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Free vibrations of stepped axially functionally graded Timoshenko beams

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Free vibrations of stepped axially functionally graded Timoshenko beams

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Abstract. This paper provides an analytical solution for free transverse vibrations of axially functionally graded beams with step changes in geometry and in material properties. The differential quadrature method using domain decomposition technique is used. Based on Timoshenko beam theory, the equations of motion are derived using Hamilton's principle. Material properties are assumed to vary along the beam in a continuous or an abrupt fashion. The combinations of classical boundary conditions (free, simply supported and clamped) are considered to determine the natural frequencies of many numerical examples. The results for different step locations with different axially functionally graded materials are presented. The phenomenon of dynamic stiffness of beams can be observed in various situations. As there are no available previous results of axially functionally graded beams with step changes, only the results for beams with no abrupt discontinuities are compared with published results. The developed differential quadrature solution has proved its simplicity and robustness to solve the problem presented in the title.

Keywords: *Free vibration, Timoshenko, stepped beam, axially functionally graded, differential quadrature method*

1 Introduction

The dynamic behaviour of stepped beam-like elements is of practical interest in many engineering applications, including civil, aerospace, shipbuilding and automobile engineering. For instance, long span bridges, tall buildings, spacecraft antennae, rotor blades and robot arm manipulators can be modeled with beam-like elements. The presence of steps in cross-section and in material properties may change the natural frequencies. As this situation may cause resonance, if the changed frequency is close to the working frequency, it is crucial to predict the change in the frequency, as well as the mode shape, in a dynamical environment. A recent literature survey on free vibration of stepped beams of functionally graded materials revealed that not many papers cover this topic. In particular, the case of beams with an abrupt change in geometrical and material properties is scarce. To the authors' knowledge, there are no natural frequency data in the literature for axially functionally graded, AFG, Timoshenko beams with step changes in cross-section and in material properties.

The classical Bernoulli-Euler beam theory adequately predicts the frequencies of vibration of lower modes of slender beams. The governing characteristic differential equation of a non-uniform beam is a fourth order ordinary differential equation in the flexural displacement with variable coefficients. Many authors have performed analysis of vibration of stepped beams based on this theory. [1-10]. Recently Mao el al. [1] employed the Adomain decomposition method to investigate the free vibration of stepped Bernoulli-Euler beams. Duan and Wang [5] and Wang and Wang [6] analyzed the free vibration of multiple-stepped Bernoulli-Euler beams.

In the case of Timoshenko beams, the governing characteristic differential equations are two differential equations coupled in terms of the flexural displacement and the angle of rotation which results from bending. [11-15]. The case of homogeneous stepped beams studies based on Timoshenko beam theory have been presented in [14-15] among other papers.

In the present paper a different point of view that adds the effect of the material inhomogeneity [16] to the step change geometry is modeled for free vibration of Timoshenko beams with various combinations of classical boundary conditions. The functionally graded material properties are assumed to vary along the beam in a linear, quadratic or cubic fashion in each beam element with an abrupt discontinuity at the stepped change geometry. Various previous studies have been reported for beams made of AFG materials with a continuous variation of the cross-sectional area (tapered beams) [11, 17-20].

The exact solution for the behavior of vibrating Timoshenko beams with variable coefficients does not exist. The problem must be analyzed by approximate procedures. The differential quadrature method, DQM, is a useful technique to solve the governing equations directly. Early references on the DQM can be found in Bellman and Casti [21], Bert and Malik [22], Laura and Gutiérrez [23] and more recent development and applications can be found in [6], [14], [15], [20], [24], [25], [26], and [27] among many others. In particular, Karami and co-workers [14] developed an accurate differential quadrature element method based on the theory of shear deformable beams. They employed it to analyze beams with non-uniform or discontinuous geometry and other complexities.

In 1991 Laura et al. [10] studied the beneficial effects which can be achieved in a straight beam by introducing step variations of the area and the second moment of area of the beam cross-section. They analyzed the possibility of obtaining a lighter structure with a higher fundamental frequency of transverse vibration and presented some experimental data and the corresponding predictions made by using a finite element code.

2 Theory

2.1 Axially functionally graded material properties

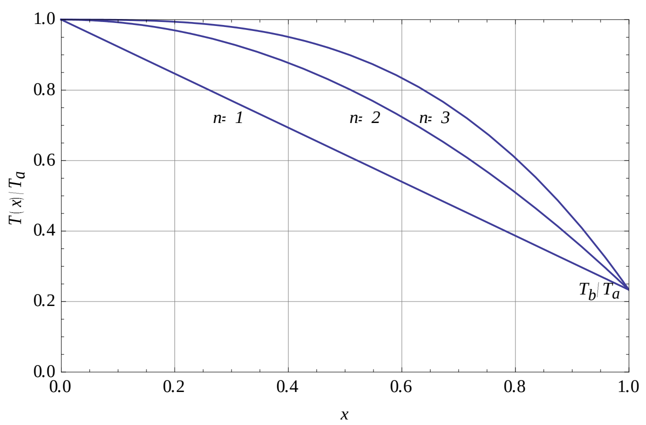


Fig. 1 Power law relation of AFG material properties. $x = \bar{x}/L$

In the present paper the free vibration of stepped AFG Timoshenko beams with different combinations of classical boundary conditions is analyzed.

The beam could have step jumps in cross-sectional area and in material properties, [16], [28]. In order to obtain the dynamic response, the beam is discretized into elements or subdomains depending on the geometrical and material discontinuities.

The inhomogeneous material, with gradient compositional variation of the constituents, varies in the longitudinal direction of the beam. Properties of AFG materials, like mass density ρ, Young's modulus E, shear modulus G, continuously vary in the axial direction.

The material properties [17-20] are assumed to vary along the beam axis $\bar{x}$ with a power law relation:

$$T(\bar{x}) = T_a + (T_a - T_a)\left(\frac{\bar{x}}{L}\right)^n \tag{1}$$

where T_a and T_b are properties of material "*a*" and material "*b*", respectively. They are the constituents of the inhomogeneous material of the beam; n is the material non-homogeneity parameter and $T(\bar{x})$ is a typical material property such as ρ, E or G.

The percentage content of material "*a*" increases as n increases. When n=1 the composition changes linearly through the length L, while n=1/2 or n=2 corresponds to a quadratic distribution, and so on. In general, any value n outside the range (1/3, 3) is not desired [16] because such a functionally graded material would contain too much of one of the constituents. (When n= 1/3 or 3, one constituent has the 75% of the total AFG material). **Fig. 1**.

2.2. Fundamental formulation

In 2001, Banerjee [29] presented a detailed derivation of the governing differential equations of motion of a Timoshenko beam of homogeneous material undergoing free natural vibration using Hamilton's principle. In the present paper, the differential equations of motion are obtained for AFG Timoshenko beams. Following the Timoshenko beam theory, the axially and shear strains for the beam could be expressed as:

$$\varepsilon = \varepsilon_x \cong -y\theta' + \frac{1}{2}\left(w'\right)^2 \tag{2}$$

$$\gamma = \gamma_{xy} = w' - \theta \tag{3}$$

$w = w\left(\bar{x}, t\right)$ and $\theta = \theta\left(\bar{x}, t\right)$ are the flexural displacement of the beam neutral axis in the y direction and the cross-section rotation, respectively. (Prime mark indicates derivative with respect to the spatial coordinate).

The strain energy due to flexure stretching and shear is given by:

$$U = \iiint_V \left(\frac{E\varepsilon^2}{2} + \frac{G\gamma^2}{2} \right) dV = \frac{1}{2}\int_A \int_0^L \left(E\varepsilon^2 + G\gamma^2 \right) d\bar{x}\, dA \tag{4}$$

where $V = AL$, $E = E\left(\bar{x}\right)$ is the Young modulus and $G = G\left(\bar{x}\right)$ is the shear modulus. $A = A\left(\bar{x}\right)$ is the cross-sectional area; $I = I\left(\bar{x}\right)$ is the second moment of area of the beam cross-section about $\bar{z}$ -axis.

Substituting equations (2) and (3) in equation (4) and integrating over the cross-sectional area A, the total strain energy is:

$$U = \frac{1}{2}\sum_{k=1}^{N_e} \int_0^{L_k} \left[EI\left(\theta'\right)^2 + \kappa AG\left(w' - \theta\right)^2 \right]_k d\bar{x}_k \tag{5}$$

where N_e is the total number of beam elements, κ is the section shape shear factor and the length of the beam L is:

$$L = \sum_{k=1}^{N_e} L_k$$

The expression of the kinetic energy is derived from the velocity components of a point at a distance y from the neutral axis.

The velocity components in the x, y and z directions are expressed as:

$$V_x = -y\dot{\theta}\,;\; V_y = 0\,;\; V_z = \dot{w} \tag{6}$$

(Superimposed dot indicates differentiation with respect to time).

Then the kinetic energy T of the beam is given by:

$$T = \frac{1}{2}\int_0^L \int_A \left(V_x^2 + V_y^2 + V_z^2 \right) \rho\, dA\, d\bar{x} = \frac{1}{2}\int_0^L \int_A \left[\left(y\dot{\theta}\right)^2 + \left(\dot{w}\right)^2 \right] \rho\, dA\, d\bar{x} \tag{7}$$

where $\rho = \rho\left(\bar{x}\right)$ is the material's density.

Simplifying expression (7), the total kinetic energy can be written as follows:

$$T=\frac{1}{2}\sum_{k=1}^{N_e}\int_0^{L_k}\left[\rho I\left(\dot{\theta}\right)^2+\rho A\left(\dot{w}\right)^2\right]_k d\bar{x}_k \tag{8}$$

The governing differential equations of motion are derived applying Hamilton's principle that states that

$$\delta\int_{t_1}^{t_2}\left(T-U\right)dt\,,$$

taken between two specified times t_1 and t_2, is stationary for a dynamic trajectory:

$$\delta\int_{t_1}^{t_2}\left(T-U\right)dt=\int_{t_1}^{t_2}\left(\delta T-\delta U\right)dt=0. \tag{9}$$

Substituting expressions (5) and (8) in eq. (9):

$$\begin{aligned}&\int_{t_1}^{t_2}\sum_{k=1}^{N_e}\int_0^{L_k}\Big[\left(\rho I\dot{\theta}\delta\dot{\theta}+\rho A\dot{w}\delta\dot{w}\right)_k-\\&\left(EI\theta'\delta\theta'+\kappa AG\left(w'-\theta\right)\left(\delta w'-\delta\theta\right)\right)_k\Big]d\bar{x}_k\,dt=0;\end{aligned} \tag{10}$$

and integrating expression (10) by parts, the expression for each beam element k is:

$$\begin{aligned}&\int_0^{L_k}\int_{t_1}^{t_2}\left[\left(EI\theta'\right)'+\kappa AG\left(w'-\theta\right)-\rho I\ddot{\theta}\right]_k\delta\theta\,dt\,d\bar{x}_k+\\&\int_0^{L_k}\int_{t_1}^{t_2}\left[\rho A\ddot{w}+\kappa AG\left(w'-\theta\right)'\right]_k\delta w\,dt\,d\bar{x}_k+\\&\int_{t_1}^{t_2}\left(-EI\theta'\right)_k\delta\theta\Big|_0^{L_k}dt+\int_{t_1}^{t_2}\left[\kappa\rho A\left(w'-\theta\right)\right]_k\delta w\Big|_0^{L_k}dt+\\&\int_0^{L_k}\left[\rho I\dot{\theta}\delta\theta+\rho A\dot{w}\delta w\right]_k\Big|_{t_1}^{t_2}d\bar{x}_k=0.\end{aligned} \tag{11}$$

Since δw and $\delta\theta$ are completely arbitrary, the governing differential equations of motion for AFG Timoshenko beam element k result in:

$$\left[\left(EI\theta'\right)'+\kappa AG\left(w'-\theta\right)-\rho I\ddot{\theta}\right]_k=0\,; \tag{12}$$

$$\left[\rho A\ddot{w}+\kappa AG\left(w'-\theta\right)'\right]_k=0\,; \tag{13}$$

for k=1, 2,…N_e.

