ERGODIC TRANSFORMS ASSOCIATED TO GENERAL AVERAGES

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Abstract. Jones and Rosenblatt started the study of an ergodic transform which is analogous to the martingale transform. In this paper we present a unified treatment of the ergodic transforms associated to positive groups induced by non-singular flows and to general means which include usual averages, Cesàro-α averages and Abel means. We prove the boundedness in L^p, $1 < p < \infty$, of the maximal ergodic transforms assuming that the semigroup is Cesàro bounded in L^p. For $p = 1$ we obtain that the maximal ergodic transforms are of weak type $(1, 1)$. Convergence results are also proved. We show some general examples of Cesàro bounded semigroups.

1. Introduction and Main Results

Let $(X, \mathcal{F}, \nu)$ be a complete σ-finite measure space. By a flow $\Gamma = \{\tau_t : t \in \mathbb{R}\}$ we mean a group of measurable transformations $\tau_t : X \to X$ such that τ_0 is the identity, $\tau_{t+s} = \tau_t \circ \tau_s (t, s \in \mathbb{R})$ and the map $(x, t) \to \tau_t x$ from $X \times \mathbb{R}$ into X is $\widetilde{\mathcal{F}}$-$\mathcal{F}$-measurable, where $\widetilde{\mathcal{F}}$ is the completion of the product σ-algebra $\mathcal{F} \otimes \mathcal{L}$ of $\mathcal{F}$ with the Lebesgue σ-algebra $\mathcal{L}$, and the completion is taken with respect to the product measure of ν on $\mathcal{F}$ and the Lebesgue measure m. The flow is said to be measure preserving if $\nu(\tau_t E) = \nu(E)$ for all $t \in \mathbb{R}$ and all $E \in \mathcal{F}$ (we shall also say that the flow preserves the measure ν). The flow is said to be nonsingular if $\nu(\tau_t E) = 0$ for all $t \in \mathbb{R}$ and all $E \in \mathcal{F}$ with $\nu(E) = 0$. It is clear that measure preserving implies nonsingular. In this paper we are mainly interested in nonsingular flows which are not necessarily measure preserving.

From now on we fix a nonsingular flow $\Gamma = \{\tau_t : t \in \mathbb{R}\}$. For each $t \in \mathbb{R}$ we consider the measures ν_t defined by $\nu_t(E) = \nu(\tau_t(E))$. These measures have the same sets of measure zero since the flow is nonsingular. If $\overline{J_t}$ is the Radon-Nikodym derivative of ν_t with respect to ν then $\int_X f(x)\, d\nu(x) = \int_X f(\tau_t x)\overline{J_t}(x)\, d\nu(x)$ for all nonnegative measurable functions f and for all integrable functions f. Moreover, $\overline{J_{t+s}}(x) = \overline{J_s}(\tau_t x)\overline{J_t}(x)$ a.e. x. It follows that the operators $S^t f(x) = \overline{J_t}(x) f(\tau_t x)$ are positive isometries in $L^1(\nu)$ and $\lim_{t\to 0} S^t = I$ in the strong operator topology [15], where I is the identity operator. Consequently, by using [6, Lemma III.11.16] (see also [23]) there exists a function $(x, t) \to J_t(x)$, measurable with respect to the product

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σ-algebra, such that, for almost every t, $J_t(x) = \overline{J_t}(x)$ a.e. x. Consequently,

$$\int_X f(x)\, d\nu(x) = \int_X f(\tau_t x) J_t(x)\, d\nu(x), \tag{1.1}$$

for almost every t. Furthermore, $J_{t+s}(x) = J_s(\tau_t x) J_t(x)$ a.e. $(x, s, t) \in X \times \mathbb{R} \times \mathbb{R}$, where in $X \times \mathbb{R} \times \mathbb{R}$ we consider the completion of the product measure. In this paper we are interested in a class of groups of positive operators $\mathcal{G} = \{T^t : t \in \mathbb{R}\}$ which contains as particular cases the groups S^t considered previously. We introduce this class in the next definition.

Definition 1.1. Let $(X, \mathcal{F}, \nu)$ be a complete σ-finite measure space. Let $\Gamma = \{\tau_t : t \in \mathbb{R}\}$ be a nonsingular flow on X. Let $g(x,t) = g_t(x)$ a positive function defined on $X \times \mathbb{R}$ such that is $\mathcal{F} \otimes \mathcal{L}$-measurable and $g_{t+s}(x) = g_s(\tau_t x) g_t(x)$ a.e. $(x, s, t) \in X \times \mathbb{R} \times \mathbb{R}$. A group $\mathcal{G} = \{T^t : t \in \mathbb{R}\}$ of positive operators induced by Γ and g is a family of linear operators acting on measurable functions, such that $T^{t+s} f = T^t(T^s f)$, $T^0 f = f$, and, for almost every t,

$$T^t f(x) = g_t(x) f(\tau_t x), \quad \text{a.e. } x. \tag{1.2}$$

We will denote by $\mathcal{G}_+$ the semigroup $\{T^t : t > 0\}$.

Throughout the paper we work only with this kind of groups. For such a group, it follows from (1.1) that if $0 < p < \infty$ and $H_t(x) = (g_t(x))^{-p} J_t(x)$ then for almost every t

$$\int_X |f(x)|^p\, d\nu(x) = \int_X |T^t f(x)|^p H_t(x)\, d\nu(x). \tag{1.3}$$

One of the classical problems in Ergodic Theory is to study the convergence of the averages

$$\mathcal{A}^+_\varepsilon f(x) = \frac{1}{\varepsilon} \int_0^\varepsilon T^t f(x) dt$$

as $\varepsilon \to 0^+$ and as $\varepsilon \to +\infty$. (In principle these averages are well defined for $f \geq 0$.) There are other kinds of averages like Cesàro-α averages

$$\mathcal{C}^+_\varepsilon f(x) = \frac{1}{\varepsilon^{1+\alpha}} \int_0^\varepsilon (\varepsilon - t)^\alpha T^t f(x) dt, \quad \alpha \geq 0,$$

or Abel means

$$\mathcal{R}^+_\varepsilon f(x) = \frac{1}{\varepsilon} \int_0^\infty \mathrm{e}^{-t/\varepsilon} T^t f(x) dt$$

which have also been studied in this ergodic setting ([2],[5],[8],[9],[17],[22], [23]). All these averages are particular cases of the "convolution" averages defined by

$$\mathcal{A}^+_{\varepsilon,\varphi} f(x) = \int_0^\infty \varphi_\varepsilon(t) T^t f(x) dt,$$

where $\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi(\frac{t}{\varepsilon})$ and φ is a nonnegative integrable decreasing function defined on $(0, +\infty)$ (the trivial case $\varphi(t) = 0$ will not be considered). If φ is the characteristic function $\chi_{(0,1)}$ of the interval $(0, 1)$ then the φ-averages $\mathcal{A}^+_{\varepsilon,\varphi} f$ are the usual ergodic averages; in this case, as we have already done, we simply write $\mathcal{A}_\varepsilon^+ f$ (this convention will be used also for other operators in this paper).

In order to prove the almost everywhere convergence of the averages $\mathcal{A}_{\varepsilon,\varphi}^+ f$, the standard approach is to consider the maximal operator

$$\mathcal{M}_\varphi^+ f(x) = \sup_{\varepsilon>0} |\mathcal{A}_{\varepsilon,\varphi}^+ f(x)|$$

and, for $f \in L^p(\nu)$, $1 < p < \infty$, to prove a dominated ergodic estimate, i.e., $\int_X |\mathcal{M}_\varphi^+ f|^p d\nu \le C \int_X |f|^p d\nu$ with a constant C independent of f. It is clear that for such an inequality to hold the averages $\mathcal{A}_{\varepsilon,\varphi}^+$ must be uniformly bounded operators in $L^p(\nu)$. This remark gives rise to the next definition.

Definition 1.2. Let $\mathcal{G}$ be a group as in Definition 1.1. Let $\mathcal{G}$ be a group as in Definition 1.1. Let φ be a nonnegative integrable decreasing function on $(0,\infty)$. We say that the semigroup $\mathcal{G}_+ = \{T^t : t > 0\}$ is φ-bounded in $L^p(\nu)$, $1 \le p < \infty$, if there exists $C > 0$ such that for all nonnegative functions $f \in L^p(\nu)$

$$\sup_{\varepsilon>0} \int_X |\mathcal{A}_{\varepsilon,\varphi}^+ f|^p \, d\nu \le C \int_X |f|^p \, d\nu. \tag{1.4}$$

If $\varphi = \chi_{(0,1)}$ then we say that the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$.

Observe that if the semigroup $\mathcal{G}_+$ is φ-bounded in $L^p(\nu)$ and $f \in L^p(\nu)$ then the averages $\mathcal{A}_{\varepsilon,\varphi}^+ f$ are well defined and (1.4) holds for all $f \in L^p(\nu)$. Obviously, the semigroup $\mathcal{G}_+$ is φ-bounded in $L^p(\nu)$ if $g_t(x) = 1$ and the flow is measure preserving.

In what follows we search the relation between φ-bounded semigroups and Cesàro bounded semigroups (this is probably known, for instance see [9] for Abel means).

Proposition 1.3. Let $\mathcal{G}$ be a group as in Definition 1.1. Let φ be a nonnegative integrable decreasing function on $(0,\infty)$ with $\int_0^\infty \varphi > 0$ and let $1 \le p < \infty$. The semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$ if and only if the semigroup $\mathcal{G}_+$ is φ-bounded in $L^p(\nu)$. Furthermore, there exists $C > 0$ such that

$$\frac{1}{C}\mathcal{M}^+ f(x) \le \mathcal{M}_\varphi^+ f(x) \le \left(\int \varphi\right) \mathcal{M}^+ f(x) \quad \text{for all measurable } f,$$

where $\mathcal{M}^+ f(x) = \sup_{\varepsilon>0} |\mathcal{A}_\varepsilon^+ f(x)|$.

Proof. It suffices to consider functions $f \ge 0$. Notice that if $\varphi(u) > 0$ for some $u > 0$ then, since φ is decreasing, we get $\mathcal{A}_{\varepsilon,\varphi}^+ f(x) \ge u\varphi(u)\mathcal{A}_{\varepsilon u}^+ f(x)$. From this inequality, we have that φ-bounded implies Cesàro-bounded and $\mathcal{M}^+ f(x) \le \frac{1}{u\varphi(u)}\mathcal{M}_\varphi^+ f(x)$. On the other hand, if φ is a simple function, $\varphi = \sum_{i=1}^n c_i \chi_{(0,b_i)}$, $c_i \ge 0$, we get that

$$\mathcal{A}_{\varepsilon,\varphi}^+ f(x) = \sum_{i=1}^n c_i b_i \mathcal{A}_{\varepsilon b_i}^+ f(x) \le \left(\sum_{i=1}^n c_i b_i\right) \mathcal{M}^+ f(x) = \left(\int \varphi\right) \mathcal{M}^+ f(x). \tag{1.5}$$

Then $\mathcal{M}^+_\varphi f(x) \le \left(\int \varphi\right) \mathcal{M}^+ f(x)$ and

$$\int_X |\mathcal{A}^+_{\varepsilon,\varphi} f|^p \, d\nu \le \left(\int \varphi\right)^p \sup_{\varepsilon>0} \int_X |\mathcal{A}^+_\varepsilon f|^p \, d\nu.$$

Consequently, Cesàro bounded implies φ-bounded for φ simple. For general φ, the assertion follows from the above inequalities and the monotone convergence theorem. □

Proposition 1.3 allows to reduce the study of the boundedness of $\mathcal{M}^+_\varphi$ to the usual ergodic maximal operator, i.e., to the case $\varphi = \chi_{(0,1)}$ corresponding to the standard ergodic averages. In the next proposition we shall show that the almost everywhere convergence of $\mathcal{A}^+_{\varepsilon,\varphi} f$ is also reduced to the standard case.

Proposition 1.4. Let $\mathcal{G}$ be a group as in Definition 1.1. Let φ be a nonnegative integrable decreasing function on $(0,\infty)$ with $\int_0^\infty \varphi > 0$ and let $1 \le p < \infty$. Assume that for a nonnegative measurable function f there exists $\lim_{\varepsilon\to+\infty} \mathcal{A}^+_\varepsilon f(x) = \ell$ and $\mathcal{M}^+ f(x) < \infty$ for some x. Then there exists $\lim_{\varepsilon\to+\infty} \mathcal{A}^+_{\varepsilon,\varphi} f(x) = \left(\int \varphi\right) \ell$. The same statement holds for the limit as ε goes to zero.

Proof. If φ is a simple function then the proof follows immediately from the first equality in (1.5). Assume now that φ has compact support. It is easy to see that there exists a sequence $\{\varphi_n\}_n$ of nonnegative decreasing simple functions such that $\varphi_n \uparrow \varphi$ and $\varphi - \varphi_n \le \psi_n$, where the functions ψ_n are nonnegative, integrable, decreasing and $\lim_{n\to\infty} \int_0^\infty \psi_n = 0$. By Proposition 1.3 we have

$$\left|\mathcal{A}^+_{\varepsilon,\varphi} f(x) - \mathcal{A}^+_{\varepsilon,\varphi_n} f(x)\right| = \left|\mathcal{A}^+_{\varepsilon,\varphi-\varphi_n} f(x)\right| \le |\mathcal{A}^+_{\varepsilon,\psi_n} f(x)| \le \left(\int \psi_n\right) \mathcal{M}^+ f(x).$$

The result follows immediately for φ with compact support since $\mathcal{M}^+ f(x)$ is finite, $\lim_{n\to\infty} \int \psi_n = 0$ and $\lim_{\varepsilon\to\infty} \mathcal{A}^+_{\varepsilon,\varphi_n} f(x) = \left(\int \varphi_n\right) \ell$.

Assume now a general φ. Let $L > 0$ and $\varphi_L = \varphi \chi_{(0,L]}$. Let $\psi_L = \varphi(L)\chi_{(0,L]} + \varphi \chi_{(L,\infty)}$. Then

$$\begin{aligned} \left|\mathcal{A}^+_{\varepsilon,\varphi} f(x) - \left(\int \varphi\right) \ell\right| &\le \left|\mathcal{A}^+_{\varepsilon,\varphi_L} f(x) - \left(\int \varphi\right) \ell\right| + |\mathcal{A}^+_{\varepsilon,\psi_L} f(x)| \\ &\le \left|\mathcal{A}^+_{\varepsilon,\varphi_L} f(x) - \left(\int \varphi\right) \ell\right| + \left(\int \psi_L\right) \mathcal{M}^+ f(x). \end{aligned}$$

Since $\lim_{L\to\infty} \int \psi_L = 0$, $\mathcal{M}^+ f(x) < \infty$ and φ_L has compact support, the general case follows from what we have already proved for functions φ with compact support. □

Now we are going to show some non trivial examples of Cesàro bounded semigroups. Some other examples and related questions are studied in §6.

Example 1.5. Let $1 \le p < \infty$ and let us consider the group $T^t f(x) = (\overline{J_t}(x))^{1/p} f(\tau_t x)$, where $\overline{J_t}$ is the Radon-Nikodym derivative considered at the beginning of the introduction. It is clear that each T^t is an isometry on $L^p(\nu)$ and, consequently, $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$. Observe that $\mathcal{G}_- = \{T^t : t < 0\}$ is also Cesàro bounded in $L^p(\nu)$.

Example 1.6. Let $X = [0, 1)$ with the Lebesgue σ-algebra. Let $d\nu = w(x)\,dx$, where $w(x) = x^\beta$ and $-1 < \beta < 0$. Consider the flow $\tau_t(x) = x + t$ (mod 1), that is, $\tau_t(x) = x + t - [x + t]$, where $[x + t]$ stands for the integer part of $x+t$. Let us consider the group $T^t f(x) = f(\tau_t x)$. The semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^1(\nu)$ (and therefore bounded in $L^p(\nu)$, $1 < p < \infty$) if and only if there exists $C > 0$ such that for all $\varepsilon > 0$

$$\frac{1}{\varepsilon}\int_0^\varepsilon w(\tau_{-t}x)dt \le Cw(x) \quad \text{a.e. } x.$$

We can see that this statement holds since $|x|^\beta$ satisfies the Muckenhoupt A_1 condition in the real line (see [10], [7] or [12]), i.e, there exists $C > 0$ such that $\frac{1}{b-a}\int_a^b |x|^\beta\,dx \le C\inf_{x\in(a,b)} |x|^\beta$, for all intervals (a, b). However, the semigroup $\mathcal{G}_- = \{T^t : t < 0\}$ is not Cesàro bounded in $L^1(\nu)$. In order to prove the last statement we observe that $\mathcal{G}_-$ is Cesàro bounded in $L^1(\nu)$ if and only if there exists $C > 0$ such that for all $\varepsilon > 0$

$$\frac{1}{\varepsilon}\int_0^\varepsilon w(\tau_t x)dt \le Cw(x) \quad \text{a.e. } x.$$

and by the continuity of w, the above condition would hold for all $x \ne 0$. If we take $\varepsilon > 0$ and $x = 1 - \varepsilon$, we have

$$\frac{1}{2\varepsilon}\int_0^{2\varepsilon} w(\tau_t x)dt \ge \frac{1}{2\varepsilon}\int_0^\varepsilon y^\beta\,dy = \frac{1}{2(\beta+1)}\varepsilon^\beta.$$

Since $w(x) = w(1 - \varepsilon) < 2$, for ε small, we see that

$$\frac{\frac{1}{2\varepsilon}\int_0^{2\varepsilon} w(\tau_t(1-\varepsilon))dt}{w(1-\varepsilon)} \ge \frac{1}{4(\beta+1)}\varepsilon^\beta$$

is so big as we wish taking ε small enough. Therefore, $\mathcal{G}_-$ is not Cesàro bounded in $L^1(\nu)$

Clearly, the measure ν is equivalent to the Lebesgue measure (the invariant measure for the flow Γ) in the sense that they have the same sets of measure zero. However ν is not comparable to the Lebesgue measure, that means that there is no constant $K > 0$ such that $(1/K)|E| \le \nu(E) \le K|E|$ for all measurable sets E, where $|E|$ is the Lebesgue measure of E: if $E = (0, b) \subset (0, 1)$ we have $\frac{\nu(E)}{|E|} = \frac{b^\beta}{\beta+1}$ and b^β is as big as we wish taking b small enough.

