Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 303 (2007) 895–908
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

Transverse vibration of Bernoulli–Euler beams carrying point masses and taking into account their rotatory inertia: Exact solution

Santiago Maiz[a,b], Diana V. Bambill[a,b,*], Carlos A. Rossit[a,b], P.A.A. Laura[a]

[a] *Departamento de Ingeniería, Instituto de Mecánica Aplicada, Universidad Nacional del Sur, Av. Alem 1253 (B8000CPB), Bahía Blanca, Argentina*
[b] *Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina*

Received 20 October 2005; received in revised form 30 May 2006; accepted 20 December 2006
Available online 26 March 2007

Abstract

The situation of structural elements supporting motors or engines attached to them is usual in technological applications. The operation of the machine may introduce severe dynamic stresses on the beam. It is important, then, to know the natural frequencies of the coupled beam-mass system, in order to obtain a proper design of the structural elements. An exact solution for the title problem is obtained in closed-form fashion, considering general boundary conditions by means of translational and rotatory springs at both ends. The model allows to analyze the influence of the masses and their rotatory inertia on the dynamic behavior of beams with all the classic boundary conditions, and also, as particular cases, to determine the frequencies of continuous beams.

1. Introduction

Extensive research has been carried out with regard to the vibration analysis of beams carrying concentrated masses at arbitrary positions and additional complexities.

Chen [1] solved analytically the problem of a vibrating simply supported beam carrying a concentrated mass at its center, introducing the mass by the Dirac delta function. Laura, Pombo and Susemihl [2], Laura, Maurizi and Pombo [3] studied the cantilever beam carrying a lumped mass at the top, obtaining analytical solution, introducing the mass in the boundary conditions. Dowell [4] in a thorough paper studied general properties of beams carrying springs and concentrated masses, making useful observations over the matter. Laura, Verniere de Irassar and Ficcadenti [5] used Rayleigh–Ritz method to study continuous beams subjected to axial forces and carrying concentrated masses. Gürgöze [6,7] used the "normal mode summation" technique to determine the fundamental frequency of cantilever beams carrying masses and torsional springs. Liu, Wu and Huang [8] used the Laplace transformation technique to formulate the frequency equation for

*Corresponding author. Tel.: +54 291 4595100; fax: +54 291 4595157.
E-mail address: dbambill@criba.edu.ar (D.V. Bambill).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.12.028

beams carrying intermediate concentrated masses. Other studies of the influence of these factors on slender beam vibrations are given in Refs. [9–17].

In most of the studies mentioned above the influence of the rotatory inertia of the attached mass is not taken into account. Laura, Filipich and Cortínez [18] considered the rotatory inertia of concentrated masses attached to beams and plates, obtaining fundamental frequencies of coupled systems by means of the Rayleigh–Ritz and Dunkerley methods. Chang [19] studied a simply supported Rayleigh beam carrying a rigidly attached centered mass. He determined the natural frequencies and normal modes of the system but he kept the position of the mass fixed.

In the present paper, we describe the determination of the natural frequencies of vibration of a Bernoulli–Euler beam with general boundary conditions at the ends, carrying a finite number of masses at arbitrary positions, having into account their rotatory inertia. The generality of this approach is based on using translational and rotational springs at both ends, which allow us to represent all the possible combinations of classical boundary conditions, as well as elastic restraints. Therefore, the purpose of this study is to present a general solution of the problem and tabulate the first five frequencies for a wide range of system parameters which may help in comparisons of approximate methods.

The model may also be used, as it is known [20], in the problem of whirling of a rotating shaft of uniform cross-section, where the masses model transmission elements or inertia wheels, since the same general equations describe both systems.

2. Mathematical procedure

As shown in Fig. 1, the model considered is a beam with concentrated masses m_1 and m_2 located at x_1 and x_2, respectively, where x is the spatial coordinate along the beam of length l. $I_1 = m_1\, r_1^2$; $I_2 = m_2\, r_2^2$ are the moments of inertia of the attached masses, where r_1 and r_2 are their radii of gyration with respect to the neutral axis of the beam; k_1 and k_2 are the translational stiffness while k_3 and k_4 are rotational stiffness.

In order to find the natural frequencies of the system one assumes that the beam deflection $v(x, t)$ may be expressed in the form

$$v(x,t) = V(x)\cos \omega t, \tag{1}$$

where ω is the natural circular frequency.

Taking into account Eq. (1), the problem under consideration is governed by the following differential equation:

$$\frac{\mathrm{d}^4 V}{\mathrm{d}x^4} - \beta^4 V = 0, \tag{2}$$

where $\beta^4 = (A\rho/EI)\omega^2$, ρ is the beam material density, A is the cross-sectional area, I is the cross-sectional moment of inertia and E is the Young's modulus.

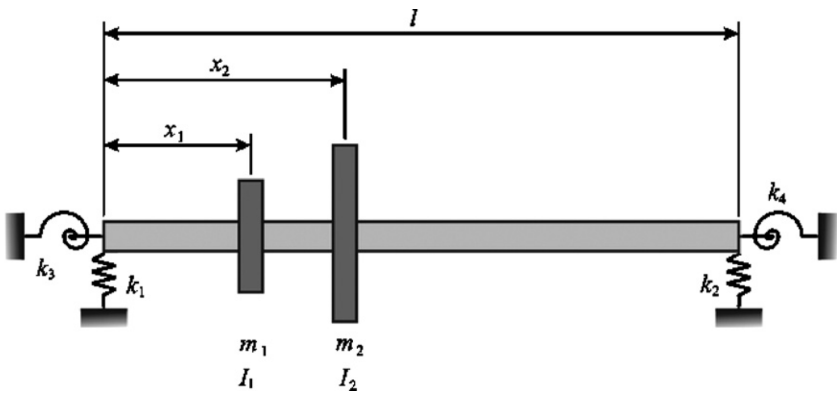

Fig. 1. Bernoulli–Euler beam elastically supported carrying two masses.