Assuming sinusoidal variation of $w\left(\bar{x},t\right)$ and $\theta\left(\bar{x},t\right)$ with circular natural frequency ω, the displacements can be written as:

$$w=\bar{W}\,e^{i\omega t}\,;\ \theta=\bar{\Theta}\,e^{i\omega t}\,; \tag{14}$$

where $\bar{W} = \bar{W}(\bar{x})$ and $\Theta = \Theta(\bar{x})$ are spatial functions. Substituting them in equations (12) and (13) the equations of motion of element k are expressed as follows:

$$\left[(EI\Theta')' + \kappa AG(\bar{W}' - \Theta) - \rho I \omega^2 \Theta \right]_k = 0; \tag{15}$$

$$\left[\rho A \omega^2 \bar{W} + \kappa AG(\bar{W}' - \Theta)' \right]_k = 0; \tag{16}$$

for k=1, 2,…N_e.

The geometrical compatibility conditions between two adjacent beam elements are:

$$\bar{W}_k(L_k) = \bar{W}_{k+1}(0);\ \Theta_k(L_k) = \Theta_{k+1}(0); \tag{17}$$

and in the form of the internal compatibility conditions of shear force Q and bending moment M:

$$Q_k(L_k) = Q_{k+1}(0);\ M_k(L_k) = M_{k+1}(0); \tag{18}$$

for k=1, 2,…N_e-1.

The external boundary conditions at the boundary ends ($\bar{x}_1$=0 and $\bar{x}_{N_e} = L_{N_e}$) are assumed as classical conditions: free (F), clamped (C) and simply supported (SS). The DQM is one of the useful procedures to solve differential system equations and it is particularly appropriate when the equations have variable coefficients. As it is known, it is a discrete approach for solving the governing equations for vibration of Timoshenko non-homogeneous beams directly. Non-dimensional expressions are introduced:

for the coordinate variable in each subdomain:

$$x = \frac{\bar{x}_k}{L_k}; \tag{19}$$

for length and displacements:

$$l_k = \frac{L_k}{L}; \tag{20}$$

$$W_k = \frac{\bar{W}_k(\bar{x}_k)}{L_k}; \tag{21}$$

$$\Psi_k = \Theta_k(\bar{x}_k); \tag{22}$$

and for the natural frequency coefficients:

$$\Omega = \sqrt{\rho_0 A_0 / \; E_0 I_0} \;\; L^2 \omega \; ; \tag{23}$$

where $\rho_0 = \rho_1(0)$; $A_0 = A_1(0)$; $E_0 = E_1(0)$; $I_0 = I_1(0)$.

Finally, from equations (15) and (16) the governing differential equations of a stepped beam can be expressed as:

$$\sum_{k=1}^{N_e} \left\{ \frac{\kappa_k s_1^2 s_k^2}{2(1+\nu)} E_k A_k \;\; \Psi_k - W_k' \;\; - \frac{s_1^2}{l_k^2} \left[E_k' I_k \Psi_k' + E_k I_k \Psi_k'' \right] - \Omega^2 \rho_k I_k \Psi_k \right\} = 0 \; ; \tag{24}$$

$$\sum_{k=1}^{N_e} \left\{ \frac{\kappa_k s_1^2}{2(1+\nu_k) l_k^2} \left[E_k' A_k \;\; W_k' + \Psi_k' \;\; + E_k A_k W_k'' \right] + \Omega^2 \rho_k A_k W_k \right\} = 0 \; ; \tag{25}$$

where $s_k = L\sqrt{A_k(0)/I_k(0)}$ is the slenderness ratio, in particular s_1 is the slenderness ratio of the beam subdomain k=1.

The compatibility conditions at adjacent subdomains k and k+1 expressed in terms of the dimensionless variables and parameters are:

$$l_k W_k(1) - l_{k+1} W_{k+1}(0) = 0 \; ; \; \Psi_k(1) - \Psi_{k+1}(0) = 0 \; ; \tag{26}$$

$$\alpha_k Q_k(1) - \alpha_{k+1} Q_{k+1}(0) = 0 \; ; \frac{\beta_k}{l_k} M_k(1) - \frac{\beta_{k+1}}{l_{k+1}} M_{k+1}(0) = 0 \; ; \tag{27}$$

with $\alpha_k = \rho_k A_k(0) / \; \rho_0 A_0$, and $\beta_k = E_k I_k(0) / \; E_0 I_0$.

3 The Differential Quadrature Method

In order to obtain the DQM analog equations of the governing equations of the stepped AFG Timoshenko beam, each beam subdomain k is discretized in a grid of p points using the Chebyshev – Gauss - Lobato expression [21-23]:

$$x_i = \;\; 1 - \cos \;\; (i-1)\pi / (p-1) \;\; / 2 \;\; , \; i = 1, 2, ..., p \; ; \tag{28}$$

where x_i is the coordinate of node i. The q order derivatives of the displacements W and ψ, at a node i of the grid, based on the quadrature rules, [22], are expressed as:

$$\left. \frac{d^{(q)} W_k}{dx^q} \right|_{x_i} = \sum_{j=1}^{p} C_{ij}^{(q)} W_{kj} \; ; \tag{29}$$

$$\left.\frac{d^{(q)}\Psi_k}{dx^q}\right|_{x_i} = \sum_{j=1}^{p} C_{ij}^{(q)}\,\Psi_{kj}\,; \tag{30}$$

where W_{kj} and Ψ_{kj} are the displacements at node j of subdomain k, and $C_{ij}^{(q)}$ are the weighting coefficients obtained using Lagrange interpolating functions:

$$\Pi(x_i) = \prod_{j=1\ \ j\neq 1}^{n} (x_i - x_j)\,; \tag{31}$$

$$C_{ij}^{(1)} = \frac{\Pi\ x_i}{(x_i - x_j)\,\Pi\ x_j}\ ,\ q=1;\ C_{ij}^{\ q} = C_{ii}^{\ q-1}\,C_{ij}^{\ 1} - \frac{C_{ij}^{\ q-1}}{x_i - x_j}\ ,\ q>1; \tag{32}$$

with $i, j = 1, 2, ...n,$ for $i \neq j$, and

$$C_{ii}^{(1)} = -\sum_{j=1\ \ j\neq i}^{n} C_{ij}^{(1)}\ ,\ q=1;\ C_{ii}^{\ q} = -\sum_{j=1\,with\ j\neq i}^{n} C_{ij}^{\ q}\ \ ,\ q>1; \tag{33}$$

$i, j = 1, 2, ...n$, for $i = j.$

Using the quadrature rules, (29-30), the differential quadrature analogs of the governing equations (24) and (25) of a node i are:

$$\begin{aligned}
&\left(\eta^2 a_k(x_i)\ \ R_k + x_i\ \ - \frac{\kappa_k}{2(1+\nu_k)}\frac{s_1^2}{l_k^2}\frac{da_k(x_i)}{dx}\right)\sum_{j=1}^{n} C_{ij}^{(1)}\ \ W_{kj} - \\
&-\left(\frac{s_1^2}{l_k^2}N_k(x_i) + \frac{\kappa_k}{2(1+\nu_k)}\frac{s_1^2}{l_k^2}a_k(x_i)\right)\sum_{j=1}^{n} C_{ij}^{(2)}\ \ W_{kj} + \\
&+\frac{\kappa_k}{2(1+\nu_k)}\frac{s_1^2}{l_k^2}a_k(x_i)\sum_{j=1}^{n} C_{ij}^{(1)}\,\Psi_{kj} + \frac{\kappa_k}{2(1+\nu_k)}\frac{s_1^2}{l_k^2}\frac{da_k(x_i)}{dx}\Psi_{ki} = \Omega^2 a_k(x_i) W_{ki};
\end{aligned} \tag{34}$$

$$\begin{aligned}
&-\frac{s_1^2 s_k^2 \kappa_k}{2(1+\nu_k)}a_k(x_i)\sum_{j=1}^{n} C_{ij}^{(1)}\,W_{kj} - \frac{s_1^2}{l_k^2}b_k(x_i)\sum_{j=1}^{n} C_{ij}^{(2)}\,\Psi_{kj} + \\
&+\left(\frac{s_1^2 s_k^2 \kappa_k}{2(1+\nu_k)}a_k(x_i) - \eta^2 b_k(x_i)\right)\Psi_{ki} - \frac{s_1^2}{l_k^2}\frac{db_k(x_i)}{dx}\sum_{j=1}^{n} C_{ij}^{(1)}\,\Psi_{kj} = \Omega^2 b_k(x_i)\Psi_{ki};
\end{aligned} \tag{35}$$

and the analog equations of internal forces at node i:

$$Q_{ki} = \frac{\kappa_{ki}}{2(1+\nu_{ki})}E_{ki}A_{ki}\left[\left(\sum_{j=1}^{p} C_{ij}^{(1)}\,W_{kj}\right) - \Psi_{ki}\right];\ M_{ki} = E_{ki}I_{ki}\left(\sum_{j=1}^{p} C_{ij}^{(1)}\,\Psi_{kj}\right). \tag{36}$$

The analog continuity equations at adjacent beam elements become:

$$l_k\,W_{kp} - l_{k+1}\,W_{k+1\,1} = 0\,;\ \Psi_{kp} - \Psi_{k+1\,1} = 0\,; \tag{37}$$

$$\alpha_k Q_{kp} - \alpha_{k+1} Q_{k+1\,1} = 0 ; \frac{\beta_k}{l_k} M_{kp} - \frac{\beta_{k+1}}{l_{k+1}} M_{k+1\,1} = 0 . \quad (38)$$

The set of analog equations derived of the governing equations (34-35), the compatibility conditions between subdomains (37-38) and the outer boundary conditions constitute the linear system of equations that allows to determine the natural frequencies of the stepped AFG Timoshenko beam.

4 Numerical results

The natural frequency coefficients, equation (23), of Timoshenko beams with various boundary conditions, different material properties and locations of the abrupt discontinuities are obtained for a range of illustrative examples. Beams of rectangular cross-section (hxb) are adopted for the numerical examples. Then the geometrical relation between height and length can be expressed as:

$$\frac{h}{L} = \frac{\sqrt{12}}{s_1} .$$

In all the numerical examples: κ=5/6.

Table 1 lists the first six frequency coefficients of a uniform homogeneous Timoshenko beam under various boundary conditions. The rate of convergence and accuracy of the proposed differential quadrature procedure can be observed. Free vibration coefficients of clamped-free (C-F), clamped-simply supported (C-SS) and free-free (F-F) beams are compared with [12], [13] and [17]. The agreement between results is excellent, and it can be concluded that the procedure proposed has adequate accuracy with 41 grid points.

Table 1 Convergence analysis: First six natural frequency coefficients of uniform homogeneous Timoshenko beams: h/L=0.35; κ=5/6; ν=0.30

B.C.	p	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	
C-F	11	3.22713	14.4689	31.5016	47.8895	62.3557	68.0104	present
	21	3.22713	14.4689	31.5025	47.9090	62.3470	67.9901	
	31	3.22713	14.4689	31.5025	47.9090	62.3470	67.9901	
	41	3.22713	14.4689	31.5025	47.9090	62.3470	67.9901	
	51	3.22713	14.4689	31.5025	47.9090	62.3470	67.9901	
		3.23	14.47	21.50	47.91	62.35	-	[12]
		3.227128	14.468928	31.502540	47.911084	62.353342	-	[13]
		3.2272	14.4729	31.5425	48.0372	-	-	[17]

C-SS	41	11.082499	27.114378	44.843534	59.203032	63.339499	76.247312	present
		11.08	27.11	44.84	59.20	63.34	-	[12]
		11.082499	27.114378	44.844585	59.203448	63.349869	-	[13]
C-C	41	13.834758	28.517925	45.665951	61.862050	68.283611	80.412094	present
		13.84	28.52	45.67	61.86	68.28	-	[12]
		13.834758	28.517926	45.667237	61.867699	68.292529	-	[13]
F-F	41	16.791957*	33.814869	51.521440	58.991998	73.739689	75.304144	present
		16.79	33.82	51.52	58.99	73.74	-	[12]
		16.791957	33.814869	51.526943	58.993336	73.763812	-	[13]

* The repeated null eigenvalues for rigid translation and rotation for the F-F case are omitted in the Table.