One may ask whether or not the flow could have the property that for some constant $K > 0$,

$$(1/K)\nu(E) \le \nu(\tau_t(E)) \le K\nu(E) \tag{1.6}$$

for all measurable sets E and all t. Then there should be a σ-finite measure μ equivalent to ν for which the flow is measure-preserving and such that μ is comparable to ν (consequently, everything would be reduced to the measure preserving case). Our present example shows that the measure μ would be the Lebesgue measure and we have already shown that it is not comparable to ν. However, we are going to give a direct proof showing that (1.6) does not hold in our example although the semigroup is Cesàro bounded in $L^1(\nu)$. In fact, for $0<\varepsilon<1$ and $I_\varepsilon=(1-\varepsilon,1)$, we have $\tau_\varepsilon(I_\varepsilon)=(0,\varepsilon)$. Therefore, $\nu(\tau_\varepsilon(I_\varepsilon))=\frac{\varepsilon^{\beta+1}}{\beta+1}$. Since $\nu(I_\varepsilon)\leq\frac{1}{2^\beta}\varepsilon$, we have

$$\frac{\nu(\tau_\varepsilon(I_\varepsilon))}{\nu(I_\varepsilon)}\geq\frac{2^\beta\varepsilon^\beta}{(\beta+1)}$$

which is so big as we wish taking ε small enough.

In what follows we state our results about the boundedness of $\mathcal{M}_\varphi^+$ and the convergence of the averages $\mathcal{A}_{\varepsilon,\varphi}^+f$ under the main assumption that the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$.

Theorem 1.7. Let $(X,\mathcal{F},\nu)$, Γ, $\{g_t:t\in\mathbb{R}\}$ and $\mathcal{G}=\{T^t:t\in\mathbb{R}\}$ be as in Definition 1.1. Let φ be a nonnegative integrable decreasing function on $(0,\infty)$ and let $1\leq p<\infty$. Assume that the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$.

(a) If $1<p<\infty$ then
- (i) There exists $C>0$ such that for all $f\in L^p(\nu)$,
$$\int_X|\mathcal{M}_\varphi^+f|^pd\nu\leq C\int_X|f|^pd\nu.$$
- (ii) For all $f\in L^p(\nu)$, $\lim_{\varepsilon\to0^+}\mathcal{A}_{\varepsilon,\varphi}f=\left(\int\varphi\right)f$ a.e. and in $L^p(\nu)$.
- (iii) For all $f\in L^p(\nu)$, the averages $\mathcal{A}_{\varepsilon,\varphi}^+f$ converge a.e. and in $L^p(\nu)$ as $\varepsilon\to+\infty$.

(b) If $p=1$ and $g_t(x)=1$ then
- (i) There exists $C>0$ such that for all $f\in L^1(\nu)$ and all $\lambda>0$
$$\nu\left(\{x\in X:|\mathcal{M}_\varphi^+f(x)|>\lambda\}\right)\leq\frac{C}{\lambda}\int_X|f|^pd\nu.$$
- (ii) For all $f\in L^1(\nu)$, $\lim_{\varepsilon\to0^+}\mathcal{A}_{\varepsilon,\varphi}^+f=\left(\int\varphi\right)f$ a.e. and in measure.
- (iii) For all $f\in L^1(\nu)$, the averages $\mathcal{A}_{\varepsilon,\varphi}^+f$ converge a.e. and in measure as $\varepsilon\to+\infty$.

Remark 1.8. The result does not hold if $p=1$ and $g_t(x)\neq1$. To show an example we work on the real line. We follow Example 2.11 in [11]. Take $g(x)=\chi_{(-\infty,1)}(x)+\frac{1}{x}\chi_{(1,+\infty)}(x)$. Let $T^tf(x)=\frac{g(x-t)}{g(x)}f(x-t)$ and $d\nu=g(x)\,dx$. It is very easy to see that the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^1(d\nu)$. Notice that the maximal operator $\mathcal{M}^+$ associated to the semigroup verifies

$$\mathcal{M}^+f(x)=\frac{1}{g(x)}M^-(fg)(x),$$

where M^- is the one-sided Hardy-Littlewood maximal function defined by $M^-f(x)=\sup_{\varepsilon>0}\frac{1}{\varepsilon}\int_{x-\varepsilon}^x|f(s)|\,ds$. Therefore, if a weak type $(1,1)$ inequality were satisfied for $\mathcal{M}^+$ with respect to ν we would have

$$\int_{\{x:\frac{1}{g(x)}M^-f(x)>\lambda\}}g(x)\,dx\leq\frac{C}{\lambda}\int_{\mathbb{R}}|f(x)|\,dx$$

If we take $f = \chi_{(0,1)}$ then $M^- f(x) = 1/x$ for $x > 1$. Taking $\lambda = 1/2$, we have $+\infty = \int_{(1,\infty)} g(x)\,dx \le 2C \int_{\mathbb{R}} |f(x)|\,dx = 2C$.

Once we know that the convergence of $\mathcal{A}^+_{\varepsilon,\varphi} f$ holds in the almost everywhere sense or in the L^p-norm then it is reasonable to try to give some information about how the convergence occurs.

Let us take any sequence $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ with $\varepsilon_{k+1} > \varepsilon_k > 0$ for all k, $\lim\limits_{k\to-\infty} \varepsilon_k = 0$ and $\lim_{k\to+\infty} \varepsilon_k = +\infty$. There is no rate of convergence of $\lim_{k\to-\infty} \mathcal{A}^+_{\varepsilon_k,\varphi} f$. However, observe that, for $N \le 0$, $\sum_{k=N}^{0} \left(\mathcal{A}_{\varepsilon_k,\varphi} f(x) - \mathcal{A}_{\varepsilon_{k-1},\varphi} f(x)\right) = \mathcal{A}_{\varepsilon_0,\varphi} f(x) - \mathcal{A}_{\varepsilon_{N-1},\varphi} f(x)$. Therefore, the limit

(1.7)

$$\lim_{N\to-\infty} \sum_{k=N}^{0} \left(\mathcal{A}_{\varepsilon_k,\varphi} f(x) - \mathcal{A}_{\varepsilon_{k-1},\varphi} f(x)\right) = \mathcal{A}_{\varepsilon_0,\varphi} f(x) - \lim_{N\to-\infty} \mathcal{A}_{\varepsilon_{N-1},\varphi} f(x)$$

exists and it is essentially equal to $\lim_{k\to-\infty} \mathcal{A}^+_{\varepsilon_k,\varphi} f$. Consequently, if we try to give some information about how the convergence of $\mathcal{A}^+_{\varepsilon_k,\varphi} f$ occurs, we can think about how is the convergence of $\displaystyle\sum_{k=-\infty}^{0} \left(\mathcal{A}_{\varepsilon_k,\varphi} f(x) - \mathcal{A}_{\varepsilon_{k-1},\varphi} f(x)\right)$. We can try to search the absolute convergence, the unconditional convergence or the existence in $L^p(\nu)$ of the square function $\sum_{k=-\infty}^{\infty} |\mathcal{A}_{\varepsilon_k,\varphi} f - \mathcal{A}_{\varepsilon_{k-1},\varphi} f|^2$. If, in particular, we take a lacunary sequence $\{\varepsilon_k\}_{k\in\mathbb{Z}}$, i.e., $\varepsilon_k > 0$ and $\frac{\varepsilon_{k+1}}{\varepsilon_k} \ge \rho > 1$ for all k, then an example in [1] shows that there exists $f \in L^\infty(\nu)$ such that the series does not converge absolutely. More precisely, let $X = [0,1)$ with the Lebesgue measure ν and, as before, consider the flow $\tau_t(x) = x + t \pmod 1$. Let $\varphi = \chi_{(0,1)}$. Then there exists $f \in L^\infty(\nu)$ such that $\sum_{k=-\infty}^{0} \left|\mathcal{A}_{\varepsilon_k,\varphi} f(x) - \mathcal{A}_{\varepsilon_{k-1},\varphi} f(x)\right| = +\infty$ a.e. (The example in [1] is for $\varphi = \chi_{(-1,1)}$, that is $\mathcal{A}_\varepsilon f(x) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(x+t)\,dt$, but the example for $\varphi = \chi_{(0,1)}$ follows immediately.) Given the cancelation properties of $\sum_{k=-\infty}^{0} \left(\mathcal{A}_{\varepsilon_k,\varphi} f(x) - \mathcal{A}_{\varepsilon_{k-1},\varphi} f(x)\right)$ and the last result, it is natural to consider the convergence of

$$\sum_{k=-\infty}^{0} \upsilon_k \left(\mathcal{A}_{\varepsilon_k,\varphi} f(x) - \mathcal{A}_{\varepsilon_{k-1},\varphi} f(x)\right),$$

where υ_k is a bounded sequence of real or complex numbers. Reasoning in the same way for $k \to +\infty$, we arrive to the problem of studying the convergence of the series $\sum_{k=-\infty}^{\infty} \upsilon_k \left(\mathcal{A}_{\varepsilon_k,\varphi} f(x) - \mathcal{A}_{\varepsilon_{k-1},\varphi} f(x)\right)$, where υ_k is a bounded sequence of real or complex numbers. We shall only study the convergence and boundedness of the last series for lacunary sequences because the expected results imply unconditional convergence of the series $\sum_{k=-\infty}^{\infty} \left(\mathcal{A}_{\varepsilon_k,\varphi} f(x) - \mathcal{A}_{\varepsilon_{k-1},\varphi} f(x)\right)$, and this fact restricts the classes of sequences for which you can expect positive results for the series (see Remark 1.15 after the statement of the results). We must point out that Jones and Rosenblatt [13] studied this problem for $\varphi = \chi_{(0,1)}$ in the setting of the periodic functions in the real line with the flow $\tau_t(x) = x + t$ and for discrete averages associated to a measure preserving transformation; the problem for general functions in the real line with the same flow and in the context of weighted spaces was studied in [3].

So far, we have only established our main aim, that is, the study of the convergence and boundedness of

$$\sum_{k=-\infty}^{+\infty} \upsilon_k \left(\mathcal{A}_{\varepsilon_k,\varphi}f(x) - \mathcal{A}_{\varepsilon_{k-1},\varphi}f(x)\right), \tag{1.8}$$

in the setting of Cesàro bounded semigroups, where ε_k is a lacunary sequence and υ_k is a bounded sequence of real or complex numbers. The natural approach is to consider the maximal operator

$$\mathcal{T}^*_\varphi f(x) = \sup_N |\mathcal{T}_{N,\varphi}f(x)|,$$

where for each $N = (N_1, N_2) \in \mathbb{Z}^2$, $N_1 \leq N_2$, $\mathcal{T}_{N,\varphi}$ is the truncation operator

$$\mathcal{T}_{N,\varphi}f(x) = \sum_{k=N_1}^{N_2} \upsilon_k \left(\mathcal{A}_{\varepsilon_k,\varphi}f(x) - \mathcal{A}_{\varepsilon_{k-1},\varphi}f(x)\right).$$

Unlike the problem of the boundedness of $\mathcal{M}^+_\varphi$, it is not immediately clear how to estimate the maximal operator $\mathcal{T}^*_\varphi$ by $\mathcal{T}^*$, i.e., by the corresponding operator associated to the standard ergodic averages or, in other words, it is not obvious how to reduce the general problem of the convergence of (1.8) to the case $\varphi = \chi_{(0,1)}$. That is because of the nature of the operator $\mathcal{T}^*_\varphi$ which is essentially a singular integral maximal operator (when we look at the real line with $\tau_t x = x + t$).

Our results are collected in the following theorems.

Theorem 1.9. Let $(X, \mathcal{F}, \nu)$, Γ, $\{g_t : t \in \mathbb{R}\}$, $\mathcal{G}$ and φ be as in Theorem 1.7. Let $1 \leq p < \infty$ and assume that the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$.

(a) If $1 < p < \infty$ then there exists $C > 0$ such that for all $f \in L^p(\nu)$, $\int_X |\mathcal{T}^*_\varphi f|^p d\nu \leq C \int_X |f|^p d\nu$.
(b) If $p = 1$ and $g_t(x) = 1$ then there exists $C > 0$ such that for all $f \in L^1(\nu)$ and all $\lambda > 0$, $\nu\left(\{x \in X : |\mathcal{T}^*_\varphi f(x)| > \lambda\}\right) \leq \frac{C}{\lambda} \int_X |f| d\nu$.

To obtain the a.e. convergence of $\mathcal{T}_{N,\varphi}f$ it suffices to prove that there exist the limits $\lim_{N\to+\infty} \mathcal{T}^1_{N,\varphi}f(x)$ and $\lim_{N\to+\infty} \mathcal{T}^2_{N,\varphi}f(x)$ a.e., where

$$\mathcal{T}^1_{N,\varphi}f(x) = \sum_{k=-N}^{0} \upsilon_k \left(\mathcal{A}^+_{\varepsilon_k,\varphi}f(x) - \mathcal{A}^+_{\varepsilon_{k-1},\varphi}f(x)\right) \qquad \text{and}$$

$$\mathcal{T}^2_{N,\varphi}f(x) = \sum_{k=1}^{N} \upsilon_k \left(\mathcal{A}^+_{\varepsilon_k,\varphi}f(x) - \mathcal{A}^+_{\varepsilon_{k-1},\varphi}f(x)\right).$$

(Here N stands for a natural number.) We shall need to assume some extra assumptions on φ but we point out that the examples in the introduction and others as the Poisson kernel, $\varphi(t) = \frac{1}{1+t^2}$, satisfy these conditions.

Theorem 1.10. Let $(X, \mathcal{F}, \nu)$, Γ, $\{g_t : t \in \mathbb{R}\}$, $\mathcal{G}$ and φ be as in Theorem 1.7. Let $1 < p < \infty$. Assume that $\int t^\beta \varphi(t)\, dt < \infty$ for all $0 < \beta < 1$ and the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$. Then for all $f \in L^p(\nu)$, the sequence $\mathcal{T}^1_{N,\varphi} f$ converges a.e. and in $L^p(\nu)$ as $N \to +\infty$.

If $g_t(x) = 1$, that is, $T^t f(x) = f(\tau_t x)$, we can prove the same result, including the case $p = 1$, with a weaker condition on φ. The key point is that in this case the operators T^t are contractions in $L^\infty(\nu)$.

Theorem 1.11. Let $(X, \mathcal{F}, \nu)$, Γ and φ be as in Theorem 1.7. Let $\mathcal{G}$ be the group defined as $T^t f(x) = f(\tau_t x)$. Let $1 \le p < \infty$. Assume that $\int_1^\infty |\log t| \varphi(t)\, dt < \infty$ and the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$.

(a) If $1 < p < \infty$ then for all $f \in L^p(\nu)$, the sequence $\mathcal{T}^1_{N,\varphi} f$ converges a.e. and in $L^p(\nu)$ as $N \to +\infty$.

(b) If $p = 1$ then for all $f \in L^1(\nu)$, the sequence $\mathcal{T}^1_{N,\varphi} f$ converges a.e. and in measure as $N \to +\infty$.

For the same class of semigroups, i.e., $g_t(x) = 1$ we have the following result for the convergence of $\mathcal{T}^2_{N,\varphi} f$.

Theorem 1.12. Let $(X, \mathcal{F}, \nu)$, Γ and φ be as in Theorem 1.7. Let $\mathcal{G}$ be the group defined as $T^t f(x) = f(\tau_t x)$. Let $1 \le p < \infty$. Assume that $\int_0^1 |\log t| \varphi(t)\, dt < \infty$ and the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$.

(a) If $1 < p < \infty$ then for all $f \in L^p(\nu)$, the sequence $\mathcal{T}^2_{N,\varphi} f$ converges a.e. and in $L^p(\nu)$ as $N \to +\infty$.

(b) If $p = 1$ then for all $f \in L^1(\nu)$, the sequence $\mathcal{T}^2_{N,\varphi} f$ converges a.e. and in measure as $N \to +\infty$.

