Perfectness of clustered graphs

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Abstract

Given a clustered graph $(G, \mathcal{V})$, that is, a graph $G = (V, E)$ together with a partition $\mathcal{V}$ of its vertex set, the selective coloring problem consists in choosing one vertex per cluster such that the chromatic number of the subgraph induced by the chosen vertices is minimum. This problem can be formulated as a covering problem with a 0-1 matrix $M(G, \mathcal{V})$. Nevertheless, we observe that, given $(G, \mathcal{V})$, it is NP-hard to check if $M(G, \mathcal{V})$ is conformal (resp. perfect). We will give a sufficient condition, checkable in polynomial time, for $M(G, \mathcal{V})$ to be conformal that becomes also necessary if conformality is required to be hereditary. Finally, we show that $M(G, \mathcal{V})$ is perfect for every partition $\mathcal{V}$ if and only if G belongs to a superclass of threshold graphs defined with a complex function instead of a real one.

Keywords: Selective coloring; partition coloring; conformal matrix; perfect matrix; threshold graph.

1 Introduction

All graphs in this paper are finite, simple and loopless. Throughout the paper, we will consider a graph $G = (V, E)$ and a partition $\mathcal{V} = \{V_1, V_2, \ldots, V_p\}$ of its vertex set into nonempty subsets. We will denote by $(G, \mathcal{V})$ the graph G together with a partition $\mathcal{V}$ of its vertex set and call it a *clustered graph*. The sets $V_1, \ldots, V_p$ are called *clusters* and $\mathcal{V}$ is called a *clustering of G*.

Let $V' \subseteq V$. We denote by $G[V']$ the graph induced by V', i.e., the graph obtained from G by deleting the vertices of $V - V'$ and all edges incident to at least one vertex of $V - V'$. Two sets $A, B \subseteq V$ are said to be *complete* (resp. *anticomplete*) to each other if every vertex in A is adjacent (resp. *non-adjacent*) to every vertex in B. A *clique* in a graph $G = (V, E)$ is a set of pairwise

adjacent vertices. The maximum size of a clique in a graph G is called the *clique number* of G and is denoted by $\omega(G)$. A *k-coloring* of G is a mapping $c : V \to \{1, \dots, k\}$ such that $c(u) \neq c(v)$ for all $uv \in E$. The smallest integer k such that G is k-colorable is called the *chromatic number* of G and is denoted by $\chi(G)$.

A *selective k-coloring* of $(G, \mathcal{V})$ is a mapping $c : V' \to \{1, \dots, k\}$, for some $V' \subseteq V$ with $|V' \cap V_i| = 1$ for all $i \in \{1, \dots, p\}$, such that $c(u) \neq c(v)$ for every edge uv of $G[V']$. The smallest integer k for which a graph G admits a selective k-coloring with respect to $\mathcal{V}$ is called the *selective chromatic number* of G and is denoted by $\chi_{sel}(G, \mathcal{V})$. It is obvious to see that $\chi_{sel}(G, \mathcal{V}) \leq \chi(G)$ for every clustering $\mathcal{V}$ of G.

The *selective coloring problem*, also called partition coloring in the literature, consists in determining the selective chromatic number of a given clustered graph $(G, \mathcal{V})$. It was introduced in 2000 for its application in network design [9]. Since then, several heuristics and exact approaches have been designed to solve it (see for instance [11, 8]). Many other interesting applications of the selective graph coloring problem related for instance to timetabling and dichotomy-based constraint encoding can be found in [5]. Recently, the selective coloring problem was shown to be NP-hard in paths, cycles and split graphs even with strong restrictions on the cardinality of the clusters [6].

A graph $G = (V, E)$ is *perfect* if, for every induced subgraph H of G, we have the min-max relation $\omega(H) = \chi(H)$. The characterization of perfect graphs together with polynomial time algorithms to recognize them and to find their clique and chromatic numbers form the *perfect graph theory*, see e.g. Schrijver's book [13].

In this paper, we will study the min-max relation associated with the selective coloring problem. The *auxiliary graph $G/\mathcal{V}$ of* $(G, \mathcal{V})$ is the graph where each vertex v_i in $G/\mathcal{V}$ corresponds to a cluster V_i in $\mathcal{V}$, for $i = 1, \dots, p$, and where two vertices in $G/\mathcal{V}$ are adjacent if and only if the corresponding clusters are complete to each other. First we will observe that the following two inequalities hold:

$$\omega(G/\mathcal{V}) \leq \chi(G/\mathcal{V}) \leq \chi_{sel}(G, \mathcal{V}) \tag{1}$$

Then we will study the clustered graphs for which one or both inequalities in (1) hold with equality. After showing that it is NP-hard to decide if a clustered graph satisfies the equality $\chi(G/\mathcal{V}) = \chi_{sel}(G, \mathcal{V})$ for all subclustering $\mathcal{V}' \subseteq \mathcal{V}$, we give a sufficient condition that can be checked in polynomial time for a clustered graph to have this property. This condition becomes necessary if the previous property is required to hold for all subclusterings. The graph G is *selective-perfect* if $\omega(G/\mathcal{V}) = \chi_{sel}(G, \mathcal{V})$ for every clustering $\mathcal{V}$. We identify the class of selective-perfect graphs as being a class that we call *i-threshold graphs* since it is a super-class of threshold graphs defined with complex numbers in $\mathbb{R} \cup \{-i, +i\}$ instead of real ones only. It strictly contains the class of threshold graphs and the class of complete bipartite graphs.

The paper is organized as follows. In Section 2, we give some more notation, definitions and preliminary results. Section 3 deals with conformality of clustered-graphs. In Section 4 we analyze selective-perfect graphs.