Table 2 presents the first six frequency coefficients of a tapered AFG Timoshenko beam under three different combinations of boundary conditions. To make a comparison with published results, the material properties are assumed to vary according to equation (1), with *n*=1, 2, 3 and 4.

$$E_k = E_k(x) = E_a\left[1+\left(\chi_{E_k}-1\right)x^n\right];\ \rho_k = \rho_k(x) = \rho_a\left[1+\left(\chi_{\rho_k}-1\right)x^n\right]; \qquad (39)$$

with $\chi_{E_k} = E_b/E_a$ and $\chi_{\rho_k} = \rho_b/\rho_a$.

Table 2 First six natural frequency coefficients of AFG Timoshenko beams, with a small taper in height: $h(x) = h_0(1-0.1x)$; h_0/L=0.35 ; κ=5/6; ν=0.30; $\chi_E = 0.35$; $\chi_\rho = 0.47$

B.C.	*n*	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	
C-F	1	3.944636	14.93640	30.57274	46.40688	60.9420	65.7584	present
		3.944636	14.93640	30.57274	46.40888	-	-	[13]
	2	3.935789	15.15333	31.22390	47.58364	62.7344	66.9431	present
		3.935789	15.15333	31.22390	47.58572	-	-	[13]
		3.9359	15.1577	31.2638	47.7164	-	-	[17]
	3	3.849497	15.19867	31.59328	48.24423	63.7301	67.5523	present
		3.849497	15.19869	31.59328	48.24669	-	-	[13]
	4	3.77127	15.1970	31.8164	48.6325	64.3432	67.9315	present
		3.771269	15.19695	31.81639	48.63501	-	-	[13]
C-SS	1	10.88465	25.56609	42.18263	58.13438	60.9556	74.1197	present
		10.88465	25.56609	42.18907	58.14309	-	-	[13]
	2	10.80070	25.61789	42.64742	58.85281	62.7800	75.2574	present
		10.80070	25.61789	42.64780	58.85946	-	-	[13]
	3	10.73937	25.63540	42.85451	59.08722	63.7788	75.7602	present
		10.73937	25.63540	42.85506	59.09377	-	-	[13]
	4	10.71567	25.66653	42.95520	59.14091	64.391	75.9913	present

		10.71567	25.66653	42.95581	59.14709	-	-	[13]
C-C	1	12.68158	26.49101	42.64171	58.65182	66.816	75.9159	present
		12.68158	26.49101	42.64203	58.66849	-	-	[13]
	2	12.46329	26.38044	42.96071	59.39162	68.058	77.0951	present
		12.46329	26.38044	42.92108	59.40234	-	-	[13]
		12.4689	26.4153	43.0904	59.6829	-	-	[17]
	3	12.37525	26.31883	43.08334	59.68936	68.5813	77.5992	present
		12.37525	26.31883	43.08388	59.69942	-	-	[13]
	4	12.36220	26.31154	43.13433	59.80551	68.8795	77.8249	present
		12.36220	26.31158	43.13499	59.81504	-	-	[13]

In the calculations, the constituents of the inhomogeneous material are assumed to be aluminum Al and zirconia ZrO_2. Their Young modulus and density are:

$$E_{Al} = 70GPa;\ \rho_{Al} = 2700kg/m^3;\ E_{ZrO_2} = 200GPa;\ \rho_{ZrO_2} = 5700kg/m^3; \qquad (40)$$

with $\nu_{Al} = \nu_{ZrO_2} = 0.30$. It can be seen that the agreement with previous published results is excellent. Tables 1 and 2 demonstrate the rate of convergence and accuracy of the approach proposed.

The results on Table 3 show the effect of an AFG material (40) on the frequency coefficients of a uniform Timoshenko beam, s_1=12.5, which is equivalent to $h_0/L \cong$ 0.28; L_1=L, for eight different combinations of boundary conditions. The material properties (40) vary according to equations (39), with n=1, 2 and 3.

Next, free vibration of stepped AFG Timoshenko beams with different boundary conditions, step locations and material properties is studied.

Three different geometrical situations, as shown in **Fig. 2**, are assumed introducing step variations of the area and the second moment of area of a rectangular beam cross-section. [1]:

Case A	$L = L_1 + L_2$;	$h_2 = h_1$;	$b_2 = \xi_b b_1$;	$A_2 = \xi_b A_1$;	$I_2 = \xi_b I_1$.
Case B	$L = L_1 + L_2$;	$h_2 = \xi_h h_1$;	$b_2 = b_1$;	$A_2 = \xi_h A_1$;	$I_2 = \xi_h{}^3 A_1$.
Case C	$L = L_1 + L_2$;	$h_2 = \xi_h h_1$;	$b_2 = \xi_b b_1$;	$A_2 = \xi_h \xi_b A_1$;	$I_2 = \xi_b \xi_h{}^3 A_1$.

Table 3 First six natural frequency coefficients of uniform cross-section AFG Timoshenko beams with various boundary conditions. h_0/L=0.28; $\chi_E = 0.35$; $\chi_\rho = 0.47$

n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
	SS-C						C-SS					
(*)	12.1785	31.2031	52.8839	75.5682	91.1848	98.6071	12.1785	31.2031	52.8839	75.5682	91.1848	98.6071

1	10.6527	28.775	49.4047	71.0515	87.9797	93.3051	12.1651	29.9625	50.4151	71.7136	84.8549	93.2976
2	10.6989	28.9102	49.9738	72.0967	89.5464	94.6851	12.0899	30.0762	50.9878	72.6812	87.2899	94.7715
3	10.7643	28.9303	50.1384	72.4798	90.0939	95.3002	12.0341	30.1315	51.2538	73.0823	88.5729	95.4611
	C-C						SS-SS					
(*)	15.6659	33.6285	54.4651	76.0997	98.6071	98.8973	8.82664	28.3570	51.2257	74.8810	88.4591	98.5858
1	14.6202	31.5549	51.3116	71.8519	93.0976	94.0592	8.28580	26.8203	48.4458	70.8139	83.4161	93.2972
2	14.3771	31.4512	51.6786	72.7317	94.5697	95.9212	8.45284	27.2194	49.237	71.9953	85.4714	94.5548
3	14.2839	31.3979	51.8204	73.097	95.176	96.7421	8.53962	27.3644	49.5312	72.4364	86.4428	95.0597
	C-F						F-C					
(*)	3.32139	16.2331	36.5346r	57.9414	79.6803	93.6481	3.32139	16.2331	36.5346	57.9414	79.6803	93.6481
1	4.02882	16.8325	35.8482	56.0353	76.3509	88.5229	2.39704	13.9273	33.4166	53.8750	74.8178	89.1143
2	4.01239	17.0959	36.6134	57.3718	78.3942	90.6695	2.47269	14.2114	33.9864	54.8942	76.3463	90.7864
3	3.91997	17.1605	37.0517	58.1304	79.5352	91.8588	2.56071	14.4004	34.1414	55.1771	76.8348	91.3604
	SS-F						F–SS					
(*)	0	13.1082	33.8752	56.692	78.8321	90.6865	0	13.1082	33.8752	56.692	78.8321	90.6865
1	0	13.3545	32.9559	54.5518	75.286	87.1940	0	11.6817	31.5971	53.1957	74.4974	84.2197
2	0	13.8453	34.0090	56.1060	77.5417	88.6983	0	12.1240	32.4552	54.417	76.2008	86.1989
3	0	13.9978	34.5127	56.9025	78.7123	89.4351	0	12.3723	32.7588	54.811	76.7682	87.1194

(*) homogeneous material $\chi_E = 200/200$=1; $\chi_\rho = 5700/5700$=1.

One of the elements of the stepped beam is assumed to have constant material properties while the other has AFG properties.

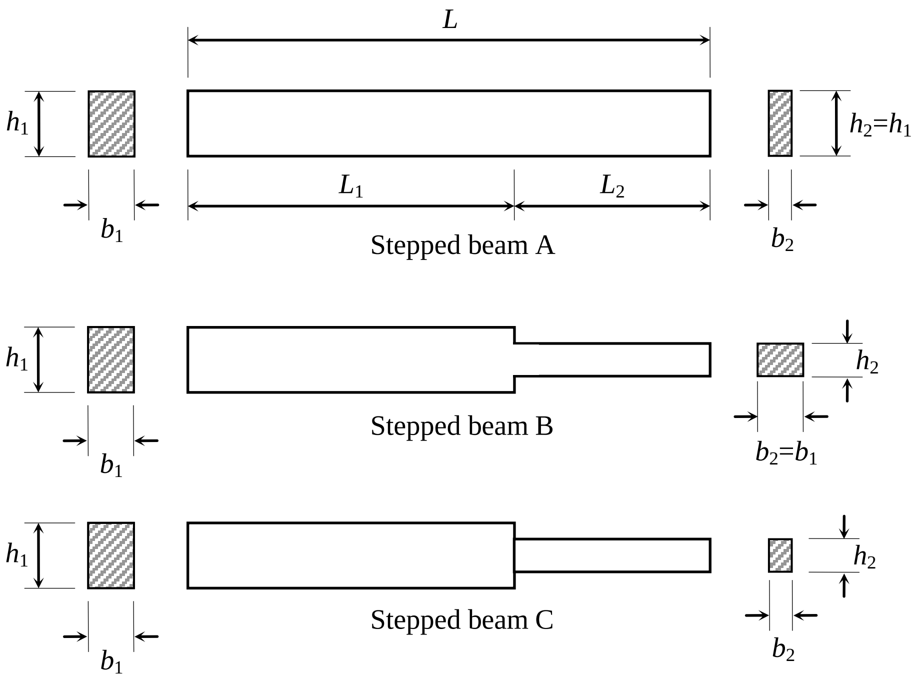


Fig. 2 Stepped AFG Timoshenko beams

The material properties, equations (39), of the portion of the beam of length l_1 are supposed to have AFG characteristics: $n_1 = 1$, 2 and 3; χ_{E_1} =70/200=0.35; χ_{ρ_1} =2702/5700=0.474 with constant cross-section A_1.While the other part of the stepped beam of length l_2, has homogeneous material: $\chi_{E_2} = 200/200 = 1$; $\chi_{\rho_2} = 5700/5700 = 1$, the cross-sectional area being constant and equal to A_2.

Tables 4, 5 and 6 present the first six natural frequency coefficients of cantilever beams of **Fig. 2** with a step located at $l_1 = L_1 / L$ =0.250, 0.370, 0.620, 0.750.

Table 4 First six natural frequency coefficients of clamped-free AFG beams with a step. h_0/L=0.0017. $b_2 = 0.5 b_1$; $h_2 = h_1$. Beam A

l_1	n_1	Ω_1	[1]	Ω_2	[1]	Ω_3	[1]	Ω_4	[1]	Ω_5	Ω_6
0.250	(*)	4.3468	4.3468	24.1602	24.1602	62.4786	62.4811	120.3563	120.365	200.7861	299.6742
	1	3.8637		23.6956		64.1197		121.8507		199.6484	298.4878
	2	4.0457		24.1105		64.2078		121.4757		199.4563	298.747
	3	4.1295		24.2318		63.9806		121.055		199.3913	299.0133
0.375	(*)	4.6338	4.6337	22.9914	22.992	61.3733	61.3763	121.9037	121.9125	198.2126	299.2762
	1	4.0354		23.6227		62.0314		120.8812		197.9171	293.4266
	2	4.2798		23.7724		61.7598		120.9824		198.3788	294.4107
	3	4.3884		23.6998		61.5064		121.1276		198.5901	294.9978
0.625	(*)	4.6338	4.6337	22.9916	22.992	61.3755	61.3763	121.9103	121.9125	198.2126	299.2554
	1	4.3126		22.2184		61.1649		117.8723		193.6996	290.8491
	2	4.5714		22.2770		61.3557		118.5899		195.4600	293.7625
	3	4.6547		22.2811		61.4775		119.0521		196.4258	295.3792
0.750	(*)	4.3469	4.3468	24.1607	24.1602	62.4806	62.4811	120.3628	120.365	200.8059	299.7157
	1	4.3801		22.0700		59.4769		117.2304		193.6328	287.011
	2	4.5772		22.4126		59.9505		118.4397		195.8656	290.6275
	3	4.6072		22.6572		60.2381		119.1656		197.1076	292.4955

(*) homogeneous material $\chi_{E_1} = \chi_{E_2} = 200/200 = 1$; $\chi_{\rho_1} = \chi_{\rho_2} = 5700/5700 = 1$.

