A Numerical Approximation based Controller for Mobile Robots with Velocity Limitation

Mario E. Serrano, Sebastián A. Godoy, Santiago Rómoli, Gustavo J. E. Scaglia

M. E. S. Author is with National Council for Scientific and Technical Research (CONICET) and National University of San Juan, Argentine (corresponding author, phone: +549 264 4211700; fax: +54 264 4227216; e-mail: serranoemanuel84@gmail.com).

S. A. G. Author is with National Council for Scientific and Technical Research (CONICET) and National University of San Juan, Argentine (e-mail: sgodoy@unsj.edu.ar).

S. R. Author is with National Council for Scientific and Technical Research (CONICET) and National University of San Juan, Argentine (e-mail: sromoli@unsj.edu.ar).

G. J. E. S. Author is with National Council for Scientific and Technical Research (CONICET) and National University of San Juan, Argentine (e-mail: gscaglia@unsj.edu.ar).

Abstract — In this paper the problem of trajectory tracking considers that the values of the control actions do not exceed a maximum allowable value is focused and the zero convergence of tracking errors is demonstrated. The control law is based on a linear algebra approach. First, the desired trajectories of some states variables are determined through analyzing the conditions for a system of linear equations to have an exact solution. Therefore, the control signals are obtained by solving the system of linear equations. The optimal controller parameters are selected through nonlinear programming, so as to prevent the saturation of the control actions. Experimental results are presented and discussed, demonstrating the good performance of the controller. Finally, the performance of the proposed controller is compared with a fuzzy controller, and all the results are validated through laboratory experimental tests.

Index Terms— Control actions constraints, Control design, Linear algebra, Mobile robots, Nonlinear Programming, Trajectory tracking.

I. Introduction

THIS WORK proposes a new approach to limit the control signals during a trajectory tracking in mobile robots. In the literature it is common to find works that use explicit saturation functions, such as the hyperbolic tangent [1, 2], or fuzzy rules [3] to limit control signals in mobile robots. In this paper, however, concepts of linear algebra and non-linear programming are adopted to achieve such limitation while keeping an efficient trajectory tracking controller operation.

In recent years, there has been an increasing amount of research on the mobile robotics field [3-5]. Mobile Robots are currently used in industry, for domestic needs (vacuum cleaners, lawn mowers, pets), in difficult-to access or dangerous places areas (space, army, nuclear-waste cleaning) and also for entertainment (competition, robot soccer).

Several controllers can be found in the literature aiming at achieving trajectory tracking. Some of such controllers are not based on a model associated with the used platform. Such case is shown in [6, 7], where neural networks or fuzzy logic are used for generating the control actions. However, these controllers are relatively few compared to model-based developments [5, 8].

In [9] an adaptive fuzzy controller for trajectory tracking in mobile robots is presented. With the purpose of accomplishing a perfect tracking of the WMR heading, velocity and position variables, a heuristic method is proposed to design the expert knowledge base as fuzzy if-then rules. In [10] the authors presents an adaptive controller to solve the tracking problem of a unicycle robot with unknown dynamic parameters. Unfortunately, [9, 10] only present simulation results.

In [11], a novel trajectory-tracking controller was presented. The robot mobile model is approximated by numerical methods and the control actions are calculated under the assumption that the reference trajectories are known. Such control action forces the system to move from its current state to the reference one; and the conditions for achieving a zero tracking error are obtained by solving a system of linear equations. This design technique has been applied successfully in several systems [5, 11-15].

Another typical problem, covered in the literature by other authors ([16, 17]) is the trajectory tracking with constraints in the control actions. In general, in robot mobile system, the linear and angular velocities constraints prevents the mobile robot from slipping and saturating the actuators.

Nonlinear system theory has been employed to solve this problem in [16]. The controller proposed by the authors is based on the backstepping method and an idea taken from the LaSalle's invariance principle. With the proposed control law, the robot can globally follow any path specified by a straight line, a circle or a path approaching the origin using a single controller.

In [17], a model-predictive trajectory-tracking control applied to a mobile robot is presented. In order to predict future system behavior, a linearized tracking-error dynamic is used, and a control law is derived from a quadratic cost function that penalizes the system tracking error and the control effort. In [3], fuzzy rules are adopted to achieve control actions limitation problems, combining the heuristic knowledge of the problem, the sector non linearity approach and the inverse kinematic of the mobile platform.

In this work, the saturation constraints in the control inputs (the linear and angular velocities) are incorporated in our controller design. A new control law based on the numerical approximation of the mobile robot model is then developed. This novel control law achieve the limitation in the control actions while keeping an efficient trajectory tracking controller operation. The trajectory tracking controller structure arises naturally derived through a handcrafted procedure that is inferred by analyzing the mathematical model of the robot. In addition, a new parameter assignment method based on nonlinear programming is proposed. The controller underlying idea for tracking the reference trajectory (x_{ref} and y_{ref}) is intuitively simple: it is based on determining the desired trajectories of the remaining state variables. They are determined through analysing the conditions for a system of linear equations to have an exact solution. Lastly, the control signals are obtained by solving a system of linear equations. The main contribution of this work is that the proposed methodology is based upon easily understandable concepts, and there is no need of complex calculations to attain the control signal.

The proposed newfangled method ensures the convergence to zero of the tracking errors and prevents the controller saturation. We also include a comparison analysis of our approach with other two trajectory tracking controllers previously published in the literature [3, 18]. The proposed controller has lower tracking errors and presents slight oscillations, which minimizes the maneuverability space needed by the vehicle. Finally, the proof of convergence to zero of the tracking error is presented in the Appendix.

The paper is organized as follows: Section 2 summarizes the kinematical model of the mobile robot. The controller design and the parameters analysis are included in Section 3. A method to choose the controller parameters based on nonlinear programming is considered in Section 4. Experimental results of the proposed controller with a mobile robot system are given in Section 5, followed by the discussions and conclusions in Section 6 and Section 7 respectively.

II. Kinematic Model Of The Mobile Robot

A nonlinear kinematic model for a mobile robot will be used [11, 14, 19, 20], and it is represented by (1),

$$\begin{cases} \dot{x} = V \cos \theta \\ \dot{y} = V \sin \theta \\ \dot{\theta} = W \end{cases} \tag{1}$$

where, V: linear velocity of the mobile robot, W: angular velocity of the mobile robot, (x, y): cartesian position, θ: mobile robot orientation. This model has been used in several recent papers such as [13, 20, 21].

(este párrafo está medio caido del cielo)Then, it aims to find the values of V and W so that the mobile robot follows a pre-established trajectory (x_{ref} and y_{ref}) with a minimum error. The values of $x(t)$, $y(t)$, $\theta(t)$, $V(t)$ and $W(t)$ at discrete time $t=nT_0$, where T_0 is the sample time and $n \in [0,1,2,\ldots)$, will be denoted as x_n, y_n, θ_n, V_n and W_n, respectively.

Remark 1: The value of the difference between the reference and the real trajectory shall be called tracking error. It is given by $e_{x,n}= x_{ref,n}-x_n$ and $e_{y,n}= y_{ref,n}-y_n$. Thus, the tracking error is represented by $\|e_n\| = (e_{x,n}^{\;2} + e_{x,n}^{\;2})^{1/2}$.

III. Controller Design

In this paper a new control law is proposed. This novel approach by trapezoidal approximation of the system model ensures an acceptable trajectory tracking and avoids exceeding the allowable limits of the control actions.

