. The linear subspaces defined by $\frac{\partial G}{\partial X_1}, \ldots, \frac{\partial G}{\partial X_k}$ intersect transversally. Hence Condition D is satisfied.

3.2. Algorithms

Let notations and assumptions be as in the previous subsection. We associate with G and $F_1, \ldots, F_p$ intrinsic invariants that control the complexity of the algorithms we are going to develop in order to solve the computational problems of minimizing locally and globally G on the set of points in $\mathbb{R}^n$ defined by the equality constraints $F_1 = 0, \ldots, F_p = 0$.

Let G and $F_1, \ldots, F_p \in \mathbb{Q}[X_1, \ldots, X_n]$ be given as outputs of an essentially division-free arithmetic circuit β in $\mathbb{Q}$ having size L. Let $d \ge 2$ be an upper bound for $\deg G, \deg F_1, \ldots, \deg F_p$.

3.2.1. The isolated local minimal point searching problem

In this subsection we shall assume that the polynomials G and $F_1, \ldots, F_p$ satisfy Conditions B and C.

We consider the task of finding all isolated local minimal points of G in $V_{\mathbb{R}} = \{F_1 = 0, \ldots, F_p = 0\}_{\mathbb{R}}$. For this purpose we search for every index sequence $\mathcal{J}$ the isolated local minimal points of G in the corresponding chart $(V_{\mathbb{R}})_{\Delta_{\mathcal{J}}}$. Let $R_{\mathcal{J}}$ be the determinant of the Jacobian of $F_1, \ldots, F_p, M_1^{\mathcal{J}}, \ldots, M_{n-p}^{\mathcal{J}}$ and let $\delta_{\mathcal{J}}$ be the maximal degree of the Zariski closures in $\mathbb{C}^n$ of all locally closed sets

$$\{F_1 = 0, \ldots, F_j = 0\}_{\Delta_{\mathcal{J}} \cdot R_{\mathcal{J}}}, \quad 1 \le j \le p, \quad \text{and}$$

$$\{F_1 = 0, \ldots, F_p = 0,\ M_1^{\mathcal{J}} = 0, \ldots, M_k^{\mathcal{J}} = 0\}_{\Delta_{\mathcal{J}} \cdot R_{\mathcal{J}}}, \quad 1 \le k \le n - p.$$

Let finally

$$\delta := \max\{\delta_{\mathcal{J}} \mid \mathcal{J} \text{ index sequence}\}.$$

We call δ the *degree* of the isolated minimum searching problem for G on $V_{\mathbb{R}} = \{F_1 = 0, \ldots, F_p = 0\}_{\mathbb{R}}$. From the Bézout Inequality we deduce

$$\delta \le (n\, d)^{O(n)}.$$

Fix for the moment an index sequence $\mathcal{J}$ and observe that the polynomials $F_1, \ldots, F_p, M_1^{\mathcal{J}}, \ldots, M_{n-p}^{\mathcal{J}}$ generate the trivial ideal or form a reduced regular sequence in $\mathbb{Q}[X_1, \ldots, X_n]_{\Delta_{\mathcal{J}} \cdot R_{\mathcal{J}}}$. Therefore we may apply the Kronecker algorithm [20, Theorems 1 and 2], to the input system

$$F_1 = 0, \ldots, F_p = 0, \qquad M_1^{\mathcal{J}} = 0, \ldots, M_{n-p}^{\mathcal{J}} = 0, \qquad \Delta_{\mathcal{J}} \cdot R_{\mathcal{J}} \neq 0$$

in order to obtain for the complex points of

$$\{F_1 = 0, \ldots, F_p = 0,\ M_1^{\mathcal{J}} = 0, \ldots, M_{n-p}^{\mathcal{J}} = 0\}_{\Delta_{\mathcal{J}} \cdot R_{\mathcal{J}}}$$

an algebraic description by univariate polynomials over $\mathbb{Q}$. There are at most $\delta_{\mathcal{J}}$ such points. For the real points among them we obtain even a description à la Thom. We now discard the points with non-zero imaginary part and evaluate the signature of the Hessian matrix $H_{\mathcal{J}}$ at each of the real points and discard the real points where the Hessian is not positive definite. The remaining real points are by Lemma 7 exactly the isolated local minimal points of G in $(V_{\mathbb{R}})_{\Delta_{\mathcal{J}}}$. Repeating this procedure for each index sequence we obtain all isolated local minimal points of G in $V_{\mathbb{R}}$.