Table 5 First six natural frequency coefficients of clamped-free beams with a step. h_0/L=0.28. $b_2 = 0.5 b_1$; $h_2 = h_1$. Beam A

l_1	n_1	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.250	(*)	4.06449	17.8630	37.1720	57.3490	79.4140	92.6848
	1	3.63133	17.2152	37.8868	58.8358	79.3544	94.5491
	2	3.79451	17.5841	38.1841	58.6410	79.0353	94.5874
	3	3.86955	17.7206	38.1507	58.3178	78.8752	94.4485
0.375	(*)	4.32412	17.2855	35.7138	58.0554	78.0655	92.5543

	1	3.78354	17.2768	37.0641	57.7466	78.9122	91.8550
	2	4.00354	17.5597	36.7862	57.7511	79.1242	91.3234
	3	4.10153	17.6018	36.4833	57.7650	79.1120	91.0414
0.625	(*)	4.32733	16.9958	36.7192	57.6770	76.8641	93.2328
	1	4.02863	16.6640	35.8378	56.0373	78.1264	91.7424
	2	4.26506	16.7748	36.0281	55.9663	78.6646	93.2880
	3	4.34232	16.7660	36.1687	55.9462	78.9014	94.1504
0.750	(*)	4.06743	17.5799	37.1502	59.1402	81.0677	89.7291
	1	4.08903	16.5377	35.4589	55.3383	75.7887	90.5304
	2	4.27107	16.7656	35.8330	56.0908	76.3512	91.5395
	3	4.30051	16.8773	36.0521	56.7025	76.6374	91.8365

(*) homogeneous material $\chi_{E_1} = \chi_{E_2} = 200/200 = 1$; $\chi_{\rho_1} = \chi_{\rho_2} = 5700/5700 = 1$.

Table 6 First six natural frequency coefficients of clamped-free beams with a step h_0/L=0.0017. $b_2 = b_1$; $h_2 = 0.5h_1$. Beam B

l_1	n_1	Ω_1	[1]	Ω_2	[1]	Ω_3	[1]	Ω_4	[1]	Ω_5	Ω_6
0.250	(*)	2.7846	2.7846	15.6249	15.6246	37.9882	37.9887	68.9783	68.9797	115.7087	177.3212
	1	2.6467		14.9821		38.2198		71.5726		118.2374	177.1747
	2	2.7016		15.3217		38.6780		71.4441		117.7290	176.9778
	3	2.7255		15.4488		38.7363		71.0993		117.2629	176.8448
0.375	(*)	3.4955	3.4954	15.5133	15.5134	37.6981	37.6984	78.7317	78.7355	127.8980	182.6741
	1	3.2168		15.4993		39.7631		78.6336		125.6686	185.0525
	2	3.3340		15.8160		39.5096		78.4952		126.3286	185.3328
	3	3.3840		15.8727		39.1949		78.4009		126.7853	185.3927
0.625	(*)	4.4912	4.4914	16.7903	16.7903	46.8926	46.8937	89.9449	89.9482	148.9669	226.3189
	1	4.1806		17.6457		45.6956		90.2874		144.5670	221.5364
	2	4.4176		17.4948		46.0733		90.4663		146.0521	223.2275
	3	4.4952		17.3331		46.3366		90.4909		147.0350	224.0949
0.750	(*)	4.3318	4.3318	21.8649	21.8650	48.1350	48.1358	99.8838	99.8900	168.7895	238.3905
	1	4.3597		20.6312		49.2036		95.5215		161.9129	236.5752
	2	4.5553		20.8023		49.3366		96.7237		163.6992	238.9683
	3	4.5855		20.9209		49.3557		97.5346		164.6782	240.1911

(*) homogeneous material $\chi_{E_1} = \chi_{E_2} = 200/200 = 1$; $\chi_{\rho_1} = \chi_{\rho_2} = 5700/5700 = 1$.

In Tables 4 and 6 a comparison is made with Mao et al. [1] when the material properties are assumed to be constant in both beam elements. It can be seen that the agreement with the present results is excellent.

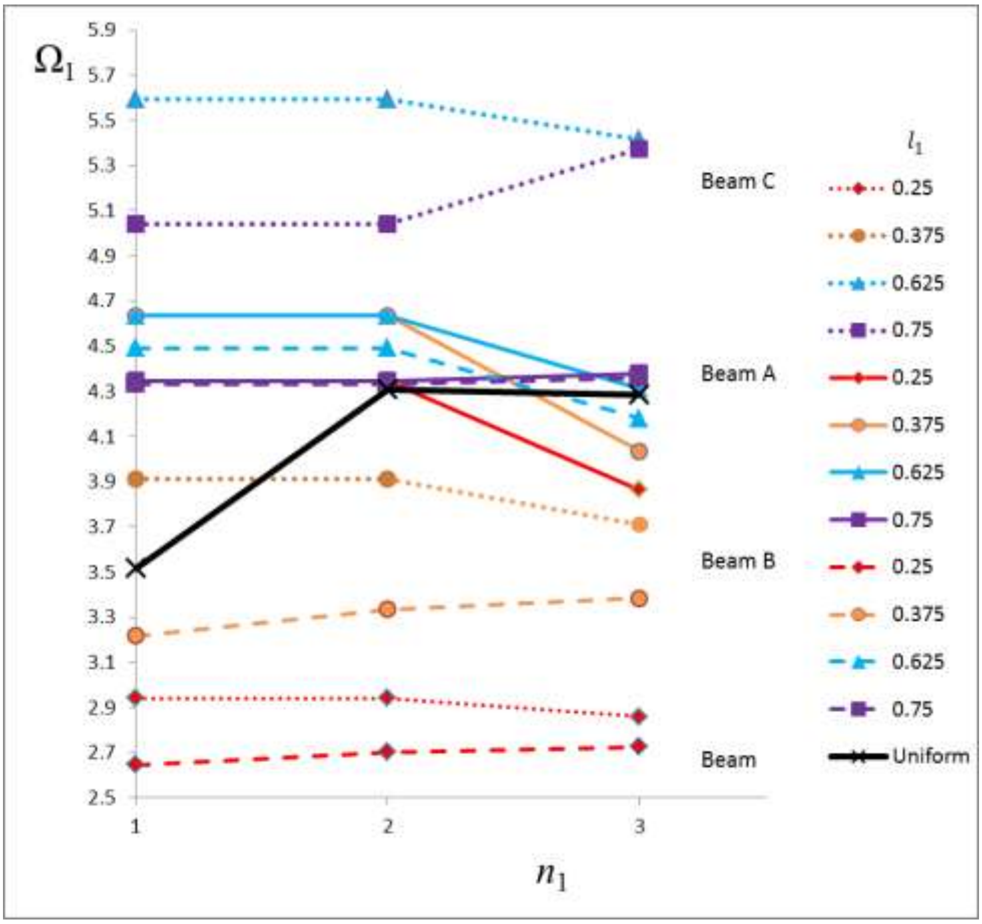


Fig. 3 Fundamental natural frequency coefficient for cantilever stepped AFG beams

Fig. 3 shows the fundamental frequency coefficients for cantilever beams, with different locations of the step. l_1 is equal to 0.25, 0.375, 0.625, 0.75 and AFG material properties for the part of the beam of length l_1 are χ_{E_1} =70/200=0.35; χ_{ρ_1} =2702/5700=0.474; and $\chi_{E_2} = 200/200 = 1$; $\chi_{\rho_2} = 5700/5700 = 1$ for the element of length l_2. $h_0 / L = 0.0017$; (Beam A, color solid line; Beam B, color dotted line; Beam C, color dashed line). The frequency coefficients of the stepped beams can be compared with the coefficients of the uniform beam of similar material properties, which is indicated by a solid black line. It can be seen that it is possible to have lighter structures with higher coefficients of fundamental frequency when the beams are of AFG materials. [10].

Hereafter there are several numerical examples of frequency coefficients of stepped Timoshenko beams with different AFG materials and eight combinations of classical boundary conditions. The step is assumed to be at l_1=0.625, and the changes in the cross-section are: $b_2 = 0.5\,b_1$; $h_2 = 0.5\,h_1$. Table 7 presents natural frequency coefficients of stepped beams of AFG materials: χ_{E_1} =70/200=0.35; χ_{ρ_1} =2702/5700=0.474; $\chi_{E_2} = 200/200 = 1$; $\chi_{\rho_2} = 5700/5700 = 1$. With $h_0/L = 0.28$

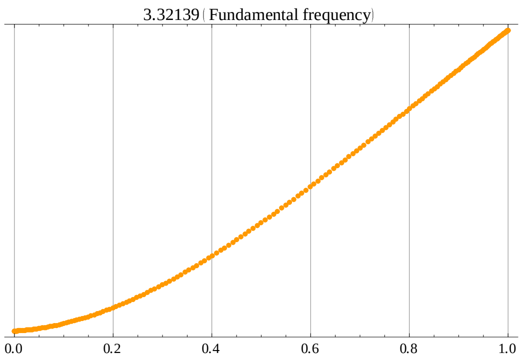
Table 7 First six natural frequency coefficients of beams of AFG materials with a step. $l_1 = 0.625$; $b_2 = 0.5\,b_1$; $h_2 = 0.5\,h_1$; h_0/L=0.28. Beam-C

B.C.	n_1	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
SS-C	(*)	7.19564	27.5087	48.6301	67.294	90.7471	94.8681
	1	8.10075	27.1250	47.9073	67.3493	87.9421	91.0087
	2	7.88432	27.7650	48.2833	68.4196	89.4862	92.1438
	3	7.74909	28.0319	48.3773	68.8784	90.3165	92.5564
	(*)	*(8.51776 **)*	*(37.4279)*	*(80.3227)*	*(129.222)*	*(211.882)*	*(277.673)*
C-SS	(*)	9.07035	28.0166	44.0097	67.7693	91.2451	97.3789
	1	10.3578	27.4970	45.4545	66.7572	88.6132	96.2024
	2	10.2765	27.7143	45.2915	67.5657	89.9042	97.5837
	3	10.1043	27.8085	45.1025	68.0746	90.5237	98.1914
	(*)	*(10.4072 **)*	*(42.1479)*	*(71.4341)*	*(141.237)*	*(195.943)*	*(295.073)*
C-C	(*)	11.0143	30.3741	49.5723	69.2024	92.5762	99.0514
	1	12.8290	30.4488	49.6334	68.9892	89.9604	96.9636
	2	12.5740	30.7034	49.6123	69.7247	91.5159	98.1565
	3	12.3046	30.7872	49.5514	70.1695	92.2960	98.6610
	(*)	*(13.6160**)*	*(49.4501)*	*(87.6013)*	*(151.384)*	*(224.476)*	*(308.164)*
C-F	(*)	5.20353	13.5788	30.9211	51.5550	70.5594	93.5004
	1	5.05507	14.4658	31.1255	51.3059	71.2779	91.3077
	2	5.28648	14.3217	31.3292	51.2914	71.9772	93.3965
	3	5.34654	14.1618	31.3901	51.2428	72.3584	94.5246
	(*)	*(5.59372**)*	*(15.9079)*	*(49.0616)*	*(87.1239)*	*(151.407)*	*(224.520)*
F-C	(*)	0.929072	9.20821	34.9496	51.9305	71.4098	94.814
	1	1.012010	10.1953	32.1938	51.6411	70.3562	90.5051
	2	0.966524	10.0134	33.4188	52.4965	72.1808	91.6169
	3	0.950797	9.89641	34.0806	52.7623	72.8943	91.9021
	(*)	*(0.947401**)*	*(11.4575)*	*(50.0108)*	*(87.6521)*	*(151.347)*	*(224.476)*
SS-SS	(*)	4.98837	25.1872	42.901	65.7235	90.6115	92.4199
	1	5.63974	24.0047	43.5573	64.9097	87.9153	89.0929
	2	5.48827	24.6502	43.7541	66.1185	89.469	90.1089
	3	5.39267	24.9659	43.7127	66.6739	90.1937	90.5524
	(*)	*(5.34078**)*	*(32.6569)*	*(63.1071)*	*(120.781)*	*(184.236)*	*(263.781)*