A. Trapezoidal controller.

Firstly, consider the Trapezoidal approximation of the kinematic model (1), as proposed here:

$$\begin{cases} x_{n+1} = x_n + \dfrac{T_0}{2}\left(V_n \cos\theta_n + V_{n+1}\cos\theta_{n+1}\right) \\ y_{n+1} = y_n + \dfrac{T_0}{2}\left(V_n \sin\theta_n + V_{n+1}\sin\theta_{n+1}\right) \\ \theta_{n+1} = \theta_n + \dfrac{T_0}{2}\left(W_n + W_{n+1}\right) \end{cases} \tag{2}$$

The system (2) can be rearranged as (3),

$$\begin{bmatrix} \cos\theta_{n+1} & 0 \\ \sin\theta_{n+1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_{n+1} \\ W_{n+1} \end{bmatrix} = \begin{bmatrix} \dfrac{2}{T_0}\left(x_{n+1} - x_n\right) - V_n\cos\theta_n \\ \dfrac{2}{T_0}\left(y_{n+1} - y_n\right) - V_n\sin\theta_n \\ \dfrac{2}{T_0}\left(\theta_{n+1} - \theta_n\right) - W_n \end{bmatrix} \tag{3}$$

From (3), the control law to move from (x_n, y_n) to (x_{n+1}, y_{n+1}) can be derived. Considering (3) and replacing $[x_{n+1}, y_{n+1}]$ by the desired trajectory $[x_{d,n+1}, y_{d,n+1}]$, system (4) can be obtained:

$$\begin{cases} V_{n+1}\cos\theta_{n+1} = \dfrac{2}{T_0}\left(x_{d,n+1}-x_n\right)-V_n\cos\theta_n \\ V_{n+1}\sin\theta_{n+1} = \dfrac{2}{T_0}\left(y_{d,n+1}-y_n\right)-V_n\sin\theta_n \end{cases} \quad (4)$$

After that, we propose the following replacements:

$$\begin{aligned} &V_n\cos\theta_n \ by\ V_{ref,n}\cos\theta_{ref,n} \\ &V_n\sin\theta_n \ by\ V_{ref,n}\sin\theta_{ref,n} \\ &W_n \ by\ W_{ez,n} \end{aligned} \quad (5)$$

where,

$$W_{ez,n} = \frac{\theta_{ez,n+1}-\theta_{ez,n}}{T_0} \quad (6)$$

The orientation θ_{ez} is the value of the robot mobile orientation (θ) required in order to the tracking errors tend to zero, for details see Appendix. Thus, W_{ez} represents the necessary angular velocity for the mobile robot reaching and following the reference trajectory.

The new variable, θ_{ez}, is calculated in each sample time by analyzing the condition for which the system (3) has an exact solution. One possible way to meet the aforementioned condition is to resolve (7):

$$\frac{\sin\theta_{n+1}}{\cos\theta_{n+1}} = \tan\theta_{n+1} = \frac{\dfrac{2}{T_0}\left(y_{d,n+1}-y_n\right)-V_{ref,n}\sin\theta_{ref,n}}{\dfrac{2}{T_0}\left(x_{d,n+1}-x_n\right)-V_{ref,n}\cos\theta_{ref,n}} \quad (7)$$

where the direction θ_{n+1} will be called $\theta_{ez,n+1}$. As shown in (3), this is a three-equation system where linear and angular velocities are unknown. Considering (4)-(7), system (3) can be expressed as follows:

$$\begin{bmatrix} \cos\theta_{ez,n+1} & 0 \\ \sin\theta_{ez,n+1} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} V_{n+1} \\ W_{n+1} \end{bmatrix} = \begin{bmatrix} \dfrac{2}{T_0}\left(x_{d,n+1}-x_n\right)-V_{ref,n}\cos\theta_{ref,n} \\ \dfrac{2}{T_0}\left(y_{d,n+1}-y_n\right)-V_{ref,n}\sin\theta_{ref,n} \\ \dfrac{2}{T_0}\left(\theta_{n+1}-\theta_n\right)-W_{ez,n} \end{bmatrix} \quad (8)$$

At time n, the mobile is at $[x_n, y_n, \theta_n]$; the desirable next state, $[x_{d,n+1}, y_{d,n+1}, \theta_{d,n+1}]$, is not necessarily the new reference state value. Consider then, this state vector ($[x_{d,n+1}, y_{d,n+1}, \theta_{d,n+1}]$) assuming an approaching proportional to the error as proposed here:

$$\begin{aligned} x_{d,n+1} &= x_{ref,n+1}-k_v\left(x_{ref,n}-x_n\right) \\ y_{d,n+1} &= y_{ref,n+1}-k_v\left(y_{ref,n}-y_n\right) \\ \theta_{n+1} &= \theta_{ez,n+1}-k_w\left(\theta_{ez,n}-\theta_n\right) \end{aligned} \quad (9)$$

With the purpose of the tracking errors tend to zero, the controller parameters must fulfill $0<k_v<1$ and $0<k_w<1$ (see Appendix). Note that:

- If $k_v = 0$, ($x_{d,n+1}= x_{ref,n+1}$), the goal is to reach the reference trajectory in one step.
- If $k_v = 1$, the error will remain constant, ($x_{d,n+1}- x_n= x_{ref,n+1}- x_{ref,n}$).

Thus, the approach proposed in (9) is applied in order to get a smooth trajectory. The same analysis can be applied to $y_{d,n+1}$ and $\theta_{d,n+1}$.

In addition, we define:

$$\begin{aligned} \Delta_x &= x_{ref,n+1}-k_v\left(x_{ref,n}-x_n\right)-x_n \\ \Delta_y &= y_{ref,n+1}-k_v\left(y_{ref,n}-y_n\right)-y_n \\ \Delta_\theta &= \theta_{ez,n+1}-k_w\left(\theta_{ez,n}-\theta_n\right)-\theta_n \end{aligned} \quad (10)$$

$$\begin{bmatrix} \cos\theta_{ez,n+1} & 0 \\ \sin\theta_{ez,n+1} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} V_{n+1} \\ W_{n+1} \end{bmatrix} = \begin{bmatrix} \dfrac{2}{T_0}\Delta_x-V_{ref,n}\cos\theta_{ref,n} \\ \dfrac{2}{T_0}\Delta_y-V_{ref,n}\sin\theta_{ref,n} \\ \dfrac{2}{T_0}\Delta_\theta-W_{ez,n} \end{bmatrix} \quad (11)$$

The system (11) is of type $\boldsymbol{Au}=\boldsymbol{b}$, with more equations than unknowns. Its solution by least squares can be obtained by solving the normal equations [22], $\boldsymbol{A}^T\boldsymbol{Au}=\boldsymbol{A}^T\boldsymbol{b}$, and thence, the proposed controller is given by (12),

$$\begin{aligned} V_{n+1} &= \left(\frac{2}{T_0}\Delta x-V_{ref,n}\cos\theta_{ref,n}\right)\cos\theta_{ez,n+1}+\left(\frac{2}{T_0}\Delta y-V_{ref,n}\sin\theta_{ref,n}\right)\sin\theta_{ez,n+1} \\ W_{n+1} &= \frac{2}{T_0}\Delta_\theta-W_{ez,n} \end{aligned} \quad (12)$$

Theorem 1. If the system behavior is ruled by (2) and the controller is designed by (7), (10) and (12), then $e_n\rightarrow 0$, $n\rightarrow\infty$ when trajectory tracking problems are considered and controller parameters fulfill $0<k_v<1$ and $0<k_w<1$.

The proof of Theorem 1 is shown in Appendix.

B. Analysis of trapezoidal controller parameters.

In this subsection, the performance of the control law proposed when its parameters vary, is discussed. The objective is to determine the conditions to be fulfilled by the controller parameters, such that the proposed control actions (V and W) do not exceed the maximum allowable values (V_{max} and W_{max}).

