NATURAL FREQUENCIES OF THIN RECTANGULAR PLATES WITH PARTIAL INTERMEDIATE SUPPORTS

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***Abstract*— In the present study, a methodology to find natural frequencies with arbitrary precision of thin rectangular plates on linear intermediate supports and mixed boundary conditions is presented. This means that the edges are total or partially supported, clamped or free, or any combination of these. The layout, number and place of linear intermediate supports are arbitrary, which allows for the analysis of a wide range of cases that include intermediate supports of different kinds: simple and multiple, straight and curved, complete (the ends coincide with the plate edges) and partial (at least one of the ends is not coincident with the plate edges). In the case of curved linear supports, the curve can be open or closed. The generalized solution is obtained using the Whole Element Method. A continuous and a discrete model of equidistant points are studied both for intermediate supports and clamped edges. In all cases, both a systematic approach to the solution and the theoretical basis, which ensures the arbitrary precision of the results, should be emphasized. In order to illustrate the accuracy and efficiency of the method described, numerical results are presented for several problems and comparison is made with previously published results in some cases and in some others with the Finite Element Method. These numerical results may be of interest to design engineers and researchers who conduct vibration studies.**

***Keywords*— plates, free vibrations, intermediate supports, exact frequencies.**

I. INTRODUCTION

Rectangular plates on intermediate supports find use in many engineering structures and other areas of practical interest, such as slabs on columns, printed circuit boards or solar panels supported at a few points. With their potential applications, the vibration of plates with internal supports and with complex boundary conditions has received considerable attention from researchers. The free flexural vibration of rectangular plates has been the subject of numerous studies, many of which have been discussed by Leissa (1969, 1981,1987). Plates involving various complexities have been considered, including the case in which the plate is supported by internal lines. For such plates, most of the reported work has been concerned with plates with internal line supports, which are straight and parallel to the edges of the plates. Some examples of theses studies are the works by Veletsos and Newmark (1956), Wu and Cheung, (1974), Elishakoff and Stemberg (1979). The literature on the vibration of rectangular plates for which the internal line supports are not parallel to the edges is sparse. Two examples of such work are that by Gorman (1979) who studied the vibration of diagonally supported rectangular plates, and that by Takahashi and Chishaki (1979), who represented the internal line support by rows of equidistant point supports and gave frequency parameters and mode shapes for a rectangular plate with an oblique line support passing through its center at various angles. Li and Gorman (1992) considered rectangular plates with free edges and intermediate linear supports along one or two diagonals with the superposition method. Other studies of rectangular plates with an oblique intermediate support were carried out by Kim (1995) and Sanzi and Laura (1989) who used the Rayleigh-Ritz method. Young and Dickinson (1993) used the Rayleigh-Ritz method to obtain the eigenvalues for the free vibration of rectangular plates with the supports lying along different types of curves, including a central circular support. Cheung and Kong (1995) used modified single-span vibrating beam functions with the finite layer method to study the vibration of shear-deformable plates with intermediate line supports. Another general study in this particular area is that by Fan and Cheung (1984), in which they used the spline strip element method to analyze plates with complex boundary conditions and point supports.

Huang and Thambiratnam (2001) used the finite step element method combined with a spring system to treat the free vibration analysis of plates on elastic intermediate supports. Saadatpour *et. al.* (2000) developed a numerical technique for the dynamic analysis of arbitrary quadrilateral –shaped plates with internal supports.

We must emphasize that in all these works the intermediate supports are complete, *i.e.*, the ends coincide with the plate edges.

More recently, Zhao and Wei (2002) used the discrete singular convolution (DSC) algorithm for the analysis of rectangular plates with non-uniform and combined boundary conditions. Finally Zhao *et al.* (2002), who made an extensive bibliography review about the problem of vibration of thin plates with internal supports, proposed a novel computational method to study the problem of plate vibration under complex and irregular internal support conditions.

Except for this last paper, the authors are not aware of any other work published in the open literature dealing with plates with partial internal line supports, which is the main objective in this work.

Technically, it is more difficult to analyze plates with irregular or partial internal supports. The problem is far more complicated to admit an analytical solution. For example, this problem is difficult to tackle by the Ritz Method if the internal support topology cannot be analytically expressed, even using penalty approaches. It is evident that the partial internal support is still a challenge for the Ritz method and it has not been solved by it yet.