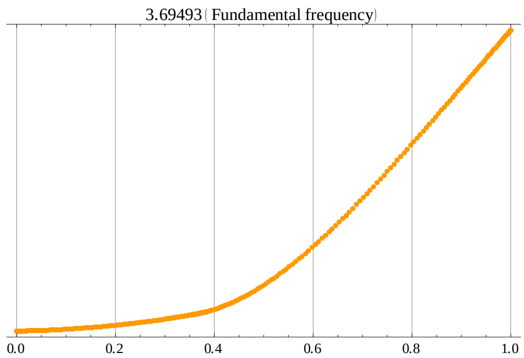
SS-F	(*)	0	11.7121	28.0948	50.5545	68.684	90.7496
	1	0	11.4088	27.8168	49.5263	69.6462	87.9270
	2	0	11.5026	28.4034	49.9353	70.6627	89.4768
	3	0	11.5485	28.6469	50.0470	71.0366	90.3489
	(*)	*(0)*	*(12.9282 **)*	*(37.2397)*	*(79.7088)*	*(129.215)*	*(211.932)*
F-SS	(*)	0	6.72053	31.5573	47.1766	70.2838	92.4005
	1	0	7.37897	28.6308	47.6707	68.3732	89.0052
	2	0	7.25495	29.6362	48.521	70.3443	90.0524
	3	0	7.17966	30.1991	48.7932	71.1526	90.255
	(*)	*(0)*	*(7.52668 **)*	*(42.5766)*	*(71.6266)*	*(141.200)*	*(195.939)*
F-F	(*)	0	0	13.5386	35.5016	54.0474	72.4241
	1	0	0	13.4427	32.8224	53.5280	72.2935
	2	0	0	13.5085	34.0342	54.4367	74.0526
	3	0	0	13.5416	34.6933	54.7046	74.6769
	(*)	*(0)*	*(0)*	*(15.2543 **)*	*(49.6032)*	*(87.1795)*	*(151.369)*

(*) homogeneous material; $\chi_{E_1} = \chi_{E_2} = 200/200 = 1$; $\chi_{\rho_1} = \chi_{\rho_2} = 5700/5700 = 1$.

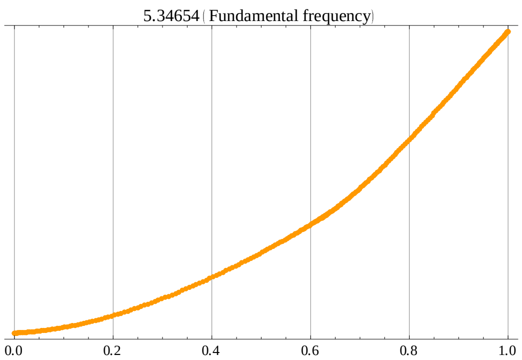
**slender beam with h_0/L=0.0017.



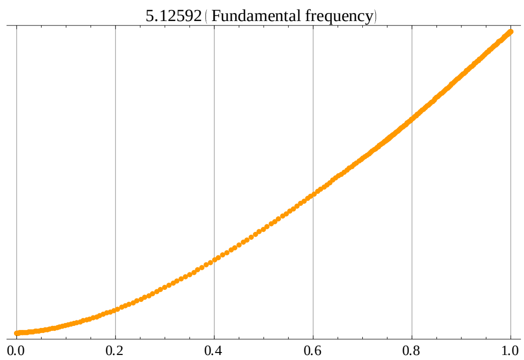
a) Uniform beam $l_1 = 1$, homogeneous material



b) Stepped beam $l_1 = 0.375$, AFG material, $n = 3$



c) Stepped beam $l_1 = 0.625$, AFG material, $n = 3$



d) Stepped beam $l_1 = 0.750$, AFG material, $n = 3$

Fig.4 Fundamental mode shapes of cantilever Timoshenko beams. h_0/L=0.28. Beam C

Fig. 4 shows the fundamental mode shapes of cantilever Timoshenko beams. **Fig. 4** a) corresponds to a uniform beam of homogeneous material. While **Figs. 4** b), c) and d) correspond to stepped beams, (case Beam C; $b_2 = 0.5\, b_1$; $h_2 = 0.5\, h_1$), with the step located at l_1 = 0.375, 0.625 and 0.750, respectively. Again the portion of the beam of length l_1 is made of AFG material, equations (40) with $n = 3$, and the span of length l_2 has homogeneous material. In general, the effect of the step on the dynamic behavior of the beam can be observed in the magnitude of the fundamental frequency coefficient and in the shape associated to this mode.

5 Conclusions

This paper examines the case of vibrations of stepped inhomogeneous beams on the basis of the Timoshenko beam theory. Different combinations of classical boundary conditions are considered. The equations of motion for the stepped AFG beams are obtained applying Hamilton´s principle.

An approximate differential quadrature model is developed since the DQM can easily be applied for any type of inhomogeneity in the axial direction (step change in geometry and/or material properties).

The results of natural frequencies of stepped Timoshenko beams made of AFG materials are provided.

The variation of the material properties and step changes plays an important role in the variations of the natural frequency coefficients. It is possible to have lighter structures with higher coefficients of fundamental frequency when the beams are of AFG materials and have step variations of the cross-sectional area, second moment of area and material properties.

Additionally, since to the authors' knowledge this technological situation has not been previously studied in the literature, the present results may be used as a means of comparison for future studies.

6 Acknowledgments

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References

1 Mao Q, Pietrzko S (2010) Free vibration analysis of stepped beams by using Adomian decomposition method. Appl Math Comput 217:3429–3441

2 Jang SK, Bert CW (1989) Free vibration of stepped beams: higher mode frequencies and effects of steps on frequency. J Sound Vib132(1):164-168

3 Naguleswaran S (2002) Natural frequencies, sensitivity and mode shape details of an Euler-Bernoulli beam with one-step change in cross-section and with ends on classical supports. J Sound Vib 252(4):751-767

4 Koplow MA, Bhattacharyya A, Mann BP (2006) Closed form solutions for the dynamic response of Euler–Bernoulli beams with step changes in cross-section. J Sound Vib 295:214–22

5 Duan G, Wang X (2013) Free vibration analysis of multiple-stepped beams by the discrete singular convolution Appl Math Comput 219:11096–11109

6 Wang X, Wang Y (2013) Free vibration analysis of multiple-stepped beams by the differential quadrature element method. Appl Math Comput 219:5802-5810

7 Singh KV, Li G, Pang SS (2006) Free vibration and physical parameter identification of non-uniform composite beams. Compos Struct74:37–50

8 Yavari A, Sarkani S, Reddy JN (2001) On nonuniform Euler–Bernoulli and Timoshenko beams with jump discontinuities: application of distribution theory. Int J Solids Struct38:8389–8406

9 Jaworski JW (2008) Dowell EH Free vibration of a cantilevered beam with multiple steps: Comparison of several theoretical methods with experiment. J Sound Vib 312:713–725

10 Laura PAA, Rossi RE, Pombo JL, Pasqua D (1991) Dynamic stiffening of straight beams of rectangular cross-section: a comparison of finite element predictions and experimental results. J Sound Vib150:174–178

11 Rajasekaran S (2013) Buckling and vibration of axially functionally graded nonuniform beams using differential transformation based dynamic stiffness approach. Meccanica 48(5):1053-1070

12 Leung AYT, Zhou WE, Lim CW, Yuen RKK, Lee U (2001) Dynamic stiffness for piecewise non-uniform Timoshenko column by power series-part I: conservative axial force. Int J Numer Math Eng 51:505–29

13 Huang Y, Yang L-E, Luo Q-Z (2013) Free vibration of axially functionally graded Timoshenko beams with non-uniform cross-section. Compos: Part B 45:1493–1498

14 Karami G, Malekzadeh P, Shahpari SA (2003) A DQEM for vibration of shear deformable nonuniform beams with general boundary conditions. Eng Struct 25:1169–1178

15 Felix DH, Rossi RE, Bambill DV (2009) Análisis de vibración libre de una viga Timoshenko escalonada, centrífugamente rigidizada, mediante el método de cuadratura diferencial. Rev Int Met Num Calc Dis Ing 25(2):111-132

16 Nakamura T, Wang T, Sampath S (2000) Determination of properties of graded materials by inverse analysis and instrumented indentation. Acta Materialia 48:4293–4306

17 Shahba A, Attarnejad R, Tavanaie Marvi M, Hajilar S (2011) Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions. Compos: Part B 42:801–808

18 Shahba A, Rajasekaran S (2012) Free vibration and stability of tapered Euler–Bernoulli beams made of axially functionally graded materials. Appl Math Model 36:3094–3111

19 Shahba A, Attarnejad R, Hajilar S A (2013) Mechanical-based solution for axially functionally graded tapered Euler-Bernoulli beams. Mech Adv Mater Struct 20: 696–707

20 Rajasekaran S (2013) Free vibration of centrifugally stiffened axially functionally graded tapered Timoshenko beams using differential transformation and quadrature methods. Appl Math Model 37:4440–4463

21 Bellman RE, Casti J (1971) Differential quadrature and long-term integration. J Math Anal Appl 34:235-238

22 Bert CW, Malik M (1997) Differential quadrature: a powerful new technique for analysis of composite structures. Compos Struct 39(3-4):179-189

23 Laura PAA, Gutiérrez RH (1993) Analysis of vibrating Timoshenko beams using the method of differential quadrature. Shock Vib Digest 1:89-93

24 Liu GR, Wu TY (2001) Vibration analysis of beams using the generalized differential quadrature rule and domain decomposition. J Sound Vib 246(3):461-481

25 Karami G, Malekzadeh P (2002) A new differential quadrature methodology for beam analysis and the associated differential quadrature element method. Comput Methods Appl Mech Eng 191: 3509–3526

26 Bambill DV, Felix DH, Rossi RE (2010) Vibration analysis of rotating Timoshenko beams by means of the differential quadrature method. Struct Eng Mech 34(2):231-245

27 Bambill DV, Rossit CA, Rossi RE, Felix DH, Ratazzi AR (2013) Transverse free vibration of non uniform rotating Timoshenko beams with elastically clamped boundary conditions. Meccanica 48(6):1289-1311

28 Su H, Banerjee JR, Cheung CW (2013) Dynamic stiffness formulation and free vibration analysis of functionally graded beams. Compos Struct 106:854-862

29 Banerjee JR (2001) Dynamic stiffness formulation and free vibration analysis of centrifugally stiffened Timoshenko beams. J Sound Vib 247(1):97-115