In this work it is not considered the design problem of a global trajectory planner. The trajectory tracking problem is addressed considering that the desired trajectory is admissible. Thus, the desired trajectory satisfies (2). Then (13), (14) and (15) are fulfilled. Here, $\theta_{ref,n+1}$, $V_{ref,n+1}$ and $W_{ref,n+1}$ are the mobile robot orientation, linear and angular velocities of the reference trajectory, which fulfills: $V_{ref,n+1}<V_{max}$ and $W_{ref,n+1}<W_{max}$.

$$\frac{\sin\theta_{ref,n+1}}{\cos\theta_{ref,n+1}} = \tan\theta_{ref,n+1} = \frac{\dfrac{2}{T_0}\left(y_{ref,n+1}-y_{ref,n}\right)-V_{ref,n}\sin\theta_n}{\dfrac{2}{T_0}\left(x_{ref,n+1}-x_{ref,n}\right)-V_{ref,n}\cos\theta_n} \quad (13)$$

$$V_{ref,n+1} = \left(\frac{2}{T_0}\left(x_{ref,n+1} - x_{ref,n}\right) - V_{ref,n}\cos\theta_{ref,n}\right)\cos\theta_{ref,n+1} + \left(\frac{2}{T_0}\left(y_{ref,n+1} - y_{ref,n}\right) - V_{ref,n}\sin\theta_{ref,n}\right)\sin\theta_{ref,n+1} \tag{14}$$

$$W_{ref,n+1} = \frac{2}{T_0}\Delta_\theta - W_{ref,n} \tag{15}$$

Subsequently, the trapezoidal controller performance is analyzed. First the mobile orientation is evaluated in (7) when $k_v \to 1^-$.

$$\lim_{k_v \to 1^-}\theta_{ez,n+1} = \lim_{k_v \to 1^-}\operatorname{atan}\frac{\frac{2}{T_0}\left(y_{ref,n+1} - k_v\left(y_{ref,n} - y_n\right) - y_n\right) - V_{ref,n}\sin\theta_{ref,n}}{\frac{2}{T_0}\left(x_{ref,n+1} - k_v\left(x_{ref,n} - x_n\right) - x_n\right) - V_{ref,n}\cos\theta_{ref,n}}$$

thus,

$$\lim_{k_v \to 1^-}\theta_{ez,n+1} = \operatorname{atan}\frac{\frac{2}{T_0}\left(y_{ref,n+1} - y_{ref,n}\right) - V_{ref,n}\sin\theta_{ref,n}}{\frac{2}{T_0}\left(x_{ref,n+1} - x_{ref,n}\right) - V_{ref,n}\cos\theta_{ref,n}} \tag{16}$$

By inspection of (13) and (16), when $k_v \to 1^-$, (17) can be immediately obtained. As can be seen, $\theta_{ez,n+1} \to \theta_{ref,n+1}$,

$$\lim_{k_v \to 1^-}\theta_{ez,n+1} = \theta_{ref,n+1} \tag{17}$$

Next, the linear velocity (V) is analyzed. Considering (17), and taken the limit of V when $k_v \to 1^-$, in (12), (18) results,

$$\lim_{k_v \to 1^-}V_{n+1} = \left(\frac{2}{T_0}\left(x_{ref,n+1} - x_{ref,n}\right) - V_{ref,n}\cos\theta_{ref,n}\right)\cos\theta_{ref,n+1} + \ldots$$
$$\ldots + \left(\frac{2}{T_0}\left(y_{ref,n+1} - y_{ref,n}\right) - V_{ref,n}\sin\theta_{ref,n}\right)\sin\theta_{ref,n+1} \tag{18}$$

By comparison of (14) and (18), when $k_v \to 1^-$, (19) can be obtained,

$$\lim_{k_v \to 1^-}V_{n+1} = V_{ref,n+1} \tag{19}$$

Finally, the angular velocity (W) is evaluated. From (12):

$$\lim_{k_w \to 1^-}W_{n+1} = \frac{2}{T_0}\left(\theta_{ref,n+1} - \theta_{ref,n}\right) - W_{ez,n} = \frac{2}{T_0}\left(\theta_{ref,n+1} - \theta_{ref,n}\right) - \frac{\theta_{ez,n+1} - \theta_{ez,n}}{T_0} \tag{20}$$

Considering (17) and replacing θ_{ez} by θ_{ref} in (20),

$$\lim_{\substack{k_v \to 1^- \\ k_w \to 1^-}}W_{n+1} = \frac{2}{T_0}\left(\theta_{ref,n+1} - \theta_{ref,n}\right) - \frac{\theta_{ref,n+1} - \theta_{ref,n}}{T_0} = \frac{2}{T_0}\left(\theta_{ref,n+1} - \theta_{ref,n}\right) - W_{ref,n} \tag{21}$$

According to (15) and (21), when $k_v \to 1^-$ and $k_w \to 1^-$, (22) results:

$$\lim_{\substack{k_v \to 1^- \\ k_w \to 1^-}}W_{n+1} = W_{ref,n+1} \tag{22}$$

Remark 2: Note that if the replacements propose in (5) is not made, then (19) and (22) are not fulfilled. Then, the control actions can exceed allowable limits for large values of e_n.

From (19) and (22), when $k_v \to 1^-$ and $k_w \to 1^-$ then, $V_n \to V_{ref,n}$ and $W_n \to W_{ref,n}$; therefore $V_n < V_{max}$ and $W_n < W_{max}$. The latter demonstrates that a value below but close to one of the parameters (k_v and k_w), makes the actual velocities of the robot close to the reference ones, and the tracking errors converge to zero. Thus, we obtain a control law capable of tracking the desired trajectory without surpassing the maximum allowable values of the velocities.

In the next section a new algorithm to select the controller parameters (k_v and k_w) so that the tracking error tends to zero, and the control actions do not exceed the allowable limits is proposed.

IV. Selection of the Controller Parameters by Nonlinear Programming.

Nonlinear programming (NLP) deals with the problem of optimizing an objective function in the presence of equality and inequality constraints, where some of the constraints or the objective function is nonlinear. Typically a NLP is posed as:

$$\textit{Minimize} \quad f(x) \quad \textit{subject to} \quad \begin{cases} g_i(x) \le 0 & \text{for i=1,...,m} \\ h_i(x) = 0 & \text{for i=1,...,p} \\ x \in X \end{cases}$$

Where f, g_1, ..., g_m, h_1, ..., h_p are functions defined on R_n, X is a subset of R_n, and x is a vector of n components x_1, ..., x_n. The above problem must be solved for the values of the variables $x_1, \ldots, x_n$ that satisfy the restrictions while minimizing the function f. The function f is usually called the objective function, or the criterion function. Each of the constraints $g_i \le 0$ for $i=1,\ldots,m$ is called an inequality constraint, and each of the constraints $h_i = 0$ for $i=1\ldots,p$ is called an equality constrain, [23].

Considering the issue addressed in this paper, the conditions to avoid the control actions saturation (12), are specified by,

$$V_{n+1} = \left(\frac{2}{T_0}\Delta x - V_{ref,n}\cos\theta_{ref,n}\right)\cos\theta_{ez,n+1} + \left(\frac{2}{T_0}\Delta y - V_{ref,n}\sin\theta_{ref,n}\right)\sin\theta_{ez,n+1} < V_{\max}$$
$$W_{n+1} = \frac{2}{T_0}\Delta\theta - W_{ez,n} < W_{\max} \tag{23}$$

Besides, to ensure that the robot movements tend to the desired trajectory, (7) must be satisfied,

$$\theta_{ez,n+1} = a\tan\frac{\frac{2}{T_0}\left(y_{ref,n+1} - k_v\left(y_{ref,n} - y_n\right) - y_n\right) - V_{ref,n}\sin\theta_{ref,n}}{\frac{2}{T_0}\left(x_{ref,n+1} - k_v\left(x_{ref,n} - x_n\right) - x_n\right) - V_{ref,n}\cos\theta_{refn}} \tag{24}$$

So as to guarantee the smooth operation of the robot and the zero convergence of the tracking errors, the controller parameters must fulfill:

$$0.94 < k_v < 1 \tag{25}$$

$$0.94 < k_w < 1 \tag{26}$$

Equations (25) and (26) guarantee that there are no sharp variations in the system. The upper limit is selected in accordance with the conditions to allow the tracking error to tend to zero, see Appendix, where the parameters must be lesser than 1 to ensure the zero errors convergence.