The general solution of the differential Eq. (2) may be written as a piecewise function:

$$\begin{aligned} V_1(x) &= C_1 \cosh \beta x + C_2 \sinh \beta x + C_3 \cos \beta x + C_4 \sin \beta x \quad \text{for} \quad 0 \leqslant x \leqslant x_1, \\ V_2(x) &= C_5 \cosh \beta x + C_6 \sinh \beta x + C_7 \cos \beta x + C_8 \sin \beta x \quad \text{for} \quad x_1 \leqslant x \leqslant x_2, \\ V_3(x) &= C_9 \cosh \beta x + C_{10} \sinh \beta x + C_{11} \cos \beta x + C_{12} \sin \beta x \quad \text{for} \quad x_2 \leqslant x \leqslant l, \end{aligned} \tag{3}$$

where C_i are constants to be determined with the boundary conditions at $x = 0$ and l and the continuity equations at x_1 and x_2 while V_1, V_2 and V_3 are, respectively, the left, central and right transverse displacements divided at the points where the concentrated masses are attached.

Introducing the following non-dimensional coordinates:

$$\eta = \frac{x}{l} \text{ one has } \eta_1 = \frac{x_1}{l} \text{ and } \eta_2 = \frac{x_2}{l}, \tag{4}$$

the boundary conditions are:

$$\left.\frac{\mathrm{d}^3 V_1(\eta)}{\mathrm{d}\eta^3}\right|_{\eta=0} = -K_1 V_1(\eta)\big|_{\eta=0}, \tag{5}$$

Table 1
First five eigenvalues ($\beta_n l$) for symmetric location of masses on a clamped–clamped beam

$\eta_1 = 0.25$; $\eta_2 = 0.75$	$\beta_n l$	$c_1 = c_2 = 0$	$c_1 = c_2 = 0.01$	$c_1 = c_2 = 0.05$	$c_1 = c_2 = 0.1$
$M_1 = M_2 = 0$	$\beta_1 l$	4.7300			
	$\beta_2 l$	7.8532			
	$\beta_3 l$	10.9956			
	$\beta_4 l$	14.1372			
	$\beta_5 l$	17.2788			
$M_1 = M_2 = 0.01$	$\beta_1 l$	4.7126	4.7125	4.7112	4.7071
	$\beta_2 l$	7.7732	7.7731	7.7723	7.7696
	$\beta_3 l$	10.8958	10.8956	10.8899	10.8714
	$\beta_4 l$	14.1150	14.1125	14.0520	13.8602
	$\beta_5 l$	17.2557	17.2513	17.1426	16.7908
$M_1 = M_2 = 0.1$	$\beta_1 l$	4.5668	4.5663	4.5554	4.5217
	$\beta_2 l$	7.1911	7.1908	7.1855	7.1671
	$\beta_3 l$	10.2346	10.2325	10.1796	9.9795
	$\beta_4 l$	13.9713	13.9472	13.3525	11.7542
	$\beta_5 l$	17.1148	17.0715	15.9720	13.5895
$M_1 = M_2 = 0.5$	$\beta_1 l$	4.0973	4.0961	4.0663	3.9755
	$\beta_2 l$	5.8984	5.8980	5.8893	5.8555
	$\beta_3 l$	9.1453	9.1356	8.8716	7.9804
	$\beta_4 l$	13.7527	13.6401	11.2437	8.5500
	$\beta_5 l$	16.9258	16.7178	12.9941	10.8372
$M_1 = M_2 = 1$	$\beta_1 l$	3.7335	3.7320	3.6959	3.5868
	$\beta_2 l$	5.1746	5.1743	5.1656	5.1306
	$\beta_3 l$	8.7418	8.7220	8.1800	6.9010
	$\beta_4 l$	13.6791	13.4578	9.8682	7.2687
	$\beta_5 l$	16.8681	16.4514	11.6278	10.2256
$M_1 = M_2 = 2$	$\beta_1 l$	3.3053	3.3037	3.2659	3.1514
	$\beta_2 l$	4.4574	4.4571	4.4491	4.4160
	$\beta_3 l$	8.4667	8.4261	7.3841	5.8827
	$\beta_4 l$	13.6312	13.1901	8.4819	6.1460
	$\beta_5 l$	16.8320	15.9869	10.6332	9.8684

$$\left.\frac{d^2 V_1(\eta)}{d\eta^2}\right|_{\eta=0} = K_3 \left.\frac{d V_1(\eta)}{d\eta}\right|_{\eta=0}, \tag{6}$$

$$\left.\frac{d^3 V_3(\eta)}{d\eta^3}\right|_{\eta=1} = K_2 V_3(\eta)\big|_{\eta=1}, \tag{7}$$

$$\left.\frac{d^2 V_3(\eta)}{d\eta^2}\right|_{\eta=1} = -K_4 \left.\frac{d V_3(\eta)}{d\eta}\right|_{\eta=1}. \tag{8}$$

The continuity equations of the beam at the position η_1 are:

$$V_1(\eta)\big|_{\eta=\eta_1} = V_2(\eta)\big|_{\eta=\eta_1}, \tag{9}$$

$$\left.\frac{d V_1(\eta)}{d\eta}\right|_{\eta=\eta_1} = \left.\frac{d V_2(\eta)}{d\eta}\right|_{\eta=\eta_1}, \tag{10}$$

$$\left.\frac{d^3 V_1(\eta)}{d\eta^3}\right|_{\eta=\eta_1} + M_1(\beta l)^4 V_1(\eta)\big|_{\eta=\eta_1} = \left.\frac{d^3 V_2(\eta)}{d\eta^3}\right|_{\eta=\eta_1}, \tag{11}$$

$$\left.\frac{d^2 V_1(\eta)}{d\eta^2}\right|_{\eta=\eta_1} - M_1 c_1^2(\beta l)^4 \left.\frac{d V_1(\eta)}{d\eta}\right|_{\eta=\eta_1} = \left.\frac{d^2 V_2(\eta)}{d\eta^2}\right|_{\eta=\eta_1} \tag{12}$$

Table 2
First five eigenvalues ($\beta_n l$) for asymmetric location of masses on a clamped–clamped beam