Under the assumptions in Theorems 1.11 and 1.12, if $\int_0^\infty |\log t| \varphi(t)\, dt < \infty$ and the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$ then the corresponding results of convergence for $\mathcal{T}_{N,\varphi} f$ hold as $N = (N_1, N_2) \to (-\infty, +\infty)$.

For general groups but for standard averages, we can obtain the a.e. convergence of $\mathcal{T}^2_{N,\varphi} f$.

Theorem 1.13. Let $(X, \mathcal{F}, \nu)$, Γ, $\{g_t : t \in \mathbb{R}\}$ and $\mathcal{G}$ be as in Theorem 1.7. Let $1 < p < \infty$ and assume that the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$. Then for all $f \in L^p(\nu)$, the sequence $\mathcal{T}^2_N f$ converges a.e. and in $L^p(\nu)$ as $N \to +\infty$.

Applying Theorems 1.10 and 1.13 we obtain the following result for general groups and standard averages.

Theorem 1.14. Let $(X, \mathcal{F}, \nu)$, Γ, $\{g_t : t \in \mathbb{R}\}$ and $\mathcal{G}$ be as in Theorem 1.7. Let $1 < p < \infty$ and assume that the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$. Then, for all $f \in L^p(\nu)$, the sequence $\mathcal{T}_N f$ converges a.e. and in $L^p(\nu)$ as $N = (N_1, N_2) \to (-\infty, +\infty)$.

Remark 1.15. As Jones and Rosenblatt remarked in [13], once we have Theorem 1.9, we can obtain, under the same assumptions, that if $1 < p < \infty$ then the square operator $Sf(x) = \left(\sum_{k=-\infty}^{\infty} |\mathcal{A}_{\varepsilon_k,\varphi} f - \mathcal{A}_{\varepsilon_{k-1},\varphi} f|^2\right)^{1/2}$ is bounded in $L^p(\nu)$. Furthermore, as in [13], Theorem 1.9 implies that the series

$$\sum_{k=-\infty}^{+\infty} \left(\mathcal{A}_{\varepsilon_k,\varphi} f(x) - \mathcal{A}_{\varepsilon_{k-1},\varphi} f(x)\right)$$

converges unconditionally in $L^p(\nu)$, $1 < p < \infty$. We remark that Johnson [14] characterized the decreasing sequences ε_k with $\lim_{k\to-\infty} \varepsilon_k = 0$ such that the series $\sum_{k=-\infty}^{0} \left(\mathcal{A}_{\varepsilon_k,\varphi} f(x) - \mathcal{A}_{\varepsilon_{k-1},\varphi} f(x)\right)$ is unconditionally convergent for all $f \in L^2(\nu)$ in the case $X = [0, 1)$ with $\tau_t(x) = x + t$ (mod 1) and $\varphi = \chi_{(0,1)}$. This characterization allows to see that the lacunary sequences are good but $\varepsilon_k = -1/k$, $k < 0$, is a bad sequence, in the sense that the series does not converge unconditionally for all $f \in L^2(\nu)$ (see [14]). Therefore, our results are not valid for $\varepsilon_k = -1/k$, $k < 0$, and we have to restrict our attention to a class of subsequences, for instance, the lacunary sequences, as we have done.

We start establishing in §2 the results for $\mathcal{T}_\varphi^*$ in the case of the real line $\tau_t x = x + t$ in weighted L^p spaces. We prove Theorem 1.7 in §3. Next, we transfer the results in §2 to the ergodic setting and prove Theorem 1.9 in §4. The proofs of Theorems 1.10, 1.11, 1.12 and 1.13 are in §5. We point out that in §3 and §4 we need the results in weighted spaces. One of the difficulties in the transference argument in §4 comes from the fact that φ has not necessarily compact support. The other problem to overcome is the a.e. convergence of the truncation operators $\mathcal{T}_{N,\varphi} f$ for functions f in the suitable dense class. Finally, in §6, we provide general examples of Cesàro bounded semigroups.

2. Theorem 1.9 in the real line for the translation flow

Let us consider $X = \mathbb{R}$ with the Lebesgue measure, the flow on $\mathbb{R}$ defined by $\tau_t(x) = x+t$ and $g_t(x) = 1$. Let φ be a nonnegative integrable decreasing function on $(0, \infty)$. The φ-averages associated to this flow are

$$A^+_{\varepsilon,\varphi} f(x) = \frac{1}{\varepsilon} \int_0^\infty f(x + t)\varphi(t/\varepsilon)\, dt.$$

In this section we consider φ extended to the whole real line with $\varphi(t) = 0$ for $t \le 0$ and we shall define $\tilde{\varphi}(t) = \varphi(-t)$. With this notation, it is clear that

$$A^+_{\varepsilon,\varphi} f(x) = \frac{1}{\varepsilon} \int_{-\infty}^\infty f(x - t)\tilde{\varphi}(t/\varepsilon)\, dt = f * \tilde{\varphi}_\varepsilon(x).$$

Notice that $\tilde{\varphi}$ is increasing in $(-\infty, 0)$. Then it is well known (otherwise, use Proposition 1.3) that the maximal function $M_\varphi f(x) = \sup_{\varepsilon>0} |A^+_{\varepsilon,\varphi} f(x)|$ is controlled by the one-sided Hardy-Littlewood maximal function $M^+ f(x) = \sup_{\varepsilon>0} \frac{1}{\varepsilon} \int_0^\varepsilon |f(x + t)|\, dt$. More precisely,

$$M_\varphi f(x) \le \left(\int_{-\infty}^\infty \varphi\right) M^+ f(x).$$

(Notice that M^+ is M_φ for $\varphi = \chi_{(0,1)}$.) It follows from this inequality that the good weights for M^+ are also good weights for M_φ (by a weight w

we mean a nonnegative measurable function defined on $\mathbb{R}$). The following results can be obtained from the theorems in [24] and [20].

(1) Assume that $w \in A_1^+$, i.e., there exists C such that

$$M^- w(x) = \sup_{\varepsilon>0} \frac{1}{\varepsilon} \int_0^\varepsilon |w(x-t)|\,dt \le Cw(x) \quad a.e.$$

(M^- is the left-sided Hardy Littlewood maximal function). Then the operator M_φ is of weak type $(1,1)$ with respect to the measure $w(x)\,dx$, that is, there exists C such that $\int_{\{x: M_\varphi f(x) > \lambda\}} w \le \frac{C}{\lambda} \int |f| w$, for all $\lambda > 0$ and all $f \in L^1(w)$.

(2) Assume that $w \in A_p^+$, i.e., there exists C such that for any three points $a < b < c$

$$\left(\int_a^b w\right)^{\frac{1}{p}} \left(\int_b^c w^{1-p'}\right)^{\frac{1}{p'}} \le C(c-a), \tag{2.1}$$

where $p + p' = pp'$. Then the operator M_φ is bounded in $L^p(w)$, $1 < p < \infty$, that is, there exists $C > 0$ such that $\int |M_\varphi f|^p w \le C \int |f|^p w$, for all $f \in L^p(w)$.

Remarks 2.1. Notice that $w \in A_p^+$, $1 < p < \infty$, if and only if (2.1) holds for $a < b < c$ with $b = (a+c)/2$. We point out that we can define A_p^- classes, reversing the orientation of the real line, and obtain the corresponding results for the maximal function associated to a function φ supported in $(-\infty, 0)$. There are many important properties of these classes of weights; in particular we shall use frequently in §6 the following result (see [24] and [20]): $w \in A_p^+$ if and only if there exist $u \in A_1^+$ and $v \in A_1^-$ such that $w = uv^{1-p}$.

Throughout the paper, we will consider a bounded sequence $v = \{v_k\}$, $k \in \mathbb{Z}$, of real or complex numbers and a lacunary sequence $\varepsilon = \{\varepsilon_k\}$ of positive numbers. We shall say that $v = \{v_k\}$ is a multiplying sequence and we shall write $\|v\|_\infty = \sup_k |v_k|$. For each $N \in \mathbb{Z}^2$, $N = (N_1, N_2)$ with $N_1 \le N_2$ we define the sum

$$T_{N,\varphi} f(x) = \sum_{k=N_1}^{N_2} v_k (A^+_{\varepsilon_k,\varphi} f(x) - A^+_{\varepsilon_{k-1},\varphi} f(x)) = K_{N,\varphi} * f(x), \tag{2.2}$$

where $K_{N,\varphi}(x) = \sum_{k=N_1}^{N_2} v_k(\tilde{\varphi}_{\varepsilon_k}(x) - \tilde{\varphi}_{\varepsilon_{k-1}}(x))$. Notice that $T_{N,\varphi}$ is the operator $\mathcal{T}_{N,\varphi}$ defined in the previous section with the flow $\tau_t(x) = x + t$ and $g_t(x) = 1$. If we need to emphasize the dependence on $\varepsilon = \{\varepsilon_k\}$ and $v = \{v_k\}$ we shall write $T_{N,\varphi,\varepsilon,v}$ and $K_{N,\varphi,\varepsilon,v}$. As usual, to prove the a.e. convergence, we shall study the boundedness of the associated maximal operator

$$T^*_\varphi f(x) = \sup_{N \in \mathbb{Z}^2} |T_{\varphi,N} f(x)|$$

in the setting of the weighted spaces $L^p(w) = \{f : \left(\int_{\mathbb{R}} |f|^p w\right)^{1/p} < \infty\}$. (If necessary, we shall write $T^*_{\varphi,\varepsilon,v}$.) Since the operators $T_{\varphi,N}$ are convolution operators with kernels $K_{\varphi,N}$ supported in $(-\infty,0)$, the study of T^*_φ $(T_{\varphi,N})$ is related to the right-sided Hardy Littlewood maximal operator M^+.

Now we can state the main result in this section.

Theorem 2.2. Let φ be a nonnegative integrable function with support in $(0,\infty)$ and decreasing in that interval. Let $\varepsilon = \{\varepsilon_k\}$ be a ρ-lacunary sequence and let $v = \{v_k\}$ be a multiplying sequence.

(i) If $1 < p < \infty$ and $w \in A_p^+$ then there exists a constant C depending only on ρ, p, $\|v\|_\infty$ and w such that $\int_{\mathbb{R}} |T^*_\varphi f|^p w \le C \left(\int \varphi\right)^p \int_{\mathbb{R}} |f|^p w$, for all functions $f \in L^p(w)$.

(ii) If $w \in A_1^+$ then there exists C depending only on ρ, $\|v\|_\infty$ and w such that $\int_{\{x\in\mathbb{R}:\ |T^*_\varphi f(x)|>\lambda\}} w \le \frac{C}{\lambda}\left(\int \varphi\right) \|f\|_{L^1(w)}$, for all $\lambda > 0$ and all $f \in L^1(w)$.

The organization of this section is as follows. Subsection 2.1 is devoted to state notations and properties of the lacunary sequences. In Subsection 2.2 we prove Theorem 2.2 (i), while Theorem 2.2 (ii) is in Subsection 2.3.

Throughout this paper, we shall use the notations introduced in this section and the letter C will mean a positive constant not necessarily the same at each occurrence.

2.1. **Lacunary sequences.** We establish in this section some properties of the ρ-lacunary sequence $\varepsilon = \{\varepsilon_k\}$. The next proposition shows that, without loss of generality, we may assume that

$$1 < \rho \le \varepsilon_{k+1}/\varepsilon_k \le \rho^2. \tag{2.3}$$

Proposition 2.3. Given the ρ-lacunary sequence $\varepsilon = \{\varepsilon_k\}$ and the multiplying sequence $v = \{v_k\}$, we can define a ρ-lacunary sequence $\eta = \{\eta_k\}$ and a multiplying sequence $u = \{u_k\}$ verifying the following properties:
(i) $1 < \rho \le \frac{\eta_{k+1}}{\eta_k} \le \rho^2$ and $||v||_\infty = ||u||_\infty$.
(ii) For all $N = (N_1, N_2)$ there exists $M = (M_1, M_2)$ with $T_{N,\varphi} = \widetilde{T}_{M,\varphi}$, where $\widetilde{T}_{M,\varphi}$ is the operator defined in (2.2) for $\eta = \{\eta_k\}$ and $u = \{u_k\}$.

The proof is exactly as in the case $\varphi = \chi_{(0,1)}$ (see [3]). It follows from this proposition that it is enough to prove all the results of this paper in the case of a ρ-lacunary sequence satisfying (2.3). For this reason, in the rest of the paper we assume that $\{\varepsilon_k\}$ satisfies (2.3) without saying it explicitly. Observe that, under this assumption, the following properties hold:

$$(1/\rho)^{2(m-n)} \le \varepsilon_n/\varepsilon_m \le (1/\rho)^{m-n}, \quad \text{for all} \quad m > n. \tag{2.4}$$

If we denote by α the smallest positive integer such that $1/\rho + (1/\rho)^\alpha \le 1$, we get from (2.4) that $\varepsilon_i + \varepsilon_m \le \varepsilon_{m+1}$ for all $m \ge i + \alpha - 1$.

2.2. **Proof of Theorem 2.2 (i).** Let us first assume that φ is simple, that is, $\varphi = \sum_{\ell=1}^s a_\ell \chi_{(-b_\ell,0)}$, $a_\ell \ge 0$. We point out that this theorem was proved in [3] for the function $\chi = \chi_{(-1,0)}$. Observe that the kernels $K_{N,\varphi}$ satisfy the following equality:

$$K_{N,\varphi} = \sum_{\ell=1}^{s} a_\ell b_\ell \sum_{k=N_1}^{N_2} v_k \left(\frac{1}{b_\ell \varepsilon_k} \chi_{(-b_\ell \varepsilon_k, 0)} - \frac{1}{b_\ell \varepsilon_{k-1}} \chi_{(-b_\ell \varepsilon_{k-1}, 0)} \right).$$

For each ℓ, the sequence $\varepsilon^\ell = \{\varepsilon_k^\ell\}$ where $\varepsilon_k^\ell = b_\ell \varepsilon_k$ is a ρ-lacunary sequence and the equality can be written as $K_{N,\varphi} = \sum_{\ell=1}^{s} a_\ell b_\ell K_{N,\chi,\varepsilon^\ell,v}$. Therefore, $T_{N,\varphi} = \sum_{\ell=1}^{s} a_\ell b_\ell T_{N,\chi,\varepsilon^\ell,v}$ and $T_\varphi^* \leq \sum_{\ell=1}^{s} a_\ell b_\ell T^*_{\chi,\varepsilon^\ell,v}$. By the results in [3], there exists a constant C depending only on ρ and $\|v\|_\infty$ such that $\int |T^*_{\chi,\varepsilon^\ell,v} f|^p w \leq C \int |f|^p w$. Thus

$$\int |T_\varphi^* f|^p w \leq C \left(\sum_{\ell=1}^{s} a_\ell b_\ell \right)^p \int |f|^p w = C \left(\int \varphi \right)^p \int |f|^p w.$$

Now, let $f \in L^1(dx) \cap L^p(w)$. Let us choose a sequence $\{\varphi_k\}$ of simple functions with support in $(0,\infty)$, decreasing in $(0,\infty)$ and such that φ_k converges to φ in the L^1-norm. Then $T_{N,\varphi_k} f$ converges to $T_{N,\varphi} f$ in the L^1-norm as $k \to +\infty$. Let us fix a positive integer M. It follows that there exists a subsequence φ_{k_j} such that $T_{N,\varphi_{k_j}} f$ converges a.e. to $T_{N,\varphi} f$ as $j \to +\infty$ for all $N = (N_1, N_2)$ such that $|N_1|, |N_2| \leq M$. Then $|T_{N,\varphi} f| \leq \liminf_{j\to+\infty} T^*_{\varphi_{k_j}} f$ almost everywhere for all $N = (N_1, N_2)$ such that $|N_1|, |N_2| \leq M$. Consequently, $\sup_{N=(N_1,N_2):|N_1|,|N_2|\leq M} |T_{N,\varphi} f| \leq \liminf_{j\to+\infty} T^*_{\varphi_{k_j}} f$, a.e. By Fatou Lemma and the result for simple functions we have

$$\begin{aligned} \int \sup_{N=(N_1,N_2):|N_1|,|N_2|\leq M} |T_{N,\varphi} f|^p w &\leq \liminf_{j\to+\infty} \int |T^*_{\varphi_{k_j}} f|^p w \\ &\leq C \liminf_{j\to+\infty} \left(\int \varphi_{k_j} \right)^p \int |f|^p w = C \left(\int \varphi \right)^p \int |f|^p w. \end{aligned}$$

Letting M tend to ∞ we obtain $\int |T_\varphi^* f|^p w \leq C \left(\int \varphi \right)^p \int |f|^p w$ for $f \in L^1(dx) \cap L^p(w)$. Now let $f \in L^p(w)$. There exists a sequence $f_n \in L^1(dx) \cap L^p(w)$ such that f_n converges to f in $L^p(w)$. Let us fix M. There is a constant C_M such that if $N = (N_1, N_2)$, $|N_1|, |N_2| \leq M$, then

$$|T_{N,\varphi} f| \leq |T_{N,\varphi}(f - f_n)| + |T_{N,\varphi}(f_n)| \leq C_M M^+(f - f_n) + T_\varphi^* f_n.$$

Therefore $\sup_{N=(N_1,N_2):|N_1|,|N_2|\leq M} |T_{N,\varphi} f| \leq C_M M^+(f - f_n) + T^* f_n$. Using that $w \in A_p^+$ and what we have already proved, we have

$$\begin{aligned} \int \sup_{N=(N_1,N_2):|N_1|,|N_2|\leq M} |T_{N,\varphi} f|^p w &\leq C_M \int |M^+(f - f_n)|^p w + C \int |T_\varphi^* f_n|^p w \\ &\leq C_M \int |f - f_n|^p w + C \left(\int \varphi \right)^p \int |f_n|^p w. \end{aligned}$$

Letting n tend to ∞ and then taking limit as $M \to +\infty$ we finish the proof.