2 Preliminaries

For a vertex $v \in V$, let $N(v)$ denote the set of vertices in G that are adjacent to v, i.e., the *neighbors* of v and let $N[v] = N(v) \cup \{v\}$. A *stable set* in a graph $G = (V, E)$ is a set $S \subseteq V$ of pairwise non-adjacent vertices. As usual P_n (resp. C_n) denotes the path (resp. the cycle) induced by n vertices. A clique on n vertices will be denoted by K_n. The complement of a graph G is denoted by $\overline{G}$. Let $\mathcal{F}$ be a set of graphs, then G is $\mathcal{F}$-free if no induced subgraph of G is isomorphic to a graph in $\mathcal{F}$. For all graph-theoretical terms not defined here, the reader is referred to [14].

Let A be a set. The *characteristic vector* $x^B \in \{0, 1\}^A$ *of* $B \subseteq A$ has a component

$$x_a^B := \begin{cases} 1 & \text{if } a \in B, \\ 0 & \text{otherwise} \end{cases}$$

for each $a \in A$. The *stable set–vertex matrix* $M(G)$ *of* $G = (V, E)$ is the matrix the rows of which are the transpose $(x^S)^{\mathsf{T}}$ of the characteristic vectors $x^S \in \{0, 1\}^V$ of all maximal stable sets S of G. Note that, for any stable set S and any clique K, since $|S \cap K| \leq 1$, we have $(x^S)^{\mathsf{T}} x^K \leq 1$. Actually a vector $x \in \{0, 1\}^V$ is the characteristic vector of a clique if and only if $(x^S)^{\mathsf{T}} x \leq 1$ for all stable sets S with $|S| = 2$. The *clique polytope of* G is the convex-hull of the vectors x^K for all cliques K of G. So the polyhedron $\{x \in \mathbb{R}^V : (x^S)^{\mathsf{T}} x \leq 1$ for all stable sets S of size 2, $x \geq \mathbf{0}\}$ contains the polyhedron $\{x \in \mathbb{R}^V : M(G)x \leq \mathbf{1}, x \geq \mathbf{0}\}$ which contains the clique polytope of G.

Let M be a 0-1 matrix. The *matrix M is perfect* if it is the stable set–vertex matrix of a perfect graph. The matrix M is *conformal* if it is the stable set–vertex matrix of some graph. For instance, the following matrix is not conformal:

$$J_3 - I_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Indeed, it should be the stable set–vertex matrix of a graph on three vertices such that every pair of vertices forms a stable set but the three vertices do not form a stable set; but this is clearly impossible.

Let M be an $m \times n$ matrix. A row i is said to be *dominated* if there exists a row k such that $M_{ij} \leq M_{kj}$ for all $1 \leq j \leq n$, with $1 \leq i \neq k \leq m$.

Theorem 2.1 (Lovász [10]) *For a 0-1 matrix M without dominated rows, the following statements are equivalent:*

  (i) M is perfect;

  (ii) $\max\{w^{\mathsf{T}} x : Mx \leq \mathbf{1}, x \geq \mathbf{0}, x \text{ integer}\} = \max\{w^{\mathsf{T}} x : Mx \leq \mathbf{1}, x \geq \mathbf{0}\}$ *for any 0-1 vector w;*

  (iii) $\max\{w^{\mathsf{T}} x : Mx \leq \mathbf{1}, x \geq \mathbf{0}, x \text{ integer}\} = \min\{y^{\mathsf{T}} \mathbf{1} : y^{\mathsf{T}} M \geq w^{\mathsf{T}}, y \geq \mathbf{0}, y \text{ integer}\}$ *for any integer vector w.*

A corollary of this theorem is the polyhedral characterization of perfect graphs.

Corollary 2.2 (Chvátal [2]) *A graph G is perfect if and only if its clique polytope is equal to $\{x \in \mathbb{R}^V : M(G)x \leq \mathbf{1}, x \geq \mathbf{0}\}$.*

One can recognize if a given matrix M is the stable set–vertex matrix of some graph, i.e., if M is conformal, in a time polynomial in the size of the matrix (see e.g. [4]). By Theorem 2.3 below, we can decide if the stable set–vertex matrix of a given graph is perfect or not in time polynomial in the size of the graph.

Theorem 2.3 (Chudnovsky et al. [1]) *The problem of deciding whether a graph is perfect or not can be solved in polynomial time.*

Threshold graphs are a subclass of perfect graphs and are defined as follows: a graph $G = (V, E)$ is a *threshold graph* if there exists a function $w : V \to \mathbb{R}$ such that $uv \in E \Leftrightarrow w(u) + w(v) > 0$. Threshold graphs are split graphs, i.e., the vertex set can be partitioned into a stable set S and a clique K as we can take $S := \{v : w(v) \leq 0\}$ and $K := \{v : w(v) > 0\}$). Furthermore, if we order the vertices of a threshold graph G as $v_1, \ldots, v_n$ such that $w(v_1) \leq \ldots \leq w(v_n)$, then, first, $N(v_i) \subseteq N(v_{i+1})$, and moreover, the permutation π given by ordering the absolute values $|w(v_i)|$ is such that $v_i v_j \in E \Leftrightarrow (i - j)(\pi(i) - \pi(j)) > 0$. See [12] for more details on threshold graphs. Chvátal and Hammer [3] showed that a graph is threshold if and only if neither G nor $\overline{G}$ contains P_4 or C_4 as an induced subgraph, which is equivalent to being $\{2K_2, P_4, C_4\}$-free.

3 Conformality of cluster matrices

Let $(G, \mathcal{V})$ be a clustered graph with clustering $\mathcal{V} = \{V_1, \ldots, V_p\}$. A *stable subclustering* of $(G, \mathcal{V})$ is a subset $\mathcal{V}' \subseteq \mathcal{V}$ of clusters V_i such that $S \cap V_i \neq \emptyset$ for some stable set S of G. The stable set S is said to be a stable set *corresponding to* the stable subclustering $\mathcal{V}'$. A set $\{\mathcal{V}'_1, \ldots, \mathcal{V}'_k\}$ of stable subclusterings *covers* $\mathcal{V}$ if $\mathcal{V} = \mathcal{V}'_1 \cup \ldots \cup \mathcal{V}'_k$.