$\eta_1 = 0.25$; $\eta_2 = 0.5$	$\beta_n l$	$c_1 = c_2 = 0$	$c_1 = c_2 = 0.01$	$c_1 = c_2 = 0.05$	$c_1 = c_2 = 0.1$
$M_1 = M_2 = 0.01$	$\beta_1 l$	4.6921	4.6921	4.6915	4.6895
	$\beta_2 l$	7.8128	7.8125	7.8061	7.7861
	$\beta_3 l$	10.8932	10.8931	10.8903	10.8810
	$\beta_4 l$	14.1262	14.1236	14.0593	13.8577
	$\beta_5 l$	17.1859	17.1836	17.1287	16.9486
$M_1 = M_2 = 0.1$	$\beta_1 l$	4.4053	4.4051	4.4003	4.3856
	$\beta_2 l$	7.4860	7.4841	7.4361	7.2818
	$\beta_3 l$	10.2227	10.2217	10.1940	10.0654
	$\beta_4 l$	14.0604	14.0336	13.3904	11.9149
	$\beta_5 l$	16.6703	16.6471	16.0376	13.7828
$M_1 = M_2 = 0.5$	$\beta_1 l$	3.7027	3.7022	3.6922	3.6606
	$\beta_2 l$	6.4814	6.4778	6.3855	6.0575
	$\beta_3 l$	9.2683	9.2606	9.0218	8.0269
	$\beta_4 l$	13.9693	13.8313	11.3901	9.4410
	$\beta_5 l$	16.0876	15.9755	12.9703	10.2982
$M_1 = M_2 = 1$	$\beta_1 l$	3.2772	3.2768	3.2658	3.2314
	$\beta_2 l$	5.7693	5.7658	5.6755	5.3312
	$\beta_3 l$	9.0003	8.9827	8.3750	6.9111
	$\beta_4 l$	13.9388	13.6573	10.2652	8.0475
	$\beta_5 l$	15.9243	15.7069	11.3195	9.8784
$M_1 = M_2 = 2$	$\beta_1 l$	2.8399	2.8394	2.8287	2.7949
	$\beta_2 l$	5.0077	5.0046	4.9240	4.5992
	$\beta_3 l$	8.8463	8.8084	7.5086	5.8790
	$\beta_4 l$	13.9185	13.3490	9.0757	6.8012
	$\beta_5 l$	15.8239	15.3957	10.2276	9.6737

and at η_2

$$V_2(\eta)\big|_{\eta=\eta_2} = V_3(\eta)\big|_{\eta=\eta_2}, \tag{13}$$

$$\left.\frac{\mathrm{d}V_2(\eta)}{\mathrm{d}\eta}\right|_{\eta=\eta_2} = \left.\frac{\mathrm{d}V_3(\eta)}{\mathrm{d}\eta}\right|_{\eta=\eta_2}, \tag{14}$$

$$\left.\frac{\mathrm{d}^3V_2(\eta)}{\mathrm{d}\eta^3}\right|_{\eta=\eta_2} + M_2(\beta l)^4 V_2(\eta)\big|_{\eta=\eta_2} = \left.\frac{\mathrm{d}^3V_3(\eta)}{\mathrm{d}\eta^3}\right|_{\eta=\eta_2}, \tag{15}$$

$$\left.\frac{\mathrm{d}^2V_2(\eta)}{\mathrm{d}\eta^2}\right|_{\eta=\eta_2} - M_2c_2^2(\beta l)^4\left.\frac{\mathrm{d}V_2(\eta)}{\mathrm{d}\eta}\right|_{\eta=\eta_2} = \left.\frac{\mathrm{d}^2V_3(\eta)}{\mathrm{d}\eta^2}\right|_{\eta=\eta_2}, \tag{16}$$

Table 3
First five eigenvalues ($\beta_n l$) for symmetric location of masses on a simply supported beam

$\eta_1 = 0.25$; $\eta_2 = 0.75$	$\beta_n l$	$c_1 = c_2 = 0$	$c_1 = c_2 = 0.01$	$c_1 = c_2 = 0.05$	$c_1 = c_2 = 0.1$
$M_1 = M_2 = 0$	$\beta_1 l$	3.1416			
	$\beta_2 l$	6.2832			
	$\beta_3 l$	9.4248			
	$\beta_4 l$	12.5664			
	$\beta_5 l$	15.7080			
$M_1 = M_2 = 0.01$	$\beta_1 l$	3.1261	3.1261	3.1257	3.1246
	$\beta_2 l$	6.2218	6.2218	6.2218	6.2218
	$\beta_3 l$	9.3790	9.3786	9.3687	9.3376
	$\beta_4 l$	12.5664	12.5644	12.5167	12.3679
	$\beta_5 l$	15.6328	15.6309	15.5845	15.4321
$M_1 = M_2 = 0.1$	$\beta_1 l$	3.0013	3.0012	2.9983	2.9892
	$\beta_2 l$	5.7745	5.7745	5.7745	5.7745
	$\beta_3 l$	9.0595	9.0559	8.9674	8.6820
	$\beta_4 l$	12.5664	12.5465	12.0741	10.8225
	$\beta_5 l$	15.1713	15.1541	14.6979	13.3007
$M_1 = M_2 = 0.5$	$\beta_1 l$	2.6393	2.6390	2.6315	2.6085
	$\beta_2 l$	4.7664	4.7664	4.7664	4.7664
	$\beta_3 l$	8.4744	8.4594	8.0892	7.1123
	$\beta_4 l$	12.5664	12.4671	10.4963	8.0784
	$\beta_5 l$	14.5617	14.4846	12.5720	10.8300
$M_1 = M_2 = 1$	$\beta_1 l$	2.3832	2.3828	2.3740	2.3469
	$\beta_2 l$	4.1920	4.1920	4.1920	4.1920
	$\beta_3 l$	8.2394	8.2114	7.5328	6.2114
	$\beta_4 l$	12.5664	12.3679	9.3276	6.8955
	$\beta_5 l$	14.3802	14.2279	11.4423	10.2253
$M_1 = M_2 = 2$	$\beta_1 l$	2.0960	2.0956	2.0864	2.0583
	$\beta_2 l$	3.6171	3.6171	3.6171	3.6171
	$\beta_3 l$	8.0730	8.0190	6.8399	5.3282
	$\beta_4 l$	12.5664	12.1712	8.0784	5.8419
	$\beta_5 l$	14.2680	13.9592	10.5691	9.8684

where

$$K_1 = \frac{k_1 l^3}{EI}, \quad K_2 = \frac{k_2 l^3}{EI}, \quad K_3 = \frac{k_3 l}{EI}, \quad K_4 = \frac{k_4 l}{EI},$$
$$M_i = \frac{m_i}{\rho A l},$$
$$c_i = \frac{r_i}{l}. \tag{17}$$

