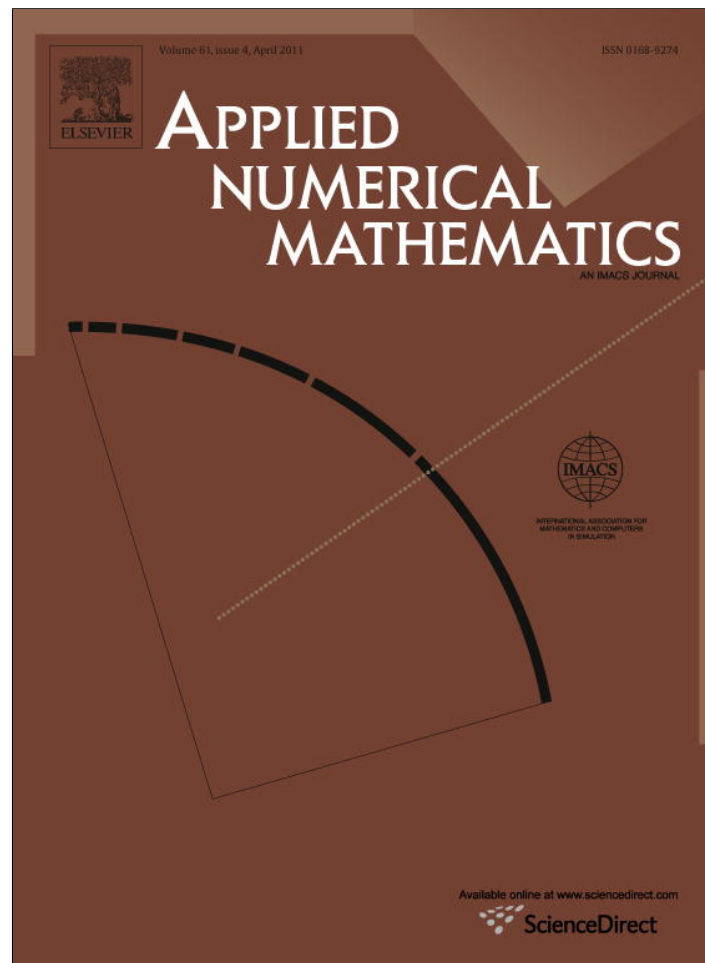


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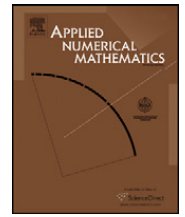
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Convergence of an adaptive Kačanov FEM for quasi-linear problems

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ABSTRACT

We design an adaptive finite element method to approximate the solutions of quasi-linear elliptic problems. The algorithm is based on a Kačanov iteration and a mesh adaptation step is performed after each linear solve. The method is thus *inexact* because we do not solve the discrete nonlinear problems exactly, but rather perform one iteration of a fixed point method (Kačanov), using the approximation of the previous mesh as an initial guess. The convergence of the method is proved for any *reasonable* marking strategy and starting from any initial mesh. We conclude with some numerical experiments that illustrate the theory.

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1. Introduction

In this paper we consider quasi-linear elliptic partial differential equations over a polygonal/polyhedral domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) of the form

$$\begin{cases} -\nabla \cdot [\alpha(\cdot, |\nabla u|^2) \nabla u] = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\alpha : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a function whose properties will be stated in Section 2 below, and $f \in L^2(\Omega)$ is given. These equations describe stationary conservation laws which frequently arise in problems of mathematical physics [17]. For example, in hydrodynamics and gas dynamics (subsonic and supersonic flows), electrostatics, magnetostatics, heat conduction, elasticity and plasticity (e.g., plastic torsion of bars), etc. Some of these examples are better modeled by variational inequalities (see [8] and the references therein), but these fall beyond the scope of this article, which attempts to set a first step towards understanding the convergence and optimality of inexact Kačanov-type iterations, in the context of adaptive finite element methods.

For a summary of convergence and optimality results of AFEM we refer the reader to the survey [12], and the references therein. We restrict ourselves to those references strictly related to our work.

Inexact adaptive finite element methods have been considered for Stokes problem in [1,10] using Uzawa's algorithm. Briefly, a Richardson iteration is applied to the infinite-dimensional Schur complement operator and in each iteration, the elliptic problem is solved up to a decreasing tolerance. In [1] linear convergence is proved and in [10] the optimality of the method in terms of degrees of freedom is proved, after adding some new refinement steps to the algorithm.

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In a recent work [9] a convergence theory for a general class of adaptive approximation algorithms for abstract nonlinear operator equations on Banach spaces has been developed. The theory is used to obtain convergence results for practical adaptive finite element methods applied to several classes of nonlinear elliptic equations. The framework can be viewed as an extension of the recent convergence results for linear problems of [11] to a general nonlinear setting.

In [2], a contraction property is proved for an adaptive algorithm based on Dörfler's marking strategy [3] for nonlinear problems of the type (1), using Orlicz norms to cope with the very mild assumptions on the nonlinear term α .

In this work we impose stronger assumptions on α , which guarantee the convergence of Kačanov's iteration [8]. More precisely, we assume that $\alpha(\cdot, \cdot)$ is decreasing with respect to its second variable (cf. (20)), and fulfills condition (3) below; which are the same assumptions stated in [8], where they consider the iteration on a fixed space, either finite- or infinite-dimensional. Our focus is the convergence analysis of the adaptive method that results from performing one mesh adaptation in each iteration of the nonlinear solver. This turns out to be a very realistic and efficient method, based on the sole assumption that a *linear* system is exactly solved in each iteration step.

This paper is organized as follows. In Section 2 we present the class of specific nonlinear problems that we study, and some of its properties. In Section 3, we present the inexact adaptive Kačanov algorithm and in Section 4 we state and prove the main result of this article, namely the convergence of the discrete solutions produced by the algorithm to the exact solution of the nonlinear problem. Finally, in Section 5, we present some numerical experiments which illustrate the theory, and explore the practical boundaries of applicability of the algorithm.

2. Setting

Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal ($d = 2$) or polyhedral ($d = 3$) domain with Lipschitz boundary. A weak formulation of (1) consists in finding $u \in H_0^1(\Omega)$ such that

$$a(u; u, v) = L(v), \quad \forall v \in H_0^1(\Omega), \tag{2}$$

where

$$a(w; u, v) = \int_{\Omega} \alpha(\cdot, |\nabla w|^2) \nabla u \cdot \nabla v, \quad \forall w, u, v \in H_0^1(\Omega),$$

and

$$L(v) = \int_{\Omega} f v, \quad \forall v \in H_0^1(\Omega).$$

We require that α is piecewise Lipschitz in its first variable, uniformly with respect to its second variable (cf. (18) below). On the other hand, we assume that α is C^1 as a function of its second variable and there exist positive constants c_a and C_a such that

$$c_a \leq \alpha(x, t^2) + 2t^2 D_2 \alpha(x, t^2) \leq C_a, \quad \forall x \in \Omega, t > 0, \tag{3}$$

where $D_2 \alpha$ denotes the partial derivative of α with respect to its second variable. The last condition is related to the well-posedness of problem (2) as we will show below. Also, it is possible to prove that (3) implies that

$$c_a \leq \alpha(x, t) \leq C_a, \quad \forall x \in \Omega, t > 0. \tag{4}$$

These same assumptions are stated in [8], where several interesting applied problems are shown to satisfy them.

Additionally, it is easy to check that the form a is linear and symmetric in its second and third variables. Also, from (4) it follows that a is bounded,

$$|a(w; u, v)| \leq C_a \|\nabla u\|_{\Omega} \|\nabla v\|_{\Omega}, \quad \forall w, u, v \in H_0^1(\Omega), \tag{5}$$

and coercive,

$$c_a \|\nabla u\|_{\Omega}^2 \leq a(w; u, u), \quad \forall w, u \in H_0^1(\Omega). \tag{6}$$

If we define $A : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ as the nonlinear operator given by

$$\langle Au, v \rangle := a(u; u, v), \quad \forall u, v \in H_0^1(\Omega),$$

then problem (2) is equivalent to the equation

$$Au = L,$$

where $L \in H^{-1}(\Omega)$ is given. Assumption (3) implies that A is Lipschitz and strongly monotone (see [17]), i.e., there exist positive constants C_A and c_A such that

$$\|Au - Av\|_{H^{-1}(\Omega)} \leq C_A \|\nabla(u - v)\|_{\Omega}, \quad \forall u, v \in H_0^1(\Omega), \tag{7}$$

and

$$\langle Au - Av, u - v \rangle \geq c_A \|\nabla(u - v)\|_{\Omega}^2, \quad \forall u, v \in H_0^1(\Omega). \tag{8}$$

As a consequence of (7) and (8), problem (2) has a unique solution and is stable [16,17].

3. Adaptive algorithm

In order to define discrete adaptive approximations to problem (2) we will consider *triangulations* of the domain Ω . Let \mathcal{T}_0 be an initial conforming triangulation of Ω , that is, a partition of Ω into d -simplices such that if two elements intersect, they do so at a full vertex/edge/face of both elements. Let \mathbb{T} denote the set of all conforming triangulations of Ω obtained from \mathcal{T}_0 by refinement using the bisection procedures presented by Stevenson [15]. These coincide (after some re-labeling) with the so-called *newest vertex* bisection procedure in two dimensions and the bisection procedure of Kossaczky in three dimensions [13].

Due to the processes of refinement used, the family \mathbb{T} is shape regular, i.e.,

$$\sup_{T \in \mathbb{T}} \sup_{T \in \mathcal{T}} \frac{\text{diam}(T)}{\rho_T} =: \kappa_{\mathbb{T}} < \infty,$$

where $\text{diam}(T)$ is the diameter of T , and ρ_T is the radius of the largest ball contained in it. Throughout this article, we only consider meshes \mathcal{T} that belong to the family \mathbb{T} , so the shape regularity of all of them is bounded by the uniform constant $\kappa_{\mathbb{T}}$ which only depends on the initial triangulation \mathcal{T}_0 [13]. Also, the diameter of any element $T \in \mathcal{T}$ is equivalent to the local mesh-size $H_T := |T|^{1/d}$, which in turn defines the global mesh-size $H_{\mathcal{T}} := \max_{T \in \mathcal{T}} H_T$.

For the discretization we consider the Lagrange finite element spaces consisting of continuous functions vanishing on $\partial\Omega$ which are piecewise linear over a mesh $\mathcal{T} \in \mathbb{T}$, i.e.,

$$\mathbb{V}_{\mathcal{T}} := \{v \in H_0^1(\Omega) \mid v|_T \in \mathcal{P}_1(T), \forall T \in \mathcal{T}\}. \tag{9}$$

We are now in position to state the adaptive loop to approximate the solution u of the problem (2).

Adaptive Algorithm. Let \mathcal{T}_0 be an initial conforming mesh of Ω and $u_0 \in \mathbb{V}_{\mathcal{T}_0}$ be an initial approximation. Let $\mathcal{T}_1 = \mathcal{T}_0$ and $k = 1$.

1. $u_k := \text{SOLVE}(u_{k-1}, \mathcal{T}_k)$.
 2. $\{\eta_k(T)\}_{T \in \mathcal{T}_k} := \text{ESTIMATE}(u_{k-1}, u_k, \mathcal{T}_k)$.
 3. $\mathcal{M}_k := \text{MARK}(\{\eta_k(T)\}_{T \in \mathcal{T}_k}, \mathcal{T}_k)$.
 4. $\mathcal{T}_{k+1} := \text{REFINE}(\mathcal{T}_k, \mathcal{M}_k, n)$.
 5. Increment k and go back to step 1.
-

We now explain each module of the last algorithm in detail.