Remark 3.1 *$(G, \mathcal{V})$ admits a selective k-coloring if and only if there exists a set of k stable subclusterings which covers $\mathcal{V}$.*

Remark 3.2 *Every stable subclustering $\mathcal{V}' \subseteq \mathcal{V}$ of $(G, \mathcal{V})$ corresponds to a stable set $S_{\mathcal{V}'}$ of $G/\mathcal{V}$. Moreover, every stable set $S_{\mathcal{V}'}$ of $G/\mathcal{V}$ of size 2 corresponds to a (not necessarily maximal) stable subclustering $\mathcal{V}'$ of $(G, \mathcal{V})$.*

We will note later that a stable set $S_{\mathcal{V}'}$ of $G/\mathcal{V}$ of size at least 3 does not necessarily correspond to a stable subclustering of $(G, \mathcal{V})$ (see the configurations in Figure 2).

By Remarks 3.1 and 3.2, it follows that $\chi(G/\mathcal{V}) \leq \chi_{sel}(G, \mathcal{V})$. This inequality also follows from the fact that $G/\mathcal{V}$ is a partial subgraph of $G[V']$ for any $V' \subseteq V$ with $|V' \cap V_i| = 1$ for all $i \in \{1, \ldots, p\}$.

Furthermore, since we trivially have $\omega(G/\mathcal{V}) \leq \chi(G/\mathcal{V})$, we conclude that the inequalities (1) hold. In what follows, we will show the linear programming duality between both problems of determining $\omega(G/\mathcal{V})$ and $\chi_{sel}(G, \mathcal{V})$. We would like to mention that the LP relaxation of [8] strengthens that of our integer formulation for the selective coloring problem.

The stable subclustering-cluster matrix $M(G, \mathcal{V})$ of $(G, \mathcal{V})$, called *cluster matrix* for short, is the matrix the rows of which are the transpose $(x^{\mathcal{V}'})^{\mathsf{T}}$ of the characteristic vectors $x^{\mathcal{V}'} \in \{0, 1\}^{\mathcal{V}}$ of all maximal stable subclusterings $\mathcal{V}' \subseteq \mathcal{V}$ of $(G, \mathcal{V})$. Let $\mathcal{S}$ be the set of maximal stable subclusterings of $(G, \mathcal{V})$. So, each column of $M(G, \mathcal{V})$ is the characteristic vector $y^{\mathcal{S}_i} \in \{0, 1\}^{\mathcal{S}}$ of the set $\mathcal{S}_i \subseteq \mathcal{S}$ of all stable subclusterings containing some cluster V_i.
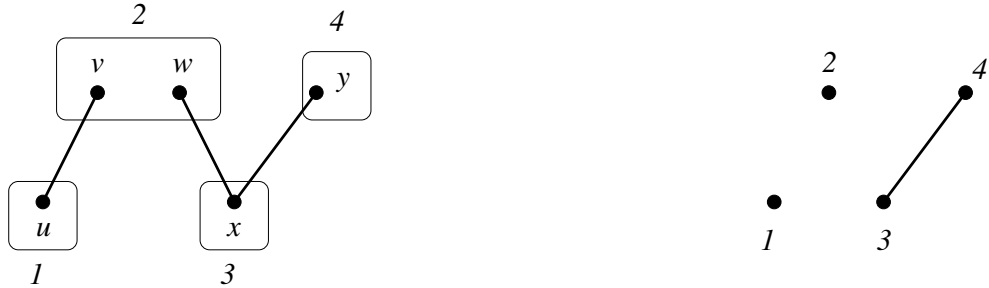

Figure 1: A clustered graph $(G, \mathcal{V})$, on the left, and its auxiliary graph $G/\mathcal{V}$, on the right.

Let us take for instance the clustered graph of Figure 1. The maximal stable subclusterings are those corresponding to the stable sets $\{u, w, y\}$, $\{u, x\}$ and $\{v, x\}$, hence:

$$M(G, \mathcal{V}) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \text{ and } M(G/\mathcal{V}) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

(Recall that $M(G/\mathcal{V})$ is the stable set–vertex matrix of the auxiliary graph $G/\mathcal{V}$). The stable subclustering corresponding to some maximal stable set of G is not necessarily itself maximal, as shown by the stable set $\{v, w, y\}$. Notice that every 0-1 matrix M is the cluster matrix of some clustered graph. Indeed, to build the clustered graph of a given matrix M, start with an empty clustered graph with as many empty clusters as columns of M; then, for each row of M, create a new stable set S with one vertex in each cluster corresponding to a 1 in the row and make S complete to $V \setminus S$.

Clearly, $M(G, \mathcal{V}) = M(G/\mathcal{V})$ if and only if every stable set S in $G/\mathcal{V}$ corresponds to a stable subclustering of $(G, \mathcal{V})$. The matrices $M(G, \mathcal{V})$ and $M(G/\mathcal{V})$ have one column per cluster respectively per vertex. Let $C \subseteq \mathcal{V}$ be a subset of clusters and observe that, by Remark 3.2, the following three propositions are equivalent: (i) $\{v_i : V_i \in C\}$ is a clique of $G/\mathcal{V}$; (ii) The submatrix of $M(G/\mathcal{V})$ induced by the columns in C has at most one 1 per row; (iii) The submatrix of $M(G, \mathcal{V})$ induced by the columns in C has at most one 1 per row. It follows that C is a clique of $G/\mathcal{V}$ if and only if $(x^{\mathcal{V}'})^{\mathsf{T}} x^C \leq 1$ for all stable subclusterings $\mathcal{V}'$, where $x^C \in \{0, 1\}^{\mathcal{V}}$ is the characteristic vector of