3. Frequency equation

Substituting Eq. (3) into Eqs. (5)–(16), taking into account Eq. (4) one obtains, after appropriate non-dimensionalization, the following system of equations expressed as

$$\begin{bmatrix} a_{1-1} & a_{1-2} & \cdot & \cdot & \cdot & a_{1-12} \\ a_{2-1} & a_{2-2} & \cdot & \cdot & \cdot & a_{2-12} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{12-1} & a_{12-2} & \cdot & \cdot & \cdot & a_{12-12} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \cdot \\ \cdot \\ \cdot \\ C_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}. \tag{18}$$

Table 4
First five eigenvalues ($\beta_n l$) for asymmetric location of masses on a simply supported beam

$\eta_1 = 0.25$; $\eta_2 = 0.5$	$\beta_n l$	$c_1 = c_2 = 0$	$c_1 = c_2 = 0.01$	$c_1 = c_2 = 0.05$	$c_1 = c_2 = 0.1$
$M_1 = M_2 = 0.01$	$\beta_1 l$	3.1185	3.1184	3.1183	3.1177
	$\beta_2 l$	6.2524	6.2523	6.2494	6.2403
	$\beta_3 l$	9.3558	9.3556	9.3509	9.3356
	$\beta_4 l$	12.5664	12.5644	12.5168	12.3684
	$\beta_5 l$	15.5961	15.5951	15.5714	15.4950
$M_1 = M_2 = 0.1$	$\beta_1 l$	2.9415	2.9414	2.9401	2.9359
	$\beta_2 l$	6.0161	6.0151	5.9914	5.9175
	$\beta_3 l$	8.8650	8.8637	8.8302	8.6981
	$\beta_4 l$	12.5664	12.5465	12.0735	10.8986
	$\beta_5 l$	14.9527	14.9422	14.6718	13.4418
$M_1 = M_2 = 0.5$	$\beta_1 l$	2.4946	2.4945	2.4916	2.4824
	$\beta_2 l$	5.3428	5.3403	5.2788	5.0881
	$\beta_3 l$	7.9643	7.9604	7.8441	7.2183
	$\beta_4 l$	12.5664	12.4664	10.5152	8.8187
	$\beta_5 l$	14.2171	14.1610	12.4431	9.6262
$M_1 = M_2 = 1$	$\beta_1 l$	2.2162	2.2161	2.2128	2.2027
	$\beta_2 l$	4.8384	4.8355	4.7649	4.5445
	$\beta_3 l$	7.6317	7.6240	7.3718	6.3101
	$\beta_4 l$	12.5664	12.3649	9.4562	7.9257
	$\beta_5 l$	14.0212	13.9073	10.8964	8.5582
$M_1 = M_2 = 2$	$\beta_1 l$	1.9256	1.9254	1.9222	1.9121
	$\beta_2 l$	4.2553	4.2525	4.1825	3.9609
	$\beta_3 l$	7.4180	7.4019	6.8212	5.4070
	$\beta_4 l$	12.5664	12.1594	8.4533	6.7602
	$\beta_5 l$	13.9050	13.6744	9.4571	8.1863

The non-triviality condition is established by solving:

$$\det(\mathbf{A}) = 0, \tag{19}$$

where **A** is the matrix of the coefficients a_{i-j} of the system and the roots ($\beta_n l$) are the eigenvalues of the problem. Eq. (18) is a complicated function of $\beta_n l$. A numerical technique, the Newton–Raphson algorithm, was used in the present paper to find the roots.

Note that the dimension of the **A** matrix is $(4+4n)\times(4+4n)$ in the case of a beam carrying n concentrated masses; therefore, if the system under study has a symmetrical configuration it will be convenient to analyze only half of the beam to calculate symmetric and antisymmetric vibration modes.

4. Numerical results

The first five eigenvalues were obtained for different combinations of classical boundary conditions, available in the literature. Also, some special configurations of beams are calculated as particular cases of the proposed model.

Table 5
First five eigenvalues ($\beta_n l$) for symmetric location of masses on a cantilever beam

$\eta_1 = 0.25$; $\eta_2 = 0.75$	$\beta_n l$	$c_1 = c_2 = 0$	$c_1 = c_2 = 0.01$	$c_1 = c_2 = 0.05$	$c_1 = c_2 = 0.1$
$M_1 = M_2 = 0$	$\beta_1 l$	1.8751			
	$\beta_2 l$	4.6941			
	$\beta_3 l$	7.8548			
	$\beta_4 l$	10.9955			
	$\beta_5 l$	14.1372			
$M_1 = M_2 = 0.01$	$\beta_1 l$	1.8669	1.8669	1.8668	1.8665
	$\beta_2 l$	4.6851	4.6850	4.6827	4.6757
	$\beta_3 l$	7.7887	7.7887	7.7869	7.7813
	$\beta_4 l$	10.9048	10.9046	10.8999	10.8850
	$\beta_5 l$	14.1171	14.1147	14.0569	13.8735
$M_1 = M_2 = 0.1$	$\beta_1 l$	1.8003	1.8002	1.7994	1.7967
	$\beta_2 l$	4.6083	4.6074	4.5867	4.5240
	$\beta_3 l$	7.3191	7.3184	7.3026	7.2516
	$\beta_4 l$	10.3067	10.3050	10.2639	10.1052
	$\beta_5 l$	13.9865	13.9634	13.3953	11.8800
$M_1 = M_2 = 0.5$	$\beta_1 l$	1.6000	1.5999	1.5976	1.5903
	$\beta_2 l$	4.3191	4.3162	4.2466	4.0495
	$\beta_3 l$	6.3836	6.3800	6.2961	6.0715
	$\beta_4 l$	9.3381	9.3312	9.1379	8.2312
	$\beta_5 l$	13.7841	13.6761	11.4015	9.2243
$M_1 = M_2 = 1$	$\beta_1 l$	1.4529	1.4528	1.4499	1.4411
	$\beta_2 l$	4.0343	4.0305	3.9408	3.6874
	$\beta_3 l$	5.9799	5.9712	5.7797	5.3853
	$\beta_4 l$	8.9843	8.9709	8.5646	7.0960
	$\beta_5 l$	13.7146	13.5026	10.1527	8.5116
$M_1 = M_2 = 2$	$\beta_1 l$	1.2838	1.2837	1.2806	1.2712
	$\beta_2 l$	3.6358	3.6319	3.5381	3.2631
	$\beta_3 l$	5.7009	5.6803	5.2724	4.6783
	$\beta_4 l$	8.7435	8.7169	7.8393	6.0303
	$\beta_5 l$	13.6691	13.2466	9.0712	8.0572

4.1. Clamped– clamped beam

This case is obtained by assuming the following values for the coefficients of translational and rotational stiffness:

$$K_1 \to \infty, \quad K_2 \to \infty, \quad K_3 \to \infty \quad \text{and} \quad K_4 \to \infty. \tag{20}$$

In Table 1, the first five eigenvalues ($\beta_n l$) for the problem of transverse vibration where there is a symmetric location of masses are shown. Results in Table 2 are for an asymmetric location of masses on the beam.