The module SOLVE. Given the conforming triangulation \mathcal{T}_k of Ω , and the solution u_{k-1} from the previous iteration, the module SOLVE outputs the solution $u_k \in \mathbb{V}_k := \mathbb{V}_{\mathcal{T}_k}$ of the *linear* problem

$$a(u_{k-1}; u_k, v_k) = L(v_k), \quad \forall v_k \in \mathbb{V}_k. \tag{10}$$

Remark 1 (Linear versus nonlinear). Notice that while SOLVE requires the solution of a linear system, the usual discretization of (2) in \mathbb{V}_k consists in finding $u_k \in \mathbb{V}_k$ such that

$$a(u_k; u_k, v_k) = L(v_k), \quad \forall v_k \in \mathbb{V}_k, \tag{11}$$

which is nonlinear. We propose here to make only one step of a fixed point iterative method to solve the nonlinear problem, and proceed to the mesh adaptation, whereas the usual approach would be to approximate the discrete nonlinear problem up to a very fine accuracy (pretending to have it exactly solved) before proceeding to mesh adaptation [6] (see also [9] for a convergence result in a general setting). Each iteration of the adaptive loop is thus much cheaper in our approach. Our theory guarantees convergence of this algorithm, and the numerical experiments of Section 5 show that the convergence is quasi-optimal, although the latter is not yet rigorously proved.

Remark 2 (Kačanov's Method). Let us consider problem (2) (resp. problem (11) with $k \in \mathbb{N}$ fixed). We denote the space $H_0^1(\Omega)$ (resp. \mathbb{V}_k) by \mathbb{V} , and the solution u (resp. u_k) by U . Given an initial approximation $U_0 \in \mathbb{V}$ of the solution U , we consider the sequence $\{U_i\}_{i \in \mathbb{N}_0}$ where $U_i \in \mathbb{V}$ is the solution of the linear problem

$$a(U_{i-1}; U_i, v) = L(v), \quad \forall v \in \mathbb{V}, i \in \mathbb{N}.$$

This is known as Kačanov's Method, and it follows [8] that the sequence $\{U_i\}_{i \in \mathbb{N}_0}$ converges to the solution U , provided $D_2\alpha(x, t) \leq 0$ for all $x \in \Omega$ and $t > 0$.

Notice that our algorithm consists in performing *only one* step of Kačanov iteration in each step of the adaptive loop.

The module ESTIMATE. Given \mathcal{T}_k , u_{k-1} and the corresponding output u_k of SOLVE, the module ESTIMATE computes and outputs the local error indicators $\{\eta_k(T)\}_{T \in \mathcal{T}_k}$ given by

$$\eta_k^2(T) := H_T^2 \|R_k\|_T^2 + H_T \|J_k\|_{\partial T}^2, \tag{12}$$

for all $T \in \mathcal{T}_k$. Here R_k denotes the *element residual* given by

$$R_k|_T := -\nabla \cdot [\alpha(\cdot, |\nabla u_{k-1}|^2) \nabla u_k] - f, \quad \forall T \in \mathcal{T}_k,$$

and J_k denotes the *jump residual* given by

$$J_k|_S := \frac{1}{2} [(\alpha(\cdot, |\nabla u_{k-1}|^2) \nabla u_k)|_T \cdot \vec{n} + (\alpha(\cdot, |\nabla u_{k-1}|^2) \nabla u_k)|_{T'} \cdot \vec{n}'],$$

for each interior side S , and $J_k|_S := 0$, if S is a side lying on the boundary of Ω . Here, T and T' denote the elements of \mathcal{T}_k sharing S , and \vec{n}, \vec{n}' are the outward unit normals of T, T' on S , respectively.

Remark 3. Note that in order to compute the local indicator $\eta_k(T)$ we need the outputs u_k and u_{k-1} of the module SOLVE from the last two iterations. The index k in η_k corresponds to the iteration counter k of the algorithm, and we should keep in mind that $\eta_k(T)$ depends both on u_k and u_{k-1} .

The residual $\mathbf{R}_k \in H^{-1}(\Omega)$ of problem (10) is given by

$$\langle \mathbf{R}_k, v \rangle := a(u_{k-1}; u_k, v) - L(v) = \int_{\Omega} (\alpha(\cdot, |\nabla u_{k-1}|^2) \nabla u_k \cdot \nabla v - f v), \tag{13}$$

for $v \in H_0^1(\Omega)$. Integrating by parts on each $T \in \mathcal{T}_k$ we have that

$$\langle \mathbf{R}_k, v \rangle = \sum_{T \in \mathcal{T}_k} \left(\int_T R_k v + \int_{\partial T} J_k v \right), \tag{14}$$

and since $\langle \mathbf{R}_k, v_k \rangle = 0$ for $v_k \in \mathbb{V}_k$, using (14) and interpolation estimates, it follows that³

$$|\langle \mathbf{R}_k, v \rangle| \lesssim \sum_{T \in \mathcal{T}_k} \eta_k(T) \|\nabla v\|_{\omega_k(T)}, \quad \forall v \in H_0^1(\Omega), \tag{15}$$

where $\omega_k(T)$ is the union of T and its neighbors in \mathcal{T}_k .

Remark 4 (A posteriori error estimates). Note that \mathbf{R}_k is the residual of the linear problem (10), and depends both on u_k and u_{k-1} . As with the local indicator $\eta_k(T)$ the index k in \mathbf{R}_k corresponds to the iteration counter k of the algorithm. The residual $\mathbf{R}(u_k)$ of u_k corresponding to the original nonlinear problem (2) is given by

$$\langle \mathbf{R}(u_k), v \rangle := a(u_k; u_k, v) - L(v), \quad \text{for } v \in H_0^1(\Omega).$$

Using (5) and (7), it is easy to check that

$$|\langle \mathbf{R}(u_k), v \rangle| \leq (C_a + C_A) \|\nabla(u_k - u_{k-1})\|_{\Omega} \|\nabla v\|_{\Omega} + |\langle \mathbf{R}_k, v \rangle|, \tag{16}$$

for $v \in H_0^1(\Omega)$. On the other hand, (8) yields

$$c_A \|\nabla(u - u_k)\|_{\Omega}^2 \leq \langle Au - Au_k, u - u_k \rangle = -\langle \mathbf{R}(u_k), u - u_k \rangle. \tag{17}$$

³ From now on, we will write $a \lesssim b$ to indicate that $a \leq Cb$ with $C > 0$ a constant depending on the data of the problem and possibly on shape regularity $\kappa_{\mathbb{T}}$ of the meshes.

Thus, from (17), (16) and (15) we obtain the following upper bound for the error:

$$\|\nabla(u - u_k)\|_{\Omega} \lesssim \left(\sum_{T \in \mathcal{T}_k} \eta_k^2(T) \right)^{\frac{1}{2}} + \|\nabla(u_k - u_{k-1})\|_{\Omega}.$$

This upper bound is computable, and can be used as a stopping criterion for the iterative algorithm. Additionally, standard techniques allow us to prove that

$$\eta_k(T) \lesssim \|\nabla(u - u_k)\|_{\omega_k(T)} + \|\nabla(u_k - u_{k-1})\|_{\omega_k(T)} + \text{H.O.T.},$$

for $T \in \mathcal{T}_k$, where H.O.T. stands for *higher order terms* (also called oscillation terms).

The next result is some kind of *stability* result for the indicators, which is a property somewhat weaker than the usual *efficiency*, but sufficient to guarantee convergence of our adaptive algorithm (see also [14,4]). In order to prove it, we assume that α is piecewise Lipschitz (over \mathcal{T}_0) in its first variable, uniformly with respect to its second variable. More precisely, we assume that $\alpha(\cdot, t) \in L^\infty(\Omega)$ and $\alpha(\cdot, t)|_T \in W^{1,\infty}(T)$ with

$$\|\alpha(\cdot, t)\|_{W^{1,\infty}(T)} \leq C_\alpha, \quad \forall T \in \mathcal{T}_0, t > 0, \tag{18}$$

where $C_\alpha > 0$ is a constant. This assumption allows $\alpha(\cdot, t)$ to jump across sides of \mathcal{T}_0 .

Proposition 1 (*Stability of the local error indicators*). *Let $\{u_k\}_{k \in \mathbb{N}}$ be the sequence of discrete solutions computed with the Adaptive Algorithm. Then, the local error indicators given by (12) are stable. More precisely, there holds*

$$\eta_k(T) \lesssim \|\nabla u_k\|_{\omega_k(T)} + H_T \|f\|_T, \quad \forall T \in \mathcal{T}_k,$$

for all $k \in \mathbb{N}$.

Proof. Let $\{u_k\}_{k \in \mathbb{N}}$ be the sequence computed with the Adaptive Algorithm. Let $k \in \mathbb{N}$ and $T \in \mathcal{T}_k$ be fixed. On the one hand, using that $u_k|_T$ is linear, and thus $\Delta u_k = 0$ inside T , we have that

$$\|R_k\|_T = \|\nabla \cdot [\alpha(\cdot, |\nabla u_{k-1}|^2) \nabla u_k] - f\|_T \leq \|\nabla[\alpha(\cdot, |\nabla u_{k-1}|^2)] \cdot \nabla u_k\|_T + \|f\|_T.$$

Since α is piecewise Lipschitz in its first variable (cf. (18)), it follows that if $\xi := \nabla u_{k-1}$ (constant over T),

$$\left| \frac{\partial}{\partial x_i} \alpha(x, |\xi|^2) \right| = \left| \frac{\partial \alpha}{\partial x_i}(x, |\xi|^2) \right| \leq C_\alpha, \quad \forall x \in T, 1 \leq i \leq d,$$

and thus,

$$\|R_k\|_T \lesssim \|\nabla u_k\|_T + \|f\|_T.$$

On the other hand, if S is a side of \mathcal{T}_k shared by the elements $T, T' \in \mathcal{T}_k$,

$$\begin{aligned} \|J_k\|_S &\leq \|(\alpha(\cdot, |\nabla u_{k-1}|^2) \nabla u_k)|_T\|_S + \|(\alpha(\cdot, |\nabla u_{k-1}|^2) \nabla u_k)|_{T'}\|_S \\ &\lesssim \|\nabla u_k|_T\|_S + \|\nabla u_k|_{T'}\|_S \lesssim H_T^{-1/2} \|\nabla u_k\|_T + H_{T'}^{-1/2} \|\nabla u_k\|_{T'} \lesssim H_T^{-1/2} \|\nabla u_k\|_{T \cup T'}, \end{aligned}$$

where we have used (4) and a scaled trace theorem. Therefore,

$$H_T^{1/2} \|J_k\|_{\partial T} \lesssim \|\nabla u_k\|_{\omega_k(T)},$$

which completes the proof. \square

The module MARK. Based on the local error indicators, the module MARK selects a subset \mathcal{M}_k of \mathcal{T}_k , using any of the so-called *reasonable* marking strategies, such as the *maximum strategy*, the *equidistribution strategy*, or *Dörfler's strategy* [13]. More precisely, we only require that the set of marked elements \mathcal{M}_k has at least one element of \mathcal{T}_k holding the largest local indicator. That is, there exists an element $T_k^{\max} \in \mathcal{M}_k$ such that

$$\eta_k(T_k^{\max}) = \max_{T \in \mathcal{T}_k} \eta_k(T). \tag{19}$$

This is what practitioners usually do in order to maximize the error reduction with a minimum computational effort.

The module REFINE. Finally, the module REFINE takes the mesh \mathcal{T}_k and the subset $\mathcal{M}_k \subset \mathcal{T}_k$ as inputs. By using the bisection rule described by Stevenson in [15], this module refines (bisects) each element in \mathcal{M}_k at least n times (where $n \geq 1$ is fixed), in order to obtain a new conforming triangulation \mathcal{T}_{k+1} of Ω , which is a refinement of \mathcal{T}_k and the output of this module. By definition, $\mathcal{T}_k \in \mathbb{T}$ for all $k \in \mathbb{N}$ and the family of meshes obtained by the Adaptive Algorithm is shape-regular. Finally, it is worth observing that the resulting spaces are nested, i.e., $\mathbb{V}_k \subset \mathbb{V}_{k+1}$; this fact will be used in the proof of Lemma 3 below.