$C \subseteq \mathcal{V}$. In other words, both matrices $M(G, \mathcal{V})$ and $M(G/\mathcal{V})$ formulate the problem of determining the clique number of $G/\mathcal{V}$ since:

$$\omega(G/\mathcal{V}) = \max\{\mathbf{1}^\mathsf{T}x : x \in P_1, x \text{ integer}\} = \max\{\mathbf{1}^\mathsf{T}x : x \in P_2, x \text{ integer}\}$$

with
$$P_1 := \{x \in \mathbb{R}^\mathcal{V} : M(G, \mathcal{V})x \le \mathbf{1}, x \ge \mathbf{0}\} \text{ and } P_2 := \{x \in \mathbb{R}^\mathcal{V} : M(G/\mathcal{V})x \le \mathbf{1}, x \ge \mathbf{0}\}.$$

Now let $R \subseteq \mathcal{S}$ be a subset of stable subclusterings of $M(G, \mathcal{V})$ and observe that: R covers $\mathcal{V}$ if and only if the submatrix of $M(G, \mathcal{V})$ induced by the rows corresponding to the stable subclusterings of R has at least one 1 per column. It follows that R covers $\mathcal{V}$ if and only if $(y^R)^\mathsf{T}y^{\mathcal{S}_i} \ge 1$ for every cluster V_i, where $y^R \in \{0, 1\}^\mathcal{S}$ is the characteristic vector of $R \subseteq \mathcal{S}$. Hence by Remark 3.1, the matrix $M(G, \mathcal{V})$ formulates the problem of determining the selective chromatic number of $(G, \mathcal{V})$ since:

$$\chi_{sel}(G, \mathcal{V}) = \min\{y^\mathsf{T}\mathbf{1} : y \in D_1, y \text{ integer}\}$$

with
$$D_1 := \{y \in \mathbb{R}^\mathcal{S} : y^\mathsf{T}M(G, \mathcal{V}) \ge \mathbf{1}^\mathsf{T}, y \ge \mathbf{0}\}.$$

Now it immediately follows from linear programming duality that $\omega(G/\mathcal{V}) \le \chi_{sel}(G, \mathcal{V})$.

Our main purpose in this paper is to find good characterizations for having inequalities in (1) holding with equality. Given a subset $\mathcal{V}' \subseteq \mathcal{V}$ of clusters, we let $(G, \mathcal{V}')$ denote the clustered graph obtained from $(G, \mathcal{V})$ by deleting all vertices in V_i for all $V_i \notin \mathcal{V}'$; also, $G/\mathcal{V}'$ is the subgraph of $G/\mathcal{V}$ induced by the vertices v_i such that $V_i \in \mathcal{V}'$. By definition, $\omega(G/\mathcal{V}') = \chi(G/\mathcal{V}')$ for all $\mathcal{V}' \subseteq \mathcal{V}$ if and only if the matrix $M(G/\mathcal{V})$ (resp. the graph $G/\mathcal{V}$) is perfect.

Moreover, one can see from the integer linear programming formulation of the selective coloring problem given above and Theorem 2.1 (by setting $M := M(G, \mathcal{V})$ and $w := y^{\mathcal{V}'}$ the characteristic vector in $\{0, 1\}^\mathcal{V}$ of the subset $\mathcal{V}' \subseteq \mathcal{V}$ of clusters) that $\omega(G/\mathcal{V}') = \chi_{sel}(G, \mathcal{V}')$ for all $\mathcal{V}' \subseteq \mathcal{V}$ if and only if the matrix $M(G, \mathcal{V})$ is perfect.

One can check if a graph or a matrix is perfect, or if a matrix is conformal but only in time polynomial in the size of the input. An interesting question is to know whether one can check these properties for the matrix $M(G, \mathcal{V})$ when the input is the clustered graph $(G, \mathcal{V})$ and not the matrix itself. Let us first investigate how to characterize conformality for $M(G, \mathcal{V})$.

Lemma 3.1 *Let $G = (V, E)$ be a graph and let $\mathcal{V}$ be a clustering of V. Then, the following statements are equivalent:*

(i) *$M(G, \mathcal{V})$ is conformal;*

(ii) *$M(G, \mathcal{V}) = M(G/\mathcal{V})$;*

(iii) *$\chi(G/\mathcal{V}') = \chi_{sel}(G, \mathcal{V}')$ for all $\mathcal{V}' \subseteq \mathcal{V}$.*

Proof: (i) $\Rightarrow$ (ii): If $M(G, \mathcal{V})$ is the stable-set matrix of a graph, then, by Remark 3.2, this graph must be the auxiliary graph $G/\mathcal{V}$.

(ii) $\Rightarrow$ (i): This immediately follows from the definitions of conformality and of the auxiliary graph $G/\mathcal{V}$.

(ii) $\Rightarrow$ (iii): If (iii) does not hold, there exists $\mathcal{V}' \subseteq \mathcal{V}$ such that $\chi_{sel}(G, \mathcal{V}') > \chi(G/\mathcal{V}')$. Hence, by the observation above, there is $\mathcal{V}'' \subseteq \mathcal{V}'$ which corresponds to a stable set of $G/\mathcal{V}'$ but is not a stable subclustering of $(G, \mathcal{V}')$. Then $\mathcal{V}''$ is contained is some maximal stable set of $G/\mathcal{V}$ but it is contained in no maximal stable subclustering of $(G, \mathcal{V})$. Hence $M(G, \mathcal{V}) \neq M(G/\mathcal{V})$, contradicting (ii).

(iii) $\Rightarrow$ (ii): If $M(G, \mathcal{V}) \neq M(G/\mathcal{V})$, there exists a stable set S in $G/\mathcal{V}$ which does not correspond to a stable subclustering in $(G, \mathcal{V})$. Thus $\chi(G/\mathcal{V}') = 1$ and $\chi_{sel}(G, \mathcal{V}') > 1$, contradicting (iii). $\square$

It turns out in the following that recognizing neither perfectness nor conformality of $M(G, \mathcal{V})$ having $(G, \mathcal{V})$ as input is easy.