4.2. Simply supported beam

In this case the translational stiffness coefficients assume a value, which approaches infinity at both ends while the rotational stiffness coefficients vanish:

$$K_1 \to \infty, \quad K_2 \to \infty, \quad K_3 = 0 \quad \text{and} \quad K_4 = 0. \tag{21}$$

The eigenvalues are shown on Tables 3 and 4 for a symmetric and asymmetric location of masses on the beam respectively.

4.2. Cantilever beam

In this case the end $\eta = 0$ is clamped and the end $\eta = 1$ is free, therefore

$$K_1 \to \infty, \quad K_2 = 0, \quad K_3 \to \infty \quad \text{and} \quad K_4 = 0. \tag{22}$$

Results of eigenvalues for two different locations of the masses are shown in Tables 5 and 6.

Table 6
First five eigenvalues ($\beta_n l$) for asymmetric location of masses on a cantilever beam

$\eta_1 = 0.25$; $\eta_2 = 0.5$	$\beta_n l$	$c_1 = c_2 = 0$	$c_1 = c_2 = 0.01$	$c_1 = c_2 = 0.05$	$c_1 = c_2 = 0.1$
$M_1 = M_2 = 0.01$	$\beta_1 l$	1.8728	1.8728	1.8727	1.8724
	$\beta_2 l$	4.6627	4.6626	4.6620	4.6602
	$\beta_3 l$	7.8141	7.8138	7.8078	7.7888
	$\beta_4 l$	10.8925	10.8924	10.8896	10.8803
	$\beta_5 l$	14.1262	14.1235	14.0592	13.8573
$M_1 = M_2 = 0.1$	$\beta_1 l$	1.8523	1.8522	1.8514	1.8490
	$\beta_2 l$	4.4279	4.4277	4.4232	4.4090
	$\beta_3 l$	7.4885	7.4866	7.4417	7.2971
	$\beta_4 l$	10.2160	10.2149	10.1879	10.0621
	$\beta_5 l$	14.0603	14.0335	13.3891	11.9090
$M_1 = M_2 = 0.5$	$\beta_1 l$	1.7711	1.7709	1.7677	1.7579
	$\beta_2 l$	3.8880	3.8875	3.8759	3.8384
	$\beta_3 l$	6.5059	6.5026	6.4207	6.1329
	$\beta_4 l$	9.2404	9.2331	9.0069	8.0333
	$\beta_5 l$	13.9690	13.8307	11.3806	9.4421
$M_1 = M_2 = 1$	$\beta_1 l$	1.6881	1.6879	1.6828	1.6676
	$\beta_2 l$	3.5984	3.5977	3.5809	3.5261
	$\beta_3 l$	5.8179	5.8151	5.7418	5.4687
	$\beta_4 l$	8.9619	8.9453	8.3707	6.9231
	$\beta_5 l$	13.9383	13.6562	10.2484	8.0525
$M_1 = M_2 = 2$	$\beta_1 l$	1.5636	1.5633	1.5565	1.5363
	$\beta_2 l$	3.3385	3.3374	3.3101	3.2221
	$\beta_3 l$	5.0967	5.0946	5.0421	4.8403
	$\beta_4 l$	8.8007	8.7651	7.5268	5.9004
	$\beta_5 l$	13.9179	13.3470	9.0692	6.8091

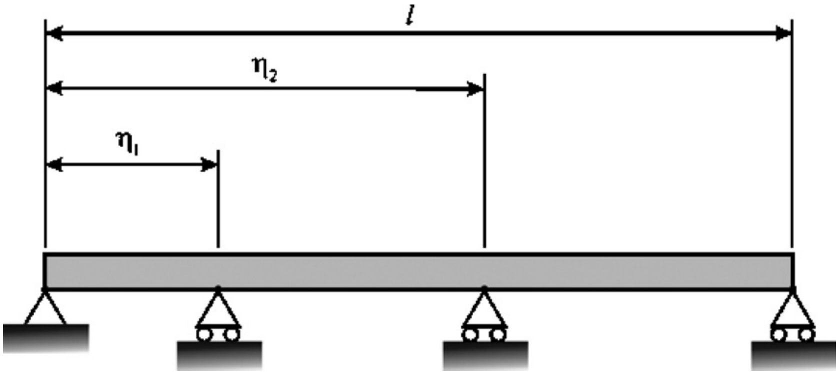

Fig. 2. Continuous beam.

Table 7
First five eigenvalues ($\beta_n l$) for two different configurations of a continuous beam

	$\beta_1 l$	$\beta_2 l$	$\beta_3 l$	$\beta_4 l$	$\beta_5 l$
$\eta_1 = 0.25$; $\eta_2 = 0.75$	7.8532	12.5664	14.1372	15.7064	20.4204
$\eta_1 = 0.25$; $\eta_2 = 0.5$	7.1711	12.5664	13.7741	16.6419	19.8539

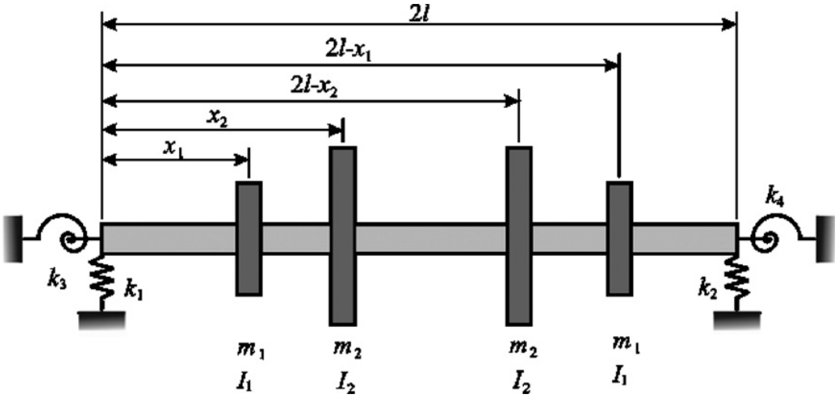

Fig. 3. Symmetrical beam with four masses.