II. FORMULATION OF THE PROBLEM

The vibrational problem is analyzed within the classical theory of thin plates (Germain-Lagrange) using the Whole Element Method (WEM). This is a direct variational method previously founded and developed for boundary value problems, as well as those with initial conditions and others governed by partial differential equations, uni, bi or tridimensional domains, conservative or not, linear or not: Rosales (1997), Filipich *et. al.* (1998, 1999), Rosales *et a*l. (1999), Filipich and Rosales, (1997, 1999a, 1999b), Rosales and Filipich (2000), Rosales *et al*. (2000a,b).

It starts from an *ad-hoc* functional (in this problem, the energetic functional) and proposing an extremizing sequence of functions belonging to a complete set in L_2. Unlike the Ritz Method in which each coordinate function must satisfy the boundary conditions, WEM requires the complete sequence to satisfy these conditions. Eventual non-satisfied conditions are taken into account by Lagrange multipliers. The sequences used are extended trigonometric series, which are systematically generated. It should be noted that the previous study of the mode shapes is not necessary.

In all cases, both a systematic approach to the solution and the theoretical basis that ensures the arbitrary precision (accuracy) of the results should be emphasized.

The results obtained with WEM are contrasted with the values obtained from the classical solution when this is available and with other ones obtained by others authors by approximate methods and the Finite Element Method (FEM) using the ALGOR® software (Algor, 1999).

It is assumed that the plate under consideration is a thin rectangular plate, lies in the *x*-*y* plane, is bounded by the edges $X = 0$, $X = a$, $Y = 0$ and $Y = b$, of uniform thickness h_p, isotropic material, uniform density ρ , Young Modulus E.

In Fig. 1, a rectangular plate with a partial intermediate support with arbitrary design is shown. Let $\mathbf{s}(t) = \left(x(t), y(t)\right)$ with $t_0 \le t \le t_1$ be the equation of the intermediate support

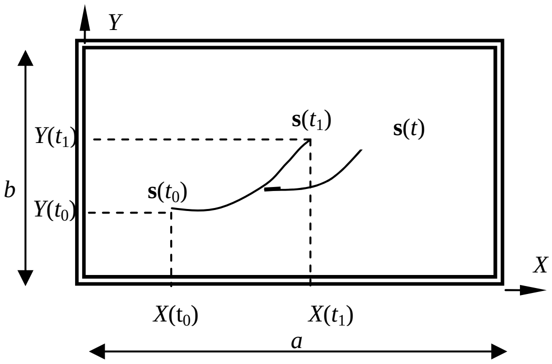


Figure 1: rectangular thin plate on intermediate linear supports

The energetic functional corresponding to the free vibrations of the thin rectangular plate, using the German-Lagrange theory, is:

$$\Im^*[w] = \left\| w'' + \lambda^2 \bar{\bar{w}} \right\|^2 + 2\lambda(1-\upsilon)\left[\left\| \bar{w}' \right\|^2 - \left(w'', \bar{\bar{w}} \right) \right] - \Omega^2 \left\| w \right\|^2 \qquad (1)$$

where $\lambda \equiv a/b$, $\Omega \equiv \omega a^2 \sqrt{\rho h_p / D}$, $D = E h_p^3 \big/ 12(1-\upsilon^2)$ w denotes the transverse displacement and υ the Poisson's ratio, being $(\cdot)' \equiv \partial w/\partial x$, $\overline{(\cdot)} \equiv \partial w / \partial y$, etc. There are also introduced the non-dimensional quantities

$$x \equiv X/a, \ y = Y/b \qquad (2)$$

in order to work in the domain $\{0 \le x \le 1, \ 0 \le y \le 1\}$.

WEM requires an extremizing sequence, which in the thin plates problem is a bidimensional sequence. Being the functional under study, quadratic and positive defined, this sequence will be minimizing. When the plate problems are tackled with direct methods, boundary conditions that involve essential functions must be imposed, where the latter are the conditions that involve the solution and its derivative functions up to the order k-1, being 2*k* the order of the differential equation that govern the problem. Thus, both the displacement function and its first order derivatives must verify uniform convergence. One of the possible sequences to use is:

$$w_{MN}(x,y)=\sum_{i=1}^{M}\sum_{j=1}^{N}A_{ij}s_i s_j+y\left(\sum_{i=1}^{M}A_{i0}s_i+b_0\right)+ x\left(\sum_{j=1}^{N}A_{0j}s_j+a_0\right)+\sum_{i=1}^{M}a_i s_i+ \sum_{j=1}^{N}b_j s_j+A_{00}xy+k_0 \quad (3)$$

where $s_i\equiv\sin(\alpha_i x)$, $s_j\equiv\sin(\alpha_j y)$, $\alpha_i\equiv i\pi$, $\alpha_j\equiv j\pi$ and A_{ij}, A_{i0}, A_{0j}, a_i, b_j, A_{00}, a_0, b_0, k_0 are the unknown coefficients, Escalante (2001).