SELECTIVE 1-COLORING
Input: A clustered graph $(G, \mathcal{V})$.
Question: Do we have $\chi_{sel}(G, \mathcal{V}) \leq 1$?

It is shown in [6] that SELECTIVE 1-COLORING is NP-complete.

SELECTIVE-COLORING-PERFECT
Input: A clustered graph $(G, \mathcal{V})$.
Question: Is the matrix $M(G, \mathcal{V})$ perfect?

Corollary 3.1 SELECTIVE-COLORING-PERFECT *is NP-hard.*

Proof: First note that $\chi_{sel}(G, \mathcal{V}) = 1$ if and only if $\omega(G/\mathcal{V}') = \chi_{sel}(G, \mathcal{V}') = 1$ for any (nonempty) subclustering $\mathcal{V}' \subseteq \mathcal{V}$. It follows that $\chi_{sel}(G, \mathcal{V}) \leq 1$ if and only if the graph $G/\mathcal{V}$ contains no edge (which can be checked in polynomial time) and the matrix $M(G, \mathcal{V})$ is perfect. $\square$

Now we conclude from Corollary 3.1 that, given a clustered graph $(G, \mathcal{V})$, deciding whether $M(G, \mathcal{V})$ is conformal is NP-hard.

SELECTIVE-COLORING-CONFORMAL
Input: A clustered graph $(G, \mathcal{V})$.
Question: Is the matrix $M(G, \mathcal{V})$ conformal?

Corollary 3.2 SELECTIVE-COLORING-CONFORMAL *is NP-hard.*

Proof: It follows from Lemma 3.1, that $M(G, \mathcal{V})$ is a perfect matrix if and only if $M(G, \mathcal{V})$ is conformal and $G/\mathcal{V}$ is perfect. On one hand, it is hard to check if $M(G, \mathcal{V})$ is perfect, and on the other hand, by Theorem 2.3, it is easy to check if $G/\mathcal{V}$ is perfect. We conclude that checking if $M(G, \mathcal{V})$ is conformal is NP-hard. $\square$

Thus it seems unlikely that a good characterization for perfectness or conformality of the cluster matrix $M(G, \mathcal{V})$ is available. We will establish in the remaining of this section a sufficient condition, based on excluded configurations, for $M(G, \mathcal{V})$ to be conformal. Moreover we will show that this condition becomes also necessary if we require the equality $M(G, \mathcal{V}) = M(G/\mathcal{V})$ not only to hold for $(G, \mathcal{V})$ but also for all induced subgraphs of G as well with the clustering induced by $\mathcal{V}$.

If H is an induced subgraph of G with clustering $\mathcal{V}$, we let $\mathcal{V}_H$ be the clustering of H induced by the clustering $\mathcal{V}$ restricted to the vertex set $V(H)$ of H. Notice that the conformality (resp. perfectness) of the matrix $M(G, \mathcal{V})$ does not imply that the matrix $M(H, \mathcal{V}_H)$ is conformal (resp. perfect). To see this, consider again the clustered graph of Figure 1: if we add a vertex x' in V_3 (with no edge incident to it) then $\mathcal{V}$ is the stable subclustering corresponding to the stable set $\{u, w, x', y\}$ and hence $M(G, \mathcal{V}) = M(G/\mathcal{V})$ is conformal and perfect.
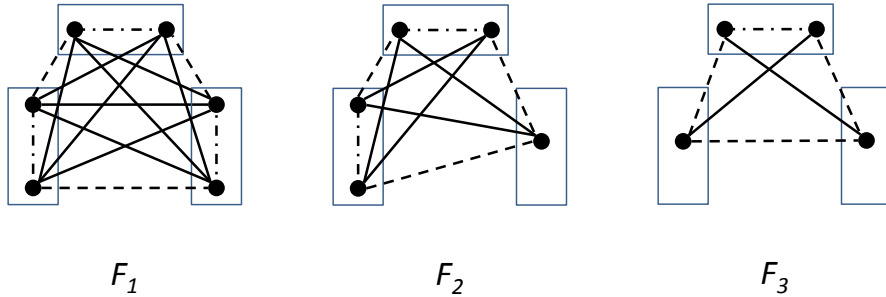

Figure 2: Forbidden configurations F_1, F_2 and F_3.

Now consider Figure 2. In each of the configurations F_1, F_2, F_3, dashed lines represent non-edges, lines represent edges and dashed-dotted lines represent possible edges, i.e., the two corresponding vertices may or may not be adjacent. The rectangles represent the clusters of a clustering. We say that the *clustered graph $(G, \mathcal{V})$ contains a configuration F_i* if $(H, \mathcal{V}_H) = F_i$ for some induced subgraph H of G, $i \in \{1, 2, 3\}$. Notice that in this case, $M(H, \mathcal{V}_H)$ is isomorphic to $J_3 - I_3$.

Theorem 3.3 *If the clustered graph $(G, \mathcal{V})$ does not contain any of the configurations F_1, F_2, F_3, then $M(G, \mathcal{V}) = M(G/\mathcal{V})$.*

Proof: Suppose that the result does not hold. Let $(G, \mathcal{V})$ be a minimal counterexample with $\mathcal{V} = \{V_1, \ldots, V_p\}$, i.e., G does not contain any of the configurations F_1, F_2, F_3 and $M(G-v, \mathcal{V}_{G-v}) =$

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$M(G - v/\mathcal{V}_{G-v})$ for every vertex v of G but $M(G, \mathcal{V}) \neq M(G/\mathcal{V})$. It follows that there exists a stable set S of $G/\mathcal{V}$ which does not correspond to a stable subclustering of $(G, \mathcal{V})$. So we may assume that $G/\mathcal{V}$ contains no edge, since we may remove all the vertices of G not belonging to any cluster of the stable subclustering corresponding to S. Hence we may assume that $(G, \mathcal{V})$ does not contain any stable subclustering of size p.