4.3. Continuous beam

One type of structural element that could be represented with this model is the continuous beam as shown in Fig. 2. Obviously, in the case of a whirling problem, the system will correspond to a rotating shaft with intermediate bearings. In this case the following parameters must be taken:

$$K_1 \to \infty, \quad K_2 \to \infty, \quad K_3 = 0, \quad K_4 = 0,$$
$$M_1 \to \infty, \quad M_2 \to \infty, \quad c_1 = 0 \quad \text{and} \quad c_2 = 0. \tag{23}$$

The frequency coefficients are shown on Table 7 for a location of the masses (intermediate supports) at $\eta_1 = 0.25$ and $\eta_2 = 0.75$ and at $\eta_1 = 0.25$ and $\eta_2 = 0.5$.

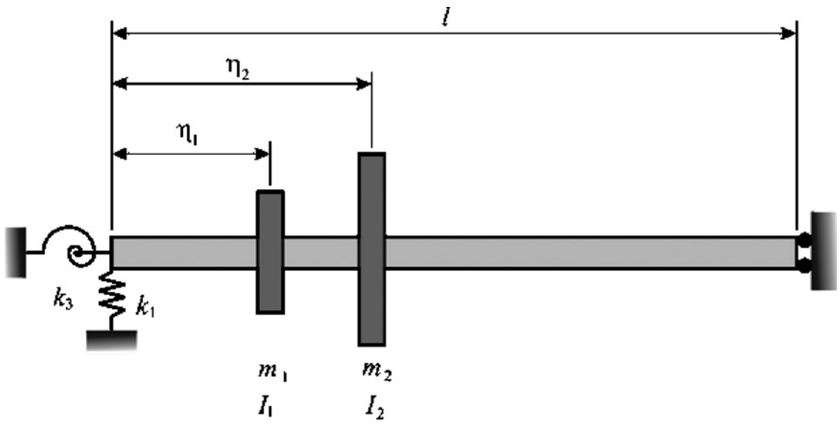

Fig. 4. Half-beam in a symmetric mode of vibration.

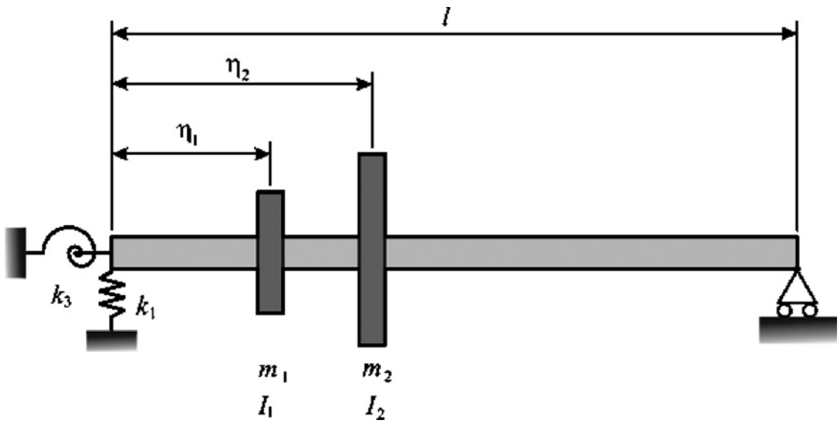

Fig. 5. Half-beam in an antisymmetric mode of vibration.

4.5. Symmetric beam configuration with four masses attached

Making use of the symmetric configuration of the beam, the case shown in Fig. 3 can be solved. As it is known, the symmetric normal modes of the system of Fig. 3 can be obtained by means of the configuration shown in Fig. 4 and the antisymmetric modes can be calculated by means of the configuration shown in Fig. 5.

The case of a clamped–clamped beam with four masses added in a symmetrical way, may be represented by the present model, assuming the following values for the stiffness coefficients:

$$K_1 \to \infty \qquad \text{and} \qquad K_3 \to \infty. \tag{24}$$

In order to obtain the symmetric modes (see Fig. 4), one must consider

$$K_2 = 0 \quad \text{and} \quad K_4 \to \infty \tag{25}$$

and for the antisymmetric modes, as shown in Fig. 5

$$K_2 \to \infty \qquad \text{and} \qquad K_4 = 0. \tag{26}$$

The first five eigenvalues of this problem are shown on Table 8, for a symmetric arbitrary location of the four masses.

Table 8
First five eigenvalues ($\beta_n l$) for a symmetric location of four masses on a clamped–clamped beam

$\eta_1 = 0.125$; $\eta_2 = 0.375$	$\beta_n l$	$c_1 = c_2 = 0$	$c_1 = c_2 = 0.01$	$c_1 = c_2 = 0.05$	$c_1 = c_2 = 0.1$
$M_1 = M_2 = 0$	$\beta_1 l$	4.7300			
	$\beta_2 l$	7.8532			
	$\beta_3 l$	10.9956			
	$\beta_4 l$	14.1372			
	$\beta_5 l$	17.2788			
$M_1 = M_2 = 0.01$	$\beta_1 l$	4.6840	4.6840	4.6826	4.6782
	$\beta_2 l$	7.7796	7.7792	7.7697	7.7399
	$\beta_3 l$	10.9328	10.9310	10.8880	10.7556
	$\beta_4 l$	13.8857	13.8851	13.8706	13.8239
	$\beta_5 l$	17.0445	17.0409	16.9556	16.6877
$M_1 = M_2 = 0.1$	$\beta_1 l$	4.3491	4.3487	4.3392	4.3099
	$\beta_2 l$	7.2352	7.2325	7.1689	6.9764
	$\beta_3 l$	10.3944	10.3819	10.0848	9.2616
	$\beta_4 l$	12.3091	12.3056	12.2171	11.8915
	$\beta_5 l$	15.7406	15.7063	14.9448	13.3483
$M_1 = M_2 = 0.5$	$\beta_1 l$	3.5945	3.5937	3.5757	3.5210
	$\beta_2 l$	5.9801	5.9753	5.8617	5.5285
	$\beta_3 l$	8.7765	8.7569	8.2499	6.9649
	$\beta_4 l$	9.6195	9.6142	9.4698	8.9339
	$\beta_5 l$	14.3751	14.1862	11.5622	9.5382
$M_1 = M_2 = 1$	$\beta_1 l$	3.1633	3.1625	3.1435	3.0865
	$\beta_2 l$	5.2591	5.2542	5.1380	4.7995
	$\beta_3 l$	7.7375	7.7194	7.2232	5.9649
	$\beta_4 l$	8.3269	8.3217	8.1776	7.6522
	$\beta_5 l$	14.0575	13.6736	9.9683	8.0941
$M_1 = M_2 = 2$	$\beta_1 l$	2.7309	2.7301	2.7119	2.6577
	$\beta_2 l$	4.5370	4.5325	4.4235	4.1072
	$\beta_3 l$	6.6774	6.6616	6.2123	5.0638
	$\beta_4 l$	7.1145	7.1097	6.9766	6.4975
	$\beta_5 l$	13.8827	13.1317	8.4936	6.8383