The lower limits in (25) and (26) represent the values of k_v and k_w when the mobile robot reaches desired trajectory. If the lower limits are close to 0 the controller corrects the tracking error quickly; however, they can lead to unwanted oscillations. These values can be set under 0.94 but is not ensured a smooth trajectory tracking, which is an objective of this paper. Taking into account these considerations, the desired minimum values for k_v and k_w are 0.94 and 0.94 respectively. The lower limits of the parameters k_v and k_w have been obtained by empirical test in the mobile robot in order to avoid unwanted oscillations.

Equations (19) and (22) show that when $k_v \to 1^-$ and $k_w \to 1^-$ then, $V(n) \to V_d(n)$ and $W(n) \to W_d(n)$. The aim of this work is to reduce k_v and k_w without exceeding the maximum values of the control actions while guaranteeing the zero error convergence. Thus, (27) proposed a cost function with only one global minimum,

$$f(k_v, k_w) = k_v^{\,2} + k_w^{\,2} \tag{27}$$

With these considerations, the problem of selection the parameters k_v and k_w is analyzed as a nonlinear programming problem. Therefore, the equations that must be solved in each sample time are:

Minimize

$$f(k_v, k_w) = k_v^{\,2} + k_w^{\,2}$$

Subject to:

$$\begin{cases} \left(\dfrac{2}{T_0}\Delta x - V_{ref,n}\cos\theta_{ref,n}\right)\cos\theta_{ez,n+1} + .. \\ \quad .. + \left(\dfrac{2}{T_0}\Delta y - V_{ref,n}\sin\theta_{ref,n}\right)\sin\theta_{ez,n+1} < V_{\max} \\ \dfrac{2}{T_0}\Delta\theta - W_{ez,n} < W_{\max} \\ 0.94 < k_v < 1 \\ 0.94 < k_w < 1 \\ \theta_{ez,n+1} = a\tan\dfrac{\dfrac{2}{T_0}\left(y_{ref,n+1} - k_v\left(y_{ref,n} - y_n\right) - y_n\right) - V_{ref,n}\sin\theta_{ref,n}}{\dfrac{2}{T_0}\left(x_{ref,n+1} - k_v\left(x_{ref,n} - x_n\right) - x_n\right) - V_{ref,n}\cos\theta_{ref,n}} \end{cases}$$

To resolve this problem of Nonlinear Programming in [24], different algorithms based on the concept of trust-region-reflective are developed. In this paper, we use this algorithm in each sample time. The values k_v and k_w found are used in (12) to find the control action that will be applied.

Remark 3: Nonlinear Programming problem is solved in each sample time using the function "*fmincon*" in MatLab®. The command *fmincon* uses one of these three algorithms: active-set, interior-point, or trust-region-reflective, which is the default option.

V. Experimental Results

To show the performance of the proposed controller several experiments and simulations were made. Some of the results are presented in this section. The experiments where performed using a PIONEER 3AT mobile robot. The PIONEER 3AT mobile robot includes an estimation system based on an odometric positioning system. Figure 1 shows the PIONEER 3AT and the laboratory facilities where the experiments were carried out.

Fig. 1. PIONEER 3AT, the robot used into the experimental environment.

A. Curvature Test.

The first one is a curvature test, as recommended in [25]: the controller performance is evaluated using different circle-shaped trajectories. Three circle-trajectories are used in this work, with different radius. The internal trajectory has a radius of *r = 1.5m*, the medium one *r = 2 m* and the last one *r=2.5m*. The initial robot position is the system origin and the reference trajectory begins in the position $(x_{ref}, y_{ref}) = (1m, 0.5m)$. The maximum allowable velocities are V_{max}=0.55m/s and W_{max}=0.5rad/s. The sample time used is $T_0=0.1s$.

The reference trajectory and the results of the controller performance are shown in Fig. 2. As can be seen, the controller reaches and follows the desired trajectory. Fig. 3 shows the plots of the tracking error in the *x-coordinate* and *y-coordinate* according to curvatures shown in Fig. 2. In Fig. 3, can be seen that both errors remain bounded and close to zero when the robot reaches the reference trajectory. The Control actions are shown in Fig. 4 and Fig. 5. The plots show the control actions calculated by the controller, the robot mobile velocities and the maximums allowable velocities. It is important to note in Fig. 4 and Fig. 5, that there is a difference between the control actions calculated and the velocities reached by the mobile robot. It is because the PIONEER 3AT has a proportional-integral-derivative (PID) velocity controller used to maintain the velocities of the mobile robot at the desired value. The values of the parameters of the controller at each sample time are shown in Fig.6.

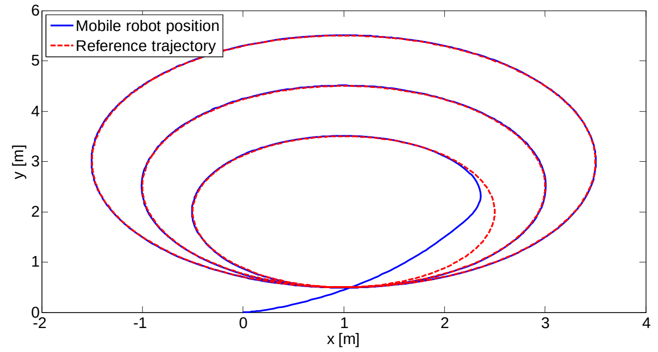

Fig. 2. Reference trajectory and the trajectory performed by the robot using the proposed controller. Continuous line represents the position of the mobile robot; and the dotted line the reference trajectory.

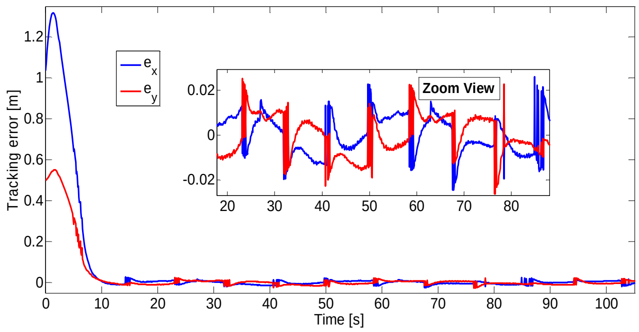

Fig. 3. Tracking errors vs. time. The figure shows how the tracking errors are close to zero when the robot reaches the trajectory.

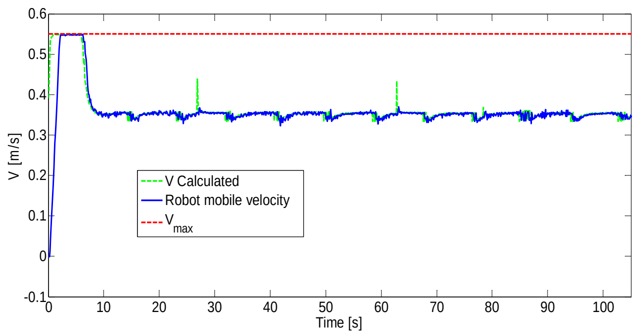

Fig. 4. Results of curvature test, control action *V* vs. time.