Introducing the boundary conditions (B.C.) for the case of a rectangular simple supported plate in (3), that is:

$$w(x,0)=w(0,y)=w(x,1)=w(1,y)=0$$

and

$$w'(x,0)=\overline{w}(0,y)=w'(x,1)=\overline{w}(1,y)=0$$

the sequence is reduced to:

$$w_{MN}(x,y)=\sum_{i=1}^{M}\sum_{j=1}^{N}A_{ij}\,s_i\,s_j \quad (4)$$

A. Intermediate Supports

To illustrate the methodology we consider a simply supported rectangular plate with an internal support. We present two different models: continuous and discrete.

A.1. Continuous Model

In this case, the expression of the extended functional to be used is:

$$\overline{\Im}(w)=\left\|w''+\lambda^2\overline{\overline{w}}\right\|^2-\Omega^2\left\|w\right\|^2 -\int_0^1 f(\varepsilon)\,w^*(\varepsilon)\,d\varepsilon \quad (5)$$

where the two first terms correspond to the energetic functional for thin plates with their edges simply supported. The last term puts in evidence the nullity of the transversal displacement on the internal support, by means of the integral that represents the virtual work of the reaction. In that integral, $f(\varepsilon)$ plays the role of a Lagrange multiplier. Next, we show the reasoning by which we obtain that integral.

Let $F(X(t),Y(t))$ be the distributed reaction on the intermediate support (unknown), $S(X(t),Y(t))$ the parametric equation of the support and $W(X,Y)$ the transversal displacement, the virtual work of the reaction on the support must be vanished, that is

$$\int_S F(X,Y)\,W(X,Y)\,dS=0 \quad (6)$$

where again, by changing to the new variables indicated in (2) we obtain

$$\int_s f^*(x,y)\,w(x,y)\,ds=0 \quad (7)$$

being

$$w=w(x,y),\ f^*=f^*(x,y)\ \text{ y }\ s=s(x,y)$$

the transversal displacement equations, the support reaction and the Cartesian equation of the support in dimensionless coordinates respectively. Equation (7) written as a function of the parameter t is:

$$\int_{t_0}^{t_1} f^*\left(x(t),y(t)\right)w(x(t),y(t))\sqrt{(a\dot{x})^2+(b\dot{y})^2}\,dt=0 \quad (8)$$

Defining:

$$\varepsilon\equiv\frac{t-t_0}{t_1-t_0}=\frac{t-t_0}{\Delta}\,; \qquad 0\le\varepsilon\le 1 \quad (9)$$

isolating *t*:

$$t=\varepsilon\,\Delta+t_0 \quad (10)$$

and replacing it in (8) we obtain:

$$\int_0^1 f(\varepsilon)\,w^*(\varepsilon)\,d\varepsilon=0 \quad (11)$$

where

$$w^*(\varepsilon)=w(x(\varepsilon\Delta+t_0),y(\varepsilon\Delta+t_0))$$

and

$$f(\varepsilon)\equiv\frac{f^*(x(\varepsilon\Delta+t_0),y(\varepsilon\Delta+t_0))}{\Delta}\cdot \sqrt{\left(a\dot{x}(\varepsilon\Delta+t_0)\right)^2+\left(b\dot{y}(\varepsilon\Delta+t_0)\right)^2}$$

that is proportional to the reaction on the internal support and in general it is not object of study.

Next, we should apply the condition to minimize equation (5), that is

$$\delta\overline{\Im}(w_{MN})=0 \quad (12)$$

However, this would yield again to $w^*_{MN}(\varepsilon)=0\ \ \forall\varepsilon$

in [0,1], which we already knew and thus is not useful. We must think, then, that the Lagrange condition (11) is actually an infinite sum, *i.e.*

$$\int_0^1 f(\varepsilon) w_{MN}[x(\varepsilon\Delta + t_0), y(\varepsilon\Delta + t_0)]\, d\varepsilon = \lim_{\substack{n\to\infty \\ \Delta_k \to 0}} \sum_{k=1}^{n} f(\varepsilon_k) w_{MN}[x(\varepsilon_k\Delta + t_0), y(\varepsilon_k\Delta + t_0)]\Delta_k \quad \left(\Delta_k = \varepsilon_{k+1} - \varepsilon_k\right) \tag{13}$$

To be consequent with WEM, we propose to expand the unknown reaction as an extended trigonometric series, that is

$$f(\varepsilon) = \begin{cases} \sum_{p=1}^{P} a_p \operatorname{sen}(p\pi\varepsilon) + \varepsilon\, a_0 + k & (a) \\ \sum_{p=0}^{P} b_p \cos(p\pi\varepsilon) & (b) \\ \sum_{p=1}^{P} c_p \operatorname{sen}(p\pi\varepsilon) & (c) \end{cases} \tag{14}$$

where a_p, a_0, k, b_p and c_p are unknown constants. As we know both (14)(a) and (14)(b) have uniform convergence for all continuous $f(\varepsilon)$ in $[0,1]$.