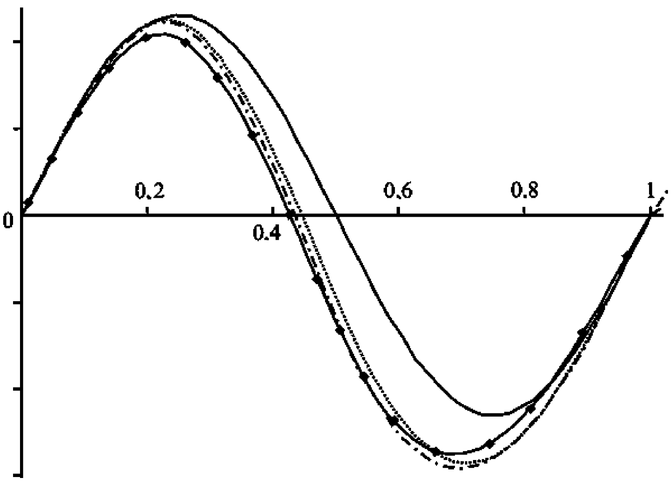

Fig. 6. Second mode shape of a simply supported beam with two masses attached. —— $M_i = 0$; · · · · · · $M_i = 0.5$; –·– $M_i = 1$; —●— $M_i = 2$ and $c_i = 0.1$.

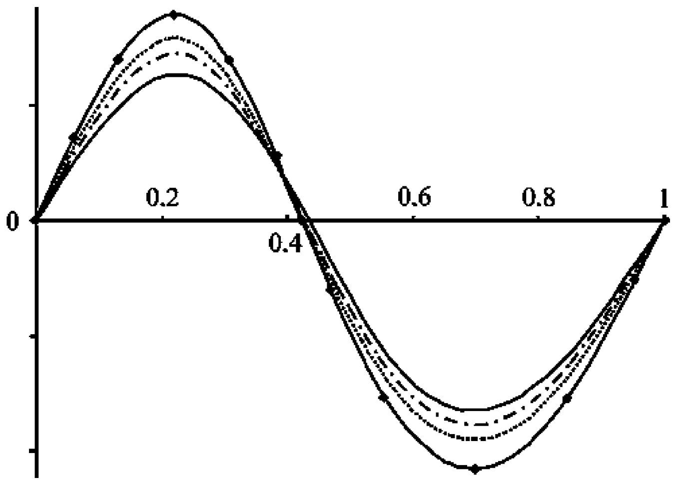

Fig. 7. Second mode shape of a simply supported beam with two masses attached. —— $c_i = 0$; ······ $c_i = 0.01$; –·– $c_i = 0.05$; —●— $c_i = 0.1$ and $M_i = 1$.

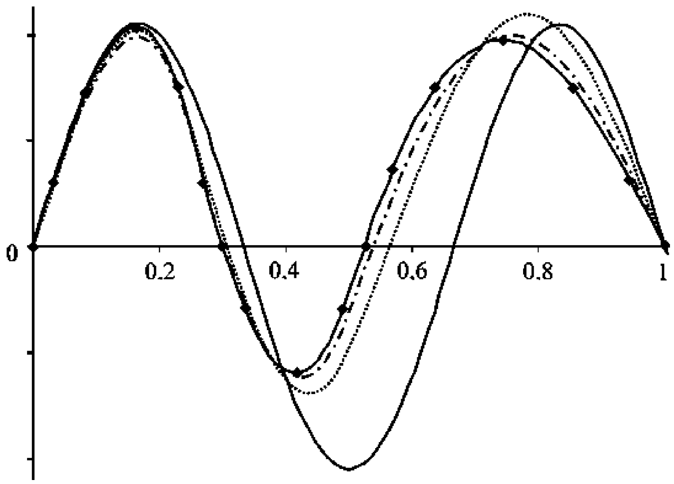

Fig. 8. Third mode shape of a simply supported beam with two masses attached. —— $M_i = 0$; ······ $M_i = 0.5$; –·– $M_i = 1$; —●— $M_i = 2$ and $c_i = 0.1$.

5. Modal shape functions

After solving for the natural frequencies, the corresponding modal shape can be determined. At each natural frequency, the matrix **A** in Eq. (17) is singular. Because of this, the constants C_i cannot be directly determined. However, a modal shape function can be obtained by setting one of the non-vanishing coefficients, say, C_1 equal to unity, and determining the others as a function of it.

In Figs. 6 and 7 shown are the second modal shape and in Figs. 8 and 9 the third modal shape of a simply supported beam with two attached masses at locations: $\eta_1 = 0.25$ and $\eta_2 = 0.5$.

The modal shape of Figs. 6 and 8 were obtained by assuming the ratio $c_i = 0.1$ and varying the mass relation for values $M_i = 0$, 0.5, 1 and 2. In Figs. 7 and 9 the second and third modal shape, assuming $M_i = 1$ and taking different values for $c_i = 0$; 0.01; 0.05 and 0.1 are shown.