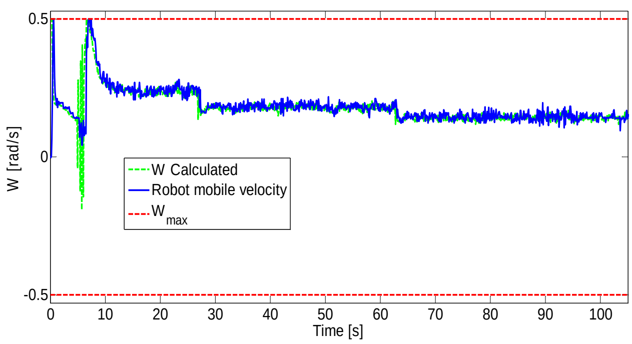

Fig. 5. Control action W vs. time during curvature test.

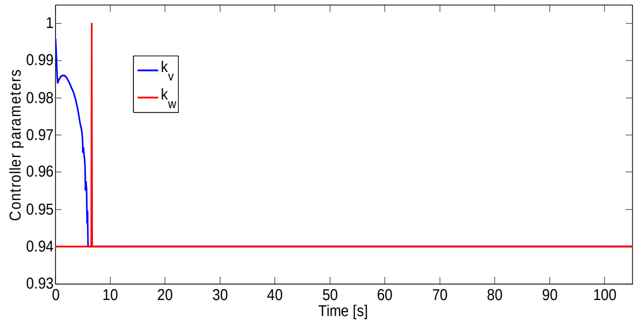

Fig. 6.Plot of the controller parameters values vs. time.

B. *Square Trajectory.*

Another proof for checking the controller performance is a square trajectory, as recommended in [25]. This strategy can be used in applications such as obstacle avoidance and contour-following. For example, if the danger of collision is large, the trajectory to be followed by the robot is modified abruptly and the robot must follow that path in order to avoid collision. Thus, the controller performance when the trajectory changes abruptly will be analyzed. The square reference trajectory is generated with constant linear velocity of $V = 0.3m/s$. The initial position of the robot is the system origin and the trajectory begins in the position $(x_{ref}(0), y_{ref}(0)) = (1m, 1m)$. The sample time used is $T_0=0.1s$ and the maximum allowable velocities are V_{max}=0.55m/s and W_{max}=0.5rad/s.

Figure 7 shows the results of the implementation. As can be seen, the controller reaches and follows the reference trajectory. In Fig. 8 it is observed that the tracking errors, when the mobile robot reaches the desired trajectory, are less than *0.01m*. However, these errors remain low compared to the mobile robot dimensions (*0.508 m* long, *0.497 m* large, *0.277 m* high). The control action values, are shown in Fig. 9 and Fig. 10. Figure 11 shows the controller parameters versus time. As can be seen, these parameters change to ensure that the tracking errors tend to zero and control actions do not exceed the maximum allowed.

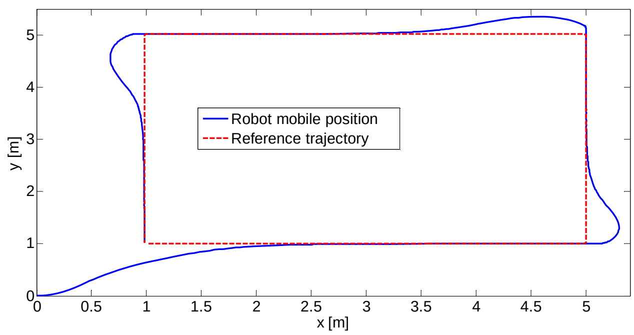

Fig. 7. Second experiment: reference trajectory and the trajectory performed by the robot.

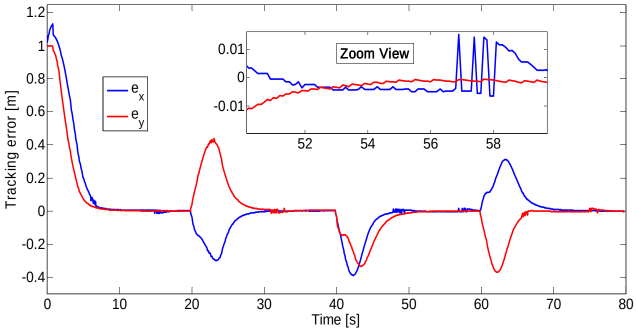

Fig. 8. Evolution of the magnitude of the tracking errors.

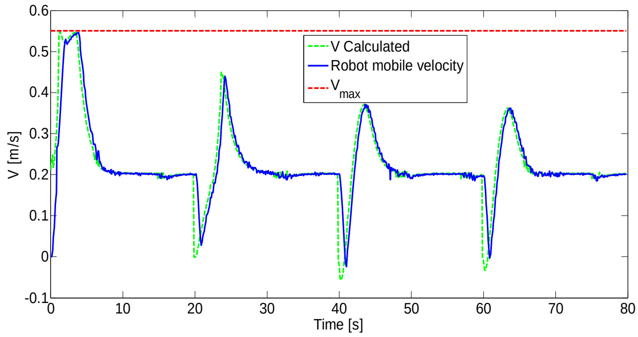

Fig. 9. Evolution of the magnitude of the velocity V during the second experiment.

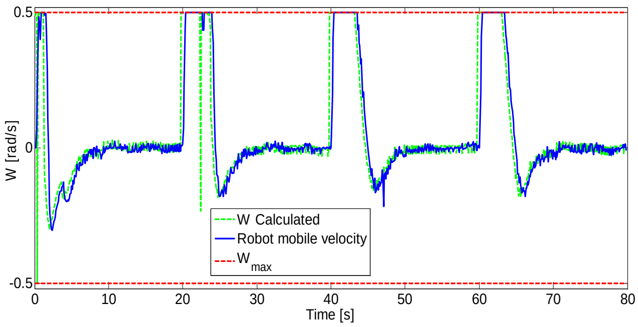

Fig. 10. Evolution of the magnitude of the velocity W during the second experiment.

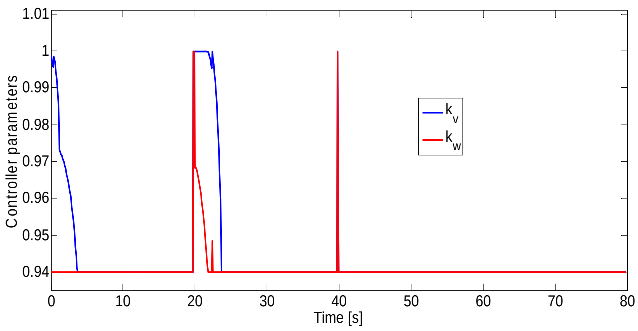

Fig. 11. Evolution of the controller parameters during the square trajectory test.

C. Controller Comparison.

To test the advantages and drawbacks of our proposal, an experimental evaluation was carried out. In order to do so, the control law proposed in (12) with a simple saturator and two controllers previously published in the scientific literature were implemented for comparison purposes on the mobile robot Pioneer 3AT. The controllers implemented for comparison are the following:

- Controller proposed in this paper when its parameters are chosen with nonlinear programming, C1 in the sequel (methodology proposed in this paper).
- Controller proposed in this paper when its parameters are fixed, and the control actions are limited with a simple saturator, C2 in the sequel.
- A non-linear trajectory tracking strategy developed by [18], C3 in the sequel.
- The controller developed in [3], C4 in the sequel.

The implementation schemes for the controllers C1 and C2 are shown in Fig. 12 and Fig. 13. The controller parameters of C2, k_v and k_w, are both equal to 0.94.