The complexity analysis of [20, Theorems 1 and 2], yields a time bound of $L \binom{n}{p} (n\,d)^{O(1)}\, \delta^2$ for the first, algebraic part of the procedure. The sign evaluations necessary to handle real algebraic points make increase the overall complexity to $L \binom{n}{p} (n\,d)^{O(1)}\, \delta^3 \leq (n\,d)^{O(n)}$. Applying [1], Lemma 10 in the spirit of [1, Section 5.1], one can show that one may find probabilistically regular matrices $A_1, \ldots, A_{n-p+1} \in \mathbb{Z}^{n \times n}$ of logarithmic heights $O(n \log dn)$ and p-minors $\mathbf{\Delta}_1, \ldots, \mathbf{\Delta}_{n-p+1}$ of $J(F_1, \ldots, F_p) \cdot A_1, \ldots, J(F_1, \ldots, F_p) \cdot A_{n-p+1}$ such that $V_{\mathbf{\Delta}_1} \cup \cdots \cup V_{\mathbf{\Delta}_{n-p+1}}$ is the regular locus of V. Thus we may improve the sequential bound above to $L\,(n\,d)^{O(1)}\, \delta^3$.

Theorem 11. *Let $n, d, p, L, \delta \in \mathbb{N}$ with $1 \leq p \leq n$ be arbitrary and let $G, F_1, \ldots, F_p \in \mathbb{Q}[X_1, \ldots, X_n]$ be polynomials of degree at most d satisfying Conditions B and C and having isolated local minimum searching degree at most δ. Suppose that $G, F_1, \ldots, F_p$ are given as outputs of an essentially division-free circuit β in $\mathbb{Q}[X_1, \ldots, X_n]$ of size L.*

There exists a uniform bounded error probabilistic algorithm $\mathcal{B}$ over $\mathbb{Q}$ which computes from the input β in time $L\,(nd)^{O(1)}\delta^3 \leq (nd)^{O(n)}$ all isolated local minimal points of G in $V_{\mathbb{R}} = \{F_1 = 0, \ldots, F_p = 0\}_{\mathbb{R}}$.

For any $n, d, p, L, \delta \in \mathbb{N}$ with $1 \leq p \leq n$ the probabilistic algorithm $\mathcal{B}$ may be realized by an algebraic computation tree over $\mathbb{Q}$ of depth $L\,(n\,d)^{O(1)}\, \delta^3 \leq (n\,d)^{O(n)}$ that depends on certain parameters which are chosen randomly.

3.2.1.1. The unrestricted case. Let us now consider the problem of searching for the isolated local minimal points in the case of unconstrained optimization. For this purpose we assume that the polynomial G satisfies the Condition D. Let H be the Hessian matrix of G. Observe that the polynomials $\frac{\partial G}{\partial X_1}, \ldots, \frac{\partial G}{\partial X_n}$ generate the trivial ideal or form a reduced regular sequence in $\mathbb{Q}[X_1, \ldots, X_n]_{\det H}$. Let δ be the maximal degree of the Zariski closure in $\mathbb{C}^n$ of the locally closed sets $\{\frac{\partial G}{\partial X_1} = 0, \ldots, \frac{\partial G}{\partial X_k} = 0\}_{\det H}$, $1 \leq k \leq n$. We call δ the *degree* of the isolated minimum searching problem for G on $\mathbb{R}^n$. The Bézout inequality implies $\delta \leq (d-1)^n \leq d^n$. Applying the Kronecker algorithm to the input system

$$\frac{\partial G}{\partial X_1} = 0, \ldots, \frac{\partial G}{\partial X_n} = 0, \quad \det H \neq 0$$

we obtain an analogous statement to Theorem 11 for the isolated minimum searching problem in the unconstrained case with Conditions B and C replaced by Condition D.

3.2.2. The global minimal point searching problem

In this subsection we shall assume that the polynomials G and $F_1, \ldots, F_p$ satisfy Condition B. The aim of the next algorithm is to compute a real algebraic point which is a global minimizer of G in $V_{\mathbb{R}} = \{F_1 = 0, \ldots, F_p = 0\}_{\mathbb{R}}$ if there exists one. Let x be a minimal point of G in $V_{\mathbb{R}}$ and let $b := G(x)$. Then x belongs to W and therefore there exists by Lemma 6 an irreducible component of W where G takes only the value b. This fact will guarantee that we are able to find a minimizer of G on $V_{\mathbb{R}}$. For an index sequence $\mathcal{J}$ and an index $1 \leq k \leq n-p$ let $\delta_{\mathcal{J},k}$ be the maximal degree of the Zariski closures in $\mathbb{C}^n$ of all locally closed sets