Remark 5 (The adaptive sequence $\{u_k\}_{k \in \mathbb{N}_0}$ is bounded). By the coercivity of a given by (6), and using the definition (10) of u_k we have that

$$\|\nabla u_k\|_{\Omega}^2 \leq \frac{1}{c_a} a(u_{k-1}; u_k, u_k) = \frac{1}{c_a} L(u_k) \leq \frac{\|L\|_{H^{-1}(\Omega)}}{c_a} \|\nabla u_k\|_{\Omega},$$

for all $k \in \mathbb{N}$, and then

$$\|\nabla u_k\|_{\Omega} \leq \frac{\|L\|_{H^{-1}(\Omega)}}{c_a}.$$

Therefore, $\{u_k\}_{k \in \mathbb{N}_0}$ is bounded in $H_0^1(\Omega)$.

4. Convergence analysis

In this section we prove the convergence of the sequence computed with the Adaptive Algorithm described in the previous section. We combine the ideas of the proof of convergence of adaptive algorithms for linear problems given in [11] and [14] with new techniques needed to overcome the difficulties arisen by the nonlinear nature of the problem treated in this paper, adapting some ideas from [5,4].

As a first step to prove the convergence of the $\{u_k\}_{k \in \mathbb{N}_0}$ to the exact solution u , we prove that $\|\nabla(u_k - u_{k+1})\|_{\Omega} \rightarrow 0$, as k tends to infinity, for which we need the following auxiliary lemma.

From now on we assume that

$$\alpha(x, t_1) \geq \alpha(x, t_2) \quad \text{whenever } 0 \leq t_1 \leq t_2 \text{ and } x \in \Omega. \tag{20}$$

Lemma 2. Let us assume that $\alpha(x, \cdot)$ is a monotone decreasing function for all $x \in \Omega$, i.e., (20) holds. Then

$$\mathcal{J}(v) - \mathcal{J}(w) \leq \frac{1}{2} (a(w; v, v) - a(w; w, w)), \quad \forall v, w \in H_0^1(\Omega),$$

where $\mathcal{J}(v) = \int_0^1 a(sv; sv, v) ds$.

Proof. Let $v, w \in H_0^1(\Omega)$. Then, a change of variables in the integral defining $\mathcal{J}(\cdot)$ yields

$$\mathcal{J}(v) - \mathcal{J}(w) = \frac{1}{2} \int_{\Omega} \left(\int_0^{|\nabla v|^2} \alpha(x, t) dt - \int_0^{|\nabla w|^2} \alpha(x, t) dt \right) dx = \frac{1}{2} \int_{\Omega} \int_{|\nabla w|^2}^{|\nabla v|^2} \alpha(x, t) dt dx.$$

Since $\alpha(x, \cdot)$ is a decreasing function for all $x \in \Omega$,

$$\frac{1}{2} \int_{\Omega} \int_{|\nabla w|^2}^{|\nabla v|^2} \alpha(x, t) dt dx \leq \frac{1}{2} \int_{\Omega} \alpha(x, |\nabla w|^2) (|\nabla v|^2 - |\nabla w|^2) dx = \frac{1}{2} (a(w; v, v) - a(w; w, w)),$$

and the assertion of the lemma follows. \square

Lemma 3. Let $\{u_k\}_{k \in \mathbb{N}_0}$ denote the sequence of discrete solutions computed with the Adaptive Algorithm. Then,

$$\lim_{k \rightarrow \infty} \|\nabla(u_k - u_{k+1})\|_{\Omega} = 0.$$

The proof of this lemma follows closely the proof of Theorem 25.L of [17], we include it here to make the article more self contained.

Proof. Let $\{u_k\}_{k \in \mathbb{N}_0}$ be the sequence obtained with the Adaptive Algorithm. Since a is coercive (cf. (6)), and linear and symmetric in its second and third variables, we have that

$$\begin{aligned}
 c_a \|\nabla(u_k - u_{k+1})\|_{\Omega}^2 &\leq a(u_k; u_k - u_{k+1}, u_k - u_{k+1}) \\
 &= a(u_k; u_k, u_k) - 2a(u_k; u_{k+1}, u_k) + a(u_k; u_{k+1}, u_{k+1}).
 \end{aligned}
 \tag{21}$$

From (10), since $u_k \in \mathbb{V}_{k+1}$, it follows that $a(u_k; u_{k+1}, u_k - u_{k+1}) = L(u_k - u_{k+1})$ and thus

$$a(u_k; u_{k+1}, u_k) = L(u_k - u_{k+1}) + a(u_k; u_{k+1}, u_{k+1}).$$

Replacing this equality in (21) and taking into account Lemma 2 we obtain

$$\begin{aligned}
 c_a \|\nabla(u_k - u_{k+1})\|_{\Omega}^2 &\leq a(u_k; u_k, u_k) - 2L(u_k - u_{k+1}) - a(u_k; u_{k+1}, u_{k+1}) \\
 &\leq 2\mathcal{J}(u_k) - 2\mathcal{J}(u_{k+1}) - 2L(u_k) + 2L(u_{k+1}) \\
 &= 2(\mathcal{F}(u_k) - \mathcal{F}(u_{k+1})),
 \end{aligned}
 \tag{22}$$

where $\mathcal{F} := \mathcal{J} - L$, and therefore, $\{\mathcal{F}(u_k)\}_{k \in \mathbb{N}_0}$ is a monotone decreasing sequence.

On the other hand, $\{\mathcal{F}(u_k)\}_{k \in \mathbb{N}_0}$ is bounded below since

$$\begin{aligned}
 \mathcal{F}(u_k) &= \int_0^1 sa(su_k; u_k, u_k) ds - L(u_k) \\
 &\geq \frac{1}{2}c_a \|\nabla u_k\|_{\Omega}^2 - \|L\|_{H^{-1}(\Omega)} \|\nabla u_k\|_{\Omega} \geq -\frac{\|L\|_{H^{-1}(\Omega)}^2}{2c_a}.
 \end{aligned}$$

Finally, from the last two assertions it follows that $\{\mathcal{F}(u_k)\}_{k \in \mathbb{N}_0}$ is convergent. Considering (22), we conclude the proof of this lemma. \square

We show now that the sequence obtained with the Adaptive Algorithm is convergent, and more precisely, that it converges to a function in the limiting space $\mathbb{V}_{\infty} := \overline{\bigcup_k \mathbb{V}_k}^{H_0^1(\Omega)}$. Note that \mathbb{V}_{∞} is a Hilbert space with the inner product inherited from $H_0^1(\Omega)$.

Theorem 4 (The adaptive sequence is convergent). *Let $\{u_k\}_{k \in \mathbb{N}_0}$ be the sequence obtained with the Adaptive Algorithm. Let $u_{\infty} \in \mathbb{V}_{\infty}$ be the only solution to*

$$u_{\infty} \in \mathbb{V}_{\infty}: \quad a(u_{\infty}; u_{\infty}, v) = L(v), \quad \forall v \in \mathbb{V}_{\infty}. \tag{23}$$

Then

$$u_k \rightarrow u_{\infty} \quad \text{in } H_0^1(\Omega).$$

Remark 6. Notice that (23) always has a solution because (7) and (8) hold on \mathbb{V}_{∞} , which is itself a Hilbert space.

Proof. Let $\{u_k\}_{k \in \mathbb{N}_0}$ be the sequence obtained with the Adaptive Algorithm. Let $u_{\infty} \in \mathbb{V}_{\infty}$ denote the solution of (23) and $\mathcal{P}_{k+1} : H_0^1(\Omega) \rightarrow \mathbb{V}_{k+1}$ be the orthogonal projection onto \mathbb{V}_{k+1} . Since A is strongly monotone (cf. (8)), using (23) and (10) we have that

$$\begin{aligned}
 c_A \|\nabla(u_k - u_{\infty})\|_{\Omega}^2 &\leq \langle Au_k - Au_{\infty}, u_k - u_{\infty} \rangle \\
 &= \langle Au_k, u_k - u_{\infty} \rangle - L(u_k - u_{\infty}) \\
 &= \langle Au_k, u_k - \mathcal{P}_{k+1}u_{\infty} \rangle + \langle Au_k, \mathcal{P}_{k+1}u_{\infty} - u_{\infty} \rangle - L(u_k - \mathcal{P}_{k+1}u_{\infty}) - L(\mathcal{P}_{k+1}u_{\infty} - u_{\infty}) \\
 &= a(u_k; u_k - u_{k+1}, u_k - \mathcal{P}_{k+1}u_{\infty}) + \langle Au_k, \mathcal{P}_{k+1}u_{\infty} - u_{\infty} \rangle - L(\mathcal{P}_{k+1}u_{\infty} - u_{\infty}) \\
 &\leq C_a \|\nabla(u_k - u_{k+1})\|_{\Omega} (\|\nabla u_k\|_{\Omega} + \|\nabla u_{\infty}\|_{\Omega}) \\
 &\quad + (C_a \|\nabla u_k\|_{\Omega} + \|L\|_{H^{-1}(\Omega)}) \|\nabla(\mathcal{P}_{k+1}u_{\infty} - u_{\infty})\|_{\Omega},
 \end{aligned}$$

for all $k \in \mathbb{N}_0$, where in the last inequality we have used (5). From Remark 5 it follows that $\{u_k\}_{k \in \mathbb{N}_0}$ is bounded in $H_0^1(\Omega)$, and using Lemma 3, together with the fact that the spaces $\{\mathbb{V}_k\}_{k \in \mathbb{N}_0}$ are nested and $\bigcup_{k \in \mathbb{N}_0} \mathbb{V}_k$ is dense in \mathbb{V}_{∞} , we conclude that $u_k \rightarrow u_{\infty}$ in $H_0^1(\Omega)$. \square

In order to show that the limiting function u_{∞} is, in fact, the solution of the problem (2) and thereby conclude that the adaptive sequence converges to the solution of this problem, we establish first two auxiliary results (see Lemma 5 and Theorem 6). We need the following

Definition 1. Given any sequence of meshes $\{\mathcal{T}_k\}_{k \in \mathbb{N}_0} \subset \mathbb{T}$, with \mathcal{T}_{k+1} a refinement of \mathcal{T}_k , for each $k \in \mathbb{N}_0$, we define

$$\mathcal{T}_k^+ := \{T \in \mathcal{T}_k \mid T \in \mathcal{T}_m, \forall m \geq k\}, \quad \mathcal{T}_k^0 := \mathcal{T}_k \setminus \mathcal{T}_k^+,$$

and

$$\Omega_k^+ := \bigcup_{T \in \mathcal{T}_k^+} \omega_k(T), \quad \Omega_k^0 := \bigcup_{T \in \mathcal{T}_k^0} \omega_k(T).$$

In words, \mathcal{T}_k^+ is the subset of the elements of \mathcal{T}_k which are never refined in the adaptive process, and \mathcal{T}_k^0 consists of the elements which are eventually refined.

It can be proved [11] that if $\chi_{\Omega_k^0}$ denotes the characteristic function of Ω_k^0 , then

$$\lim_{k \rightarrow \infty} \|h_k \chi_{\Omega_k^0}\|_{L^\infty(\Omega)} = 0, \tag{24}$$

where $h_k \in L^\infty(\Omega)$ denotes the piecewise constant mesh-size function satisfying $h_{k|_T} := H_T$, for all $T \in \mathcal{T}_k$.