By Remark 3.2, since $M(G, \mathcal{V}) \neq M(G/\mathcal{V})$, it follows that $\mathcal{V}$ has at least three different clusters. Suppose first that G has exactly three clusters, namely V_1, V_2 and V_3. Then there is a non-edge between V_i and V_j, for each pair i, j with $i \neq j$, and no stable set of size three intersecting V_1, V_2 and V_3. Since G is minimal, there are exactly three non-edges between the three clusters. But they span either six, five or four vertices, and each case induces either configuration F_1, F_2 or F_3, a contradiction.

Thus we may assume now that G has p clusters, namely $V_1, \ldots, V_p$, with $p > 3$.

Claim 1. $|V_i| \geq 2$ for every $i \in \{1, \ldots, p\}$.

Suppose on the contrary and without loss of generality that $V_1 = \{v\}$. Since $G/\mathcal{V}$ contains no edge, it follows that $V_i \setminus N(v)$ is nonempty for every $i \in \{2, \ldots, p\}$. Since the counterexample is minimal, for each subset I of $\{2, \ldots, p\}$ with $|I| = p - 2$, there is a stable subclustering of $(G, \mathcal{V})$ containing V_1 and each of the clusters V_i with $i \in I$. In particular, since $V_1 = \{v\}$, all the vertices of the stable sets corresponding to this stable subclustering are in $V \setminus N(v)$. Since $p > 3$, for each pair of distinct indices i, j in $\{2, \ldots, p\}$, there is a non-edge between $V_i \setminus N(v)$ and $V_j \setminus N(v)$. Finally, there is no stable set S of $G - v$ intersecting $V_i \setminus N(v)$ for every $i \in \{2, \ldots, p\}$, otherwise $S \cup \{v\}$ would be a stable set of G intersecting each of the clusters $V_1, \ldots, V_p$. On the other hand, there is a stable subclustering intersecting $V_2, \ldots, V_p$ by the minimality of G, so $N[v]$ is non-empty. Hence, the subgraph of G obtained by deleting $N[v]$ is a counterexample, a contradiction to the minimality of G. $\diamondsuit$

Claim 2. No vertex in V_i is complete to V_j, for $i \neq j$.

Suppose on the contrary and without loss of generality that $v \in V_1$ is complete to V_p. By Claim 1, $V_1 \setminus \{v\}$ is non-empty. Since V_1 is not complete to V_p (recall that $G/\mathcal{V}$ contains no edge), it follows that $V_1 \setminus \{v\}$ is not complete to V_p. Consider $j \in \{2, \ldots, p-1\}$. Since $p > 3$ and by minimality of G, there is a stable subclustering of G containing V_1, V_j and V_p. Then any stable set S corresponding to this stable subclustering intersects in fact $V_1 \setminus \{v\}$, V_j and V_p, since v is complete to V_p. So, the auxiliary graph $G - v/\mathcal{V}_{G-v}$ has no edge. Since there is no stable subclustering of $(G, \mathcal{V})$ of size p, there is no stable subclustering of $(G - v, \mathcal{V}_{G-v})$ of size p neither. Hence, $G - v$ is a counterexample, a contradiction to the minimality of G. $\diamondsuit$

Now, let v be a vertex in V_1. By Claim 1, $V_1 \setminus \{v\}$ is non-empty, so the auxiliary graph $G - v/\mathcal{V}_{G-v}$ has p vertices. Moreover, by Claim 2, it contains no edge. Since there is no stable subclustering of $(G, \mathcal{V})$ of size p, there is no stable subclustering of $(G - v, \mathcal{V}_{G-v})$ of size p neither. So, G would not be a minimal counterexample, a contradiction. $\square$

Corollary 3.4 $\chi(H/\mathcal{V}_H) = \chi_{sel}(H, \mathcal{V}_H)$, for every induced subgraph H of G, if and only if $(G, \mathcal{V})$

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does not contain any of the configurations F_1, F_2, F_3.

Proof: By noting that $M(F_i/\mathcal{V}_{F_i})$ is (1 1 1) and $M(F_i, \mathcal{V}_{F_i}) = J_3 - I_3$ for $i \in \{1, 2, 3\}$, Theorem 3.3 allows us to conclude that $M(H, \mathcal{V}_H) = M(H/\mathcal{V}_H)$, for every induced subgraph H of G, if and only if $(G, \mathcal{V})$ does not contain any of the configurations F_1, F_2, F_3. Hence the corollary holds. $\square$

4 Selective-perfect graphs

In this section, we will introduce the notion of perfectness related to selective coloring and give a complete characterization of selective-perfect graphs. It turns out that these graphs form a class containing both threshold graphs and complete bipartite graphs.

Definition 4.1 *A graph $G = (V, E)$ is* selective-perfect *if its cluster matrix $M(G, \mathcal{V})$ is perfect for every clustering $\mathcal{V}$.*

Thus $G = (V, E)$ is selective-perfect if and only if the two inequalities of (1) hold with equality for every clustering $\mathcal{V}$ of V. In order to characterize selective-perfect graphs, let us introduce i-threshold graphs (where i is the imaginary unit of complex numbers), a superclass of threshold graphs.

Definition 4.2 *A graph $G = (V, E)$ is an i-threshold graph if one can assign a complex number $w(v) \in \mathbb{R} \cup \{-i, +i\}$ to each vertex $v \in V$ such that:*

$$uv \in E \iff \Re(w(u) + w(v)) - \Im(w(u))\Im(w(v)) > 0.$$

We remind the reader that if we only allow real weights ($w(v) \in \mathbb{R}, \forall v \in V$), we get exactly the definition of threshold graphs [3]. Concerning i-threshold graphs, we can make the following easy observations.

Observations 4.1 *Let $G = (V, E)$ be an i-threshold graph. Let $V_- = \{v \in V : w(v) = -i\}$ and $V_+ = \{v \in V : w(v) = +i\}$.*

   (a) The class of i-threshold graphs is closed under taking induced subgraphs.