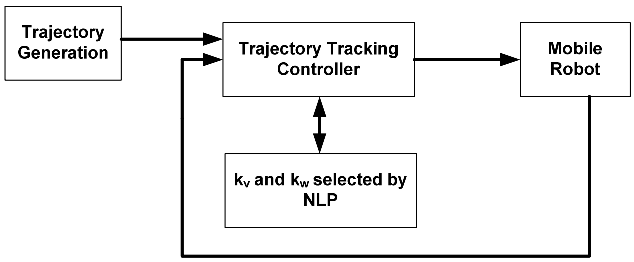

Fig. 12. General architecture of the proposed controller.

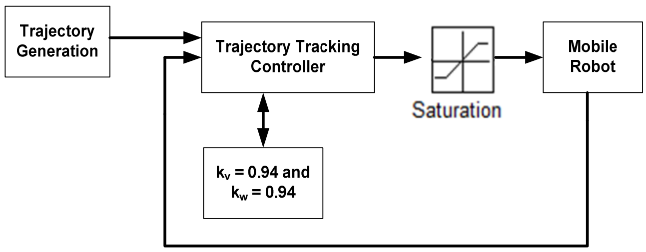

Fig. 13.General architecture of C2. The controller parameters are fixed and the control action actions are limited by a simple saturator in comparison of C1.

The designing details of the controllers C3 and C4 can be found in its respective references, and only the experimental results without a theoretical analysis of the controllers' properties are shown here. For those, [18] and [3], offer a deep insight into the controller design.

In order to compare the controllers performance, the integrated squared error (ISE) is used [26, 27]. An idea widely used in the literature is to consider the cost incurred by the error. Then, we define a cost function represented for the combination of the ISE in x-coordinate and the y-coordinate as shown in (28),

$$C^{\Phi} = C_x^{\Phi} + C_y^{\Phi} = T_0 . \sum_{i=0}^{\#\Phi} \left(\left(x_{(i)} - x_{ref(i)} \right)^2 + \left(y_{(i)} - y_{ref(i)} \right)^2 \right) \qquad (28)$$

In this subsection the robot should follow an eight-shaped trajectory. The initial conditions for the robot mobile position is the system origin and the trajectory begins in the position

$(x_{ref}(0),\ y_{ref}(0)) = (1m,-1m)$. The sample time used is $T_0=0.1s$ and the maximum allowable velocities are set in V_{max}=0.55m/s and W_{max}=0.5rad/s.

Fig.14 shows the reference trajectory and the results obtained by implementing the controllers C1, C2, C3 and C4. It shows that, all controllers reach and follow the desired trajectory without unexpected oscillations. Figure 15 and Fig. 16 show the absolute values of the tracking errors for the *x*-coordinate and *y*-coordinate respectively. It is observed that the proposed controller has better performance. The controller C4 presents a similar performance that C1, but the lowest cost is obtained by C1 as can be seen in Fig. 17.

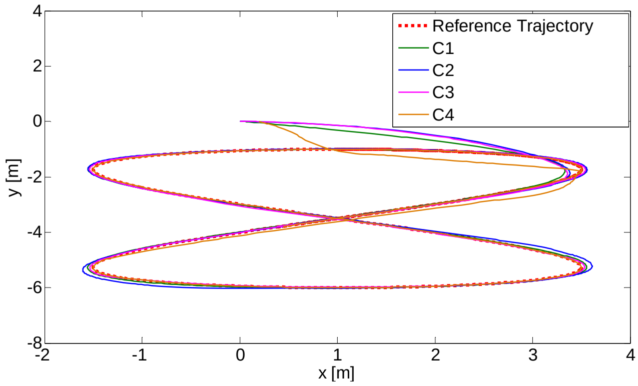

Fig. 14. Reference trajectory and the robot position for all controllers.

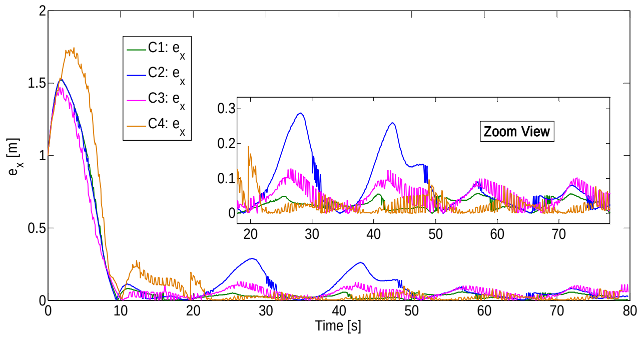

Fig. 15. Absolute value of tracking error in *x*-coordinate.

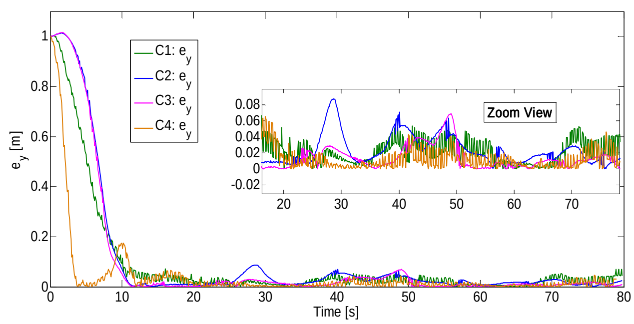

Fig. 16. Absolute value of tracking error in *y*-coordinate.

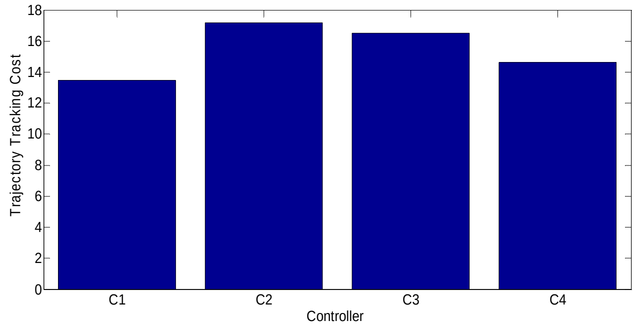

Fig. 17. Results of controller's comparison: trajectory tracking cost of each controller when the robot follow an eight-shaped trajectory.

Analyzing the above plots, one can conclude that the proposed trajectory tracking controller with velocity limitation via nonlinear programming has superior performance, compared to the others controllers.

VI. Discussions

- In our work, the controller tuning is performed by an optimization function that ensure a faster convergence avoiding the actuator's saturation. This represents a great advantage compared to previous controllers based on linear algebra published in the literature [11, 20]. In [11, 20] the controllers parameters are fixed and are chosen by empirical tests, which does not ensure that the control actions would not saturate the actuators.
- As stated in Section 5(c), when we compare our methodology with other controllers of the bibliography (C3 and C4) the tracking cost decreases, while it is avoided that the control actions exceed the saturation limits.
- Compared to [1] the proposed methodology is based on basic mathematical concepts and easy to understand. This represents a great advantage when it is desired apply the methodology in a system of different nature.
- The proposed controller is easy to implement, making it suitable for its application in low-profile processors, and its control inputs are the linear and angular velocities, common to most commercial robots.

VII. Conclusion

An efficient control law for trajectory tracking in mobile robots subject to saturation in the control actions has been presented. The conditions for synthesizing the control actions able to minimize tracking errors were obtained by analyzing a system of linear equations. In addition, the developed methodology for the controller design can be applied to other types of systems.

A contribution of this work involves the application of a method to find the parameters of the controller. This method can be used to find values of the control actions that do not exceed the actuators saturation limits. The values found by our approach maintain control actions below the saturation values while the tracking errors tend to zero.