Since the error indicators are stable (cf. Proposition 1), using the convergence proved in the last theorem and (24) we can establish the following

Lemma 5 (Indicator on marked elements). Let $\{\{\eta_k(T)\}_{T \in \mathcal{T}_k}\}_{k \in \mathbb{N}_0}$ be the sequence of local error indicators computed with the Adaptive Algorithm, and let $\{\mathcal{M}_k\}_{k \in \mathbb{N}_0}$ be the sequence of subsets of marked elements over each mesh. Then,

$$\lim_{k \rightarrow \infty} \max_{T \in \mathcal{M}_k} \eta_k(T) = 0.$$

Proof. Let $\{\{\eta_k(T)\}_{T \in \mathcal{T}_k}\}_{k \in \mathbb{N}_0}$ and $\{\mathcal{M}_k\}_{k \in \mathbb{N}_0}$ be as in the assumptions. For each $k \in \mathbb{N}_0$, we select $T_k \in \mathcal{M}_k$ such that $\eta_k(T_k) = \max_{T \in \mathcal{M}_k} \eta_k(T)$. Using Proposition 1 we have that

$$\eta_k(T_k) \lesssim \|\nabla u_k\|_{\omega_k(T_k)} + \|f\|_{T_k} \lesssim \|\nabla u_k - \nabla u_\infty\|_\Omega + \|\nabla u_\infty\|_{\omega_k(T_k)} + \|f\|_{T_k}, \tag{25}$$

where $u_\infty \in \mathbb{V}_\infty$ is the solution of (23). On the one hand, the first term in the right-hand side of (25) tends to zero due to Theorem 4. On the other hand, since $T_k \in \mathcal{M}_k \subset \mathcal{T}_k^0$, from (24) it follows that

$$|T_k| \leq |\omega_k(T_k)| \lesssim H_{T_k}^d \leq \|h_k \chi_{\Omega_k^0}\|_{L^\infty(\Omega)}^d \rightarrow 0, \quad \text{as } k \rightarrow \infty,$$

and the last two terms in the right-hand side of (25) also tend to zero. \square

Using the upper bound (15), the stability of the indicators (Proposition 1), the facts that $\{u_k\}_{k \in \mathbb{N}_0}$ is bounded (cf. Remark 5) and the marking strategy is reasonable (cf. (19)), and Lemma 5, we now prove the following important result.

Theorem 6 (Weak convergence of the residual). If $\mathbf{R}_k \in H^{-1}(\Omega)$ is given by (13), then

$$\lim_{k \rightarrow \infty} \langle \mathbf{R}_k, v \rangle = 0, \quad \text{for all } v \in H_0^1(\Omega).$$

Proof. We prove first the result for $v \in H^2(\Omega) \cap H_0^1(\Omega)$, and then extend it to $H_0^1(\Omega)$ by density. Let $p \in \mathbb{N}$ and $k > p$. By Definition 1 we have that $\mathcal{T}_p^+ \subset \mathcal{T}_k^+ \subset \mathcal{T}_k$. Let $v_k \in \mathbb{V}_k$ be the Lagrange's interpolant of v . Since $\langle \mathbf{R}_k, v_k \rangle = 0$, using (15), and Cauchy–Schwartz inequality we have that

$$\begin{aligned} |\langle \mathbf{R}_k, v \rangle| &= |\langle \mathbf{R}_k, v - v_k \rangle| \lesssim \sum_{T \in \mathcal{T}_k} \eta_k(T) \|\nabla(v - v_k)\|_{\omega_k(T)} \\ &= \sum_{T \in \mathcal{T}_p^+} \eta_k(T) \|\nabla(v - v_k)\|_{\omega_k(T)} + \sum_{T \in \mathcal{T}_k \setminus \mathcal{T}_p^+} \eta_k(T) \|\nabla(v - v_k)\|_{\omega_k(T)} \\ &\lesssim \eta_k(\mathcal{T}_p^+) \|\nabla(v - v_k)\|_{\Omega_p^+} + \eta_k(\mathcal{T}_k \setminus \mathcal{T}_p^+) \|\nabla(v - v_k)\|_{\Omega_k^0}, \end{aligned}$$

where, for any $\mathcal{E} \subset \mathcal{T}_k$ we hereafter denote $(\sum_{T \in \mathcal{E}} \eta_k^2(T))^{1/2}$ by $\eta_k(\mathcal{E})$. Taking into account Proposition 1 and the boundedness of the discrete solutions (cf. Remark 5) we have that $\eta_k(\mathcal{T}_k \setminus \mathcal{T}_p^+) \leq \eta_k(\mathcal{T}_k) \lesssim 1$, and therefore,

$$|\langle \mathbf{R}_k, v \rangle| \lesssim (\eta_k(\mathcal{T}_p^+) + \|h_p \chi_{\Omega_p^0}\|_{L^\infty(\Omega)}) |v|_{H^2(\Omega)},$$

due to interpolation estimates.

In order to prove that $\langle \mathbf{R}_k, v \rangle \rightarrow 0$ as $k \rightarrow \infty$ we now let $\varepsilon > 0$ be arbitrary. Due to (24), there exists $p \in \mathbb{N}$ such that

$$\|h_p \chi_{\Omega_p^0}\|_{L^\infty(\Omega)} < \varepsilon.$$

On the other hand, since $\mathcal{T}_p^+ \subset \mathcal{T}_k^+ \subset \mathcal{T}_k$ and the marking strategy is reasonable (cf. (19)),

$$\eta_k(\mathcal{T}_p^+) \leq (\#\mathcal{T}_p^+)^{1/2} \max_{T \in \mathcal{T}_p^+} \eta_k(T) \leq (\#\mathcal{T}_p^+)^{1/2} \max_{T \in \mathcal{M}_k} \eta_k(T).$$

Now, by Lemma 5, we can select $K > p$ such that $\eta_k(\mathcal{T}_p^+) < \varepsilon$, for all $k > K$.

Summarizing, we have proved that

$$\lim_{k \rightarrow \infty} \langle \mathbf{R}_k, v \rangle = 0, \quad \text{for all } v \in H^2(\Omega) \cap H_0^1(\Omega).$$

Finally, since $H^2(\Omega) \cap H_0^1(\Omega)$ is dense in $H_0^1(\Omega)$, this limit is also zero for all $v \in H_0^1(\Omega)$. \square

As a consequence of Theorem 6 we now prove that u_∞ is the solution of problem (2).

Theorem 7 (The limiting function is the solution). *If u_∞ denotes the solution of (23), then u_∞ is the solution of problem (2), i.e.,*

$$a(u_\infty; u_\infty, v) = L(v), \quad \forall v \in H_0^1(\Omega).$$

Proof. Let u_∞ be the solution of (23). If $v \in H_0^1(\Omega)$, and $\{u_k\}_{k \in \mathbb{N}_0}$ denotes the sequence of discrete solutions computed with the Adaptive Algorithm, then

$$\begin{aligned} |a(u_\infty; u_\infty, v) - L(v)| &= |a(u_\infty; u_\infty, v) - L(v) - a(u_k; u_{k+1}, v) + a(u_k; u_{k+1}, v)| \\ &\leq |a(u_\infty; u_\infty, v) - a(u_k; u_{k+1}, v)| + |\langle \mathbf{R}_{k+1}, v \rangle| \\ &\leq |a(u_\infty; u_\infty, v) - a(u_k; u_k, v)| + |a(u_k; u_k - u_{k+1}, v)| + |\langle \mathbf{R}_{k+1}, v \rangle| \\ &\leq \|Au_\infty - Au_k\|_{H^{-1}(\Omega)} \|\nabla v\|_\Omega + C_a \|\nabla(u_k - u_{k+1})\|_\Omega \|\nabla v\|_\Omega + |\langle \mathbf{R}_{k+1}, v \rangle| \\ &\leq C_A \|\nabla(u_\infty - u_k)\|_\Omega \|\nabla v\|_\Omega + C_a \|\nabla(u_k - u_{k+1})\|_\Omega \|\nabla v\|_\Omega + |\langle \mathbf{R}_{k+1}, v \rangle|, \end{aligned}$$

where we have used that A is Lipschitz (cf. (7)) and a is bounded (cf. (5)). Using Theorem 4, Lemma 3 and Theorem 6 it follows that the right-hand side in the last inequality tends to zero as k tends to infinity. \square

As an immediate consequence of Theorems 4 and 7 we finally obtain the main result of this article.

Theorem 8 (Main result). *Let $\{u_k\}_{k \in \mathbb{N}_0}$ denote the sequence of discrete solutions computed with the Adaptive Algorithm. If $\alpha(\cdot, \cdot)$ satisfies assumptions (3) and (20), then $\{u_k\}_{k \in \mathbb{N}_0}$ converges to the solution u of problem (2).*

We conclude this section with a couple of remarks.

Remark 7. The problem given by (1) is a particular case of the more general problem:

$$\begin{cases} -\nabla \cdot [\alpha(\cdot, |\nabla u|_{\mathcal{A}}^2) \mathcal{A} \nabla u] = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $f \in L^2(\Omega)$ satisfy the properties assumed in the previous sections, and $\mathcal{A} : \Omega \rightarrow \mathbb{R}^{d \times d}$ is symmetric for all $x \in \Omega$, and uniformly positive definite, i.e., there exist constants $\underline{a}, \bar{a} > 0$ such that

$$\underline{a} |\xi|^2 \leq \mathcal{A}(x) \xi \cdot \xi \leq \bar{a} |\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^d.$$

If \mathcal{A} is piecewise Lipschitz over an initial conforming mesh \mathcal{T}_0 of Ω , i.e., there exists $C_{\mathcal{A}} > 0$ such that

$$\|\mathcal{A}(x) - \mathcal{A}(y)\|_2 \leq C_{\mathcal{A}} |x - y|, \quad \forall x, y \in T, \forall T \in \mathcal{T}_0,$$

then the convergence results previously presented also hold for this problem.

Remark 8. We have assumed the use of linear finite elements for the discretization (see (9)). It is important to notice that the only place where we used this is in Proposition 1. The rest of the steps of the proof hold regardless of the degree of the finite element space. The use of linear finite elements is customary in nonlinear problems, because they greatly simplify the analysis. The numerical experiments of the next section show a competitive performance of the adaptive method for any tested polynomial degree (up to four).

5. Numerical experiments

We conclude this article reporting on the behavior of the adaptive algorithm for some particular nonlinear problems. In the first subsection we study the convergence rate in terms of degrees of freedom for an exact solution and different functions $\alpha(\cdot, \cdot)$. In the second subsection we show the performance of the algorithm when approximating an unknown solution of a prescribed curvature equation.

5.1. Exact solution

Let us consider the problem

$$\begin{cases} -\nabla \cdot [\alpha(|\nabla u|^2)\nabla u] = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \quad (26)$$

where $\Omega \subset \mathbb{R}^2$ is the L-shaped domain given in Fig. 1. In the following examples, in order to study experimentally the behavior of the Adaptive Algorithm, we consider (26) with different choices of the function α , defining f and g in each case so that the solution of the problem is the function u depicted in Fig. 1, given in polar coordinates by

$$u(r, \varphi) = r^{\frac{2}{3}} \sin\left(\frac{2}{3}\varphi\right). \quad (27)$$

We consider the Adaptive Algorithm using different marking strategies, namely, *global refinement*, *maximum strategy* with $\theta = 0.7$ and *Dörfler's strategy* with $\theta = 0.5$ (see [13]).

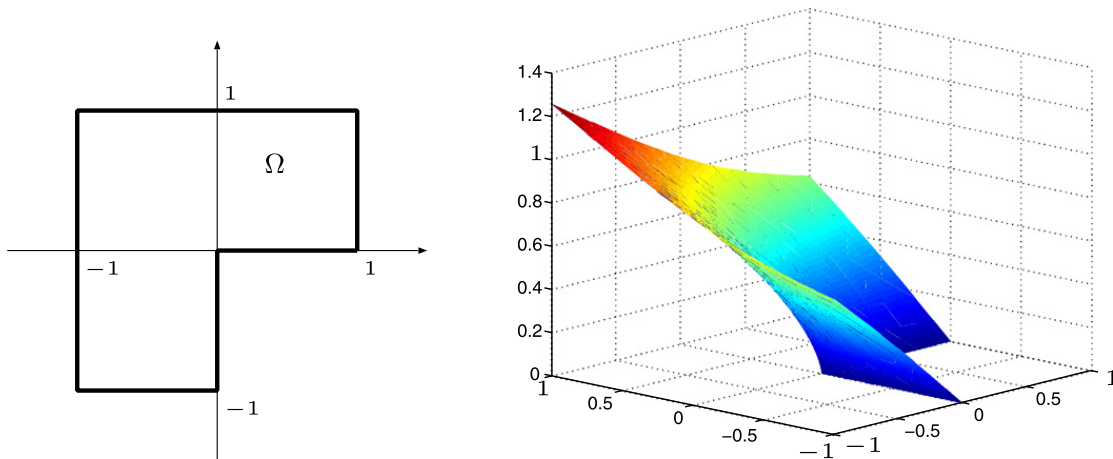


Fig. 1. The domain Ω where the problem (26) is posed and the function u which is the solution of the problem in each example.