2.3. **Proof of Theorem 2.2 (ii).** As in [3] for the function $\chi = \chi_{(-1,0)}$, the theorem follows from Theorem 2.2 (i) and the following lemma.

Lemma 2.4. Let a be supported on $I = (x^*, x^*+h)$ and such that $\int_I a = 0$ and let $w \in A_1^+$. If $A = \rho^{2(\alpha+1)}$ there exists C independent of x^*, h and a, such that

$$\int_{z<x^*-Ah} T_\varphi^* a(z)w(z)\,dz \le C\int_I |a(z)|w(z)\,dz.$$

PROOF. We start pointing out that this result was proved in [3] for the function $\chi = \chi_{(-1,0)}$. Second, as in [3], it suffices to prove

$$\int_{z<-\varepsilon_{i+\alpha}} T_\varphi^* a(z)w(z)\,dz \le C\int_I |a(z)|w(z)\,dz,$$

assuming that $I = (0, \varepsilon_i)$.

First we prove the inequality for simple functions. Let $\varphi = \sum_{\ell=1}^{s} a_\ell \chi_{(-b_\ell,0)}$, $a_\ell \ge 0$. For each ℓ, let us pick $\varepsilon_{t(\ell)}^\ell$ such that $\varepsilon_{t(\ell)-1}^\ell < \varepsilon_i \le \varepsilon_{t(\ell)}^\ell$. Obviously, a has support in $(0, \varepsilon_{t(\ell)}^\ell)$ and $T_\varphi^* a(z) \le \sum_{\ell=1}^s a_\ell b_\ell T^*_{\chi,\varepsilon^\ell,v} a(z)$. Therefore,

$$\begin{aligned}\int_{z<-\varepsilon_{i+\alpha}} T_\varphi^* a(z)w(z)\,dz &\le \sum_{\ell=1}^{s} a_\ell b_\ell \int_{z<-\varepsilon_{i+\alpha}} T^*_{\chi,\varepsilon^\ell,v} a(z)w(z)\,dz\\ &= \sum_{\ell=1}^{s} a_\ell b_\ell \int_{z<-\varepsilon^\ell_{t(\ell)+\alpha}} T^*_{\chi,\varepsilon^\ell,v} a(z)w(z)\,dz\\ &\quad+ \sum_{\ell=1}^{s} a_\ell b_\ell \int_{-\varepsilon^\ell_{t(\ell)+\alpha}<z<-\varepsilon_{i+\alpha}} T^*_{\chi,\varepsilon^\ell,v} a(z)w(z)\,dz.\end{aligned}$$

By the result in [3] for the function $\chi = \chi_{(-1,0)}$ (notice that $\varepsilon^\ell = \{\varepsilon_k^\ell\}$ is a ρ- lacunary sequence), we have that

$$\sum_{\ell=1}^{s} a_\ell b_\ell \int_{z<-\varepsilon^\ell_{t(\ell)+\alpha}} T^*_{\chi,\varepsilon^\ell,v} a(z)w(z)\,dz \le C\left(\int\varphi\right)\int |a(z)|w(z)\,dz.$$

Now we estimate the term $\sum_{\ell=1}^{s} a_\ell b_\ell \int_{-\varepsilon^\ell_{t(\ell)+\alpha}<z<-\varepsilon_{i+\alpha}} T^*_{\chi,\varepsilon^\ell,v} a(z)w(z)\,dz$. Since $\varepsilon^\ell_{t(\ell)-1} < \varepsilon_i$, we have that $\varepsilon^\ell_{t(\ell)+\alpha} \le \varepsilon_{i+2\alpha+2}$. Therefore

$$\int_{-\varepsilon^\ell_{t(\ell)+\alpha}<z<-\varepsilon_{i+\alpha}} T^*_{\chi,\varepsilon^\ell,v} a(z)w(z)\,dz \le \sum_{m=i+\alpha}^{m=i+2\alpha+1} \int_{-\varepsilon_{m+1}}^{-\varepsilon_m} T^*_{\chi,\varepsilon^\ell,v} a(z)w(z)\,dz.$$

Let us fix ℓ and $N \in \mathbb{Z}^2$. Notice that $|T_{N,\chi,\varepsilon_k^\ell,v_k} a(z)|$ is bounded by

$$\sum_k \left|\int_I v_k\left(\frac{1}{\varepsilon_k^\ell}\chi_{(-\varepsilon_k^\ell,0)}(z-u) - \frac{1}{\varepsilon_{k-1}^\ell}\chi_{(-\varepsilon_{k-1}^\ell,0)}(z-u)\right)a(u)\,du\right|.$$

If $z \in (-\varepsilon_{m+1}, -\varepsilon_m)$ and $u \in I$, then $z-u \in (-\varepsilon_{m+2}, -\varepsilon_m)$ and the k-terms in the above sum such that $\varepsilon_k^\ell \le \varepsilon_m$ or $\varepsilon_{m+2} \le \varepsilon_{k-1}^\ell$ are zero. In fact, in the first case because $(-\varepsilon_k^\ell, 0) \cap (-\varepsilon_{m+2}, -\varepsilon_m) = \emptyset$ and, in the second case, because $\chi_{(-\varepsilon_k^\ell,0)}(z-u) = \chi_{(-\varepsilon_{k-1}^\ell,0)}(z-u) = 1$ and $\int_I a = 0$. Then, we only have at most 4 terms in the above sum and in these cases $\varepsilon_k^\ell \approx \varepsilon_m$. So that $|T_{N,\chi,\varepsilon_k^\ell,v_k} a(z)| \le C\frac{1}{\varepsilon_m}\int_I |a(u)|\,du$ and

$$\int_{-\varepsilon_{m+1}}^{-\varepsilon_m} T^*_{\chi,\varepsilon^\ell,v} a(z) w(z)\,dz \le C\frac{1}{\varepsilon_m}\int_I |a(u)|\,du \int_{-\varepsilon_{m+1}}^{-\varepsilon_m} w(z)\,dz.$$

Since $w \in A_1^+$, we have that

$$\int_{-\varepsilon_{m+1}}^{-\varepsilon_m} T^*_{\chi,\varepsilon^\ell,v} a(z) w(z)\,dz \le C\int_I |a(u)|w(u)\,du.$$

Hence

$$\begin{aligned}\int_{-\varepsilon^\ell_{t(\ell)+\alpha}<z<-\varepsilon_{i+\alpha}} T^*_{\chi,\varepsilon^\ell,v} a(z)w(z)\,dz &\le C\sum_{m=i+\alpha}^{m=i+2\alpha+1}\int_I |a(u)|w(u)\,du\\ &= C\int_I |a(u)|w(u)\,du.\end{aligned}$$

Then

$$\sum_{\ell=1}^s a_\ell b_\ell \int_{-\varepsilon^\ell_{t(\ell)+\alpha}<z<-\varepsilon_{i+\alpha}} T^*_{\chi,\varepsilon^\ell,v} a(z)w(z)\,dz \le C\left(\int\varphi\right)\int_I |a(u)|w(u)\,du,$$

and we are done for simple functions.

For the general case, using the notations introduced in the proof of Theorem 2.2 (i) (with $f = a$), we have that

$$\begin{aligned}&\int_{z<-\varepsilon_{i+\alpha}} \sup_{N:|N_1|,|N_2|\le M} |T_{N,\varphi} a(z)|w(z)\,dz \le \liminf_{j\to+\infty}\int_{z<-\varepsilon_{i+\alpha}} T^*_{\varphi_{k_j}} a(z)w(z)\,dz\\ &\le C\liminf_{j\to+\infty}\left(\int\varphi_{k_j}\right)\int_I |a(z)|w(z)\,dz = C\left(\int\varphi\right)\int_I |a(z)|w(z)\,dz.\end{aligned}$$

Letting M tend to ∞, we are done. □

3. Proof of Theorem 1.7

Proof of (a)(ii) and (b)(ii) for $\varphi = \chi_{(0,1)}$. We shall follow the idea in [16] (see Theorem 2.5 in page 11). Since the semigroup is Cesàro bounded in $L^p(\nu)$, we have that for all $f \in L^p(\nu)$ and for a.e. $x \in X$ the functions $f^x(s) = T^s f(x)$ are locally integrable. It follows that for a.e. x

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_0^\varepsilon f^x(s+t)\,dt = f^x(s) \quad \text{a.e. } s.$$

Since, for a.e. s and for a.e. t, $f^x(s+t) = g_s(x)g_t(\tau_s x)f(\tau_{s+t}x)$ and $f^x(s) = g_s(x)f(\tau_s x)$ we have

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_0^\varepsilon g_t(\tau_s x)f(\tau_{s+t}x)\,dt = f(\tau_s x) \quad \text{a.e. } s \ge 0.$$

Let $E = \left\{(x,s) : s \ge 0, \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_0^\varepsilon g_t(\tau_s x)f(\tau_{s+t}x)\,dt = f(\tau_s x)\right\}$. This set is measurable in the product space. Let $N = X\times[0,\infty)\setminus E$ and let $N_x =$

$\{s \geq 0 : (x,s) \in N\}$. Then for almost every x, the Lebesgue measure $|N_x| = 0$. Therefore, $|N| = 0$ and for a.e. $s \geq 0$ the set $N^s = \{x \in X : (x,s) \in N\}$ has measure zero. Notice that $N^s = \tau_{-s}(N^0)$. Let $s > 0$ such that $\nu(N^s) = 0$. Since the transformations are nonsingular then we obtain $\nu(N^0) = 0$. That means that for a.e. x, $\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_0^\varepsilon g_t(x) f(\tau_t x)\, dt = f(x)$. as we wished to prove.

Before continuing with the proof, we notice that what we have just proved asserts that $\mathcal{M}^+ f(x) = \sup_{0<\varepsilon\in\mathbb{Q}} \mathcal{A}_\varepsilon^+|f|(x)$ a.e. for each measurable function. Therefore $\mathcal{M}^+ f$ is measurable. It is proved in a similar way that $\mathcal{M}_\varphi^+ f$ is measurable when φ is a simple function. Finally, for general φ, $\mathcal{M}_\varphi^+ f$ is measurable since is the limit of $\mathcal{M}_{\varphi_n}^+ f$, where φ_n is a sequence of simple functions.

Proof of (b)(i). As we said at the beginning of the introduction, the functions $H_t(x)$ are measurable with respect to the σ-algebra product and for a.e. t

$$\int_X f(x)\, d\nu(x) = \int_X f(\tau_t x) H_t(x)\, d\nu(x) \tag{3.1}$$

for all nonnegative functions and all $f \in L^1(\nu)$ (see (1.3) and keep in mind that we assume $g_t(x) = 1$ in the case $p = 1$). Since the flow is Cesàro bounded in $L^1(\nu)$, we have by Tonelli's Theorem that

$$\frac{1}{\varepsilon}\int_0^\varepsilon \int_X f(\tau_t x)\, d\nu\, dt \leq C \int_X f(x)\, d\nu,$$

for every $\varepsilon > 0$ and each measurable function $f \geq 0$. But, for almost all t, $\int_X f(\tau_t x)\, d\nu = \int_X f(x) H_{-t}(x)\, d\nu$. Therefore

$$\int_X f(x) \left(\frac{1}{\varepsilon}\int_0^\varepsilon H_{-t}(x)\, dt\right) d\nu \leq C \int_X f(x)\, d\nu$$

for all nonnegative measurable functions f, which implies $\frac{1}{\varepsilon}\int_0^\varepsilon H_{-t}(x)\, dt \leq C$ a.e. x. Since the function on the left-hand side is $\mathcal{F}$-measurable and τ_s is nonsingular we have that for all s, $\frac{1}{\varepsilon}\int_0^\varepsilon H_{-t}(\tau_s x)\, dt \leq C$ a.e. x. Multiplying by $H_s(x)$, $\frac{1}{\varepsilon}\int_0^\varepsilon H_{-t}(\tau_s x) H_s(x)\, dt \leq C H_s(x)$ a.e. x. Therefore, for almost every s,

$$\frac{1}{\varepsilon}\int_0^\varepsilon H_{-t}(\tau_s x) H_s(x)\, dt = \frac{1}{\varepsilon}\int_0^\varepsilon H_{s-t}(x)\, dt \leq C H_s(x) \quad \text{a.e. } x.$$

Notice that the set

$$E = \left\{(x,s) \in X \times \mathbb{R} : \frac{1}{\varepsilon}\int_0^\varepsilon H_{-t}(\tau_s x) H_s(x)\, dt > C H_s(x)\right\}$$

is measurable in the completion of the product-σ-algebra $\mathcal{F} \otimes \mathcal{L}$ and the last statement implies that the completion of the product measure of E is zero. Then it follows that for almost every $x \in X$

$$\frac{1}{\varepsilon}\int_{s-\varepsilon}^s H_t(x)\, dt = \frac{1}{\varepsilon}\int_0^\varepsilon H_{s-t}(x)\, dt \leq C H_s(x) \quad \text{for a.e. } s,$$

or, in other words, for almost every x the functions $t \to H_t(x)$ satisfy A_1^+ with a constant independent of x.

Since $\mathcal{M}_\varphi^+ \leq \left(\int_0^\infty \varphi\right) \mathcal{M}^+$, it is enough to prove the weak type $(1,1)$ inequality for $\mathcal{M}^+$ and we shall do it by transference arguments. We can assume that $f \geq 0$. For each $\eta > 0$, let us consider $\mathcal{M}_\eta^+ f(x) = \sup_{0<\varepsilon\leq\eta} \mathcal{A}_\varepsilon^+ f(x)$. Let $\lambda > 0$ and $E_\lambda = \{x \in X : \mathcal{M}_\eta^+ f(x) > \lambda\}$. Let us fix $R > 0$. Then, using (3.1),

$$\begin{aligned}\nu(E_\lambda) &= \frac{1}{R}\int_0^R \int_X \chi_{E_\lambda}(\tau_t x) H_t(x)\, d\nu(x)\, dt \\ &= \int_X \frac{1}{R}\int_0^R \chi_{E_\lambda}(\tau_t x) H_t(x)\, dt\, d\nu(x).\end{aligned}$$

If we define $f^x(t) = f(\tau_t x)$, we have that if $R > 0$, $t \leq R$, and $\chi_{E_\lambda}(\tau_t x) = 1$ then $M^+(f^x \chi_{[0,R+\eta]})(t) > \lambda$. Therefore

$$\nu(E_\lambda) \leq \int_X \frac{1}{R}\int_{\{t: M^+(f^x\chi_{[0,R+\eta]})(t)>\lambda\}} H_t(x)\, dt\, d\nu(x).$$

Since, for almost every x, the functions $t \to H_t(x)$ satisfy A_1^+ with a constant independent of x and A_1^+ characterizes the weak-type $(1,1)$ inequality of M^+ (see §2), we obtain

$$\begin{aligned}\nu(E_\lambda) &\leq \frac{C}{\lambda}\int_X \frac{1}{R}\int_0^{R+\eta} f^x(t) H_t(x)\, dt\, d\nu(x) \\ &= \frac{C}{\lambda R}\int_0^{R+\eta}\int_X f(\tau_t x) H_t(x)\, d\nu(x)\, dt \\ &= \frac{C}{\lambda R}\int_0^{R+\eta}\int_X f(x)\, d\nu(x)\, dt = \frac{C(R+\eta)}{\lambda R}\int_X f(x)\, d\nu(x).\end{aligned}$$

Letting R and then η tend to infinity we obtain the inequality that we wished to prove.