$$\{F_1 = 0, \ldots, F_j = 0\}_{\Delta_{\mathcal{J}}}, \quad 1 \leq j \leq p, \quad \text{and}$$

$$\{F_1 = 0, \ldots, F_p = 0,\ M_1^{\mathcal{J}} = 0, \ldots, M_{k'}^{\mathcal{J}} = 0\}_{\Delta_{\mathcal{J}}}, \quad 1 \leq k' \leq k$$

and all generic dual polar varieties of

$$\{F_1 = 0, \ldots, F_p = 0,\ M_1^{\mathcal{J}} = 0, \ldots, M_k^{\mathcal{J}} = 0\}_{\Delta_{\mathcal{J}}}.$$

Let finally

$$\delta := \max\{\delta_{\mathcal{J},k} \mid \mathcal{J} \text{ index sequence},\ 1 \leq k \leq n-p\}.$$

We call δ the *degree* of the global minimum searching problem for G on $V_{\mathbb{R}}$. From the Bézout inequality we deduce

$$\delta \leq (n\,d)^{O(n)}.$$

We construct now recursively in $1 \leq k \leq n-p$ an ascending chain of finite sets Y_k of real algebraic points of $V_{\mathbb{R}}$ such that $G(Y_k)$ contains the set $B_{k+1} := G(D_{k+1})$ (see Section 3.1 for the definition of B_{k+1}).

In order to construct Y_1 we apply for any index sequence $\mathcal{J}$ the algorithm of Theorem 1 to the system $F_1 = 0, \ldots, F_p = 0,\ \Delta_{\mathcal{J}} \neq 0$. The algorithm returns a finite set Y_1 of algebraic points of $V_{\mathbb{R}}$.

Let now $2 \leq k \leq n-p$ and suppose that we have already constructed Y_{k-1} subject to the condition $B_k \subset G(Y_{k-1})$. We apply now for any index sequence $\mathcal{J}$ the same algorithm to the system

$$F_1 = 0, \ldots, F_p = 0, \qquad M_1^{\mathcal{J}} = 0, \ldots, M_k^{\mathcal{J}} = 0, \qquad \Delta_{\mathcal{J}} \neq 0.$$

In this way we obtain finitely many algebraic points of $V_{\mathbb{R}}$ which together with Y_{k-1} form Y_k.

Let us consider an arbitrary irreducible component C of W of dimension $n-p-k$ on which, by virtue of Lemma 6, the constant value of G does not belong to $G(Y_{k-1})$. Then $B_k \subset G(Y_{k-1})$ implies $C \cap D_k = \emptyset$.

Let us fix a generic vector $a := (a_0, a_1, \ldots, a_n)$ of $\mathbb{Q}^{n+1}$. Since $C_{\mathbb{R}}$ is closed, there exists a point x of $C_{\mathbb{R}}$ which realizes the distance of $\left(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\right)$ to $C_{\mathbb{R}}$. The point x belongs to $V_{\mathbb{R}}$ and therefore there exists an index sequence $\mathcal{J}$ with $\Delta_{\mathcal{J}}(x) \neq 0$. From $C \cap D_k = \emptyset$ we deduce $x \notin D_k$ and from $\Delta_{\mathcal{J}}(x) \neq 0$ and $x \in W$ we conclude $x \in V_{\Delta_{\mathcal{J}}} \cap \{M_1^{\mathcal{J}} = 0, \ldots, M_k^{\mathcal{J}} = 0\}$. Hence x belongs to

$$(V_{\mathbb{R}})_{\Delta_{\mathcal{J}}} \cap \{M_1^{\mathcal{J}} = 0, \ldots, M_k^{\mathcal{J}} = 0\}_{\mathbb{R}} \setminus (D_k)_{\mathbb{R}}$$