   (b) $G[V_- \cup V_+]$ is a complete bipartite graph with bipartition $\{V_-, V_+\}$.

   (c) $G[V \setminus (V_- \cup V_+)]$ is a threshold graph with clique $K = \{v \in V : w(v) > 0\}$ and stable set $S = \{v \in V : w(v) \leq 0\}$.

   (d) $V_- \cup V_+$ is complete to K and anticomplete to S.

Notice that if $V_- = \emptyset$ or $V_+ = \emptyset$, then G is a threshold graph, since all the vertices v in $V_- \cup V_+$ have the same weight, say w.l.o.g. $w(v) = -i$, so one can reset $w(v) := 0$ for $v \in V_- \cup V_+$. Recall that if G is a threshold graph, then $\chi(G) = \omega(G)$, Furthermore, if G is an i-threshold graph but not a threshold graph (i.e. $V_-, V_+ \neq \emptyset$), then $\chi(G) = |K| + 2 = \omega(G)$ (this easily follows from Observations (b), (c) and (d)). Hence i-threshold graphs are perfect. Let us now start analyzing the relation between i-threshold graphs and selective-perfect graphs.

Let G be an i-threshold graph with clustering $\mathcal{V} = \{V_1, \ldots, V_p\}$ and weight function w. Also, let V_-, V_+, K, S be as defined above. For each cluster $V_j \in \mathcal{V}$, $j = 1, \ldots, p$, we define a complex value $z_j \in \mathbb{R} \cup \{-i, +i\}$ as follows:

(1) If $V_j \cap S \neq \emptyset$, set $z_j := \min\{w(v_\ell) : v_\ell \in V_j \cap S\} \leq 0$,
(2) else if $V_j \cap V_- \neq \emptyset$ and $V_j \cap V_+ \neq \emptyset$, set $z_j := 0$,
(3) else if $V_j \cap V_+ \neq \emptyset$, set $z_j := +i$,
(4) else if $V_j \cap V_- \neq \emptyset$, set $z_j := -i$,
(5) else $V_j \subseteq K$, set $z_j := \min\{w(v_\ell) : v_\ell \in V_j\} > 0$.

Then we obtain the following.

Lemma 4.1 *V_j is complete to V_ℓ if and only if $\Re(z_j + z_\ell) - \Im(z_j)\Im(z_\ell) > 0$.*

Proof: If V_j satisfies (1) or (2), then it can only be complete to V_ℓ if V_ℓ satisfies (5); hence the lemma holds in this case. If V_j satisfies (3), then it can only be complete to V_ℓ if V_ℓ satisfies (4) or (5); hence the lemma holds in this case. Similarly, it holds if V_j satisfies (4). Finally, it clearly holds if V_j satisfies (5). $\square$

The following corollary is a consequence of Lemma 4.1.

Corollary 4.1 *Let $G = (V, E)$ be an i-threshold graph with clustering $\mathcal{V}$.*

(α) *The auxiliary graph $G/\mathcal{V}$ is i-threshold.*

(β) *$S_{\mathcal{V}'}$ is a stable set of $G/\mathcal{V}$ if and only if $\mathcal{V}'$ is a stable subclustering of $(G, \mathcal{V})$.*

Proof: (α) immediately follows from Lemma 4.1. To see that (β) holds, first assume that $S_{\mathcal{V}'}$ is a stable set of $G/\mathcal{V}$. Notice that $S_{\mathcal{V}'}$ cannot contain two vertices corresponding to two clusters V_j, V_ℓ such that V_j satisfies (3) and V_ℓ satisfies (4). So we may assume without loss of generality that $S_{\mathcal{V}'}$ does not contain any vertex corresponding to a cluster satisfying (4). Let us now construct a stable set S of $(G, \mathcal{V})$ corresponding to the stable subclustering $\mathcal{V}'$. Suppose that $V_j \in \mathcal{V}'$. If V_j satisfies (1), (3) or (5) we select a vertex in V_j with value z_j. If V_j satisfies (2), we choose any vertex of V_j with weight $+i$. This clearly gives us a stable set S in G. Finally, by Remark 3.2, if $\mathcal{V}'$ is a stable subclustering of $(G, \mathcal{V})$, then clearly the set of vertices in $G/\mathcal{V}$ corresponding to the clusters of $\mathcal{V}'$ form a stable set. $\square$

Remark 4.1 *If a graph* $G = (V, E)$ *contains an induced subgraph isomorphic to a graph in* $\{2K_2, P_4, \overline{P_2 \cup P_3}, \overline{3K_2}\}$, *then one can find a clustering* $\mathcal{V}$ *of* V *such that* $(G, \mathcal{V})$ *contains one of the configurations* F_1, F_2, F_3.

We are now ready to prove the main result of this section.

Theorem 4.2 *Let* $G = (V, E)$ *be a graph. Then the following statements are equivalent:*

   (i) G *is selective-perfect;*

   (ii) $\omega(H/\mathcal{V}_H) = \chi_{sel}(H, \mathcal{V}_H)$ *for every induced subgraph* H *and for every clustering* $\mathcal{V}$ *of* G;

  (iii) G *is* $\{2K_2, P_4, \overline{P_2 \cup P_3}, \overline{3K_2}\}$-*free;*

  (iv) G *is an i-threshold graph.*

Proof: $(i) \Rightarrow (ii)$: Suppose that there exists a clustering $\mathcal{V} = (V_1, \ldots, V_p)$ of V and an induced subgraph H of G such that $\chi_{sel}(H, \mathcal{V}_H) \neq \omega(H/\mathcal{V}_H)$. For every cluster $V_i \in \mathcal{V}$ such that $V_i \cap V(H) \neq \emptyset$, we define $V_i' = V_i \cap V(H)$. This gives us a clustering $(V_{i_1}', \ldots, V_{i_q}')$ of $V(H)$. Let $V_{i_{q+1}}' = V \setminus V(H)$ so $\mathcal{V}' = (V_{i_1}', \ldots, V_{i_{q+1}}')$ is a clustering of V. Now define a weight vector w for the dual linear programs of Theorem 2.1(iii) as follows: $w_{i_j} = 1$ for $j = 1, \ldots, q$ and $w_{i_{q+1}} = 0$. Thus the maximum is equal to $\omega(H/\mathcal{V}_H)$ and the minimum is equal to $\chi_{sel}(H, \mathcal{V}_H)$, but since $\chi_{sel}(H, \mathcal{V}_H) \neq \omega(H/\mathcal{V}_H)$, $M(G, \mathcal{V}')$ is not perfect.