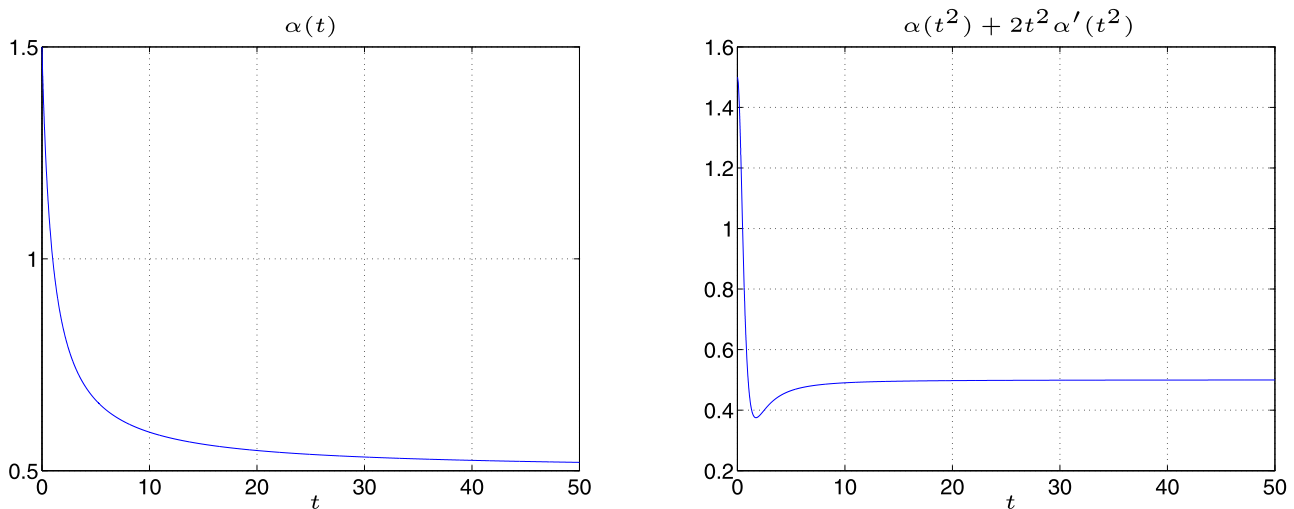


Fig. 2. The function $\alpha(t) = \frac{1}{1+t} + \frac{1}{2}$, of Example 1 satisfies the properties we require to guarantee the convergence.

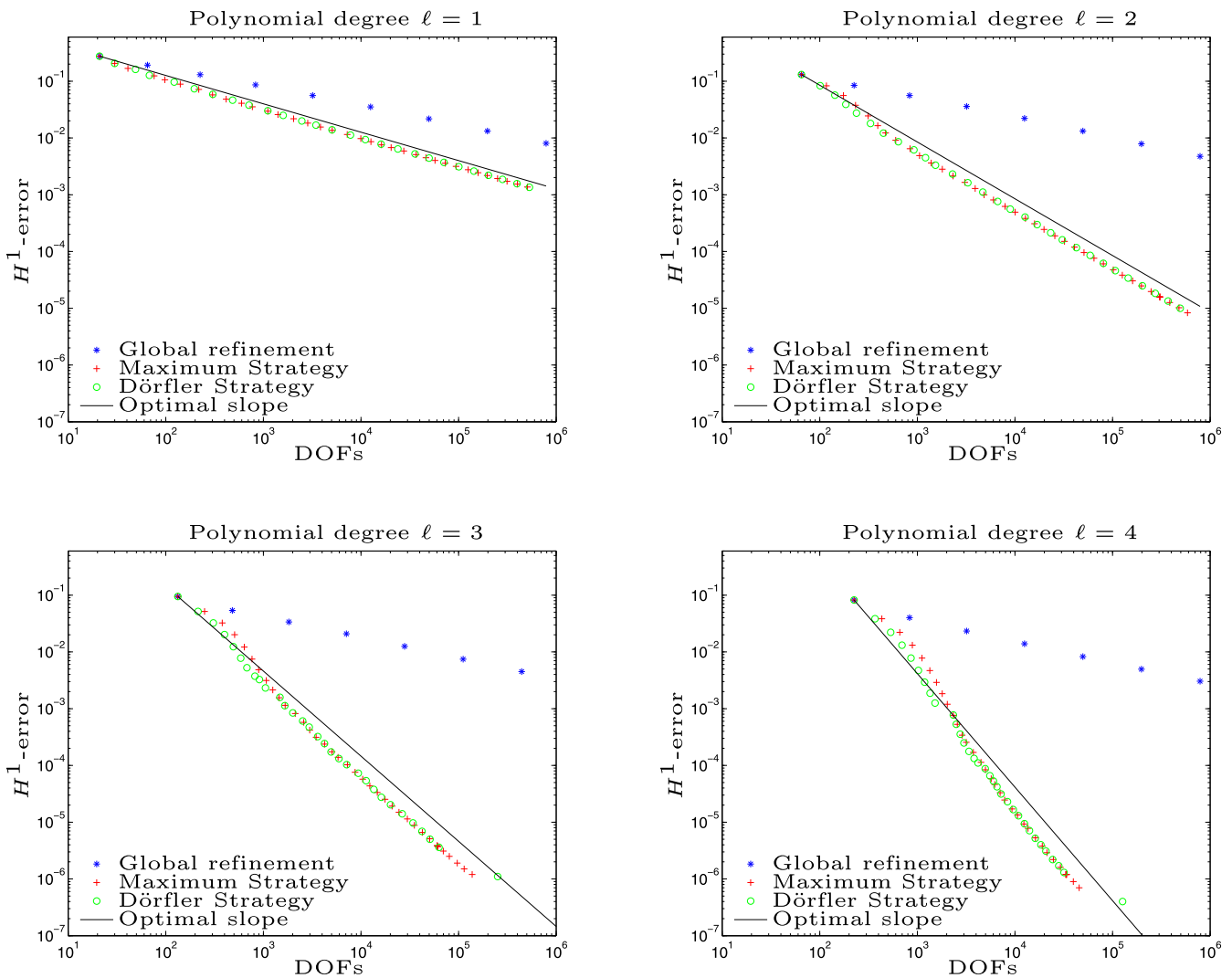


Fig. 3. Error versus DOFs for Example 1. We present the $H^1(\Omega)$ -error between the exact solution and discrete solutions, versus the number of degrees of freedom (DOFs) used to represent each of them. We note that the convergence rate is optimal for the considered adaptive strategies, but not for global refinement, due to the fact that the solution u is not sufficiently smooth. In this case, $\alpha(t) = \frac{1}{1+t} + \frac{1}{2}$ satisfies all the properties established to guarantee the convergence with linear finite elements. The numerical experiments suggest that the method converges with optimal rate for any polynomial degree.

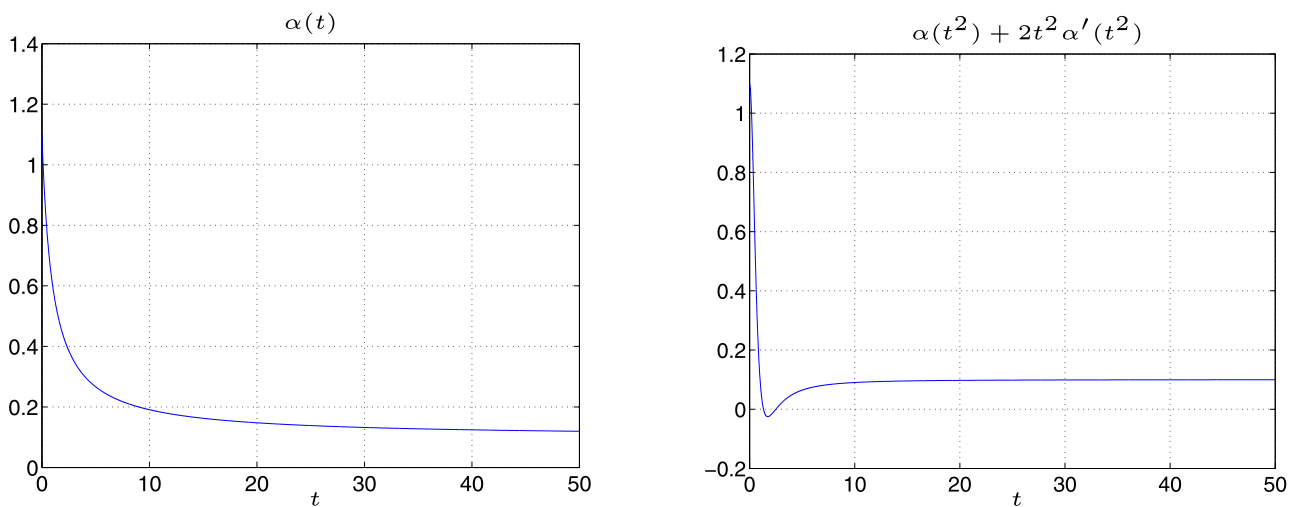


Fig. 4. The function $\alpha(t) = \frac{1}{1+t} + \frac{1}{10}$ of Example 2 does not satisfy $\alpha(t^2) + 2t^2\alpha'(t^2) > 0$ for all $t > 0$.