and Condition B implies now that x is $(F_1, \ldots, F_p, M_1^{\mathcal{J}}, \ldots, M_k^{\mathcal{J}})$-regular. Therefore there exists just one irreducible component of $\{F_1 = 0, \ldots, F_p = 0, M_1^{\mathcal{J}} = 0, \ldots, M_k^{\mathcal{J}} = 0\}$ which passes through x. This component is necessarily of dimension $n-p-k$ and contains C. It is therefore identical with C. Putting all this information together, we conclude that x is a local minimizer of the distances of $(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0})$ to the points of the real trace of $\{F_1 = 0, \ldots, F_p = 0, M_1^{\mathcal{J}} = 0, \ldots, M_k^{\mathcal{J}} = 0\}$. The point x belongs therefore to the $(n-p-k)$th generic dual polar variety of $\{F_1 = 0, \ldots, F_p = 0, M_1^{\mathcal{J}} = 0, \ldots, M_k^{\mathcal{J}} = 0\}$ associated with a. Hence x becomes computed by our algorithm. This implies $x \in Y_k$. Since C was an arbitrary irreducible component of W of dimension $n-p-k$ on which the constant value of G does not belong to $G(Y_{k-1})$, we conclude that $G(Y_k)$ contains the set B_{k+1}.

Applying this argument inductively we see that $G(W) \subset G(Y_{n-p})$ holds. We suppose now that G reaches a global minimum on $\{F_1 = 0, \ldots, F_p = 0\}_{\mathbb{R}}$. Then Y_{n-p} must contain a global minimal point of G in $V_{\mathbb{R}} = \{F_1 = 0, \ldots, F_p = 0\}_{\mathbb{R}}$ which is an element, say y, of Y_{n-p} with $G(y) = \min_{x \in Y_{n-p}} G(x)$.

The complexity analysis of the algorithm of Theorem 1 yields a time bound of $L \binom{n}{p} (n\,d)^{O(1)} \delta^2$ for the first algebraic part of the procedure. The sign evaluations necessary to handle real algebraic points make increase the overall complexity to $L \binom{n}{p} (n\,d)^{O(1)}\, \delta^3 \leq (n\,d)^{O(n)}$. We may use the same argumentation as in Theorem 11 in order to improve this bound to $L\,(n\,d)^{O(1)}\, \delta^3$. We obtain now the following complexity result.

Theorem 12. *Let $n, d, p, L, \delta \in \mathbb{N}$ with $1 \leq p \leq n$ be arbitrary and let $G, F_1, \ldots, F_p \in \mathbb{Q}[X_1, \ldots, X_n]$ be polynomials of degree at most d satisfying Condition B and having global minimum searching degree at most δ. Suppose that $G, F_1, \ldots, F_p$ are given as outputs of an essentially division-free circuit β in $\mathbb{Q}[X_1, \ldots, X_n]$ of size L.*

There exists a uniform bounded error probabilistic algorithm $\mathcal{C}$ over $\mathbb{Q}$ which computes from the input β in time $L\,(nd)^{O(1)}\,\delta^3 \leq (nd)^{O(n)}$ a global minimal point of G in $V_{\mathbb{R}} = \{F_1 = 0, \ldots, F_p = 0\}_{\mathbb{R}}$ if there exists one.

For any $n, d, p, L, \delta \in \mathbb{N}$ with $1 \leq p \leq n$ the probabilistic algorithm $\mathcal{C}$ may be realized by an algebraic computation tree over $\mathbb{Q}$ of depth $L\,(nd)^{O(1)}\delta^3 \leq (nd)^{O(n)}$ that depends on certain parameters which are chosen randomly.

Mutatis mutandis, with Condition D replacing Condition B, the same statement holds true for the unconstrained optimization problem. The degree of the minimum searching problem in this case is the maximal degree the closed sets $\{\frac{\partial G}{\partial X_1} = 0, \ldots, \frac{\partial G}{\partial X_k} = 0\}$ $1 \leq k \leq n$ and all generic dual polar varieties of them.

The reader should observe that Theorem 12 does not answer the question whether G reaches a global minimum on $\{F_1 = 0, \ldots, F_p = 0\}_{\mathbb{R}}$ and can only be applied when this existence problem is already solved. For this problem we refer to [23].

4. Conclusion

Together with [33] this paper represents only a first attempt to introduce the viewpoint of intrinsic quasi-polynomial complexity to the field of polynomial optimization. For this purpose we used some restrictive conditions which allow us to apply our algorithmic tools. In the future we shall relax these restrictions and extend the algorithmic tools and the list of real problems which can be treated in this way.

Here we want to point to another modern approach to global polynomial optimization based on the so called relaxation which reduces the task under consideration to semi-definite programming. As in our situation this method requires that certain conditions, which involve the Karush–Kuhn–Tucker ideal, become satisfied. Moreover, no complexity bounds are available at this moment for this method. On the other hand we rely on tools of real polynomial equation solving which restricts the generality of our complexity results. For details about the relaxation approach we refer to [32,15,36,26,8].