The proposed controller was implemented in a commercial robot PIONEER 3AT, and experimental results were presented, showing that the robot is capable of tracking a desired trajectory with a small distance error. The proposed controller was also compared with others controllers with saturation in control signals, and experimental results showed that it has a better performance.

The proposed controller provides an appropriate value for robot velocity commands, avoiding saturation values of control signals, while keeping a good performance of the control system. Finally, the convergence to zero of tracking errors was demonstrated in Appendix.

ACKNOWLEDGMENTS

This work was partially funded by the National Council for Scientific and Technical Research (CONICET) and the National University of San Juan, Argentine.

APPENDIX

Theorem 1: If the system behavior is ruled by (2) and the controller is designed by (7), (10) and (12), then $e_n \to 0, n \to \infty$ when trajectory tracking problems are considered and the controller parameters fulfill $0<k_v<1$ and $0< k_w<1$.

Remark 4: consider the next geometric progression,

$$a_1 = ka_0$$
$$a_2 = ka_1 = k^2 a_0$$
$$\vdots$$
$$a_{n+1} = ka_n = k^{n+1} a_0$$

Then, if $0 < k < 1$ and $n \to 0$ (with n Є N), then $a_n \to 0$.

The proof of convergence to zero of the tracking errors starts with the variable θ.

Considering the orientation from (2) and the control action from (12),

$$\theta_{n+1} = \theta_n + \frac{T_0}{2}\left(W_n + W_{n+1}\right) \tag{A.1}$$

$$W_{n+1} = \frac{2}{T_0}\left(\theta_{ez,n+1} - k_w\left(\theta_{ez,n} - \theta_n\right) - \theta_n\right) - \frac{\theta_{ez,n+1} - \theta_{ez,n}}{T_0} \tag{A.2}$$

By replacing the control action W_{n+1} given by (A.2) and the Euler approximation of W_n in (A.1), the following expression is found:

$$\theta_{n+1} = \theta_n + \frac{T_0}{2}\left(\frac{\theta_{n+1} - \theta_n}{T_0} + \frac{2}{T_0}\left(\theta_{ez,n+1} - k_w\left(\theta_{ez,n} - \theta_n\right) - \theta_n\right) - \frac{\theta_{ez,n+1} - \theta_{ez,n}}{T_0}\right) \tag{A.3}$$

After some simple operations, it yields:

$$\theta_{ez,n+1} - \theta_{n+1} - k_w\left(\theta_{ez,n} - \theta_n\right) - \frac{1}{2}\left(\theta_{ez,n+1} - \theta_{n+1}\right) + \frac{1}{2}\left(\theta_{ez,n} - \theta_n\right) = 0 \tag{A.4}$$

Thence,

$$\frac{e_{\theta,n+1}}{2} - \left(k_w - \frac{1}{2}\right)e_{\theta,n} = 0 \tag{A.5}$$

$$e_{\theta,n+1} - \left(2k_w - 1\right)e_{\theta,n} = 0 \tag{A.6}$$

Thus, in order to makes the error in (A.6) tends asymptotically to zero,

$$\left(2k_w - 1\right) < 1 \Rightarrow 0 < k_w < 1 \tag{A.7}$$

Then if $0 < k_w < 1$ and $n \to \infty$ (with n Є N), then $e_{\theta,n+1} \to 0$ (see Remark 4).

Now, the convergence analysis of e_x and e_y is developed below. From the corresponding equation of the system (2),

$$x_{n+1} = x_n + \frac{T_0}{2}\left(V_n \cos\theta_n + V_{n+1}\cos\theta_{n+1}\right) \tag{A.8}$$

By using the Taylor interpolation rule, the functions $cos\theta_{n+1}$ can be expressed as,

$$\cos\theta_{n+1} = \cos\theta_{ez,n+1} - \sin\underbrace{\left(\theta_{ez,n+1} + \lambda\left(\theta_{ez,n+1} - \theta_n\right)\right)}_{\theta_{\lambda,n}}\underbrace{\left(\theta_{n+1} - \theta_{ez,n+1}\right)}_{-e_{\theta,n+1}}; 0<\lambda<1 \tag{A.9}$$

Where $\theta_{\lambda,n}$ is an interpolation point between θ_{n+1} and $\theta_{ez,n+1}$. Thus, (A.8) will be:

$$x_{n+1} = x_n + \frac{T_0}{2}\left(V_n\cos\theta_n + V_{n+1}\cos\theta_{ez,n+1} + \underbrace{V_{n+1}\sin\theta_{\lambda,n}}_{f_{\lambda,n}} e_{\theta,n+1}\right) \tag{A.10}$$

Then, considering the control action V_{n+1} (12) and multiplying by $cos\theta_{ez,n+1}$,

$$V_{n+1}\cos\theta_{ez,n+1} = \left(\left(\frac{2}{T_0}\Delta x - V_{ref,n}\cos\theta_{ref,n}\right)\cos\theta_{ez,n+1} + \left(\frac{2}{T_0}\Delta y - V_{ref,n}\sin\theta_{ref,n}\right)\sin\theta_{ez,n+1}\right)\cos\theta_{ez,n+1} \tag{A.11}$$

From (30),

$$\frac{2}{T_0}\Delta y - V_{ref,n}\sin\theta_{ref,n} = \left(\frac{2}{T_0}\Delta x - V_{ref,n}\cos\theta_{ref,n}\right)\frac{\sin\theta_{ez,n+1}}{\cos\theta_{ez,n+1}} \tag{A.12}$$

Considering (A.11) and (A.12), after some simple operations, it yields,

$$V_{n+1}\cos\theta_{ez,n+1} = \left(\frac{2}{T_0}\Delta x - V_{ref,n}\cos\theta_{ref,n}\right)\cos^2\theta_{ez,n+1} + \left(\frac{2}{T_0}\Delta x - V_{ref,n}\cos\theta_{ref,n}\right)\sin^2\theta_{ez,n+1} \tag{A.13}$$

leading to,

$$V_{n+1}\cos\theta_{ez,n+1} = \left(\frac{2}{T_0}\Delta x - V_{ref,n}\cos\theta_{ref,n}\right) \tag{A.14}$$

Taking into account (A.10) and (A.14), it can be shown that:

$$x_{n+1} = x_n + \frac{T_0}{2}\left(V_n\cos\theta_n + \left(\frac{2}{T_0}\Delta x - V_{ref,n}\cos\theta_{ref,n}\right) + \underbrace{V_{n+1}\sin\theta_{\lambda,n}}_{f_{\lambda,n}} e_{\theta,n+1}\right) \tag{A.15}$$

Next, the following replacements are considered:

$$V_n\cos\theta_n = \frac{x_{n+1} - x_n}{T_0} \text{ and } V_{ref,n}\cos\theta_{ref,n} = \frac{x_{ref,n+1} - x_{ref,n}}{T_0} \tag{A.16}$$

According to (A.15),

$$x_{ref,n+1} - x_{n+1} - k_v\left(x_{ref,n} - x_n\right) + \frac{x_{n+1}}{2} - \frac{x_{ref,n+1}}{2} - \frac{x_n}{2} + \frac{x_{ref,n}}{2} + \frac{T_0}{2} f_{\lambda,n} e_{\theta,n+1} = 0 \quad (A.17)$$

Thence,

$$e_{x,n+1} - k_v e_{x,n} - \frac{e_{x,n+1}}{2} + \frac{e_{x,n}}{2} + \frac{T_0}{2} f_{\lambda,n} e_{\theta,n+1} = 0 \quad (A.18)$$

$$\frac{e_{x,n+1}}{2} - e_{x,n}\left(k_v - \frac{1}{2}\right) + \frac{T_0}{2} f_{\lambda,n} e_{\theta,n+1} = 0 \quad (A.19)$$

$$e_{x,n+1} - e_{x,n}\left(2k_v - 1\right) + T_0 f_{\lambda,n} e_{\theta,n+1} = 0 \quad (A.20)$$