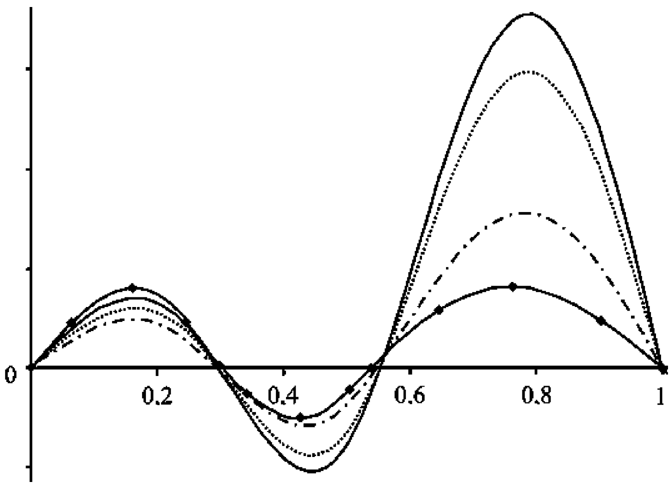

Fig. 9. Third mode shape of a simply supported beam with two masses attached. —— $c_i = 0$; ······ $c_i = 0.01$; –·– $c_i = 0.05$; —●— $c_i = 0.1$ and $M_i = 1$.

6. Conclusions

Usually when the effect of attached masses on vibrating beams is studied, only the translational inertia of the mass is considered. In those cases, it is observed in general, that natural frequencies decrease with respect to the values of the bare beam, except for the cases in which the masses are located at nodal points of the corresponding normal mode (see the case of the fourth eigenvalue depicted in Tables 3 and 4, when the rotatory inertia is not taken into account: $c_1 = c_2 = 0$).

On the other hand, when the model takes into account the rotatory inertia of the mass too, all the natural frequencies of vibration decrease. The influence of the rotatory inertia is larger on the upper frequencies. This effect may be observed in Figs. 7 and 9. In Fig. 9, the variation of c_i for the third modal shape has a larger effect than the one observed in Fig. 7 for the second mode shape.

The effect of the translatory inertia has its highest influence over a natural frequency, when the mass is located at an antinode of the corresponding normal mode. In that situation the rotatory inertia has no effect (as it occurs with the second eigenvalue shown in Table 3).

The effect of the rotatory inertia has its highest influence when the mass is located at a node of the normal mode (Table 3 and 4, fourth eigenvalue).

Acknowledgments

The present work has been sponsored by Secretaría General de Ciencia y Tecnología of Universidad Nacional del Sur at the Department of Engineering, and by CONICET Research and Development Program.

References

[1] Y. Chen, On the vibration of beams or rods carrying a concentrated mass, *Journal of Applied Mechanics* 30 (1963) 310–311.
[2] P.A.A. Laura, J.L. Pombo, E.L. Susemihl, A note on the vibration of a clamped–free beam with a mass at the free end, *Journal of Sound and Vibration* 37 (1974) 161–168.
[3] P.A.A. Laura, M.J. Maurizi, J.L. Pombo, A note on the dynamics analysis of an elastically restrained-free beam with a mass at the free end, *Journal of Sound and Vibration* 41 (1975) 397–405.
[4] E.H. Dowell, On some general properties of combined dynamical systems, *Transactions of the ASME* 46 (1979) 206–209.
[5] P.A.A. Laura, P.L. Verniere de Irassar, G.M. Ficcadenti, A note of transverse vibration of continuous beams subjected to an axial force and carrying concentrated masses, *Journal of Sound and Vibration* 86 (2) (1983) 279–284.
[6] M. Gürgöze, A note on the vibrations of restrained beams and rods with point masses, *Journal of Sound and Vibration* 96 (4) (1984) 461–468.

[7] M. Gürgöze, On the vibration of restrained beams and rods with heavy masses, *Journal of Sound and Vibration* 100 (4) (1985) 588–589.
[8] W.H. Liu, J.R. Wu, C.C. Huang, Free vibrations of beams with elastically restrained edges and intermediate concentrated masses, *Journal of Sound and Vibration* 122 (2) (1988) 193–207.
[9] A.H. Register, A note on the vibration of generally restrained end loaded beams, *Journal of Sound and Vibration* 172 (4) (1994) 561–571.
[10] S. Kukla, B. Posiadala, Free vibrations of beams with elastically mounted masses, *Journal of Sound and Vibration* 175 (4) (1994) 557–564.
[11] M.A. De Rosa, C. Franciosi, M.J. Maurizi, On the dynamics behaviour of slender beams with elastic ends carrying a concentrated mass, *Computers and Structures* 58 (6) (1995) 1145–1159.
[12] M. Gürgöze, On the eigenfrequencies of cantilevered beams carrying tip mass and a spring mass in span, *International Journal of Mechanical Engineering Sciences* 38 (12) (1996) 1295–1306.
[13] C.A. Rossit, P.A.A. Laura, Transverse vibrations of a cantilever beam with a spring mass system attached on the free end, *Ocean Engineering* 28 (2001) 933–939.
[14] C.A. Rossit, P.A.A. Laura, Transverse normal modes of vibration of a cantilever Timoshenko beam with a mass elastically mounted at the free end, *Journal of the Acoustical Society of America* 110 (6) (2001) 2837–2840.
[15] S. Naguleswaran, Transverse vibrations of an Euler–Bernoulli uniform beam carrying several particles, *International Journal of Mechanical Science* 44 (2002) 2463–2478.
[16] M.A. De Rosa, N.M. Auciello, M.J. Maurizi, The use of Mathematica in the dynamics analysis of a beam with a concentrated mass and dashpot, *Journal of Sound and Vibration* 263 (2003) 219–226.
[17] H. Su, J.R. Banerjee, Exact natural frequencies of structures consisting of two part beam-mass systems, *Structural Engineering and Mechanics* 19 (5) (2005) 551–566.
[18] P.A.A. Laura, C.P. Filipich, V.H. Cortínez, Vibrations of beams and plates carrying concentrated masses, *Journal of Sound and Vibration* 117 (3) (1987) 459–465.
[19] C.H. Chang, Free vibration of a simply supported beam carrying a rigid mass at the middle, *Journal of Sound and Vibration* 237 (4) (2000) 733–744.
[20] E. Volterra, E.C. Zachmanoglou, *Dynamics of Vibrations*, Charles E. Merrill Books, Inc., Columbus, OH, 1965.