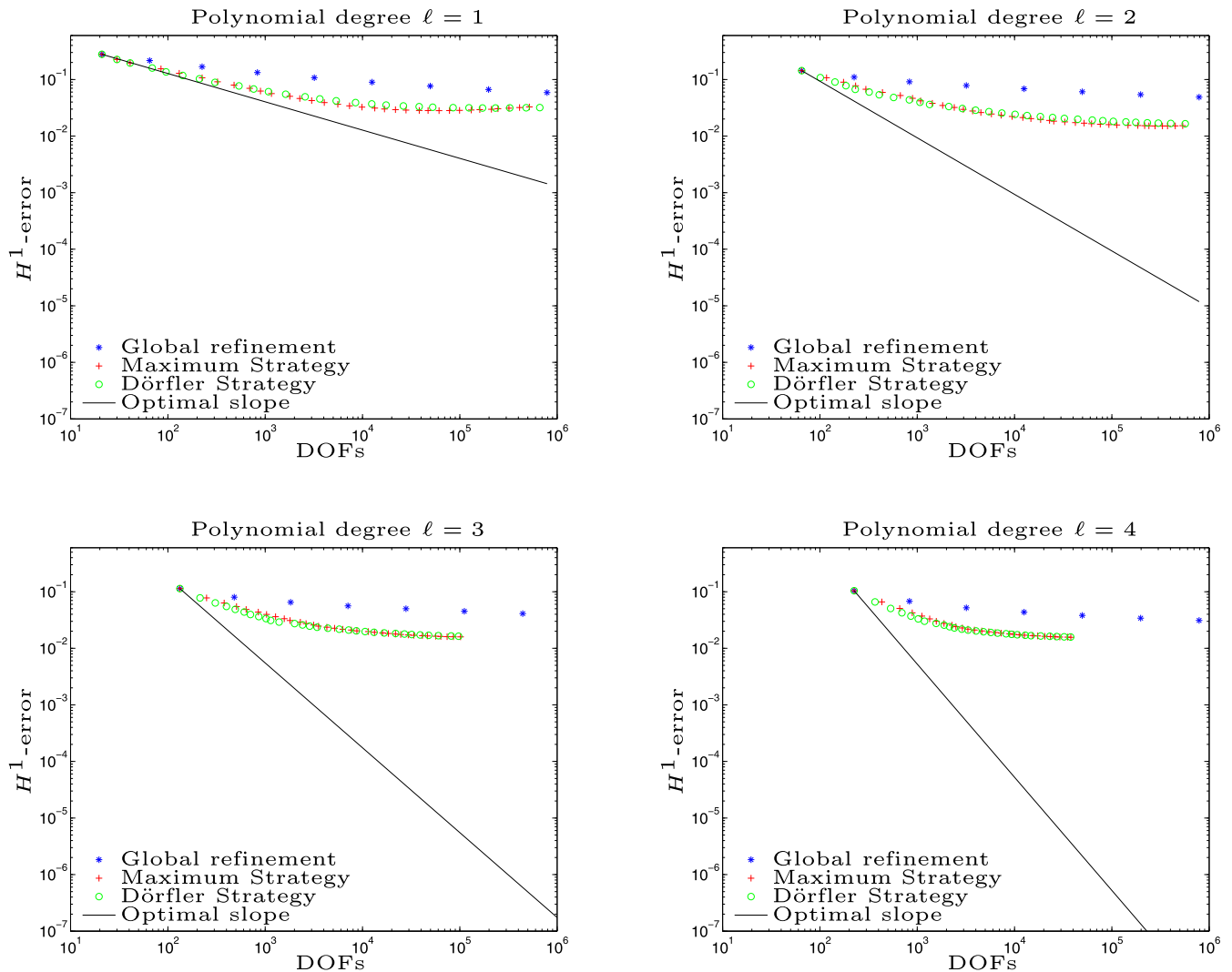


Fig. 5. Error versus DOFs for Example 2. We present the $H^1(\Omega)$ -error between the exact and discrete solutions, versus DOFs. We observe that the method does not converge, but the error stagnates around 10^{-2} . Since $\alpha(t) = \frac{1}{1+t} + \frac{1}{10}$ does not satisfy the condition which guarantees uniqueness of solutions, and based on the fact that the a posteriori error indicator does tend to zero (see Fig. 6) we conclude that the method converges to *another solution* of the same problem, different from the one given by (27).

We implemented the Adaptive Algorithm using the finite element toolbox ALBERTA [13]. We iterated the algorithm until the global error indicator was below 10^{-6} or the number of degrees of freedom exceeded 5×10^5 . We tested the limits of our theory by trying with some functions α which did not satisfy all the assumptions of the theoretical results above.

Example 1 (Optimal convergence rate when α satisfies the hypotheses). As a first example, in order to study experimentally the rate of convergence of the Adaptive Algorithm, we consider

$$\alpha(t) = \frac{1}{1+t} + \frac{1}{2}, \quad t > 0,$$

which satisfies the hypotheses to guarantee the convergence (see Fig. 2), i.e., α is a C^1 -function, and there exist positive constant c_a and C_a such that

$$c_a \leq \alpha(t^2) + 2t^2\alpha'(t^2) \leq C_a, \quad \forall t > 0, \tag{28}$$

and

$$\alpha \text{ is monotone decreasing, i.e., } \alpha'(t) \leq 0 \text{ for all } t > 0. \tag{29}$$

In Fig. 3 we plot the $H^1(\Omega)$ -error versus the number of degrees of freedom (DOFs), for finite elements of degree $\ell = 1, 2, 3, 4$. In this case, the rate of the convergence is optimal for adaptive strategies, that is, $\|u - u_k\|_{H^1(\Omega)} = O(\text{DOFs}_k^{-\ell/2})$.

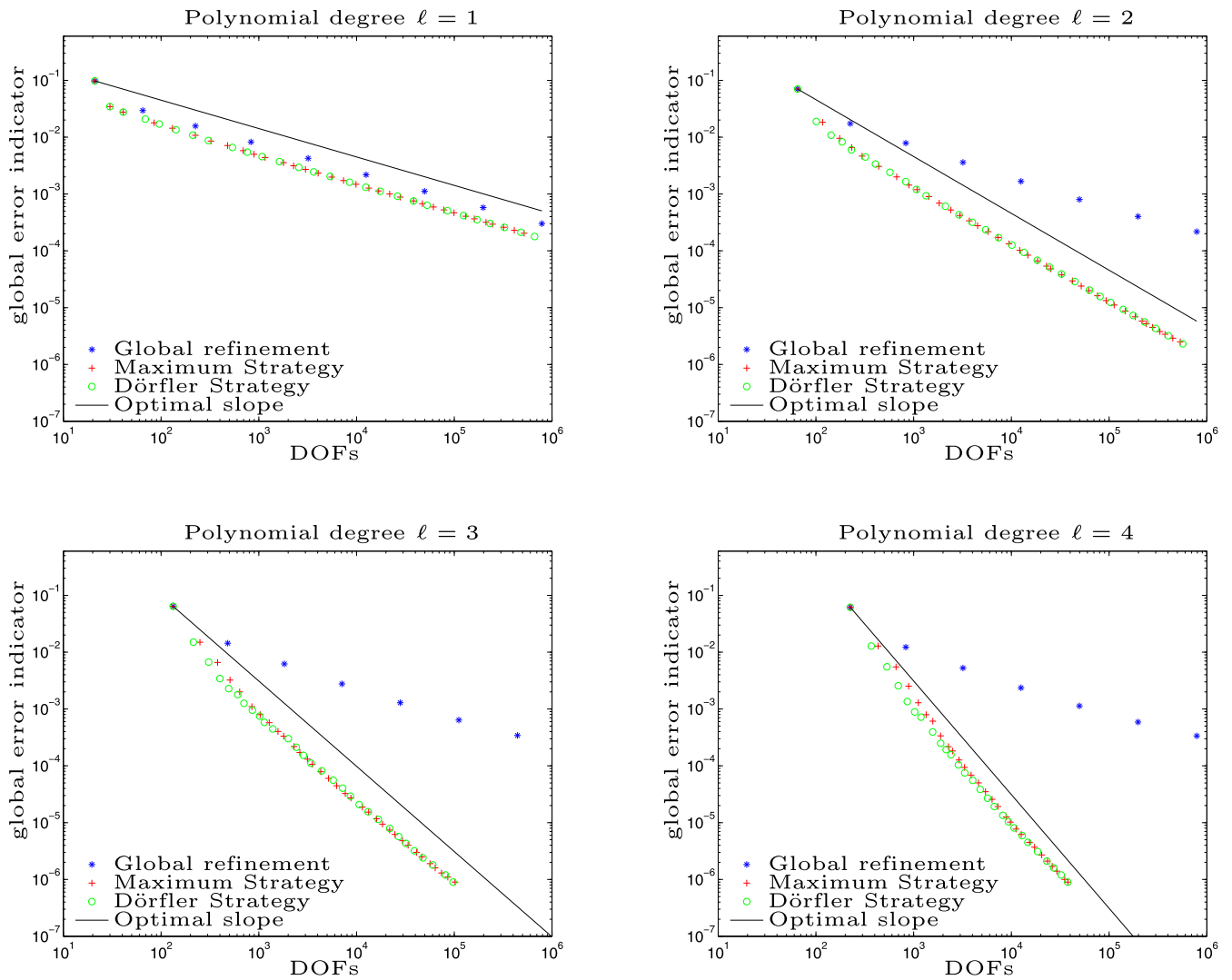


Fig. 6. Indicator versus DOFs for Example 2. We present the global error indicator $\eta_k(\mathcal{T}_k)$ versus DOFs used to represent each discrete solution. We note that for the adaptive strategies the global error indicator decreases with the optimal rate for the H^1 -error, although the error does not tend to zero (see Fig. 5). In this case, $\alpha(t) = \frac{1}{1+t} + \frac{1}{10}$ does not satisfy the condition which guarantees uniqueness of solutions. It seems that the method converges to *another solution* of the problem.

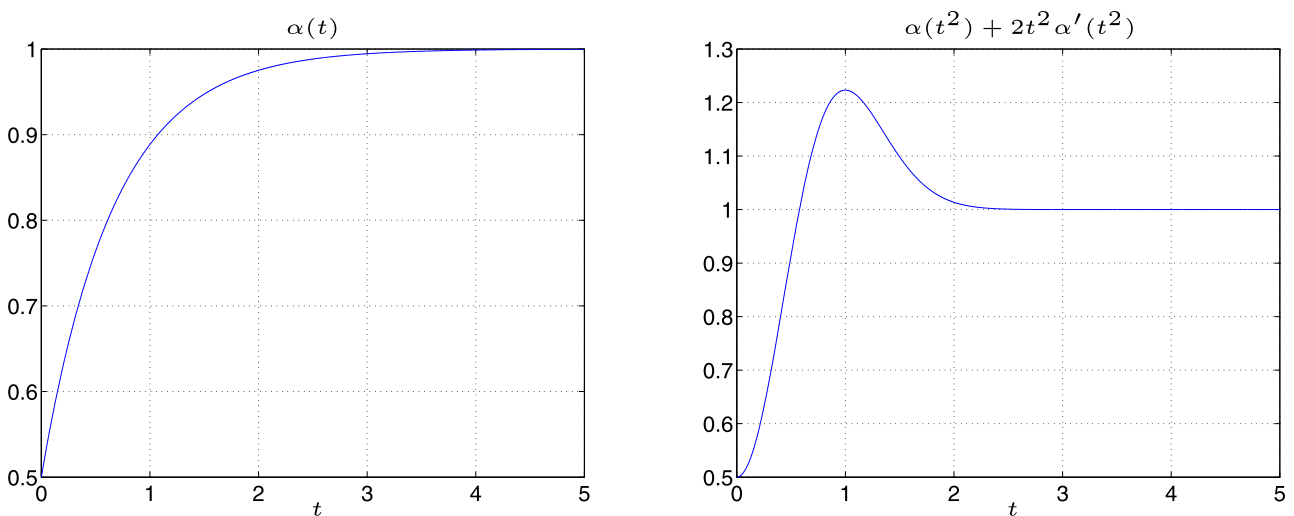


Fig. 7. The function $\alpha(t) = -\frac{1}{2}e^{-\frac{3}{2}t} + 1$ of Example 3 satisfies the hypothesis (28) but is monotone increasing.