References

[1] B. Bank, M. Giusti, J. Heintz, G. Lecerf, G. Matera, P. Solernó, Degeneracy loci and polynomial equation solving, Found. Comput. Math. (2013) in press. Eprint arXiv1306.3390 [math.AG].

[2] B. Bank, M. Giusti, J. Heintz, G.M. Mbakop, Polar varieties and efficient real elimination, Math. Z. 238 (2001) 115–144.

[3] B. Bank, M. Giusti, J. Heintz, L.M. Pardo, Generalized polar varieties and an efficient real elimination procedure, Kybernetika 40 (2004) 519–550.

[4] B. Bank, M. Giusti, J. Heintz, L.M. Pardo, Generalized polar varieties: geometry and algorithms, J. Complexity 21 (2005) 377–412.

[5] B. Bank, M. Giusti, J. Heintz, M. Safey El Din, E. Schost, On the geometry of polar varieties, Appl. Algebra Eng. Commun. Comput. 21 (2010) 33–83.

[6] S. Basu, R. Pollack, M.-F. Roy, On the combinatorial and algebraic complexity of quantifier elimination, J. ACM 43 (1996) 1002–1045.

[7] S. Basu, R. Pollack, M.-F. Roy, Algorithms in Real Algebraic Geometry, second ed. (English), in: Algorithms and Computation in Mathematics, vol. 10, Springer, Berlin, 2006.

[8] M.A. Bucero, B. Mourrain, Certified relaxation for polynomial optimization on semi-algebraic sets, 2013, Eprint arXiv:1307.6426.

[9] P. Bürgisser, M. Clausen, M.A. Shokrollahi, Algebraic Complexity Theory, with the Collaboration of Thomas Lickteig, in: Grundlehren der Mathematischen Wissenschaften, vol. 315, Springer-Verlag, Berlin etc., 1997.

[10] A. Cafure, G. Matera, Fast computation of a rational point of a variety over a finite field, Math. Comp. 75 (2006) 2049–2085.

[11] J.F. Canny, Some algebraic and geometric computations in PSPACE, in: ACM Symposium on Theory of Computing, STOC, 1988, pp. 460–467.

[12] M. Coste, M.-F. Roy, Thom's lemma, the coding of real algebraic numbers and the computation of the topology of semi-algebraic sets, J. Symbolic. Comput. 5 (1988) 121–129.

[13] F. Cucker, H. Laneau, B. Mishra, P. Pedersen, M.-F. Roy, NC algorithms for real algebraic numbers, Appl. Algebra Eng. Commun. Comput. 3 (2) (1992) 79–98.

[14] M. Demazure, Catastrophes et Bifurcations, Ellipses, Paris, 1989.