Proof of (a)(i). We shall start proving that our assumption, the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$, implies that

$$\text{(3.2)} \qquad \begin{gathered}\text{for almost every } x \in X, \text{ the function } t \to H_t(x) \\ \text{belongs to } A_p^+, \text{ with a constant independent of } x.\end{gathered}$$

We shall use the ideas of Rubio de Francia about factorization of weights [21]. By hypothesis, there exists $C > 0$ independent of $\varepsilon > 0$ and f such that

$$\int_X |\mathcal{A}_{2\varepsilon}^+ f|^p d\nu \leq C \int_X |f|^p d\nu \quad \text{for all } f \in L^p(\nu)$$

and, consequently,

$$\int_X |(\mathcal{A}_{2\varepsilon}^+)^* f|^{p'} d\nu \leq C \int_X |f|^{p'} d\nu \quad \text{for all } f \in L^{p'}(\nu),$$

where $(\mathcal{A}^+_{2\varepsilon})^*$ is the adjoint operator of $\mathcal{A}^+_{2\varepsilon}f$. Notice that if T^*_t is the formal adjoint of T_t then for almost all t

$$T^*_t h(x) = \frac{J_{-t}(x)}{g_{-t}(x)} h(\tau_{-t}x) \qquad \text{and} \tag{3.3}$$

$$H_t(x) = ((T^{-t})^* h^p)(x)(T^t h^{p'})^{1-p}(x) \quad \text{a.e. } x \tag{3.4}$$

for any function $h > 0$, $h \in L^{pp'}(\nu)$. For $h \in L^{pp'}(\nu)$, we define $Q_\varepsilon h = \left(\mathcal{A}^+_{2\varepsilon}|h|^{p'}\right)^{1/p'}$ and $P_\varepsilon h = \left((\mathcal{A}^+_{2\varepsilon})^*|h|^p\right)^{1/p}$. Then Q_ε, P_ε and $R_\varepsilon = Q_\varepsilon + P_\varepsilon$ are bounded from $L^{pp'}(\nu)$ into $L^{pp'}(\nu)$ with constants independent of $\varepsilon > 0$. Let us fix $C > 0$ such that $||R_\varepsilon h||_{L^{pp'}(\nu)} \le C||h||_{L^{pp'}(\nu)}$, for all $h \in L^{pp'}(\nu)$ and all $\varepsilon > 0$. For fixed $h > 0$, $h \in L^{pp'}(\nu)$, and $\varepsilon > 0$, let $G(x) = \sum_{j=0}^\infty \frac{R^{(j)}_\varepsilon h(x)}{(2C)^j}$, where $R^{(j)}_\varepsilon$ is the j-th iteration of R_ε. Then, $G \in L^{pp'}(\nu)$, $h \le G$ a.e., $R_\varepsilon G \le 2CG$ a.e. and, as a consequence, $P_\varepsilon G \le 2CG$ a.e. and $Q_\varepsilon G \le 2CG$ a.e., i.e., there exists $C > 0$ such that,

$$\mathcal{A}^+_{2\varepsilon} G^{p'} \le C G^{p'} \quad \text{a.e.} \qquad \text{and} \tag{3.5}$$

$$(\mathcal{A}^+_{2\varepsilon})^* G^p \le C G^p \quad \text{a.e.} \tag{3.6}$$

Since the operators T^t are linear and positive, we get from (3.5) that for $s \le t \le s + \varepsilon$,

$$\begin{aligned} CT^t G^{p'}(x) \ge T^t\left(\mathcal{A}^+_{2\varepsilon} G^{p'}\right)(x) &= \frac{1}{2\varepsilon}\int_0^{2\varepsilon} T^{t+s} G^{p'}(x)ds \\ &= \frac{1}{2\varepsilon}\int_t^{2\varepsilon+t} T^u G^{p'}(x)du \ge \frac{1}{2\varepsilon}\int_{s+\varepsilon}^{s+2\varepsilon} T^u G^{p'}(x)du. \end{aligned}$$

Raising to $1 - p < 0$, multiplying by $(T^{-t})^* G^p(x)$ and using (3.4) we have for almost all t

$$CH_t(x) \le \left(\frac{1}{2\varepsilon}\int_{s+\varepsilon}^{s+2\varepsilon} T^u G^{p'}(x)du\right)^{1-p} (T^{-t})^* G^p(x) \quad \text{a.e. } x,$$

where the exceptional set depends on ε and t. Integrating over any measurable set $A \subset X$,

$$C\int_A H_t(x)\,d\nu \le \int_A \left(\frac{1}{2\varepsilon}\int_{s+\varepsilon}^{s+2\varepsilon} T^u G^{p'}(x)du\right)^{1-p} (T^{-t})^* G^p(x)\,d\nu$$

for a.e. $t \in [s, s + \varepsilon]$. Integrating over the interval $[s, s + \varepsilon]$ and applying Fubini's Theorem, we obtain

$$\begin{aligned} &C\int_A\int_s^{s+\varepsilon} H_t(x)\,dt\,d\nu \\ &\qquad \le \int_A \left(\frac{1}{2\varepsilon}\int_{s+\varepsilon}^{s+2\varepsilon} T^u G^{p'}(x)du\right)^{1-p} \left(\int_s^{s+\varepsilon}(T^{-t})^* G^p(x)\,dt\right) d\nu. \end{aligned}$$

Since A is any measurable subset we have
(3.7)

$$C\int_s^{s+\varepsilon} H_t(x)\,dt \le \left(\frac{1}{2\varepsilon}\int_{s+\varepsilon}^{s+2\varepsilon} T^u G^{p'}(x)du\right)^{1-p}\left(\int_s^{s+\varepsilon}(T^{-t})^* G^p(x)\,dt\right) \text{a.e. } x,$$

where the exceptional set depends on s and ϵ. On the other hand, since the adjoints $(T^{-t})^*$ are also linear and positive, arguing in the same way we get from (3.6) that
(3.8)

$$C\int_{s+\varepsilon}^{s+2\varepsilon} (H_t(x))^{1-p'}dt \le \left(\frac{1}{2\varepsilon}\int_s^{s+\varepsilon}(T^{-u})^* G^p(x)du\right)^{1-p'}\int_{s+\varepsilon}^{s+2\varepsilon} T^t G^{p'}(x)dt \quad \text{a.e. } x,$$

where the exceptional set depends on s and ϵ. From (3.8) and (3.7), we get

$$\int_s^{s+\varepsilon} H_t(x)dt\left(\int_{s+\varepsilon}^{s+2\varepsilon}(H_t(x))^{1-p'}dt\right)^{p-1} \le C\varepsilon^p, \quad \text{a.e. } x, \tag{3.9}$$

where the exceptional set depends on s and ϵ. Then, for almost every x,

$$\int_a^b H_t(x)dt\left(\int_b^c (H_t(x))^{1-p'}dt\right)^{p-1} \le C(c-a)^p, \tag{3.10}$$

for all rational numbers $a < c$ and $b = (a + c)/2$. Now it is clear that the same holds for all real numbers $a < c$ and $b = (a + c)/2$. Therefore, (3.2) holds (see Remarks 2.1).

Now, let us prove *(a)(i)*. Since $\mathcal{M}^+ f(x) \le \mathcal{M}^+(|f|)(x)$, we can assume that $f \ge 0$. For each $\eta > 0$, let us consider $\mathcal{M}^+_\eta f(x) = \sup_{0<\varepsilon\le\eta}\mathcal{A}^+_\varepsilon f(x)$. From the positivity of T^t we have that $T^t\mathcal{M}^+_\eta f(x) = \mathcal{M}^+_\eta(T^t f)(x)$. If we define $f^x(t) = T^t f(x)$, we have that for all $R > 0$ and all $t \le R$
(3.11)

$$\begin{aligned}\mathcal{M}^+_\eta(T^t f)(x) &= \sup_{0<\varepsilon\le\eta}\frac{1}{\varepsilon}\int_0^\varepsilon T^{s+t}f(x)ds = \sup_{0<\varepsilon\le\eta}\frac{1}{\varepsilon}\int_0^\varepsilon f^x(s+t)ds \\ &= \sup_{0<\varepsilon\le\eta}\frac{1}{\varepsilon}\int_0^\varepsilon f^x\chi_{[0,R+\eta]}(s+t)ds \le M^+(f^x\chi_{[0,R+\eta]})(t),\end{aligned}$$

where M^+ is the one-sided Hardy-Littlewood maximal operator in $\mathbb{R}$. Then, by (1.3), Fubini's theorem, (3.2) and the fact that A^+_p implies boundedness of the one-sided Hardy-Littlewood maximal operator, we get that for each

$R > 0$,

$$\begin{aligned}
\int_X (\mathcal{M}_\eta^+ f(x))^p d\nu(x) &= \frac{1}{R}\int_0^R \int_X |T^t \mathcal{M}_\eta^+ f(x)|^p H_t(x) d\nu(x) dt \\
&\leq \int_X \frac{1}{R}\int_0^R |M^+(f^x \chi_{[0,R+\eta]})(t)|^p H_t(x)\, dt\, d\nu(x) \\
&\leq \frac{C}{R}\int_X \int_0^{R+\eta} |f^x(t)|^p H_t(x) dt d\nu(x) \qquad (3.12) \\
&= \frac{C}{R}\int_0^{R+\eta}\int_X |T^t f(x)|^p H_t(x) d\nu(x) dt \\
&= \frac{C}{R}\int_0^{R+\eta}\int_X |f(x)|^p d\nu(x) dt = C\frac{R+\eta}{R}\int_X |f(x)|^p d\nu(x).
\end{aligned}$$

Letting, first R, and then η, go to infinity we obtain

$$\int_X (\mathcal{M}^+ f(x))^p d\nu(x) \leq C \int_X |f(x)|^p d\nu(x).$$

Proof of (a)(ii) and (b)(ii). Since the maximal operator is bounded in $L^p(\nu)$ ($p > 1$) or of weak type $(1,1)$, it is enough to prove the a.e. convergence. By Proposition 1.4 it suffices to show the a.e. convergence in the standard case $\varphi = \chi_{(0,1)}$ which has already been proved at the beginning of this section.

Proof of (a)(iii). Since the maximal operator is bounded in $L^p(\nu)$ it is enough to prove the a.e. convergence in a dense class. As before, it is enough to prove it in the standard case $\varphi = \chi_{(0,1)}$. To find the dense class we proceed almost as in Lemma 4.2 of [4]. We shall need some results which are interesting by itself.

Lemma 3.1. Assume that we are in the conditions of Theorem 1.7. If $1 \leq r < p$ let $\mathcal{G}_r = \{S^t : t \in \mathbb{R}\}$ be the one-parameter group of positive operators defined by $S^t f(x) = (g_t(x))^r f(\tau_t x)$ for all $f \geq 0$. Then there exists r, $1 < r < p$ such that the semigroup $\mathcal{G}_{r,+} = \{S^t : t > 0\}$ is Cesàro bounded in $L^{p/r}(\nu)$. Furthermore, the maximal operator associated to $\mathcal{G}_{r,+}$ is bounded in $L^{p/r}(\nu)$.

Proof. We have already seen in the proof of *(a)(i)* that $\mathcal{G}_+$ Cesàro bounded implies that for almost every x the functions $t \to H_t(x)$ belong to A_p^+ with a constant independent of x (actually, the implication is an equivalence, see Remark 3.5). Then by the properties of A_p^+ classes, we have that there exists r, $1 < r < p$, such that for almost every x the function $t \to H_t(x)$ belongs to $A_{p/r}^+$ with a constant independent of x (see [24] and [19]). We notice that

$$\int_X |S^t f(x)|^{p/r} H_t(x) d\nu(x) = \int_X |f(x)|^{p/r} d\nu(x). \qquad (3.13)$$

for all $f \in L^{p/r}(\nu)$ and all $f \geq 0$. Again, by the proof of *(a)(i)*, applied to the semigroup $\mathcal{G}_{r,+}$, we obtain that the maximal operator associated to $\mathcal{G}_{r,+}$ is bounded in $L^{p/r}(\nu)$ and, therefore, the semigroup is Cesàro bounded in $L^{p/r}(\nu)$. □

Lemma 3.2. Assume that we are in the conditions of Theorem 1.7 and $1 < p < \infty$. Then, for all $f \in L^p(\nu)$,

(a) $\lim_{\varepsilon\to+\infty}[\mathcal{A}_\varepsilon^+ f(x) - \mathcal{A}_\gamma^+ (\mathcal{A}_\varepsilon^+ f)(x)] = 0$ a.e. for all $\gamma > 0$.

(b) $\lim_{\varepsilon\to+\infty} ||\mathcal{A}_\varepsilon^+ f - \mathcal{A}_\gamma^+ (\mathcal{A}_\varepsilon^+ f)||_p = 0$ for all $\gamma > 0$.

Proof. First, notice that

$$\mathcal{A}_\varepsilon^+ f(x) - \mathcal{A}_\gamma^+ \left(\mathcal{A}_\varepsilon^+ f\right)(x) = \frac{1}{\gamma}\int_0^\gamma \left(\mathcal{A}_\varepsilon^+ f(x) - T^s(\mathcal{A}_\varepsilon^+ f)(x)\right) ds. \tag{3.14}$$

Let us fix $s > 0$, $0 < s < \gamma$ and $\varepsilon > \gamma$. Then we have

$$\begin{aligned} \left|\mathcal{A}_\varepsilon^+ f(x) - T^s(\mathcal{A}_\varepsilon^+ f)(x)\right| &= \left|\frac{1}{\varepsilon}\int_0^\varepsilon T^t f(x)dt - \frac{1}{\varepsilon}\int_s^{s+\varepsilon} T^t f(x)dt\right| \\ &= \left|\frac{1}{\varepsilon}\int_0^s T^t f(x)dt - \frac{1}{\varepsilon}\int_\varepsilon^{s+\varepsilon} T^t f(x)dt\right| \\ &\le \frac{1}{\varepsilon}\int_0^\gamma T^t|f|(x)dt + \frac{1}{\varepsilon}\int_\varepsilon^{\gamma+\varepsilon} T^t|f|(x)dt. \end{aligned} \tag{3.15}$$

Therefore, by (3.14),

(3.16)
$$\left|\mathcal{A}_\varepsilon^+ f(x) - \mathcal{A}_\gamma^+ \left(\mathcal{A}_\varepsilon^+ f\right)(x)\right| \le \frac{1}{\varepsilon}\int_0^\gamma T^t|f|(x)dt + \frac{1}{\varepsilon}\int_\varepsilon^{\gamma+\varepsilon} T^t|f|(x)dt.$$

It is clear that $\lim_{\varepsilon\to\infty} \frac{1}{\varepsilon}\int_0^\gamma T^t|f|(x)dt = 0$ for a.e. x, since the function $f^x(t) = T^t f(x)$ is locally integrable for almost every x. To control the other term we use Lemma 3.1. Let $p > r > 1$ and let $\Gamma_\gamma = \{S^t : t \in \mathbb{R}\}$ be as in that lemma and let $\widetilde{\mathcal{M}}^+$ be the maximal operator associated to $\mathcal{G}_{r,+} = \{S^t : t > 0\}$. By Lemma 3.1, $\widetilde{\mathcal{M}}^+$ is bounded from $L^{p/r}(\nu)$ into $L^{p/r}(\nu)$. Consequently, $\widetilde{\mathcal{M}}^+(|f|^r)(x) < \infty$ a.e. for $f \in L^p(\nu)$. It follows that $\frac{1}{\varepsilon}\int_\varepsilon^{\gamma+\varepsilon} T^t|f|(x)dt$ tends to 0 a.e. as ε goes to infinity since

(3.17)
$$\begin{aligned} \frac{1}{\varepsilon}\int_\varepsilon^{\gamma+\varepsilon} T^t|f|(x)dt &\le \frac{1}{\varepsilon}\left(\int_\varepsilon^{\gamma+\varepsilon} (T^t|f|(x))^r dt\right)^{1/r} \gamma^{1/r'} \\ &\le \frac{(\gamma+\varepsilon)^{1/r}}{\varepsilon}\left(\frac{1}{\gamma+\varepsilon}\int_0^{\gamma+\varepsilon} S^t(|f|^r)(x)dt\right)^{1/r} \gamma^{1/r'} \\ &\le \frac{(\gamma+\varepsilon)^{1/r}\gamma^{1/r'}}{\varepsilon}[\widetilde{\mathcal{M}}^+(|f|^r)(x)]^{1/r}. \end{aligned}$$

Therefore *(a)* is completely proved.

To prove *(b)* we observe that $|\mathcal{A}_\varepsilon^+ f - \mathcal{A}_\gamma^+(\mathcal{A}_\varepsilon^+ f)| \le \mathcal{M}^+ f + \mathcal{M}^+(\mathcal{A}_\gamma^+ f)$. It follows from statement *(a)(i)* in Theorem 1.7 that $\mathcal{M}^+ f + \mathcal{M}^+(\mathcal{A}_\gamma^+ f) \in L^p(\nu)$. Then *(b)* follows from *(a)* and the dominated convergence theorem. □

The next theorem follows from Lemma 3.2 using a standard argument. We include it for the sake of completeness.

Theorem 3.3. Assume that we are in the conditions of Theorem 1.7 with $1 < p < \infty$. Let $A = \{f \in L^p(\nu) : \mathcal{A}_\gamma^+ f = f \text{ for all } \gamma > 0\}$ and let B be the linear manifold generated by $\{f - \mathcal{A}_\gamma^+ f : f \in L^p(\nu), \gamma > 0\}$. Then, $A \oplus \bar{B} = L^p(\nu)$, where $\bar{B}$ stands for the closure of B and $A \oplus \bar{B} = \{f + g : f \in A, g \in \bar{B}\}$. In particular $A \oplus B$ is dense in $L^p(\nu)$.

Proof. We first prove that $\{\mathcal{A}_\varepsilon^+ f\}$ is weakly convergent as ε goes to infinity for all $f \in L^p(\nu)$.