Figure 1
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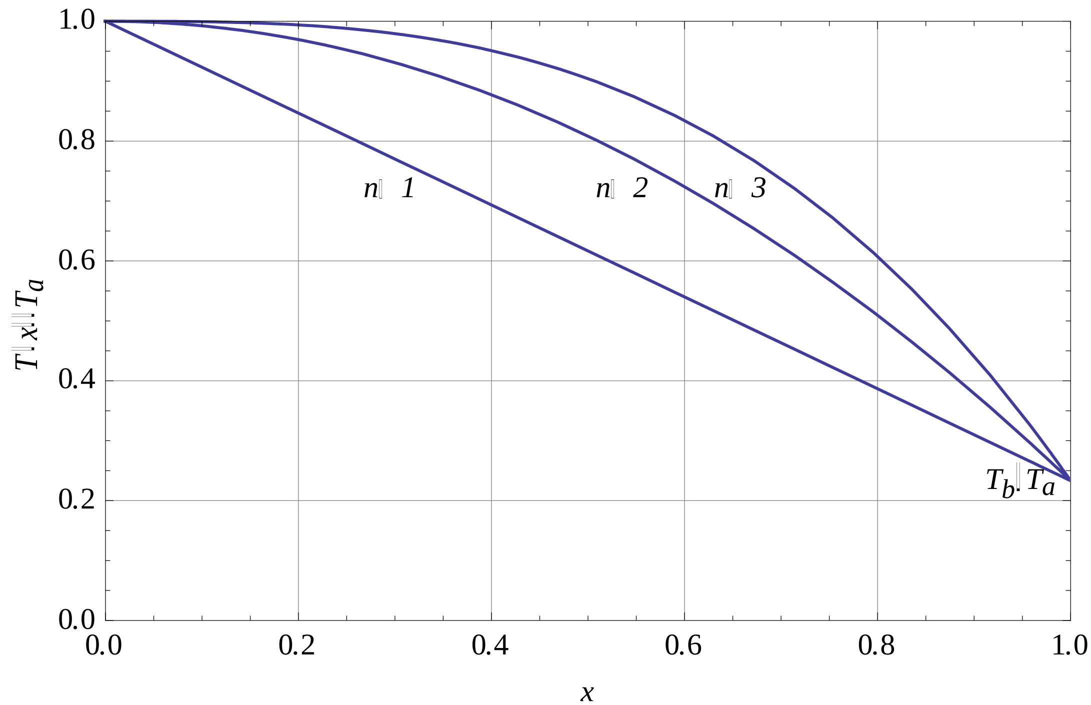

Fig. 1 Power law relation of AFG material properties. $x = \bar{x}/L$

Figure 2

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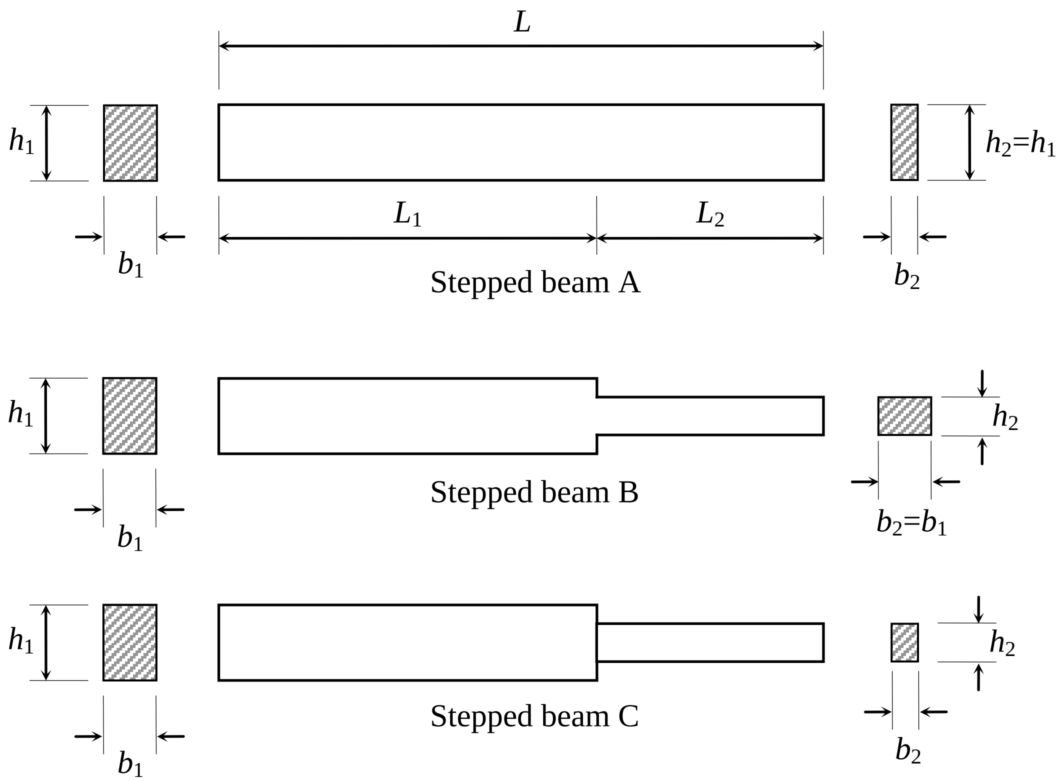

Fig. 2 Stepped AFG Timoshenko beams

Figure 3

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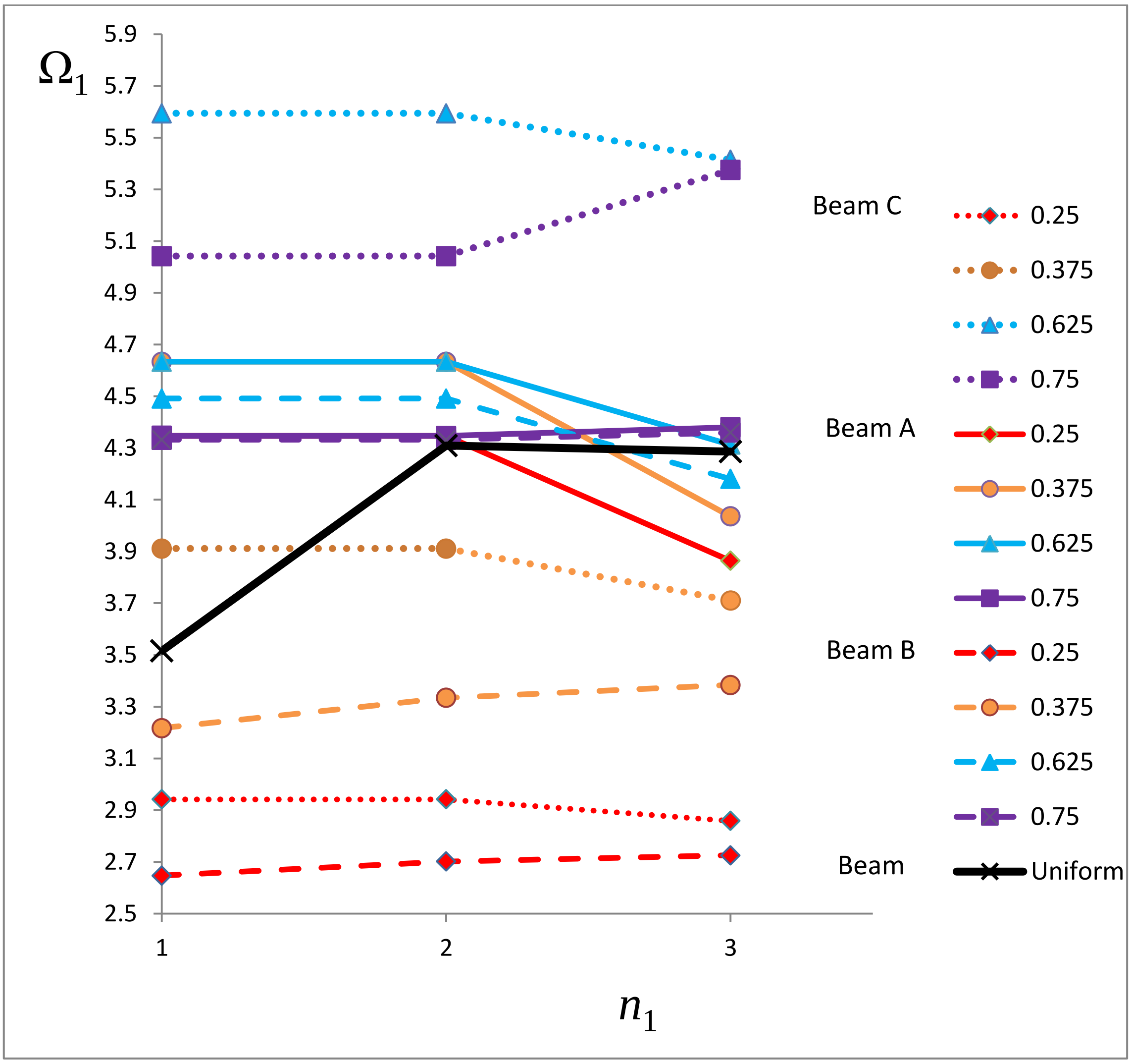

Fig. 3 Fundamental natural frequency coefficient for cantilever AFG stepped beams

Figure 4
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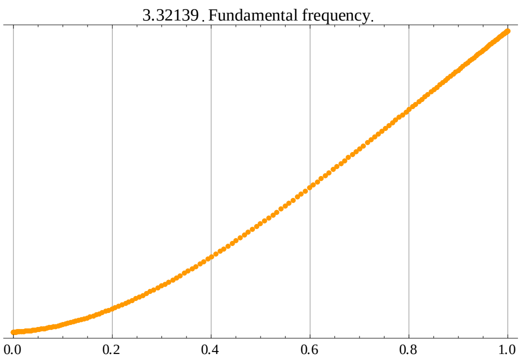
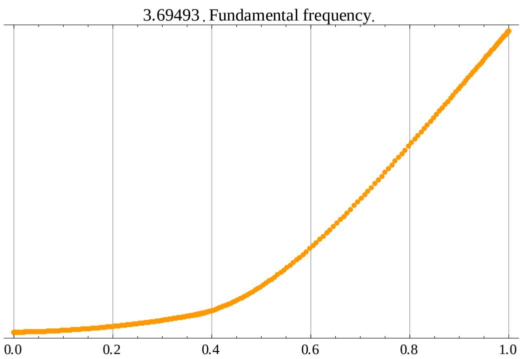
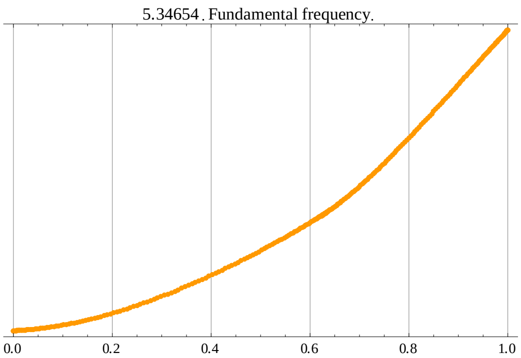


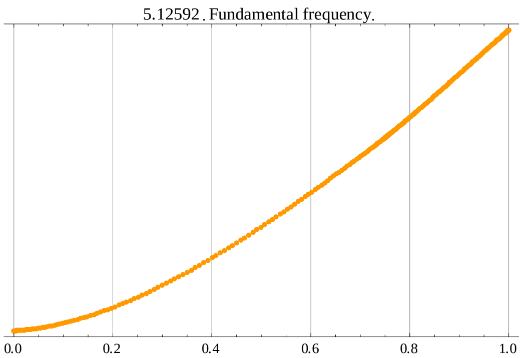
a) Uniform beam $l_1 = 1$, homogeneous material