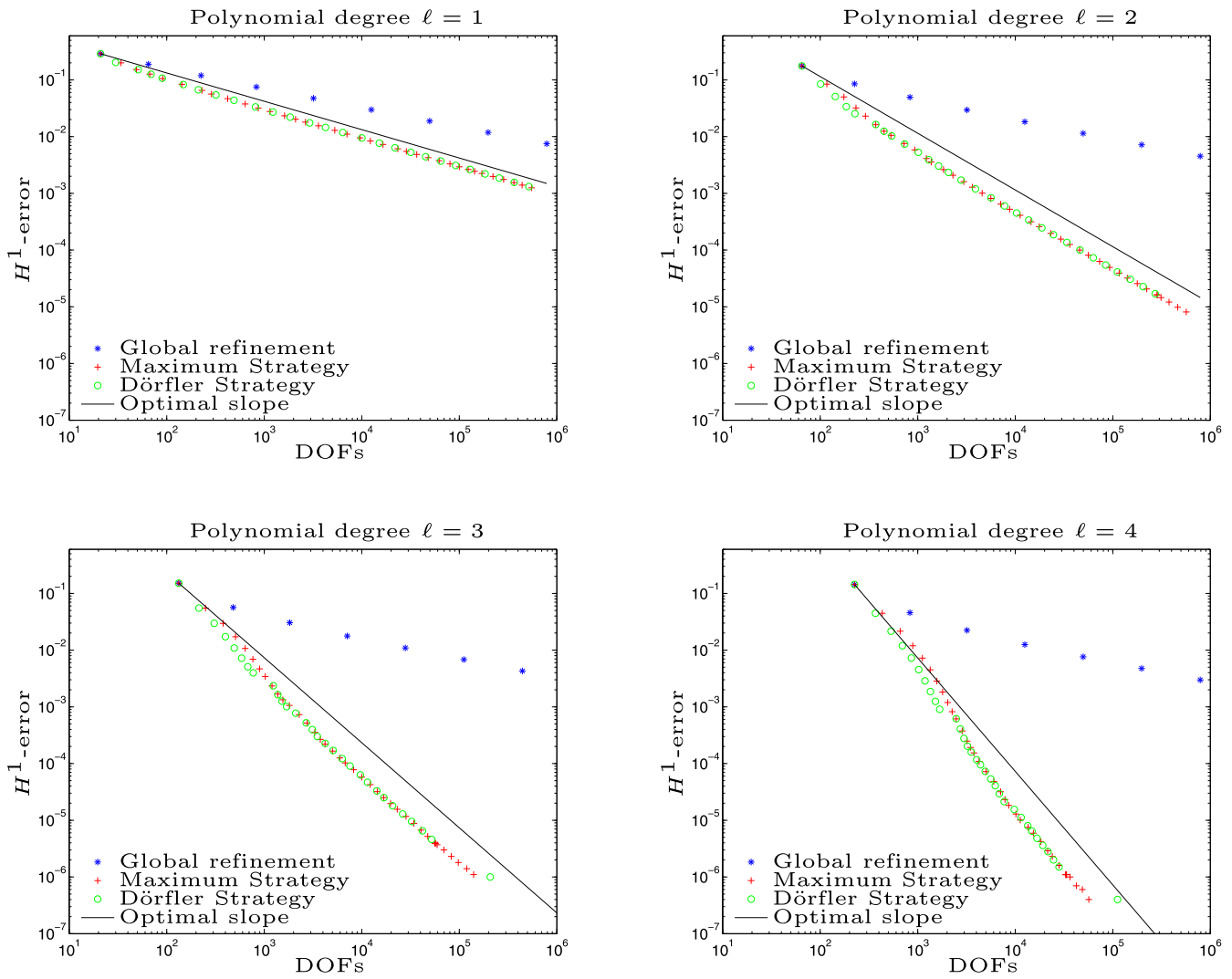


Fig. 8. Error versus DOFs for Example 3. We present the $H^1(\Omega)$ -error between the exact and discrete solutions, versus DOFs. We note that the convergence rate is optimal for the considered adaptive strategies, although the function $\alpha(t) = -\frac{1}{2}e^{-\frac{3}{2}t} + 1$ is not monotone decreasing. This could mean that this hypothesis is not necessary for the convergence of the Adaptive Algorithm, which performs better than expected by our theory.

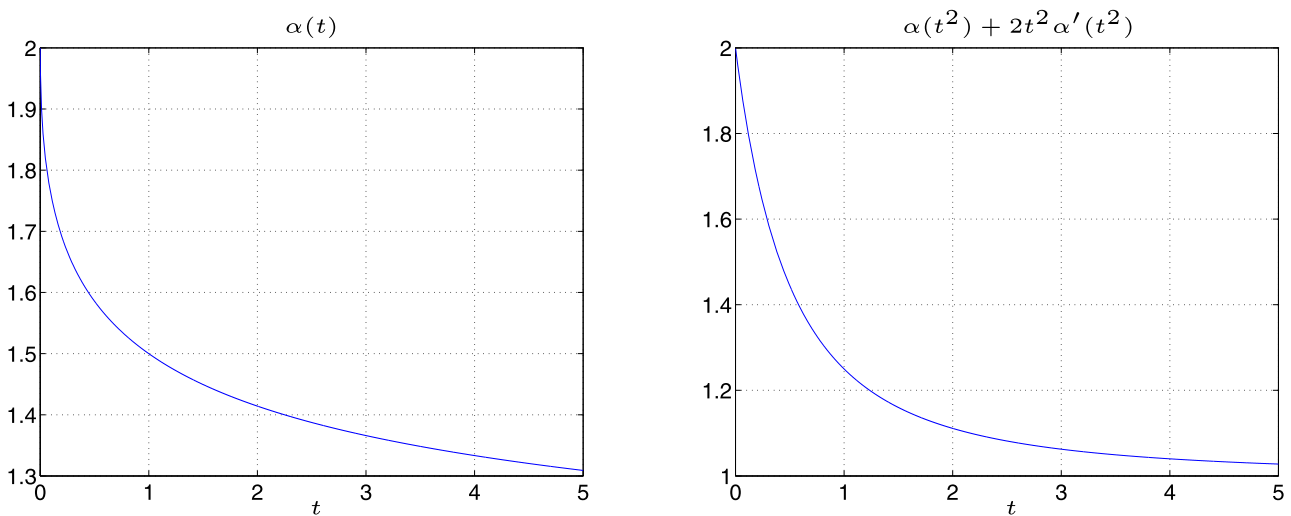


Fig. 9. The function $\alpha(t) = 2 - \frac{\sqrt{t}}{1+\sqrt{t}}$ of Example 4 satisfies our assumptions which do not require that α is Lipschitz continuous.

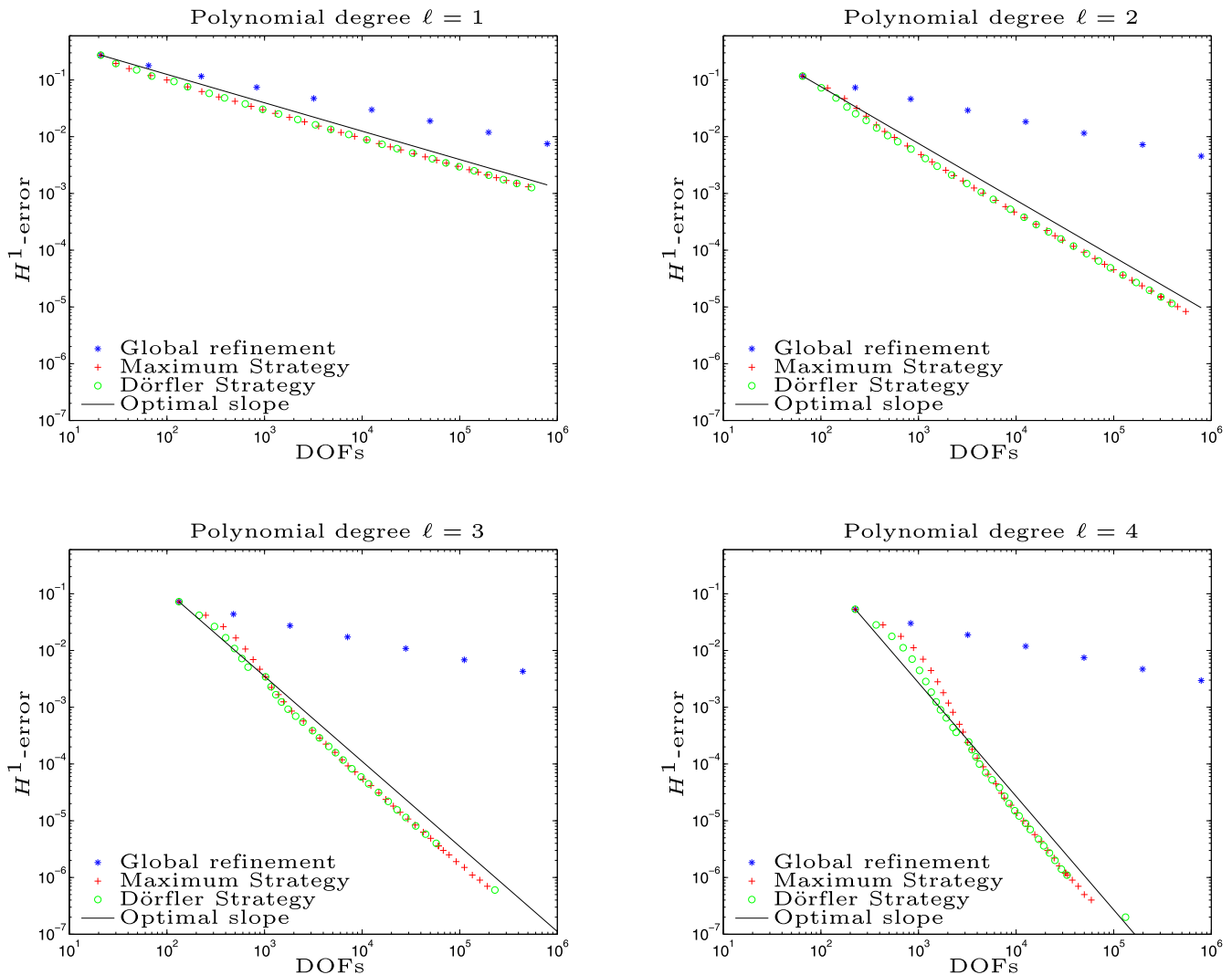


Fig. 10. Error versus DOFs for Example 4. We present the $H^1(\Omega)$ -error between the exact and discrete solutions, versus DOFs. We note that the convergence rate is optimal for the considered adaptive strategies, even though the function $\alpha(t) = 2 - \frac{\sqrt{t}}{1+\sqrt{t}}$ has an infinite derivative at 0. This example falls inside the theory, and the fact that α is not Lipschitz continuous does not destroy the optimality of the sequence of discrete solutions.

For global refinement, the observed order of convergence is $\text{DOFs}_k^{-1/3}$ for all tested polynomial degrees, due to the fact that the solution u belongs to $H^{1+\delta}(\Omega)$, for all $0 < \delta < \frac{2}{3}$, and does not belong to $H^{1+2/3}(\Omega)$.

Note that, although the theory only guarantees the plain convergence for linear elements (cf. Theorem 8), the numerical results suggest that the method works for any polynomial degree (see Remark 8), and the convergence rate is optimal.

Example 2 (About the hypothesis (28)). We consider the function

$$\alpha(t) = \frac{1}{1+t} + \frac{1}{10}, \quad t > 0,$$

which is monotone decreasing, i.e., satisfies (29), but not (28), as it is shown in Fig. 4. Since (28) guarantees the well-posedness of problem (26) (uniqueness and stability), we could be facing an example with multiple solutions.

In Fig. 5 we plot the $H^1(\Omega)$ -error versus the number degrees of freedom, for different polynomial degrees. For $\ell = 3$ and $\ell = 4$ the algorithm stopped with an indicator below the desired tolerance 10^{-6} , although the error is around 10^{-2} in all cases. On the other hand, as we can see in Fig. 6, the global error indicator decreases with optimal rate for the adaptive strategies, indicating that the adaptive algorithm may be converging to another solution of the nonlinear problem, different from the one given by (27).

Based on this remark, it seems that the adaptive algorithm converges to a solution u_1 such that $\|u - u_1\|_{H^1(\Omega)} \approx 10^{-2}$. We recall that α does not satisfy the condition (28) which guarantees the uniqueness of solutions.