[15] J. Demmel, J. Nie, V. Powers, Representations of positive polynomials on noncompact semialgebraic sets via KKT ideals, J. Pure Appl. Algebra 209 (2007) 189–200.
[16] C. Durvye, G. Lecerf, A concise proof of the Kronecker polynomial system solver from scratch, Expo. Math. 26 (2008) 101–139.
[17] W. Fulton, Intersection Theory, second ed. (English), in: Ergebnisse der Mathematik und ihrer Grenzgebiete 3, Folge 2, Springer, Berlin, 1998, xiii.
[18] M. Giusti, J. Heintz, K. Hägele, J.E. Morais, J.L. Montaña, L.M. Pardo, Lower bounds for diophantine approximations, J. Pure Appl. Algebra 117–118 (1997) 277–317.
[19] M. Giusti, J. Heintz, J.E. Morais, J. Morgenstern, L.M. Pardo, Straight-line programs in geometric elimination theory, J. Pure Appl. Algebra 124 (1998) 101–146.
[20] M. Giusti, G. Lecerf, B. Salvy, A Gröbner free alternative for polynomial system solving, J. Complexity 17 (2001) 154–211.
[21] A. Greuet, Optimisation globale algébrique et variétés: théorie, algorithmes et implantations, Ph.D. Thesis, Université Versailles Saint-Quentin, 2013.
[22] A. Greuet, F. Guo, M. Safey El Din, L. Zhi, Global optimization of polynomials restricted to a smooth variety using sums of squares, J. Symbolic. Comput. 47 (2012) 883–901.
[23] A. Greuet, M. Safey El Din, Deciding reachability of the infimum of a multivariate polynomial, in: ISSAC 2011 Proceedings, San Jose, 2011.
[24] A. Greuet, M. Safey El Din, Probabilistic algorithm for polynomial optimization over a real algebraic set, 2013, Eprint arXiv:1307.8281.
[25] D. Grigor'ev, N. Vorobjov, Solving systems of polynomial inequalities in subexponential time, J. Symbolic. Comput. 5 (1988) 37–64.
[26] H.V. Hà, T.S. Pham, Solving polynomial optimization problems via the truncated tangency variety and sums of squares, J. Pure Appl. Algebra 213 (11) (2009) 2167–2176.
[27] J. Heintz, Definability and fast quantifier elimination in algebraically closed fields, Theoret. Comput. Sci. 24 (1983) 239–277.
[28] J. Heintz, G. Matera, A. Waissbein, On the time–space complexity of geometric elimination procedures, Appl. Algebra Eng. Commun. Comput. 11 (2001) 239–296.
[29] J. Heintz, M.-F. Roy, P. Solernó, On the complexity of semialgebraic sets, in: G.X. Ritter (Ed.), IFIP Information Processing, vol. 89, Elsevier, 1989, pp. 293–298.
[30] G. Jeronimo, D. Perrucci, A probabilistic symbolic algorithm to find the minimum of a polynomial function on a basic closed semialgebraic set, 2013, Eprint arXiv:1304.5558.
[31] G. Jeronimo, D. Perrucci, E. Tsigaridas, On the minimum of a polynomial function on a basic closed semialgebraic set and applications, SIAM J. Optim. (2011) in press.
[32] J.B. Lasserre, Global optimization with polynomials and the problem of moments, SIAM J. Optim. 11 (3) (2001) 796–817.
[33] C. Le Guernic, M. Safey El Din, On the practical computation of one point in each connected component of a semi-algebraic set defined by a polynomial system of equations and non-strict inequalities, INRIA Rapport de Recherche 5079 (2004), in: L. Gonzalez-Vega (Ed.) Proceedings of EACA, Santander, Spain, 2004.
[34] H. Matsumura, Commutative Ring Theory, in: Paperback Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge etc., 1989. Transl. from the Japanese by M. Reid.
[35] D. Mumford, The Red Book of Varieties and Schemes, in: Lecture Notes in Mathematics, vol. 1358, Springer-Verlag, Berlin etc., 1988.
[36] J. Nie, An exact Jacobian SDP relaxation for polynomial optimization, Math. Program. 137 No. 1–2 (A) (2013) 225–255.
[37] P. Pedersen, M.-F. Roy, A. Szpirglas, Counting real zeroes in the multivariate case, in: Frédéric Eyssette, et al. (Eds.), Computational Algebraic Geometry, Papers from a conference, held in Nice, France, April 21–25, 1992, in: Prog. Math., vol. 109, Birkhäuser, Boston, 1993, pp. 203–224.
[38] J. Renegar, A faster PSPACE algorithm for the existential theory of the reals, in: Proc. 29th Annual IEEE Symposium on the Foundation of Computer Science, 1988, pp. 291–295.
[39] J. Renegar, On the computational complexity and geometry of the first order theory of the reals, J. Symbolic. Comput. 13 (1992) 255–352.
[40] M.-F. Roy, A. Szpirglas, Complexity of the computations with real algebraic numbers, J. Symbolic. Comput. 10 (1990) 39–51.
[41] M. Safey El Din, Computing the global optimum of a multivariate polynomial over the reals, in: D. Jeffrey (Ed.), ISSAC 2008 Proceedings, Austria, Hagenberg, 2008.
[42] M. Safey El Din, Practical and theoretical issues for the computation of generalized critical values of a polynomial mapping, in: D. Kapur (Ed.), ASCM 2007, in: Lecture Notes in Computer Science, vol. 5081, Springer, Berlin, 2008.
[43] M. Safey El Din, E. Schost, Polar varieties and computation of one point in each connected component of a smooth real algebraic set, in: J.R. Sendra (Ed.), Proc. ISSAC 2003, ACM Press, 2003, pp. 224–231.
[44] Igor R. Shafarevich, Basic Algebraic Geometry. 1: Varieties in Projective Space, Springer-Verlag, Berlin, 1994.
[45] W. Vogel, Lectures on Results on Bézout's Theorem. Notes by D.P. Patil, in: Lectures on Mathematics and Physics, Mathematics, vol. 74, Tata Institute of Fundamental Research, Springer Verlag, Berlin etc., 1984.