b) Stepped beam $l_1 = 0.375$, AFG material, $n = 3$



c) Stepped beam $l_1 = 0.625$, AFG material, $n = 3$



d) Stepped beam $l_1 = 0.750$, AFG material, $n = 3$

Fig.4 Fundamental mode shapes of cantilever Timoshenko beams. h_0/L=0.28. Beam C

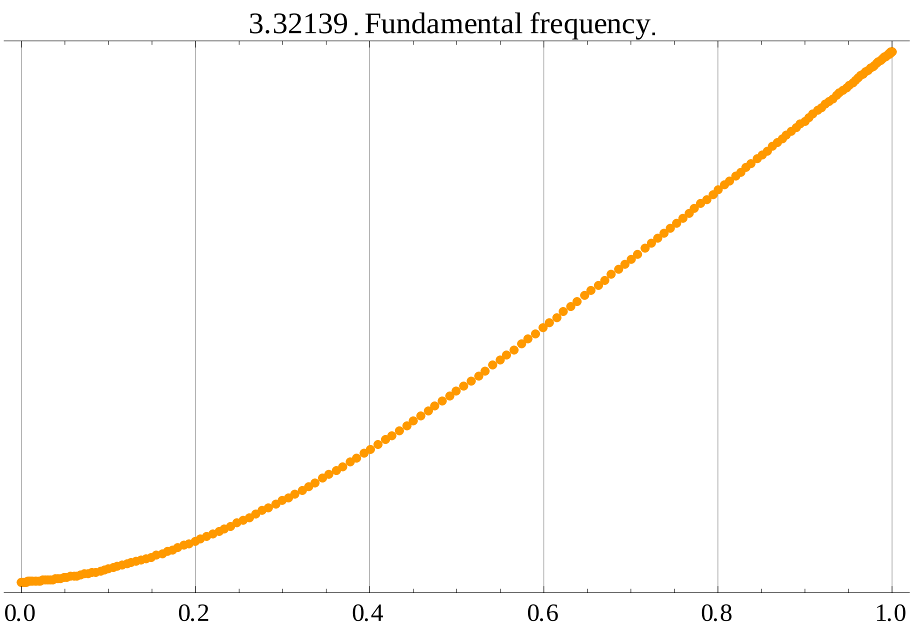


Fig.4 a) Uniform beam $l_1 = 1$, homogeneous material

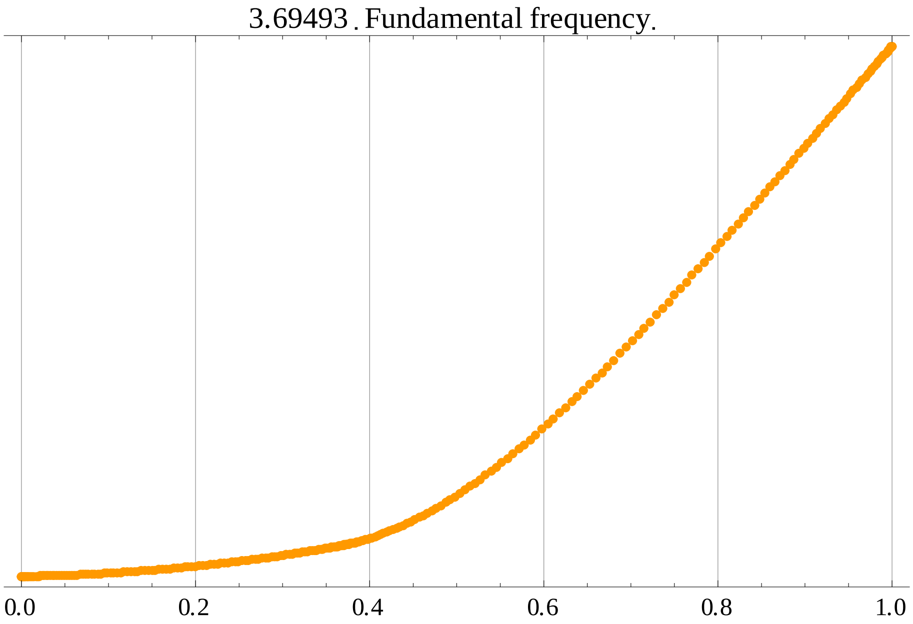


Fig.4 b) Stepped beam $l_1 = 0.375$, AFG material, $n = 3$

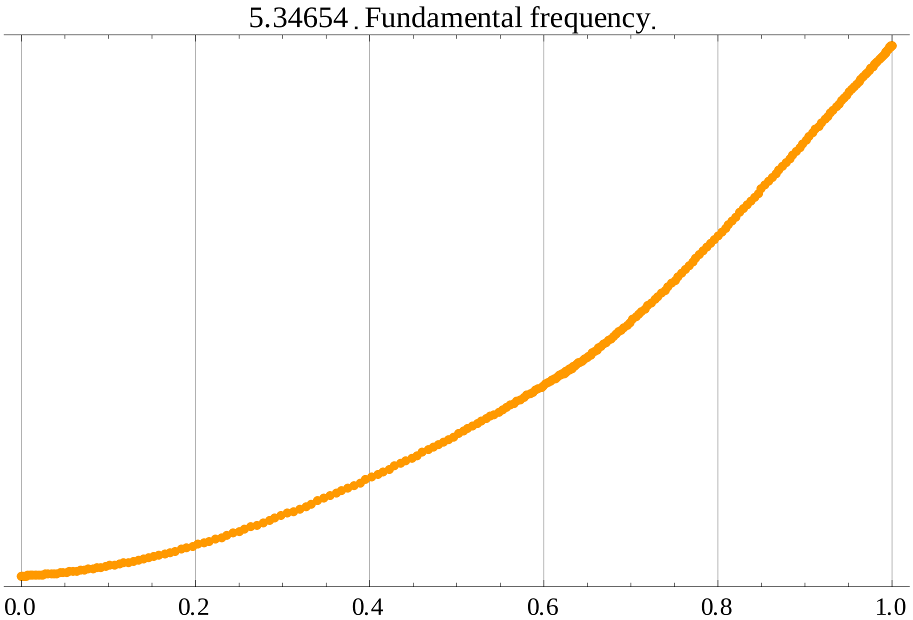


Fig.4 c) Stepped beam $l_1 = 0.625$, AFG material, $n = 3$

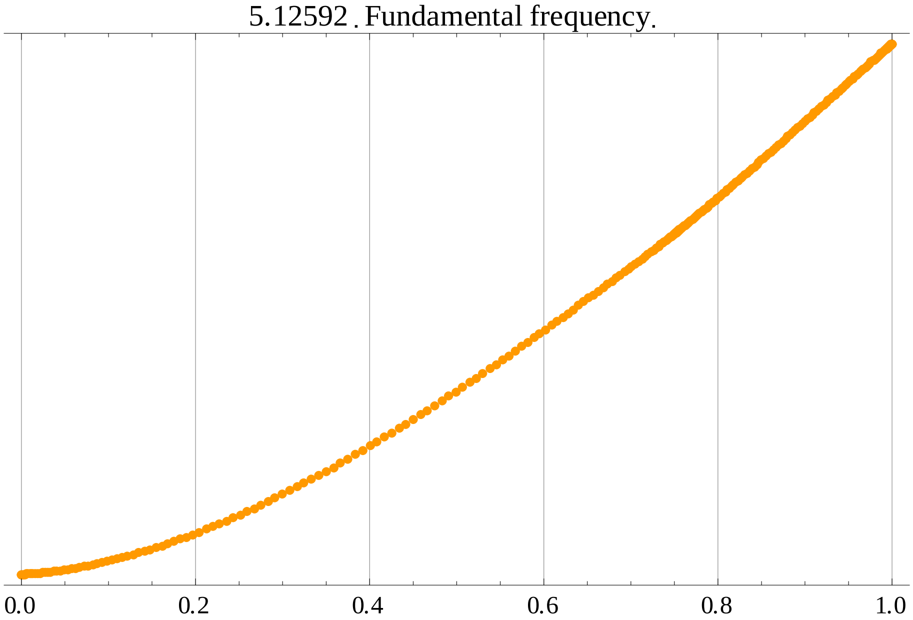


Fig. 4 d) Stepped beam $l_1 = 0.750$, AFG material, $n = 3$

Table
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Free vibrations of stepped axially functionally graded Timoshenko beams: **Tables**

List of tables

Table 1 Convergence analysis: First six natural frequency coefficients of uniform homogeneous Timoshenko beams: h/L =0.35; κ=5/6; ν=0.30

B.C.	p	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	
C-F	11	3.22713	14.4689	31.5016	47.8895	62.3557	68.0104	present
	21	3.22713	14.4689	31.5025	47.9090	62.3470	67.9901	
	31	3.22713	14.4689	31.5025	47.9090	62.3470	67.9901	
	41	3.22713	14.4689	31.5025	47.9090	62.3470	67.9901	
	51	3.22713	14.4689	31.5025	47.9090	62.3470	67.9901	
		3.23	14.47	21.50	47.91	62.35	-	[12]
		3.227128	14.468928	31.502540	47.911084	62.353342	-	[13]
		3.2272	14.4729	31.5425	48.0372	-	-	[17]
C-SS	41	11.082499	27.114378	44.843534	59.203032	63.339499	76.247312	present
		11.08	27.11	44.84	59.20	63.34	-	[12]
		11.082499	27.114378	44.844585	59.203448	63.349869	-	[13]
C-C	41	13.834758	28.517925	45.665951	61.862050	68.283611	80.412094	present
		13.84	28.52	45.67	61.86	68.28	-	[12]
		13.834758	28.517926	45.667237	61.867699	68.292529	-	[13]
F-F	41	16.791957*	33.814869	51.521440	58.991998	73.739689	75.304144	present
		16.79	33.82	51.52	58.99	73.74	-	[12]
		16.791957	33.814869	51.526943	58.993336	73.763812	-	[13]

* The repeated null eigenvalues for rigid translation and rotation for the F-F case are omitted in the Table.

Table 2 First six natural frequency coefficients of AFG Timoshenko beams, with a small taper in height: $h(x) = h_0(1-0.1x)$; h_0 / L =0.35 ; κ=5/6; ν=0.30; $\chi_E = 0.35$; $\chi_\rho = 0.47$

B.C.	n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	
C-F	1	3.944636	14.93640	30.57274	46.40688	60.9420	65.7584	present
		3.944636	14.93640	30.57274	46.40888	-	-	[13]
	2	3.935789	15.15333	31.22390	47.58364	62.7344	66.9431	present
		3.935789	15.15333	31.22390	47.58572	-	-	[13]
		3.9359	15.1577	31.2638	47.7164	-	-	[17]
	3	3.849497	15.19867	31.59328	48.24423	63.7301	67.5523	present
		3.849497	15.19869	31.59328	48.24669	-	-	[13]
	4	3.77127	15.1970	31.8164	48.6325	64.3432	67.9315	present
		3.771269	15.19695	31.81639	48.63501	-	-	[13]
C-SS	1	10.88465	25.56609	42.18263	58.13438	60.9556	74.1197	present
		10.88465	25.56609	42.18907	58.14309	-	-	[13]
	2	10.80070	25.61789	42.64742	58.85281	62.7800	75.2574	present
		10.80070	25.61789	42.64780	58.85946	-	-	[13]
	3	10.73937	25.63540	42.85451	59.08722	63.7788	75.7602	present
		10.73937	25.63540	42.85506	59.09377	-	-	[13]
	4	10.71567	25.66653	42.95520	59.14091	64.391	75.9913	present
		10.71567	25.66653	42.95581	59.14709	-	-	[13]
C-C	1	12.68158	26.49101	42.64171	58.65182	66.816	75.9159	present
		12.68158	26.49101	42.64203	58.66849	-	-	[13]
	2	12.46329	26.38044	42.96071	59.39162	68.058	77.0951	present
		12.46329	26.38044	42.92108	59.40234	-	-	[13]
		12.4689	26.4153	43.0904	59.6829	-	-	[17]
	3	12.37525	26.31883	43.08334	59.68936	68.5813	77.5992	present
		12.37525	26.31883	43.08388	59.69942	-	-	[13]
	4	12.36220	26.31154	43.13433	59.80551	68.8795	77.8249	present
		12.36220	26.31158	43.13499	59.81504	-	-	[13]

Table 3 First six natural frequency coefficients of uniform cross-section AFG Timoshenko beams with various boundary conditions. h_0 / L =0.28; $\chi_E = 0.35$; $\chi_\rho = 0.47$

n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
	SS-C						C-SS					
(*)	12.1785	31.2031	52.8839	75.5682	91.1848	98.6071	12.1785	31.2031	52.8839	75.5682	91.1848	98.6071
1	10.6527	28.775	49.4047	71.0515	87.9797	93.3051	12.1651	29.9625	50.4151	71.7136	84.8549	93.2976
2	10.6989	28.9102	49.9738	72.0967	89.5464	94.6851	12.0899	30.0762	50.9878	72.6812	87.2899	94.7715
3	10.7643	28.9303	50.1384	72.4798	90.0939	95.3002	12.0341	30.1315	51.2538	73.0823	88.5729	95.4611
	C-C						SS-SS					
(*)	15.6659	33.6285	54.4651	76.0997	98.6071	98.8973	8.82664	28.3570	51.2257	74.8810	88.4591	98.5858
1	14.6202	31.5549	51.3116	71.8519	93.0976	94.0592	8.28580	26.8203	48.4458	70.8139	83.4161	93.2972
2	14.3771	31.4512	51.6786	72.7317	94.5697	95.9212	8.45284	27.2194	49.237	71.9953	85.4714	94.5548
3	14.2839	31.3979	51.8204	73.097	95.176	96.7421	8.53962	27.3644	49.5312	72.4364	86.4428	95.0597
	C-F						F-C					
(*)	3.32139	16.2331	36.5346r	57.9414	79.6803	93.6481	3.32139	16.2331	36.5346	57.9414	79.6803	93.6481
1	4.02882	16.8325	35.8482	56.0353	76.3509	88.5229	2.39704	13.9273	33.4166	53.8750	74.8178	89.1143
2	4.01239	17.0959	36.6134	57.3718	78.3942	90.6695	2.47269	14.2114	33.9864	54.8942	76.3463	90.7864
3	3.91997	17.1605	37.0517	58.1304	79.5352	91.8588	2.56071	14.4004	34.1414	55.1771	76.8348	91.3604
	SS-F						F–SS					
(*)	0	13.1082	33.8752	56.692	78.8321	90.6865	0	13.1082	33.8752	56.692	78.8321	90.6865
1	0	13.3545	32.9559	54.5518	75.286	87.1940	0	11.6817	31.5971	53.1957	74.4974	84.2197
2	0	13.8453	34.0090	56.1060	77.5417	88.6983	0	12.1240	32.4552	54.417	76.2008	86.1989
3	0	13.9978	34.5127	56.9025	78.7123	89.4351	0	12.3723	32.7588	54.811	76.7682	87.1194

(*) homogeneous material $\chi_E = 200/200$ =1; $\chi_\rho = 5700/5700$ =1.