Let $f \in L^p(\nu)$. By hypothesis, $\sup_{\varepsilon>0} ||\mathcal{A}_\varepsilon^+ f||_{L^p(\nu)} \leq C||f||_{L^p(\nu)}$. This gives that the set $\{\mathcal{A}_\varepsilon^+ f : \varepsilon > 0\}$ is bounded in $L^p(\nu)$. Therefore there exists a sequence $\{\varepsilon_k\} \to \infty$ such that $\{\mathcal{A}_{\varepsilon_k}^+ f\}$ is weakly convergent. If we suppose that $\{\mathcal{A}_\varepsilon^+ f\}$ is not weakly convergent as ε goes to infinity, then there exist another sequence $\{\eta_k\} \to \infty$ and $g_1, g_2 \in L^p(\nu)$, $g_1 \neq g_2$, such that $\{\mathcal{A}_{\varepsilon_k}^+ f\}$ converges weakly to g_1 and $\{\mathcal{A}_{\eta_k}^+ f\}$ converges weakly to g_2. The continuity of $\mathcal{A}_\gamma^+$ gives that $\{\mathcal{A}_{\varepsilon_k}^+ f - \mathcal{A}_\gamma^+(\mathcal{A}_{\varepsilon_k}^+ f)\}$ converges weakly to $g_1 - \mathcal{A}_\gamma^+ g_1$. On the other hand, by part (b) of Lemma 3.2, $\{\mathcal{A}_{\varepsilon_k}^+ f - \mathcal{A}_\gamma^+(\mathcal{A}_{\varepsilon_k}^+ f)\}$ converges to 0 in $L^p(\nu)$. Therefore, $g_1 \in A$. The same argument gives that $g_2 \in A$ and, as a consequence, $0 \neq g_1 - g_2 \in A$.

We shall prove now that $g_1 - g_2 \in \bar{B}$. If $g_1 - g_2 \notin \bar{B}$, then there exists a linear functional $\Lambda : L^p(\nu) \to \mathbb{R}$, such that $\Lambda(\bar{B}) = 0$ and $\Lambda(g_1 - g_2) = 1$. It follows that $\Lambda g = \Lambda(\mathcal{A}_\gamma^+ g)$ for all $g \in L^p(\nu)$ and all $\gamma > 0$. In particular, $\Lambda(\mathcal{A}_{\varepsilon_k}^+ f) = \Lambda f$. On the other hand, $\{\Lambda(\mathcal{A}_{\varepsilon_k}^+ f)\}$ converges to Λg_1 in $\mathbb{R}$. Then $\Lambda g_1 = \Lambda f$. In analogous way we get that $\Lambda g_2 = \Lambda f$. It follows that $1 = \Lambda(g_1 - g_2) = \Lambda g_1 - \Lambda g_2 = 0$, which is a contradiction. This proves that $g_1 - g_2 \in \bar{B}$.

Let us prove now that $||\mathcal{A}_\varepsilon^+ g||_{L^p(\nu)} \to 0$ as ε tends to infinity, for all $g \in \bar{B}$. If $g = g_0 - \mathcal{A}_\gamma^+ g_0$ for some $g_0 \in L^p(\nu)$ and $s > 0$, this follows from part (b) of Lemma 3.2, and therefore it holds for any $g \in B$. Let now fix $g \in \bar{B}$. For any $\delta > 0$, there exists $g_0 \in B$ such that $||g - g_0||_{L^p(\nu)} < \delta$. As a consequence,

$$\begin{aligned} ||\mathcal{A}_\varepsilon^+ g||_{L^p(\nu)} &\leq ||\mathcal{A}_\varepsilon^+ g - \mathcal{A}_\varepsilon^+ g_0||_{L^p(\nu)} + ||\mathcal{A}_\varepsilon^+ g_0||_{L^p(\nu)} \\ &= ||\mathcal{A}_\varepsilon^+ (g - g_0)||_{L^p(\nu)} + ||\mathcal{A}_\varepsilon^+ g_0||_{L^p(\nu)} \leq C\delta + ||\mathcal{A}_\varepsilon^+ g_0||_{L^p(\nu)}. \end{aligned}$$

Since $||\mathcal{A}_\varepsilon^+ g_0||_{L^p(\nu)} \to 0$ as ε tends to infinity ($g_0 \in B$) and δ is any positive number we conclude that $||\mathcal{A}_\varepsilon^+ g||_{L^p(\nu)} \to 0$ as ε tends to infinity .

We have already seen that $g_1 - g_2 \in \bar{B}$. Then $\{\mathcal{A}_\varepsilon^+(g_1 - g_2)\}$ converges to 0 in $L^p(\nu)$. On the other hand, $g_1 - g_2 \in A$ which gives that $\mathcal{A}_\varepsilon^+(g_1 - g_2) = g_1 - g_2$. Then $g_1 - g_2 = 0$, against $g_1 \neq g_2$. Therefore, $\{\mathcal{A}_\varepsilon^+ f\}$ is weakly convergent as ε goes to infinity. (The preceding argument also proves that $A \cap \bar{B} = \{0\}$.)

We shall prove now that $A \oplus \bar{B} = L^p(\nu)$. Let Pf be the weak limit of $\{\mathcal{A}_\varepsilon^+ f\}$ as ε tends to infinity. Then $f = Pf + (f - Pf)$. From the continuity of $\mathcal{A}_\gamma^+$ and part *(b)* of Lemma 3.2, it follows that $\mathcal{A}_\gamma^+(Pf) = Pf$ for all $\gamma > 0$, that is, $Pf \in A$. If we suppose that $f - Pf \notin \bar{B}$, then there exists a linear functional $\Lambda : L^p(\nu) \to \mathbb{R}$, such that $\Lambda(\bar{B}) = 0$ and $\Lambda(f - Pf) = 1$. But Pf is the weak limit of $\mathcal{A}_\varepsilon^+ f$ and therefore $\Lambda(Pf) = \lim_{\varepsilon\to\infty} \Lambda(\mathcal{A}_\varepsilon^+ f)$.

However, we have that $\Lambda(\mathcal{A}_\varepsilon^+ f) = \Lambda f$. Therefore $\Lambda(f - Pf) = 0$ which is a contradiction. $\square$

Now we can conclude the proof of *(a)(iii)* in Theorem 1.7. Since the maximal operator is bounded in $L^p(\nu)$ it is enough to prove the a.e. convergence in the dense class $D_1 = A \oplus B$. If $f \in A$ it is obvious. For $f \in B$, part (a) of Lemma 3.2 proves that $\{\mathcal{A}_\varepsilon^+ f\}$ converges to 0 a.e. as ε tends to infinity.

Remark 3.4. The set A in Theorem 3.3 equals to the set $\{f \in L^p(\nu) : T^s f = f \text{ for all } s > 0\}$, since it follows from (3.15) and (3.17) that $A_\varepsilon^+ f - T^s(A_\varepsilon^+ f)$ tends to zero a.e. as ε tends to $+\infty$ for all $f \in L^p(\nu)$.

Proof of (b)(ii) and (iii). Since the flow is Cesàro bounded in $L^1(\nu)$ and $g_t(x) = 1$, it follows that is Cesàro bounded in $L^p(\nu)$ for $1 < p < \infty$. Then the averages converge a.e. as $\varepsilon \to 0$ and as $\varepsilon \to +\infty$ for $f \in L^p(\nu) \cap L^1(\nu)$ which is a dense set in $L^1(\nu)$. This fact together with the weak type $(1,1)$ inequality of $\mathcal{M}^+$ gives the almost everywhere convergence and the convergence in measure of the φ-averages for all $f \in L^1(\nu)$.

Remark 3.5. It follows from the proof of Theorem 1.7 that the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$ if and only if

$$\begin{aligned}&\text{for almost every } x \in X, \text{ the function } t \to H_t(x)\\ &\text{belongs to } A_p^+, \text{ with a constant independent of } x,\end{aligned} \tag{3.18}$$

where the functions $H_t(x)$ are defined in (1.3).

4. Proof of Theorem 1.9

We shall use transference arguments and we shall only prove (a) since the proof of (b) is similar. We point out that the support of φ is not necessarily bounded. For that reason, we have had to modify slightly the usual transference arguments. As before, we shall use the notation $f^x(t) = T^t f(x)$.

For each natural M, we consider the set $Q_M = \{N \in \mathbb{Z}^2 : N = (N_1, N_2), N_1 \leq N_2, |N_1| \leq M, |N_2| \leq M\}$ and the operator $\mathcal{T}_{\varphi,M}^* f(x) = \sup_{N \in Q_M} |\mathcal{T}_{\varphi,N} f(x)|$.

Let $L > 0$ and $\varphi_L = \varphi \chi_{(0,L]}$. Then $\mathcal{T}_{\varphi,M}^* \leq \mathcal{T}_{\varphi_L,M}^* + \mathcal{T}_{\varphi-\varphi_L,M}^*$. Since the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$ we know that for almost every x the functions $t \to H_t(x)$ belong to A_p^+ with a constant independent of x.

Then, for each $R>0$, (1.3) and Theorem 2.2 give

$$\begin{aligned}\int_X [\mathcal{T}^*_{\varphi_L,M} f(x)]^p d\nu(x) &= \frac{1}{R}\int_0^R \int_X [T^t\left(\mathcal{T}^*_{\varphi_L,M} f\right)(x)]^p H_t(x) d\nu(x) dt \\ &\le \int_X \frac{1}{R}\int_0^R \left|T^*_{\varphi_L}\left(f^x \chi_{(0,L\varepsilon_M+R)}\right)(t)\right|^p H_t(x) dt d\nu(x) \\ &\le C\left(\int \varphi_L\right)^p \int_X \frac{1}{R}\int_0^{L\varepsilon_M+R} |f^x(t)|^p H_t(x) dt d\nu(x) \\ &= C\left(\int \varphi_L\right)^p \frac{1}{R}\int_0^{L\varepsilon_M+R}\int_X |T^t f(x)|^p H_t(x) d\nu(x) dt \\ &= C\left(\int \varphi_L\right)^p \frac{L\varepsilon_M+R}{R}\int_X |f(x)|^p d\nu(x).\end{aligned}$$

Letting R go to infinity, we obtain

$$\int_X [\mathcal{T}^*_{\varphi_L,M} f(x)]^p d\nu(x) \le C\left(\int \varphi_L\right)^p \int_X |f(x)|^p d\nu(x),$$

with a constant independent of M and L.

Let $\psi_L = \varphi(L)\chi_{(0,L]} + \varphi\chi_{(L,\infty)}$. By Proposition 1.3, $T^*_{\varphi-\varphi_L,M} f(x) \le 4M\mathcal{M}^+_{\psi_L} f(x) \le 4M||\psi_L||_1 \mathcal{M}^+ f(x)$. Then,

$$\int_X [T^*_{\varphi-\varphi_L,M} f(x)]^p\, d\nu(x) \le CM^p||\psi_L||_1^p \int_X |f(x)|^p\, d\nu(x).$$

Therefore

$$\begin{aligned}\int_X [\mathcal{T}^*_{\varphi,M} f(x)]^p d\nu(x) &\le \int_X [\mathcal{T}^*_{\varphi_L,M} f(x)]^p d\nu(x) + \int_X [T^*_{\varphi-\varphi_L,M} f(x)]^p\, d\nu(x) \\ &\le C\left[\left(\int \varphi_L\right)^p + CM^p||\psi_L||_1^p\right]\int_X |f(x)|^p\, d\nu(x).\end{aligned}$$

Since $||\psi_L||_1 \to 0$ when $L\to\infty$, we have

$$\int_X [\mathcal{T}^*_{\varphi,M} f(x)]^p d\nu(x) \le C\left(\int \varphi\right)^p \int_X |f(x)|^p d\nu(x)$$

Finally, letting M go to ∞ we are done.

5. Proof of Theorems 1.10, 1.11, 1.12 and 1.13

5.1. **Proof of Theorem 1.10.** Since $\mathcal{T}^*_\varphi$ is of strong type (p,p) (Theorem 1.9) it suffices to prove the a.e. convergence for f in the set $D=\{\mathcal{A}^+_\gamma g : g\in L^p(\nu), \gamma>0\}$ which is dense in $L^p(\nu)$ (see Theorem 1.7). Assume that $f\in D$, i.e., $f=\mathcal{A}^+_\gamma g$, for some $g\in L^p(\nu)$ and some $\gamma>0$. In this case

$$\begin{aligned}\left|\mathcal{A}^+_{\varepsilon_k,\varphi} f(x) - \mathcal{A}^+_{\varepsilon_{k-1},\varphi} f(x)\right| &\le \left|\mathcal{A}^+_{\varepsilon_k,\varphi}(\mathcal{A}^+_\gamma g)(x) - \mathcal{A}^+_\gamma g(x)\right| \\ &\quad + \left|\mathcal{A}^+_\gamma g(x) - \mathcal{A}^+_{\varepsilon_{k-1},\varphi}(\mathcal{A}^+_\gamma g)(x)\right|.\end{aligned}$$

We can deal with both terms in the same way. We only write the details for the first one. We may assume that $\int \varphi = 1$. Since $\varepsilon_k \to 0$ as $k \to -\infty$, there exists $k_0 \le 0$ such that $\varepsilon_{k_0} < \gamma^2$ and $\varepsilon_{k_0+1} < 1$. Therefore,

$$
\begin{aligned}
&|\mathcal{A}^+_{\varepsilon_k,\varphi}(\mathcal{A}^+_\gamma g)(x) - \mathcal{A}^+_\gamma g(x)| \\
&\quad = \frac{1}{\gamma\varepsilon_k} \left| \int_0^\infty \varphi(t/\varepsilon_k) \left[\int_0^\gamma T^{s+t} g(x)\, ds - \int_0^\gamma T^s g(x)\, ds \right] dt \right| \\
&\quad \le \frac{1}{\gamma\varepsilon_k} \int_0^{\sqrt{\varepsilon_k}} \varphi(t/\varepsilon_k) \left| \int_t^{t+\gamma} T^s g(x)\, ds - \int_0^\gamma T^s g(x)\, ds \right| dt \\
&\qquad + \frac{1}{\gamma\varepsilon_k} \int_{\sqrt{\varepsilon_k}}^\infty \varphi(t/\varepsilon_k) \left| \int_t^{t+\gamma} T^s g(x)\, ds - \int_0^\gamma T^s g(x)\, ds \right| dt = I_k + II_k.
\end{aligned}
$$

It will suffice to prove that

$$
\sum_{k=-\infty}^{k_0} I_k < \infty \quad \text{and} \quad \sum_{k=-\infty}^{k_0} II_k < \infty \tag{5.1}
$$

for almost every x. We start with II_k. We have

$$
\begin{aligned}
II_k &= \frac{1}{\gamma\varepsilon_k} \int_{\sqrt{\varepsilon_k}}^\infty \varphi(t/\varepsilon_k) \left| \int_t^{t+\gamma} T^s g(x)\, ds - \int_0^\gamma T^s g(x)\, ds \right| dt \\
&\le \frac{1}{\gamma\varepsilon_k} \int_{\sqrt{\varepsilon_k}}^\infty \varphi(t/\varepsilon_k) \left(\int_t^{t+\gamma} |T^s g(x)|\, ds \right) dt \\
&\quad + \frac{1}{\gamma\varepsilon_k} \int_{\sqrt{\varepsilon_k}}^\infty \varphi(t/\varepsilon_k) \left(\int_0^\gamma |T^s g(x)|\, ds \right) dt = II'_k + II''_k.
\end{aligned} \tag{5.2}
$$

Now

$$
\begin{aligned}
II''_k &\le \frac{\mathcal{M}^+ g(x)}{\varepsilon_k} \int_{\sqrt{\varepsilon_k}}^\infty \varphi(t/\varepsilon_k)\, dt = \mathcal{M}^+ g(x) \int_{1/\sqrt{\varepsilon_k}}^\infty \varphi(t)\, dt \\
&\le \mathcal{M}^+ g(x) \frac{1/\sqrt{\varepsilon_k}}{1/\sqrt{\varepsilon_k} - 1/\sqrt{\varepsilon_{k+1}}} \int_{1/\sqrt{\varepsilon_{k+1}}}^{1/\sqrt{\varepsilon_k}} \frac{1}{s} \int_s^\infty \varphi(t)\, dt\, ds \\
&\le C\mathcal{M}^+ g(x) \int_{1/\sqrt{\varepsilon_{k+1}}}^{1/\sqrt{\varepsilon_k}} \frac{1}{s} \int_s^\infty \varphi(t)\, dt\, ds.
\end{aligned} \tag{5.3}
$$

Therefore, for almost every x,

(5.4)

$$
\begin{aligned}
\sum_{k=-\infty}^{k_0} II''_k &\le C\mathcal{M}^+ g(x) \sum_{k=-\infty}^{k_0} \int_{1/\sqrt{\varepsilon_{k+1}}}^{1/\sqrt{\varepsilon_k}} \frac{1}{s} \int_s^\infty \varphi(t)\, dt\, ds \\
&\le C\mathcal{M}^+ g(x) \int_1^{+\infty} \frac{1}{s} \int_s^\infty \varphi(t)\, dt\, ds \\
&= C\mathcal{M}^+ g(x) \int_1^{+\infty} (\log t) \varphi(t)\, dt \le C\mathcal{M}^+ g(x) \int_1^{+\infty} t^{1/2} \varphi(t)\, dt < \infty.
\end{aligned}
$$