Now, applying the same reasoning to the y-coordinate, and taking into account that:

$$e_{\theta,n+1} = \left(2k_w - 1\right) e_{\theta,n} \xrightarrow{0<k_w<1} e_{\theta,n+1} \to 0, n \to \infty$$

it yields, in compact form to:

$$\begin{bmatrix} e_{x,n+1} \\ e_{y,n+1} \end{bmatrix} = \begin{bmatrix} 2k_v - 1 & 0 \\ 0 & 2k_v - 1 \end{bmatrix} \begin{bmatrix} e_{x,n} \\ e_{y,n} \end{bmatrix} + \underbrace{\begin{bmatrix} -T_0 f_{\lambda,n} \\ T_0 f_{\psi,n} \end{bmatrix} e_{\theta,n+1}}_{\text{bounded nonlinearity} \to 0} \quad (A.21)$$

where,

$$f_{\psi,n} = V_{n+1} \cos\theta_{\psi,n}$$

Thus, for $0 < k_v < 1$ the error (A.21) tends asymptotically to zero.

Remark 5: Equation (A.21) is a linear system with a nonlinearity that tends to zero. It can be shown that the nonlinearity is bounded in the same manner as shown for other functions in [20]. If $0 < k_x < 1$ and $0 < k_y < 1$ then $e_{x,n} \to 0$ and $e_{y,n} \to 0$ when $n \to \infty$ ([20], appendix A (A31), (A35)-(A41)).

REFERENCES

[1] Andaluz, V., Roberti F., Toibero, J. M., and R. Carelli, "Adaptive unified motion control of mobile manipulators," *Control Engineering Practice,* Vol. 20, pp. 1337-1352 (2012).

[2] Martins, F. N., Celeste,W. C., Carelli, R., Sarcinelli-Filho, M. and T. F. Bastos-Filho, "An adaptive dynamic controller for autonomous mobile robot trajectory tracking," *Control Engineering Practice,* Vol. 16, pp. 1354-1363 (2008).

[3] Resende, C. Z., Carelli, R. and M. Sarcinelli-Filho, "A nonlinear trajectory tracking controller for mobile robots with velocity limitation via fuzzy gains," *Control Engineering Practice,* Vol. 21, pp. 1302-1309 (2013).

[4] Serrano,M. E., Scaglia, G. J. E., Rómoli,, S., Mut, V. and S. A. Godoy, "Trajectory Tracking Controller based on Numerical Approximation under Control Actions Constraints," presented at the Congreso Bienal de IEEE Argentina, San Carlos de Bariloche, Argentina. pp. 37-42 (2014).

[5] Auat, F. A. and G. Scaglia, "Trajectory Tracking Controller Design for Unmanned Vehicles: A New Methodology," *Journal of Field Robotics,* Vol. 31, pp. 861–887 (2013).

[6] Wang, K., "Near-optimal tracking control of a nonholonomic mobile robot with uncertainties," *International Journal of Advanced Robotic Systems,* Vol. 9, pp 1-11 (2012).

[7] Yang, S. X., Zhu, A., Yuan, G. and M.-H. Meng, "A bioinspired neurodynamics-based approach to tracking control of mobile robots," *Industrial Electronics, IEEE Transactions on,* Vol. 59, pp. 3211-3220 (2012).

[8] Yoo, S. J., "Approximation-based adaptive control for a class of mobile robots with unknown skidding and slipping," *International Journal of Control, Automation and Systems,* Vol. 10, pp. 703-710 (2012).

[9] Keighobadi, J. and M. B. Menhaj, "From nonlinear to fuzzy approaches in trajectory tracking control of wheeled mobile robots," *Asian J. Control,* Vol. 14, pp. 960-973 (2012).

[10] Miao, Z. and Y. Wang, "Adaptive Control for Simultaneous Stabilization and Tracking of Unicycle Mobile Robots," *Asian J. Control,* Vol. 17, No 6, p. 2277-2288 (2015).

[11] Scaglia, G., Montoya, L. Q., Mut, V. and F. di Sciascio, "Numerical methods based controller design for mobile robots," *Robotica,* Vol. 27, pp. 269-279 (2009).

[12] Rosales, A., Scaglia, G., Mut, V. and F. Di Sciascio, "Trajectory tracking of mobile robots in dynamic environments-a linear algebra approach," *Robotica,* Vol. 27, pp. 981-997 (2009).

[13] Rosales, A., Scaglia, G. Mut, V. and F. Di Sciascio, "Formation control and trajectory tracking of mobile robotic systems - A Linear Algebra approach," *Robotica,* Vol. 29, pp. 335-349 (2011).

[14] Serrano, M. E., Scaglia, G. J. E., Godoy, S. A., Mut, V. and O. A. Ortiz, "Trajectory Tracking of Underactuated Surface Vessels: A Linear Algebra Approach," *IEEE Transactions on Control Systems Technology,* Vol. 22, No 3, p. 1103-1111 (2013).

[15] Gandolfo, D., Rosales, C., Patiño, D., Scaglia, G. and M. Jordan, "Trajectory tracking control of a pvtol aircraft based on linear algebra theory," *Asian J. Control,* Vol. 16, pp. 1849-1858 (2014).

[16] Lee, T.-C., Song, K.-T., Lee, C. and C. Teng, "Tracking control of unicycle-modeled mobile robots using a saturation feedback controller," *Control Systems Technology, IEEE Transactions on,* Vol. 9, pp. 305-318 (2001).

[17] Klančar, G. and I. Škrjanc, "Tracking-error model-based predictive control for mobile robots in real time," *Robotics and Autonomous Systems,* Vol. 55, pp. 460-469 (2007).

[18] Serrano, M. E., Scaglia, G. J. E., Cheein, F. A., Mut, V. and O. A. Ortiz, "Trajectory-tracking controller design with constraints in the control signals: a case study in mobile robots," *Robotica,* Vol. 33, No 10, p. 2186-2203 (2014).

[19] Auat, F. A. and G. Scaglia, "Laser-Based Trespassing Prediction in Restrictive Environments: A Linear Approach," *Sensors,* Vol. 12, pp. 11870-11887 (2012).

[20] Scaglia, G., Rosales, A., Quintero, L., Mut, V. and R. Agarwal, "A linear-interpolation-based controller design for trajectory tracking of mobile robots," *Control Engineering Practice,* Vol. 18, pp. 318-329 (2010).

[21] Toibero, J. M., Roberti, F. and R. Carelli, "Stable contour-following control of wheeled mobile robots," *Robotica,* Vol. 27, pp. 1-12 (2009).

[22] Strang, G., *Linear Algebra and Its Applications.* Thomson–Brooks, Cole, Belmont, CA, USA (2005).

[23] Bazaraa, M. S., Sherali, H. D. and C. M. Shetty, *Nonlinear programming: theory and algorithms*, John Wiley & Sons (2013).

[24] Powell, M. and Y. Yuan, "A trust region algorithm for equality constrained optimization," *Mathematical Programming,* Vol. 49, pp. 189-211 (1990).

[25] Roth, S. A. and P. Batavia, "Evaluating Path Tracker Performance for Outdoor Mobile Robots," *Automation Technology for Off-Road Equipment,* American Society of Agricultural and Biological Engineers, Chicago, Illinois, USA, pp. 399-397 (2002).

[26] Cha, P. D. and J. I. Molinder, *Fundamentals of Signals and Systems with CD-ROM: A Building Block Approach*, Cambridge University Press (2006).

[27] Marti K., *Stochastic optimization methods*, Berlin, Springer (2005).