$(ii) \Rightarrow (iii)$: This follows from Corollary 3.4 and Remark 4.1.

$(iii) \Rightarrow (iv)$: Let G be $\{2K_2, P_4, \overline{P_2 \cup P_3}, \overline{3K_2}\}$-free. If G is C_4-free, then G is a threshold graph and thus (iv) holds. So we may assume now that G contains an induced 4-cycle. Let C be such an induced 4-cycle with vertex set $\{v_1, v_2, v_3, v_4\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Clearly, since G is $\{P_4, \overline{P_2 \cup P_3}\}$-free, it follows that no vertex in $V \setminus V(C)$ is adjacent to either exactly one vertex of C, or exactly three vertices of C, or exactly two consecutive vertices in C. Thus every vertex in $V \setminus V(C)$ having at least one neighbor in C must be adjacent to either all the vertices of C or to exactly two non-consecutive vertices in C.

Let V_{1234} be the set of vertices adjacent to all the vertices of C and let V_{13} (resp. V_{24}) be the set of vertices adjacent to v_1, v_3 (resp. v_2, v_4) and non-adjacent to v_2, v_4 (resp. v_1, v_3). Since G is $\overline{3K_2}$-free, it follows that $G[V_{1234}]$ is a clique. Now consider a vertex $v \in V_{13}$. We claim that $N(v_2) = N(v)$. Suppose by contradiction that the claim does not hold. Thus there exists a vertex w such that v is adjacent to w and v_2 is non-adjacent to w (the case when v_2 is adjacent to w and v is non-adjacent to w can be handled similarly). Since G is P_4-free it follows that w must be adjacent to both v_1 and v_3 (otherwise $G[w, v, v_1, v_2]$ (resp. $G[w, v, v_3, v_2]$) is isomorphic to P_4). But now $G[v, w, v_1, v_3, v_4]$ is isomorphic to $\overline{P_2 \cup P_3}$, a contradiction. Thus w is adjacent to v_4. But now $G[v_1, v_2, v_3, v_4, w]$ is isomorphic to $\overline{P_2 \cup P_3}$, a contradiction. Thus we conclude that $N(v_2) = N(v)$ for every vertex $v \in V_{13}$. By symmetry it follows that $N(v_3) = N(v)$ for every vertex $v \in V_{24}$. Hence $V_{13} \cup V_{24} \cup \{v_1, v_2, v_3, v_4\}$ induce a complete bipartite graph. We set $W = V_{13} \cup V_{24} \cup \{v_1, v_2, v_3, v_4\}$

and $U = V \setminus W$.

Now consider the graph $G[U \setminus V_{1234}]$. If $G[U \setminus V_{1234}]$ contains an edge, say xy, then $G[x, y, v_1, v_2]$ is isomorphic to $2K_2$, a contradiction. Thus $G[U \setminus V_{1234}]$ is a stable set. We conclude that $G[U]$ is a threshold graph with clique $K = V_{1234}$ and stable sets $S = U \setminus V_{1234}$.

$(iv) \Rightarrow (i)$: This follows from Corollary 4.1 and the fact that i-threshold graphs are perfect. $\qquad\square$

Theorem 4.2 and Corollary 4.1 imply the following.

Corollary 4.3 *Let $G = (V, E)$ be a graph. Then it can be decided in polynomial-time whether G is selective-perfect, and if G is selective-perfect, one can determine a selective $\chi_{sel}(G, \mathcal{V})$-coloring in polynomial time, for any clustering $\mathcal{V}$ of V.*

Proof: It immediately follows from Theorem 4.2 that it can be decided in polynomial-time whether G is selective-perfect. Now, given a selective-perfect graph G and a clustering $\mathcal{V}$ of its vertex set, we obtain a $\chi_{sel}(G, \mathcal{V})$-coloring as follows. We construct the auxiliary graph $G/\mathcal{V}$ which, by Corollary 4.1, is an i-threshold graph. Clearly we can obtain an optimal coloring of $G/\mathcal{V}$ in polynomial time. Finally by choosing in each cluster V_j a vertex v with $w(v) = z_j$ and coloring it with the same color as V_j in $G/\mathcal{V}$, we obtain an optimal selective-coloring of $(G, \mathcal{V})$. $\qquad\square$

5 Conclusion

We showed that the superclass of threshold graphs obtained by allowing the weight function to take values in $\mathbb{R} \cup \{-i, +i\}$ instead of $\mathbb{R}$ only, namely the i-threshold graphs, plays an important role within a generalization of the graph coloring problem, namely the selective graph coloring problem.

As future work, the class of *complex threshold graphs* defined as in Definition 4.2 but with the weight taking values in the whole complex set $\mathbb{C}$ instead of $\mathbb{R} \cup \{-i, +i\}$ only, could be investigated. Notice that this graph class is also closed under taking induced subgraphs, but it seems to be a quite large graph class. For instance, it contains $2K_2$ and even C_5 (take value sets $\{2 - i, 2 - i, 1 - 4i, -3 + i\}$ and $\{10, -9 + 3i, -11 + 7i, -11 - 7i, -9 - 3i\}$, respectively) and thus it is not perfect anymore.

Acknowledgments

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