On the other hand, to control II'_k we use Lemma 3.1. Let $p > r > 1$ and let $\mathcal{G}_r = \{S^t : t \in \mathbb{R}\}$ be as in that lemma and let $\widetilde{\mathcal{M}}^+$ be the maximal operator associated to $\mathcal{G}_{r,+} = \{S^t : t > 0\}$. By Lemma 3.1, $\widetilde{\mathcal{M}}^+$ is bounded from $L^{p/r}(\nu)$ into $L^{p/r}(\nu)$. Consequently, $\widetilde{\mathcal{M}}^+(|g|^r)(x) < \infty$ a.e. for $g \in L^p(\nu)$. Applying Hölder's inequality, we have

$$\begin{aligned} II'_k &\leq \frac{1}{\gamma^{1/r}\varepsilon_k}\left(\widetilde{\mathcal{M}}^+|g|^r(x)\right)^{1/r}\int_{\sqrt{\varepsilon_k}}^{\infty}(t+\gamma)^{1/r}\varphi(t/\varepsilon_k)\,dt \\ &\leq \frac{C}{\varepsilon_k}\left(\widetilde{\mathcal{M}}^+|g|^r(x)\right)^{1/r}\int_{\sqrt{\varepsilon_k}}^{\infty}\varphi(t/\varepsilon_k)\,dt \\ &+ \frac{C}{\gamma^{1/r}\varepsilon_k}\left(\widetilde{\mathcal{M}}^+|g|^r(x)\right)^{1/r}\int_{\sqrt{\varepsilon_k}}^{\infty}t^{1/r}\varphi(t/\varepsilon_k)\,dt. \end{aligned}$$

By the lacunarity of the sequence and the property of φ, we have for a.e. x

$$\begin{aligned} &\sum_{k=-\infty}^{k_0} II'_k \\ &\leq C\left(\widetilde{\mathcal{M}}^+|g|^r(x)\right)^{1/r}\left(\sum_{k=-\infty}^{k_0}\int_{1/\sqrt{\varepsilon_k}}^{\infty}\varphi(t)\,dt + \sum_{k=-\infty}^{k_0}\frac{\varepsilon_k^{1/r}}{\gamma^{1/r}}\int_{1/\sqrt{\varepsilon_k}}^{\infty}t^{1/r}\varphi(t)\,dt\right) \\ &\leq C\left(\widetilde{\mathcal{M}}^+|g|^r(x)\right)^{1/r}\left(\int_1^{\infty}(\log t)\varphi(t)\,dt + \sum_{k=-\infty}^{k_0}\frac{\varepsilon_k^{1/r}}{\gamma^{1/r}}\int_1^{\infty}t^{1/r}\varphi(t)\,dt\right) \\ &< +\infty. \end{aligned}$$

So far, we have proved the second inequality in (5.1). To prove the first inequality, we notice that

$$\begin{aligned} I_k &= \frac{1}{\gamma\varepsilon_k}\int_0^{\sqrt{\varepsilon_k}}\varphi(t/\varepsilon_k)\left|\int_t^{t+\gamma}T^sg(x)\,ds - \int_0^{\gamma}T^sg(x)\,ds\right|\,dt \\ &\leq \frac{1}{\gamma\varepsilon_k}\int_0^{\sqrt{\varepsilon_k}}\varphi(t/\varepsilon_k)\left(\int_0^t|T^sg(x)|\,ds + \int_{\gamma}^{t+\gamma}|T^sg(x)|\,ds\right)\,dt \\ &\leq \frac{1}{\gamma\varepsilon_k}\int_0^{\sqrt{\varepsilon_k}}\varphi(t/\varepsilon_k)\left(\int_0^{\sqrt{\varepsilon_k}}|T^sg(x)|\,ds + \int_{\gamma}^{\sqrt{\varepsilon_k}+\gamma}|T^sg(x)|\,ds\right)\,dt \\ &= I'_k + I''_k. \end{aligned}$$

Now

$$\begin{aligned} I'_k &\leq \frac{\sqrt{\varepsilon_k}}{\gamma\varepsilon_k}\mathcal{M}^+g(x)\int_0^{\sqrt{\varepsilon_k}}\varphi(t/\varepsilon_k)\,dt \\ &\leq \frac{\sqrt{\varepsilon_k}}{\gamma}\mathcal{M}^+g(x)\int_0^{\infty}\varphi(t)\,dt = \frac{\sqrt{\varepsilon_k}}{\gamma}\mathcal{M}^+g(x). \end{aligned}$$

So that, $\sum_{k=-\infty}^{k_0} I'_k \leq \frac{1}{\gamma}\left(\sum_{k=-\infty}^{k_0}\sqrt{\varepsilon_k}\right)\mathcal{M}^+g(x) < \infty$ a.e.

To control I''_k we use again Lemma 3.1. Let $p > r > 1$ and let $\Gamma_\gamma = \{S^t : t \in \mathbb{R}\}$ be as in that lemma and let $\widetilde{\mathcal{M}}^+$ be the maximal operator associated to $\mathcal{G}_{r,+} = \{S^t : t > 0\}$. By Lemma 3.1, $\widetilde{\mathcal{M}}^+$ is bounded from $L^{p/r}(\nu)$ into $L^{p/r}(\nu)$. Consequently, $\widetilde{\mathcal{M}}^+(|g|^r)(x) < \infty$ a.e. for $g \in L^p(\nu)$.

Applying Hölder's inequality, we have

$$
\begin{aligned}
I''_k &= \frac{1}{\gamma}\left(\int_0^{1/\sqrt{\varepsilon_k}} \varphi(t)\,dt\right)\left(\int_\gamma^{\sqrt{\varepsilon_k}+\gamma} |T^s g(x)|\,ds\right) \\
&\le \frac{1}{\gamma}\left(\int_0^{1/\sqrt{\varepsilon_k}} \varphi(t)\,dt\right)\left(\int_\gamma^{\sqrt{\varepsilon_k}+\gamma} |T^s g(x)|^r\,ds\right)^{1/r} (\sqrt{\varepsilon_k})^{1/r'} \\
&\le \frac{1}{\gamma}\left(\int_0^{1/\sqrt{\varepsilon_k}} \varphi(t)\,dt\right)(\sqrt{\varepsilon_k}+\gamma)^{1/r}(\sqrt{\varepsilon_k})^{1/r'}\left(\widetilde{\mathcal{M}}^+(|g|^r)(x)\right)^{1/r} \qquad (5.5) \\
&\le \frac{(2\gamma)^{1/r}}{\gamma}\left(\int_0^{1/\sqrt{\varepsilon_k}} \varphi(t)\,dt\right)(\sqrt{\varepsilon_k})^{1/r'}\left(\widetilde{\mathcal{M}}^+(|g|^r)(x)\right)^{1/r} \\
&\le \frac{2^{1/r}}{\gamma^{1/r'}}\left(\int_0^{\infty} \varphi(t)\,dt\right)(\sqrt{\varepsilon_k})^{1/r'}\left(\widetilde{\mathcal{M}}^+(|g|^r)(x)\right)^{1/r}
\end{aligned}
$$

Therefore $\sum_{k=-\infty}^{k_0} I''_k < \infty$ a.e. This finishes the proof of (5.1).

5.2. **Proof of Theorem 1.11.** It suffices to prove the a.e. convergence in the case $p > 1$. The other statements follow from the results already proved and standard arguments. We also notice that it is enough to prove the a.e. convergence for f in the set $\widetilde{D} = \{\mathcal{A}^+_\varepsilon g : g \in L^p(\nu)\cap L^\infty(\nu), \varepsilon > 0\}$ which is dense in $L^p(\nu)$ (see Theorem 1.7). So, we take $f = \mathcal{A}^+_\varepsilon g$, $g \in L^p(\nu)\cap L^\infty(\nu)$, and we follow the proof of Theorem 1.10 except for the estimates for II'_k (see (5.2)). Now we estimate this term in the following way:

$$
II'_k = \frac{1}{\gamma\varepsilon_k}\int_{\sqrt{\varepsilon_k}}^\infty \varphi(t/\varepsilon_k)\left(\int_t^{t+\gamma} |T^s g(x)|\;ds\right)dt \le \frac{||g||_\infty}{\varepsilon_k}\int_{\sqrt{\varepsilon_k}}^\infty \varphi(t/\varepsilon_k)\,dt.
$$

The proof follows with the same computations as in (5.3) and (5.4).

5.3. **Proof of Theorem 1.12.** As before, it suffices to prove the a.e. convergence in the case $p > 1$. To prove the convergence of $\mathcal{T}^2_N f(x)$, it is enough to establish it for $f \in A \oplus \widetilde{B}$, where $A = \{f \in L^p(\nu) : f(\tau_t x) = f(x) \text{ for all } t > 0\}$ and $\widetilde{B}$ is the linear manifold generated by

$$
\{f(x) - \mathcal{A}^+_\gamma f(x) : f \in L^p(\nu)\cap L^\infty(\nu), \gamma > 0\},
$$

since it follows from Theorem 3.3 that $A \oplus \widetilde{B}$ is dense in $L^p(\nu)$. If $f \in A$ there is nothing to prove. Suppose $f = g - \mathcal{A}^+_\gamma g$, $g \in L^p(\nu)\cap L^\infty(\nu)$, $\gamma > 0$. Then

$$
\left|\mathcal{A}^+_{\varepsilon_k,\varphi} f(x) - \mathcal{A}^+_{\varepsilon_{k-1},\varphi} f(x)\right| \le \left|\mathcal{A}^+_{\varepsilon_k,\varphi}(g - \mathcal{A}^+_\gamma g)(x)\right| + \left|\mathcal{A}^+_{\varepsilon_{k-1},\varphi}(g - \mathcal{A}^+_\gamma g)(x)\right|.
$$

Again, we can deal with both terms in the same way. Since $\varepsilon_k \to \infty$ as $k \to \infty$, there exists k_0 such that for all $k \ge k_0 - 1$ we have that $\varepsilon_k > \gamma$ and

$\varepsilon_k > 1$. Therefore, for $k \geq k_0$,

$$\begin{aligned}
&\left|\mathcal{A}^+_{\varepsilon_k,\varphi}(g - \mathcal{A}^+_\gamma g)(x)\right| \\
&\qquad = \left|\int_0^\infty T^t g(x)\varphi_{\varepsilon_k}(t)\,dt - \frac{1}{\gamma}\int_0^\gamma \int_0^\infty T^{t+s} g(x)\varphi_{\varepsilon_k}(t)\,dt\,ds\right| \\
&\qquad = \left|\int_0^\infty T^t g(x)\varphi_{\varepsilon_k}(t)\,dt - \frac{1}{\gamma}\int_0^\gamma \int_s^\infty T^{t} g(x)\varphi_{\varepsilon_k}(t-s)\,dt\,ds\right| \\
&\qquad \leq \frac{1}{\gamma}\int_0^\gamma \frac{1}{\varepsilon_k}\int_0^s |T^t g(x)|\varphi(t/\varepsilon_k)\,dt\,ds \\
&\qquad\quad + \frac{1}{\gamma}\int_0^\gamma \frac{1}{\varepsilon_k}\int_s^\infty |T^t g(x)||\varphi(t/\varepsilon_k) - \varphi(t/\varepsilon_k - s/\varepsilon_k)|\,dt\,ds.
\end{aligned}$$

Notice that, by the hypothesis on the function φ,

$$\begin{aligned}
\int_s^\infty |\varphi(t/\varepsilon_k) - \varphi((t-s)/\varepsilon_k)|\,dt &= \int_s^\infty [\varphi((t-s)/\varepsilon_k) - \varphi(t/\varepsilon_k)]\,dt \\
&= \int_0^s \varphi(t/\varepsilon_k)\,dt.
\end{aligned}$$

Using that $||T^t g||_\infty \leq ||g||_\infty$, we have

$$\left|\mathcal{A}^+_{\varepsilon_k,\varphi}(g - \mathcal{A}^+_\gamma g)(x)\right| \leq \frac{2}{\gamma}||g||_\infty \int_0^\gamma \frac{1}{\varepsilon_k}\int_0^s \varphi(t/\varepsilon_k)\,dt\,ds$$

For $s < \gamma$,

$$\begin{aligned}
\frac{1}{\varepsilon_k}\int_0^s \varphi(t/\varepsilon_k)\,dt &= \int_0^{s/\varepsilon_k} \varphi(t)\,dt \\
&\leq \frac{1/\varepsilon_{k-1}}{1/\varepsilon_{k-1} - 1/\varepsilon_k}\int_{\gamma/\varepsilon_k}^{\gamma/\varepsilon_{k-1}} \frac{1}{u}\int_0^u \varphi(t)\,dt\,du \\
&\leq C\int_{\gamma/\varepsilon_k}^{\gamma/\varepsilon_{k-1}} \frac{1}{u}\int_0^u \varphi(t)\,dt\,du.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{k=k_0}^\infty \left|\mathcal{A}^+_{\varepsilon_k,\varphi}(g - \mathcal{A}^+_\gamma g)(x)\right| &\leq C||g||_\infty \sum_{k=k_0}^\infty \int_{\gamma/\varepsilon_k}^{\gamma/\varepsilon_{k-1}} \frac{1}{s}\int_0^s \varphi(t)\,dt\,ds \\
\leq C||g||_\infty \int_0^1 \frac{1}{s}\int_0^s \varphi(t)\,dt\,ds &= C||g||_\infty \int_0^1 |\log t|\varphi(t)\,dt < \infty,
\end{aligned}$$

and we are done.

5.4. **Proof of Theorem 1.13.** It suffices to prove the a.e. convergence. To prove the a.e. convergence of $\mathcal{T}^2_N f(x)$, it is enough to establish it for functions $f \in A \oplus B$, where A and B are the sets in the proof of Theorem 3.3, that is, $A = \{f \in L^p(\nu) : f(\tau_t x) = f(x) \text{ for all } t > 0\}$ and B is the linear manifold generated by $\{f(x) - \mathcal{A}^+_\gamma f(x) : f \in L^p(\nu), \gamma > 0\}$. If $f \in A$ there is nothing to prove. Suppose $f = g - \mathcal{A}^+_\gamma g$, $g \in L^p(\nu)$, $\gamma > 0$. Then

$$\left|\mathcal{A}^+_{\varepsilon_k} f(x) - \mathcal{A}^+_{\varepsilon_{k-1}} f(x)\right| \leq \left|\mathcal{A}^+_{\varepsilon_k}(g - \mathcal{A}^+_\gamma g)(x)\right| + \left|\mathcal{A}^+_{\varepsilon_{k-1}}(g - \mathcal{A}^+_\gamma g)(x)\right|.$$

Again, we can deal with both terms in the same way. Since $\varepsilon_k \to \infty$ as $k \to \infty$, there exists k_0 such that for all $k \geq k_0 - 1$ we have that $\varepsilon_k > \gamma$. Therefore, for $k \geq k_0$, using (3.16),
(5.6)

$$\left|\mathcal{A}^+_{\varepsilon_k}(g - \mathcal{A}^+_\gamma g)(x)\right| \leq \frac{1}{\varepsilon_k}\int_0^\gamma T^t|g|(x)dt + \frac{1}{\varepsilon_k}\int_{\varepsilon_k}^{\gamma+\varepsilon_k} T^t|g|(x)dt = I_k + II_k.$$

It is clear that $\sum_{k\geq k_0} I_k \leq \sum_{k\geq k_0} \frac{\gamma}{\varepsilon_k}\mathcal{M}^+g(x) < \infty$ a.e. since the function $g \in L^p(\nu)$. To control the other term we use again Lemma 3.1. Let $p > r > 1$ and let $\mathcal{G}_r = \{S^t : t \in \mathbb{R}\}$ be as in that lemma and let $\widetilde{\mathcal{M}}^+$ be the maximal operator associated to $\mathcal{G}_{r,+} = \{S^t : t > 0\}$. By Lemma 3.1, $\widetilde{\mathcal{M}}^+$ is bounded from $L^{p/r}(\mu)$ into $L^{p/r}(\mu)$. Consequently, $\widetilde{\mathcal{M}}^+(|g|^r)(x) < \infty$ a.e. for $g \in L^p(\mu)$. It follows that

$$\begin{aligned} II_k &\leq \frac{1}{\varepsilon_k}\left(\int_{\varepsilon_k}^{\gamma+\varepsilon_k} (T^t|g|(x))^r dt\right)^{1/r} \gamma^{1/r'} \\ &\leq \frac{(\gamma+\varepsilon_k)^{1/r}}{\varepsilon_k}\left(\frac{1}{\gamma+\varepsilon_k}\int_0^{\gamma+\varepsilon_k} S^t(|g|^r)(x)dt\right)^{1/r} \gamma^{1/r'} \\ &\leq \frac{(\gamma+\varepsilon_k)^{1/r}\gamma^{1/r'}}{\varepsilon_k}[\widetilde{\mathcal{M}}^+(|g|^r)(x)]^{1/r} \leq \frac{2^{1/r}\gamma^{1/r'}}{\varepsilon_k^{1/r'}}[\widetilde{\mathcal{M}}^+(|g|^r)(x)]^{1/r}. \end{aligned}$$

Therefore $\sum_{k\geq k_0} II_k \leq \sum_{k\geq k_0} \frac{2^{1/r}\gamma^{1/r'}}{\varepsilon_k^{1/r'}}[\widetilde{\mathcal{M}}^+(|g|^r)(x)]^{1/r} < \infty$ a.e. Consequently, $\sum_{k\geq k_0}\left|\mathcal{A}^+_{\varepsilon_k,\varphi}(g - \mathcal{A}^+_\gamma g)(x)\right| \leq \sum_{k\geq k_0}(I_k + II_k) < \infty$ a.e., as we wished to prove.