Example 3 (About the hypothesis (29)). We consider

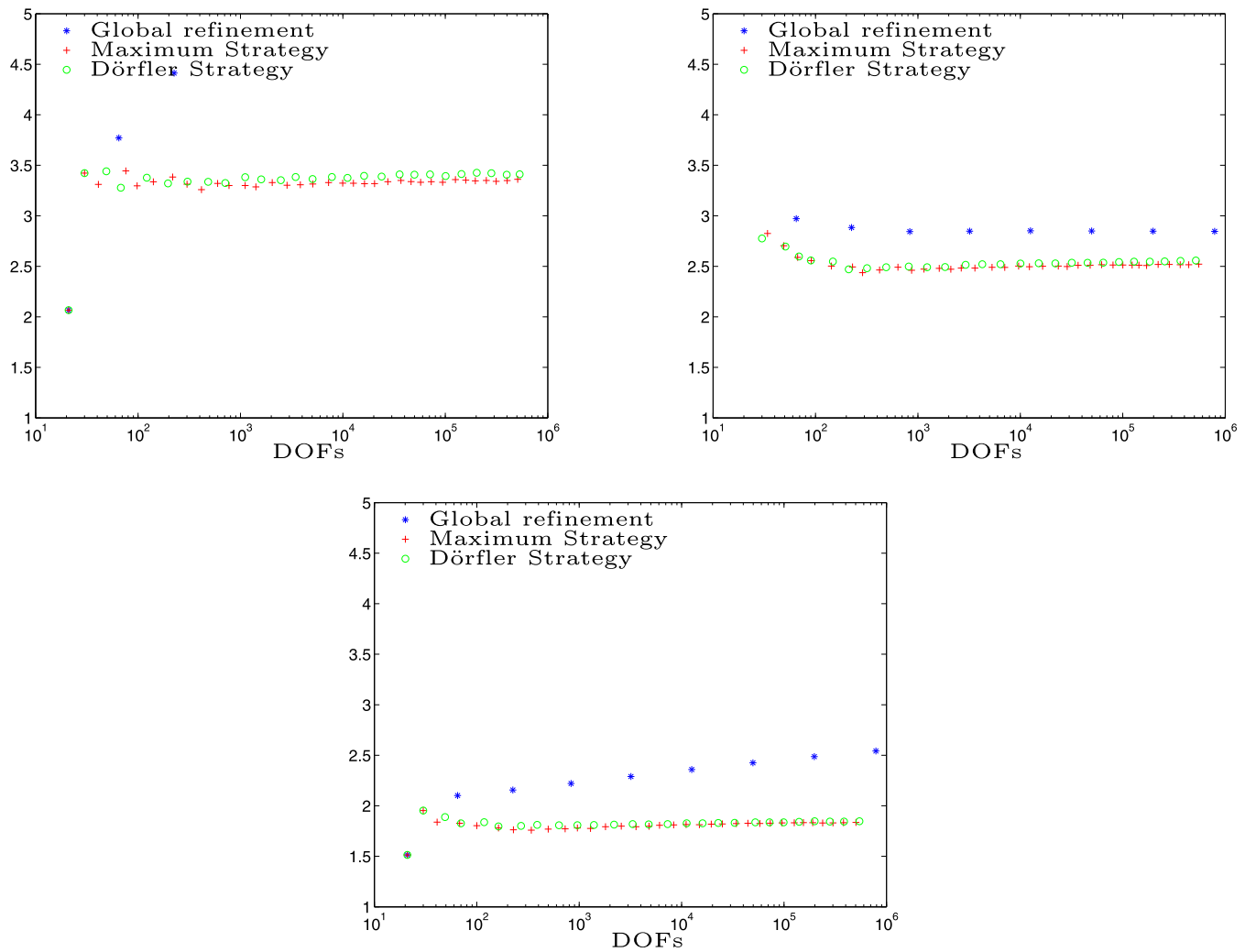


Fig. 11. Reliability of global indicator $\eta_k(\mathcal{T}_k)$. We present plots of the quotient $\|\nabla(u - u_k)\|_\Omega / \eta_k(\mathcal{T}_k)$ versus DOFs for Example 1 (top left), 3 (top right) and 4 (bottom), for polynomial degree $\ell = 1$. Even though the global indicator $\eta_k(\mathcal{T}_k)$ is not an upper bound for the error, it behaves as such when using selective refinement strategies (Maximum or Dörfler), but it seems that it underestimates the error when uniform global refinement is performed in each step of the algorithm. The behavior for higher polynomial degrees $\ell = 2, 3, 4$ is similar.

$$\alpha(t) = -\frac{1}{2}e^{-\frac{3}{2}t} + 1, \quad t > 0,$$

which satisfies the hypothesis (28) related to the well-posedness of the problem but not (29), because α is monotone increasing (see Fig. 7).

In Fig. 8 we plot $H^1(\Omega)$ -error versus the number of degrees of freedom, for finite elements of degrees $\ell = 1, 2, 3, 4$. Note that in this case the optimal convergence rate $\text{DOFs}^{-\ell/2}$ is still observed for the adaptive algorithm. This is an indication that the assumption (29) about α being monotone decreasing can be superfluous, and only an artificial requirement for the presented proof (see Lemma 2). A more detailed study about this hypothesis is subject of future research, and beyond the scope of this article.

Example 4. Finally, we consider an extreme case, with

$$\alpha(t) = 2 - \frac{\sqrt{t}}{1 + \sqrt{t}}, \quad t > 0,$$

which satisfies (28) and (29), but $\lim_{t \rightarrow 0^+} \alpha'(t) = -\infty$, as can be observed in Fig. 9. This means that α is not Lipschitz continuous. Since we only require that $|t\alpha'(t)|$ is bounded (cf. (28)), it still satisfies the assumptions of the convergence theory, and optimality is observed regardless of the fact that $\lim_{t \rightarrow 0^+} \alpha'(t) = -\infty$.

Fig. 10 shows $H^1(\Omega)$ -error versus DOFs, for polynomial degrees $\ell = 1, 2, 3, 4$. We obtain optimal convergence rate $\text{DOFs}^{-\ell/2}$ for the adaptive strategies. Thus, even when α is not Lipschitz, the convergence rate is optimal. As a consequence we conclude that it should not be necessary to make additional assumptions in order to prove optimality.

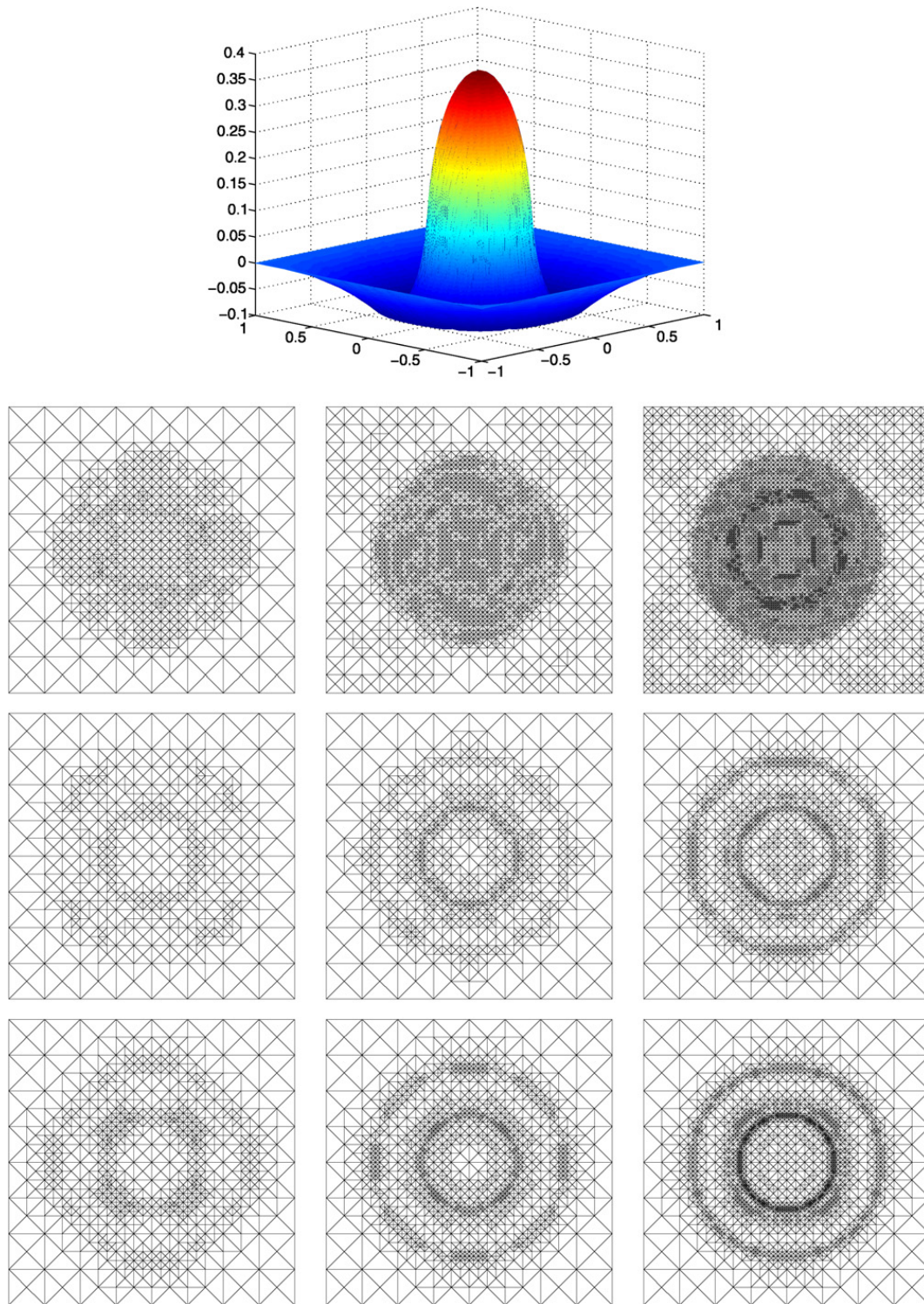


Fig. 12. Adaptive meshes obtained when solving the prescribed mean curvature equation (30) with polynomials of degree 1 (top), 2 (middle), 3 (bottom). The meshes correspond to iteration count 10 (left), 15 (middle) and 20 (right). It is worth observing the higher grading presented by the meshes corresponding to higher polynomial degree.

Error versus Indicator. As we pointed out in Remark 4 the residual \mathbf{R}_k is not an upper bound for the error. Nevertheless, we tested the practical behavior of $\eta_k(\mathcal{T}_k)$ as an error indicator in the Examples 1, 3 and 4, when the solution converges to the known solution (27). Our findings for polynomial degree $\ell = 1$ are reported in Fig. 11, the behavior is similar for degrees $\ell = 2, 3, 4$. We observe that when adaptive refinement is used in each step of the algorithm, using for example the maximum strategy or Dörfler's strategy, the ratio $\|\nabla(u - u_k)\|_{\Omega} / \eta_k(\mathcal{T}_k)$ remains bounded above and below, indicating that it may be reasonable to use $\eta_k(\mathcal{T}_k)$ in a stopping criterion for the algorithm. However, the indicator $\eta_k(\mathcal{T}_k)$ underestimates the error $\|\nabla(u - u_k)\|_{\Omega}$ when the global refinement strategy is used in Examples 1 and 4. A reliable error indicator for the stopping criterion is $\eta_k(\mathcal{T}_k) + \|\nabla(u_k - u_{k-1})\|_{\Omega}$ which is a computable upper bound for the error.

5.2. Unknown solution

In this section we use the Adaptive Algorithm to approximate a solution to a prescribed mean curvature problem. We consider the problem

$$\begin{cases} -\nabla \cdot \left[\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right] = f & \text{in } \Omega = (-1, 1) \times (-1, 1), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (30)$$

with

$$f(x) = \begin{cases} 5 & \text{if } |x| \leq 1/3, \\ -3 & \text{if } 1/3 < |x| \leq 2/3, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\alpha(t) = 1/\sqrt{1+t}$ corresponding to this equation does not fulfill (28) because $\alpha(t), \alpha'(t) \rightarrow 0$ as $t \rightarrow \infty$. Nevertheless, for many right-hand side functions f , as the one stated above, the solution belongs to $W^{1,\infty}(\Omega)$. This implies that $|\nabla u|$ is bounded and α could be replaced by a function satisfying (28) without changing the solution. This is not needed in practice, but is rather a theoretical tool for proving that the assumptions hold in some sense.

We experimented with several right-hand sides f and observed that the method performs robustly whenever $u \in W^{1,\infty}(\Omega)$. When the solution has an unbounded gradient, the method produces a sequence with a maximum value increasing to infinity. This is a drawback of our method, since it cannot be used to approximate singular solutions (not belonging to $W^{1,\infty}(\Omega)$), if α does not satisfy (28).

We present in Fig. 12 a picture of the solution obtained with our method and several meshes at different iteration steps, for polynomial degrees 1, 2 and 3. It is worth observing the stronger grading obtained for higher polynomial degrees. Since the solution belongs to $H^2(\Omega)$ a sequence of quasi-uniform meshes would be optimal for linear finite elements [7], although the adaptive method refines a bit more where the curvature of the solution is higher. When using polynomial degree $\ell = 2, 3$ adaptivity plays a more essential role, since the solution being approximated does not belong to $H^3(\Omega)$, whence quasi-uniform meshes would not be optimal. The adaptive method for polynomial degrees $\ell = 2, 3$ concentrates the refinement around the discontinuities of the right-hand side function f .

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