Table 4 First six natural frequency coefficients of clamped-free AFG beams with a step. h_0/L=0.0017. $b_2 = 0.5b_1$; $h_2 = h_1$. Beam A

l_1	n_1	Ω_1	[1]	Ω_2	[1]	Ω_3	[1]	Ω_4	[1]	Ω_5	Ω_6
0.250	(*)	4.3468	4.3468	24.1602	24.1602	62.4786	62.4811	120.3563	120.365	200.7861	299.6742
	1	3.8637		23.6956		64.1197		121.8507		199.6484	298.4878
	2	4.0457		24.1105		64.2078		121.4757		199.4563	298.747
	3	4.1295		24.2318		63.9806		121.055		199.3913	299.0133
0.375	(*)	4.6338	4.6337	22.9914	22.992	61.3733	61.3763	121.9037	121.9125	198.2126	299.2762
	1	4.0354		23.6227		62.0314		120.8812		197.9171	293.4266
	2	4.2798		23.7724		61.7598		120.9824		198.3788	294.4107
	3	4.3884		23.6998		61.5064		121.1276		198.5901	294.9978
0.625	(*)	4.6338	4.6337	22.9916	22.992	61.3755	61.3763	121.9103	121.9125	198.2126	299.2554
	1	4.3126		22.2184		61.1649		117.8723		193.6996	290.8491
	2	4.5714		22.2770		61.3557		118.5899		195.4600	293.7625
	3	4.6547		22.2811		61.4775		119.0521		196.4258	295.3792
0.750	(*)	4.3469	4.3468	24.1607	24.1602	62.4806	62.4811	120.3628	120.365	200.8059	299.7157
	1	4.3801		22.0700		59.4769		117.2304		193.6328	287.011
	2	4.5772		22.4126		59.9505		118.4397		195.8656	290.6275
	3	4.6072		22.6572		60.2381		119.1656		197.1076	292.4955

(*) homogeneous material $\chi_{E_1} = \chi_{E_2} = 200/200$ =1; $\chi_{\rho_1} = \chi_{\rho_2} = 5700/5700$ =1.

Table 5 First six natural frequency coefficients of clamped-free beams with a step. h_0/L=0.28. $b_2 = 0.5b_1$; $h_2 = h_1$. Beam A

l_1	n_1	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.250	(*)	4.06449	17.8630	37.1720	57.3490	79.4140	92.6848
	1	3.63133	17.2152	37.8868	58.8358	79.3544	94.5491
	2	3.79451	17.5841	38.1841	58.6410	79.0353	94.5874
	3	3.86955	17.7206	38.1507	58.3178	78.8752	94.4485
0.375	(*)	4.32412	17.2855	35.7138	58.0554	78.0655	92.5543
	1	3.78354	17.2768	37.0641	57.7466	78.9122	91.8550
	2	4.00354	17.5597	36.7862	57.7511	79.1242	91.3234
	3	4.10153	17.6018	36.4833	57.7650	79.1120	91.0414
0.625	(*)	4.32733	16.9958	36.7192	57.6770	76.8641	93.2328
	1	4.02863	16.6640	35.8378	56.0373	78.1264	91.7424
	2	4.26506	16.7748	36.0281	55.9663	78.6646	93.2880
	3	4.34232	16.7660	36.1687	55.9462	78.9014	94.1504
0.750	(*)	4.06743	17.5799	37.1502	59.1402	81.0677	89.7291
	1	4.08903	16.5377	35.4589	55.3383	75.7887	90.5304
	2	4.27107	16.7656	35.8330	56.0908	76.3512	91.5395
	3	4.30051	16.8773	36.0521	56.7025	76.6374	91.8365

(*) homogeneous material $\chi_{E_1} = \chi_{E_2} = 200/200$ =1; $\chi_{\rho_1} = \chi_{\rho_2} = 5700/5700$ =1.

Table 6 First six natural frequency coefficients of clamped-free beams with a step h_0/L=0.0017. $b_2 = b_1; h_2 = 0.5h_1$. Beam B

l_1	n_1	Ω_1	[1]	Ω_2	[1]	Ω_3	[1]	Ω_4	[1]	Ω_5	Ω_6
0.250	(*)	2.7846	2.7846	15.6249	15.6246	37.9882	37.9887	68.9783	68.9797	115.7087	177.3212
	1	2.6467		14.9821		38.2198		71.5726		118.2374	177.1747
	2	2.7016		15.3217		38.6780		71.4441		117.7290	176.9778
	3	2.7255		15.4488		38.7363		71.0993		117.2629	176.8448
0.375	(*)	3.4955	3.4954	15.5133	15.5134	37.6981	37.6984	78.7317	78.7355	127.8980	182.6741
	1	3.2168		15.4993		39.7631		78.6336		125.6686	185.0525
	2	3.3340		15.8160		39.5096		78.4952		126.3286	185.3328
	3	3.3840		15.8727		39.1949		78.4009		126.7853	185.3927
0.625	(*)	4.4912	4.4914	16.7903	16.7903	46.8926	46.8937	89.9449	89.9482	148.9669	226.3189
	1	4.1806		17.6457		45.6956		90.2874		144.5670	221.5364
	2	4.4176		17.4948		46.0733		90.4663		146.0521	223.2275
	3	4.4952		17.3331		46.3366		90.4909		147.0350	224.0949
0.750	(*)	4.3318	4.3318	21.8649	21.8650	48.1350	48.1358	99.8838	99.8900	168.7895	238.3905
	1	4.3597		20.6312		49.2036		95.5215		161.9129	236.5752
	2	4.5553		20.8023		49.3366		96.7237		163.6992	238.9683
	3	4.5855		20.9209		49.3557		97.5346		164.6782	240.1911

(*) homogeneous material $\chi_{E_1} = \chi_{E_2} = 200/200$ =1; $\chi_{\rho_1} = \chi_{\rho_2} = 5700/5700$ =1.

Table 7 First six natural frequency coefficients of beams of AFG materials with a step. $l_1 = 0.625$; $b_2 = 0.5\, b_1$; $h_2 = 0.5\, h_1$; h_0/L=0.28. Beam-C

B.C.	n_1	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
SS-C	(*)	7.19564	27.5087	48.6301	67.294	90.7471	94.8681
	1	8.10075	27.1250	47.9073	67.3493	87.9421	91.0087
	2	7.88432	27.7650	48.2833	68.4196	89.4862	92.1438
	3	7.74909	28.0319	48.3773	68.8784	90.3165	92.5564
	(*)	*(8.51776 **)*	*(37.4279)*	*(80.3227)*	*(129.222)*	*(211.882)*	*(277.673)*
C-SS	(*)	9.07035	28.0166	44.0097	67.7693	91.2451	97.3789
	1	10.3578	27.4970	45.4545	66.7572	88.6132	96.2024
	2	10.2765	27.7143	45.2915	67.5657	89.9042	97.5837
	3	10.1043	27.8085	45.1025	68.0746	90.5237	98.1914
	(*)	*(10.4072 **)*	*(42.1479)*	*(71.4341)*	*(141.237)*	*(195.943)*	*(295.073)*
C-C	(*)	11.0143	30.3741	49.5723	69.2024	92.5762	99.0514
	1	12.8290	30.4488	49.6334	68.9892	89.9604	96.9636
	2	12.5740	30.7034	49.6123	69.7247	91.5159	98.1565
	3	12.3046	30.7872	49.5514	70.1695	92.2960	98.6610
	(*)	*(13.6160**)*	*(49.4501)*	*(87.6013)*	*(151.384)*	*(224.476)*	*(308.164)*
C-F	(*)	5.20353	13.5788	30.9211	51.5550	70.5594	93.5004
	1	5.05507	14.4658	31.1255	51.3059	71.2779	91.3077
	2	5.28648	14.3217	31.3292	51.2914	71.9772	93.3965
	3	5.34654	14.1618	31.3901	51.2428	72.3584	94.5246
	(*)	*(5.59372**)*	*(15.9079)*	*(49.0616)*	*(87.1239)*	*(151.407)*	*(224.520)*
F-C	(*)	0.929072	9.20821	34.9496	51.9305	71.4098	94.814
	1	1.012010	10.1953	32.1938	51.6411	70.3562	90.5051
	2	0.966524	10.0134	33.4188	52.4965	72.1808	91.6169
	3	0.950797	9.89641	34.0806	52.7623	72.8943	91.9021
	(*)	*(0.947401**)*	*(11.4575)*	*(50.0108)*	*(87.6521)*	*(151.347)*	*(224.476)*
SS-SS	(*)	4.98837	25.1872	42.901	65.7235	90.6115	92.4199
	1	5.63974	24.0047	43.5573	64.9097	87.9153	89.0929
	2	5.48827	24.6502	43.7541	66.1185	89.469	90.1089
	3	5.39267	24.9659	43.7127	66.6739	90.1937	90.5524
	(*)	*(5.34078**)*	*(32.6569)*	*(63.1071)*	*(120.781)*	*(184.236)*	*(263.781)*
SS-F	(*)	0	11.7121	28.0948	50.5545	68.684	90.7496
	1	0	11.4088	27.8168	49.5263	69.6462	87.9270
	2	0	11.5026	28.4034	49.9353	70.6627	89.4768
	3	0	11.5485	28.6469	50.0470	71.0366	90.3489
	(*)	*(0)*	*(12.9282 **)*	*(37.2397)*	*(79.7088)*	*(129.215)*	*(211.932)*
F-SS	(*)	0	6.72053	31.5573	47.1766	70.2838	92.4005

	1	0	7.37897	28.6308	47.6707	68.3732	89.0052
	2	0	7.25495	29.6362	48.521	70.3443	90.0524
	3	0	7.17966	30.1991	48.7932	71.1526	90.255
	(*)	*(0)*	*(7.52668 **)*	*(42.5766)*	*(71.6266)*	*(141.200)*	*(195.939)*
F-F	(*)	0	0	13.5386	35.5016	54.0474	72.4241
	1	0	0	13.4427	32.8224	53.5280	72.2935
	2	0	0	13.5085	34.0342	54.4367	74.0526
	3	0	0	13.5416	34.6933	54.7046	74.6769
	(*)	*(0)*	*(0)*	*(15.2543 **)*	*(49.6032)*	*(87.1795)*	*(151.369)*

(*) homogeneous material; $\chi_{E_1} = \chi_{E_2} = 200/200$ =1; $\chi_{\rho_1} = \chi_{\rho_2} = 5700/5700$ =1.

**slender beam with $h0/L$=0.0017.