6. Examples of Cesàro bounded semigroups

The aim of this section is to provide more examples of Cesàro bounded semigroups. We shall follow the arguments in [18]. Given a nonsingular flow $\Gamma = \{\tau_t : t \in \mathbb{R}\}$, we are going to study in the first place the groups $T^t f(x) = f(\tau_t x)$ and in the second place the general groups $T^t f(x) = g_t(x) f(\tau_t x)$. We shall use frequently Remark 3.5.

6.1. **The group $T^t f(x) = f(\tau_t x)$.** We start giving examples in the basic setting of the interval $[0, 1)$.

Example 6.1. Let $X = [0, 1)$ with the Lebesgue σ-algebra. Let $d\nu = w(x)\,dx$, where $w(x) = x^\beta$. Consider the flow $\tau_t x = x + t$ (mod 1) Let us consider the group $T^t f(x) = f(\tau_t x)$. In this case $H_t(x) = \frac{w(\tau_t x)}{w(x)}$. In Example 1.6 we have seen that if $-1 < \beta \leq 0$ then the function $t \to H_t(x)$ belongs to A_1^+ with a constant independent of x. Therefore, by Remark 3.5 $\mathcal{G}_+$ is Cesàro bounded in $L^1(\nu)$. We already know (see Example 1.6) that the semigroup $\mathcal{G}_- = \{T^t : t < 0\}$ is not Cesàro bounded in $L^1(\nu)$ for $\beta < 0$.

In the same way, if $w(x) = (1-x)^\beta$ and $-1 < \beta \leq 0$ then the function $t \to H_t(x)$ belongs to A_1^- with a constant independent of x. It follows from the theory of one-sided weights, see Remarks 2.1, that if $0 \leq \beta < p - 1$, $p > 1$, and $w(x) = (1-x)^\beta$ or $w(x) = x^\beta$, $-1 < \beta \leq 0$, then $t \to H_t(x)$

belongs to A_p^+ with a constant independent of x. Therefore, in those cases, the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$. As in Example 1.6 we can see that the semigroup $\mathcal{G}_- = \{T^t : t < 0\}$ is not Cesàro bounded in $L^p(\nu)$ for $\beta \neq 0$.

We can see that in the above example the flow preserves a measure μ (the Lebesgue measure) which is equivalent to ν (that means $\nu(E) = 0$ if and only if $\mu(E) = 0$). One may ask if this is always the case when we consider Cesàro bounded semigroups of the form $T^t f(x) = f(\tau_t x)$. The answer is affirmative when the measure ν is finite. We state it as a theorem.

Theorem 6.2. Let $(X, \mathcal{F}, \nu)$ be a finite measure space and let $\Gamma = \{\tau_t : t \in \mathbb{R}\}$ be a nonsingular flow on X. Let $\mathcal{G} = \{T^t : t \in \mathbb{R}\}$ be the group defined as $T^t f(x) = f(\tau_t x)$. Let $1 \leq p < \infty$. If the semigroup $\mathcal{G}_+$ is Cesàro bounded in $L^p(\nu)$ then there exists a finite measure μ such that the flow preserves the measure μ and μ is equivalent to ν.

The proof is as the proof of the corresponding result in [18] (see the proof of Theorem 1 in [18, page 545]). Therefore, we do not include it.

6.2. **Non trivial examples of Cesàro bounded general semigroups.** Consider a σ-finite measure space $(X, \mathcal{F}, \nu)$ and a nonsingular flow $\{\tau_t : t \in \mathbb{R}\}$ on X. Remind that the transformation τ_t is ergodic if $\tau_{-t}(E) = E$ for a measurable set E implies that $\nu(E) = 0$ or $\nu(X \setminus E) = 0$.

Let $\mathcal{G} = \{T^t : t \in \mathbb{R}\}$ the group defined by $T^t f(x) = (\overline{J_t}(x))^{1/p} f(\tau_t x)$, $1 \leq p < \infty$. Clearly, each T^t is an isometry on $L^p(\nu)$. Therefore $\mathcal{G}^+ = \{T^t : t > 0\}$ is Cesàro bounded in $L^p(\nu)$. Our next result shows non trivial examples of Cesàro bounded semigroups, in the sense that the operators T^t, $t > 0$, are not isometries, moreover they are not uniformly bounded.

Theorem 6.3. Let $(X, \mathcal{F}, \nu)$ be a non-atomic finite measure space and let $\Gamma = \{\tau_t : t \in \mathbb{R}\}$ be a nonsingular flow on X. Assume that τ_t is ergodic for some t with respect to ν. Let $1 \leq p < \infty$. Then there exists a group of positive operators $\mathcal{S}_p = \{T^t : t \in \mathbb{R}\}$ induced by the flow, acting on measurable functions, such that

(1) the semigroup $\mathcal{S}_p^+ = \{T^t : t > 0\}$ is Cesàro bounded in $L^p(\nu)$ and
(2) the semigroup $\mathcal{S}_p^+$ is not uniformly bounded in $L^p(\nu)$, that is, there is no $C > 0$ such that $\int_X |T^t f|^p d\nu \leq C \int_X |f|^p d\nu$ for all $t > 0$ and all $f \in L^p(\nu)$.

Notice that the result is a generalization of Theorem 7 in [18].

Proof. We start by proving the case $p = 1$. We shall do it in two steps. We shall use again the ideas of Rubio de Francia.

1) Let $p = 1$ and let us assume that there exists a finite measure μ equivalent to ν such that it is preserved by the flow Γ. Let $\mathcal{M}_\mu^-$ be the maximal operator defined by $\mathcal{M}_\mu^- f(x) = \sup_{\varepsilon>0} \frac{1}{\varepsilon} \int_{-\varepsilon}^0 |f(\tau_t x)|\, dt$. We know that $\mathcal{M}_\mu^-$ is bounded on $L^2(\mu)$, i.e., there exists a constant $A > 0$ such that $||\mathcal{M}_\mu^- f||_{2,\mu} \leq$

$A||f(x)||_{2,\mu}$. Let $f>0$, $f\in L^2(\mu)$, $f\notin L^\infty(\mu)$. Let $w=\sum_{i=0}^\infty \frac{1}{A^i}\left(\mathcal{M}_\mu^-\right)^{(i)} f$, where $\left(\mathcal{M}_\mu^-\right)^{(i)}$ is the i-th iteration of $\mathcal{M}_\mu^-$. Then $w\ge f>0$, $w\in L^2(\mu)$, w is finite a.e., $w\notin L^\infty(\mu)$ and $\mathcal{M}_\mu^- w\le \sum_{i=0}^\infty \frac{1}{A^i}\left(\mathcal{M}_\mu^-\right)^{(i+1)} w\le Aw$ a.e., what implies that

$$\text{(6.1)}\qquad \begin{array}{l}\text{for a.e. } x \text{ the functions} \quad t\to w(\tau_t x)\\ \text{belong to } A_1^+ \text{ with a uniform constant.}\end{array}$$

Let $u=\frac{d\mu}{d\nu}$ the Radon-Nikodym derivative of μ with respect to ν. Let $g_t(x)=\frac{u(x)w(x)}{u(\tau_t x)w(\tau_t x)}$. and let $\mathcal{S}_1=\{T^t: t\in\mathbb{R}\}$ be the group defined as $T^t f(x)=g_t(x)f(\tau_t x)$. Since $J_t(x)=\frac{u(x)}{u(\tau_t x)}$, it follows from (6.1) that the semigroup $\mathcal{S}_1=\{T^t: t>0\}$ is Cesàro bounded in $L^1(\nu)$ (this can be seen directly or applying Remark 3.5).

Now we shall assume that the semigroup $\mathcal{S}_1$ is uniformly bounded in $L^1(d\nu)$ and we shall reach a contradiction. It is clear that if it is uniformly bounded in $L^1(d\nu)$ then there exists a constant $C>0$ such that

$$\int_X \frac{u(x)w(x)}{u(\tau_t x)w(\tau_t x)} f(\tau_t x)d\nu\le C\int_X f(x)d\nu=\int_X f(\tau_t x)\frac{u(x)}{u(\tau_t x)}\,d\nu.$$

for all $f\ge 0$. That inequality implies, for all $t>0$, $w(x)\le Cw(\tau_t x)$ a.e. x, which implies that $w\in L^\infty(d\nu)$, applying the ergodicity of some τ_t, which is a contradiction.

2) Let $p=1$ and let us assume that there is no finite measure μ equivalent to ν such that is preserved by the flow Γ. Let $1<q<\infty$ and let q' be the conjugate exponent of q. We consider the group $\mathcal{G}_{q'}=\{T^t: t\in\mathbb{R}\}$ where $T^t f(x)=(\overline{J_t}(x))^{1/q'}f(\tau_t x)$. Since the operators T^t are positive isometries on $L^{q'}(\nu)$ then the maximal operator $\mathcal{M}_{q'}^- f(x)=\sup_{\varepsilon>0}\frac{1}{\varepsilon}\int_{-\varepsilon}^0 |T^t f(x)|\,dt$ is bounded on $L^{q'}(\nu)$, i.e., there exists $A>0$ such that $||\mathcal{M}_{q'}^- f||_{q',\nu}\le A||f||_{q',\nu}$. Let $f>0$, $f\in L^{q'}(\nu)$. Let $w=\sum_{i=0}^\infty\frac{1}{A^i}\left(\mathcal{M}_{q'}^-\right)^{(i)}f$, where $\left(\mathcal{M}_{q'}^-\right)^{(i)}$ is the i-th iteration of $\mathcal{M}_{q'}^-$. Then $w\ge f>0$, $w\in L^{q'}(\nu)$, w is finite a.e. and $\mathcal{M}_{q'}^- w\le\sum_{i=0}^\infty\frac{1}{A^i}\left(\mathcal{M}_{q'}^-\right)^{(i+1)}w\le Aw$ a.e., what means that

$$\text{(6.2)}\qquad \begin{array}{l}\text{for a.e. } x \text{ the functions} \quad t\to (J_t(x))^{1/q'}\,w(\tau_t x)\\ \text{belong to } A_1^+ \text{ with a uniform constant.}\end{array}$$

Let $\overline{g_t}(x)=\frac{w(x)}{w(\tau_t x)}\left(\overline{J_t}(x)\right)^{1/q}$ and let $\mathcal{S}_1=\{T^t: t\in\mathbb{R}\}$ be the group defined as $T^t f(x)=\overline{g_t}(x)f(\tau_t x)$. As before, it follows from (6.2) that the semigroup $\mathcal{S}_1=\{T^t: t>0\}$ is Cesàro bounded in $L^1(\nu)$ (this can be seen directly or applying Remark 3.5).

Now we shall assume that the semigroup $\mathcal{S}_1$ is uniformly bounded in $L^1(d\nu)$ and we shall reach a contradiction. It is clear that if it is uniformly bounded in $L^1(d\nu)$ then there exists a constant $C>0$ such that

$$\int_X \frac{w(x)}{w(\tau_t x)}\left(\overline{J_t}(x)\right)^{1/q} f(\tau_t x)d\nu\le C\int_X f(x)d\nu=\int_X f(\tau_t x)\overline{J_t}(x)\,d\nu.$$

for all $f \geq 0$. That inequality implies, for all $t > 0$,

$$w(x) \leq Cw(\tau_t x)\left(\overline{J_t}(x)\right)^{1/q'} \quad \text{a.e. } x.$$

Raising to q', multiplying by $\chi_A(\tau_t x)$, where A is any measurable set, and integrating on X, we have

$$\int_X \chi_A(\tau_t x) w^{q'}(x)\, d\nu = \int_{\tau_{-t}A} w^{q'}\, d\nu \leq C \int_A w^{q'}\, d\nu = \int_X \chi_A(x) w^{q'}(x)\, d\nu$$

for all $t > 0$. This condition implies that if $\widetilde{T}^t f(x) = f(\tau_t x)$ and $d\widetilde{\nu} = w^{q'}\, d\nu$ then the semigroup $\{\widetilde{T}^t : t > 0\}$ is Cesàro bounded in $L^1(d\widetilde{\nu})$. By Theorem 6.2, we have that there exists a finite measure μ equivalent to $\widetilde{\nu}$ and consequently, equivalent to ν, and such that μ is preserved by the flow. That is a contradiction.

3) Let $p > 1$ and let us assume that there exists a finite measure μ equivalent to ν such that it is preserved by the flow Γ. As in the case $p = 1$ but using the maximal operator $\mathcal{M}_\mu^+ f(x) = \sup_{\varepsilon>0} \frac{1}{\varepsilon}\int_0^\varepsilon |f(\tau_t x)|\, dt$, there exists $w \in L^2(\mu)$, $w \notin L^\infty(\mu)$ such that $\mathcal{M}_\mu^+ w \leq Aw$ a.e., what means that for a.e. x the functions $t \to w(\tau_t x)$ belong to A_1^- with a uniform constant. Let $v = w^{1-p}$. Then it is well known (see Remarks 2.1) that

$$\text{for a.e. } x \text{ the functions} \quad t \to v(\tau_t x) \text{ belong to } A_p^+ \text{ with a uniform constant.} \tag{6.3}$$

Let $u = \frac{d\mu}{d\nu}$ the Radon-Nikodym derivative of μ with respect to ν. Let $g_t(x) = \left(\frac{u(x)v(x)}{u(\tau_t x)v(\tau_t x)}\right)^{1/p}$ and let $\mathcal{S}_p = \{T^t : t \in \mathbb{R}\}$ be the group defined as $T^t f(x) = g_t(x) f(\tau_t x)$. It follows from (6.3) that the semigroup $\mathcal{S}_p = \{T^t : t > 0\}$ is Cesàro bounded in $L^p(\nu)$ (this can be seen applying Remark 3.5).

Now we shall assume that the semigroup $\mathcal{S}_p$ is uniformly bounded in $L^p(d\nu)$ and we shall reach a contradiction. It is clear that if it is uniformly bounded in $L^p(d\nu)$ then there exists a constant $C > 0$ such that

$$\int_X \frac{u(x)v(x)}{u(\tau_t x)v(\tau_t x)} |f(\tau_t x)|^p d\nu \leq C \int_X |f(x)|^p d\nu = \int_X |f(\tau_t x)|^p \frac{u(x)}{u(\tau_t x)}\, d\nu.$$

for all $f \geq 0$. As in the case $p = 1$, that inequality implies that $w \in L^\infty(d\nu)$, which is a contradiction.

4) Let $p > 1$ and let us assume that there is no finite measure μ equivalent to ν such that is preserved by the flow Γ. As in the case 2), using $\mathcal{M}_{q'}^- f(x) = \sup_{\varepsilon>0} \frac{1}{\varepsilon}\int_{-\varepsilon}^0 |T^t f(x)|\, dt$, we have a function $w \in L^{q'}(\nu)$ such that $\mathcal{M}_{q'}^- w \leq Aw$ a.e., what means that for a.e. x the functions $t \to (J_t(x))^{1/q'} w(\tau_t x)$ belong to A_1^- with a uniform constant. Then

$$\text{for a.e. } x \text{ the functions} \quad t \to \left((J_t(x))^{1/q'} w(\tau_t x)\right)^{1-p} \text{ belong to } A_p^+ \text{ with a uniform constant.} \tag{6.4}$$

Let $\overline{g_t}(x) = \left(\frac{w(x)}{w(\tau_t x)}\right)^{\frac{1-p}{p}} \left(\overline{J_t}(x)\right)^{\frac{1}{p}+\frac{1}{p'q'}}$ and let $\mathcal{S}_p = \{T^t : t \in \mathbb{R}\}$, where $T^t f(x) = \overline{g_t}(x) f(\tau_t x)$. As before, it follows from (6.4) that the semigroup $\mathcal{S}_p = \{T^t : t > 0\}$ is Cesàro bounded in $L^p(\nu)$.

Now we see that that the semigroup $\mathcal{S}_p$ is not uniformly bounded in $L^p(d\nu)$. Proceeding as in case 2), we obtain, for all $t > 0$, $w(\tau_t x) \left(\overline{J_t}(x)\right)^{1/q'} \leq Cw(x)$ a.e. x and, for any measurable set A,

$$\int_X \chi_A(\tau_{-t}x) w^{q'}(x)\, d\nu \leq C \int_X \chi_A(x) w^{q'}(x)\, d\nu$$

for all $t > 0$. This condition implies that if $\widetilde{T}^t f(x) = f(\tau_{-t}x)$ and $d\widetilde{\nu} = w^{q'}\, d\nu$ then the semigroup $\{\widetilde{T}^t : t > 0\}$ is Cesàro bounded in $L^1(d\widetilde{\nu})$. By Theorem 6.2, there exists a finite measure μ equivalent to $\widetilde{\nu}$ (consequently, equivalent to ν) such that μ is preserved by the flow. That is a contradiction. $\square$

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