Then, the extended functional in case we adopt the series given by (14)(c) for $f(\varepsilon)$ is:

$$\overline{\mathfrak{I}}_{MN} = \left\| w''_{MN} + \lambda^2 \overline{\overline{w}}_{MN} \right\|^2 - \Omega^2 \left\| w_{MN} \right\|^2 - \sum_{p=0}^{P} b_p \int_0^1 \cos(\alpha_p \varepsilon)\, w^*_{MN}(\varepsilon)\, d\varepsilon \tag{15}$$

being $\alpha_p = p\pi$, with $p = 0,1,\dots,P$

Now, applying the condition (12) and integrating it we yield to

$$\sum_{i=1}^{M}\sum_{j=1}^{N} \delta A_{ij} \left\{ \frac{A_{ij}\Delta_{ij}}{4} - \sum_{p=0}^{P} b_p K_{pij} \right\} - \sum_{p=0}^{P} \delta b_p \left\{ \sum_{i=1}^{M}\sum_{j=1}^{N} A_{ij} K_{pij} \right\} = 0 \tag{16}$$

where:

$$\Delta_{ij} = \left(\alpha_i^2 + \lambda^2 \alpha_j^2\right)^2 - \Omega^2 \tag{17}$$

and

$$K_{pij} = \int_0^1 \Big(\cos(\alpha_p \varepsilon) \operatorname{sen}(\alpha_i x[t_0 + \varepsilon\Delta]) \operatorname{sen}(\alpha_j y[t_0 + \varepsilon\Delta]) \Big)\, d\varepsilon \tag{18}$$

For equation (16) to be satisfied for any arbitrary variations of δA_{ij} y δb_p the following equations must be verified:

$$\begin{cases} \frac{A_{ij}\Delta_{ij}}{4} - \sum_{p=0}^{P} b_p K_{pij} = 0 & \begin{pmatrix} i = 1,2,\dots,M \\ j = 1,2,\dots,N \end{pmatrix} \quad (a) \\ \sum_{i=1}^{M}\sum_{j=1}^{N} A_{ij} K_{pij} = 0 & (p = 0,1,2,\dots,P) \quad (b) \end{cases} \tag{19}$$

Isolating A_{ij} from (19)(a) and replacing it in (19)(b) we obtain

$$\sum_{i=1}^{M}\sum_{j=1}^{N} \left(\frac{4}{\Delta_{ij}} \sum_{p=0}^{P} b_p K_{pij} \right) K_{qij} = 0 \quad (q = 0,1,2,\dots,P) \tag{20}$$

Defining

$$K_{pq} \equiv \sum_{i=1}^{M}\sum_{j=1}^{N} \frac{K_{pij}\, K_{qij}}{\Delta_{ij}} \quad (p, q = 0,1,2,\dots,P) \tag{21}$$

equation (20) can be written as follows:

$$\sum_{p=0}^{P} b_p\, K_{pq} = 0 \quad (q = 0,1,2,\dots,P) \tag{22}$$

The fundamental equation (22) represents a homogeneous linear system of $(P+1)$ order from which $(P+1)$ frequencies $\Omega_k\ (k = 0,1,2,\dots,P)$ can be obtained.

A.2 Discrete Model

The second alternative that we propose in order to tackle the intermediate support is a discrete model considering it as an equidistant set of points.

Let $S(t) = \left(X(t), Y(t)\right)$ with $t_0 \le t \le t_1$ be again the parametric equation of the intermediate linear support, P intermediate points are considered, whose dimensionless coordinates are

$$P_k = \left(x_k(t), y_k(t)\right) = \left(x_k(t_0 + \frac{k}{P-1}\Delta), y_k(t_0 + \frac{k}{P-1}\Delta \right)$$

with $k = 0,1,2,...,(P-1)$ and $\Delta = t_1 - t_0$

As the support has been replaced by the set of equidistant points, the condition of null transversal displacement is required for each of them. That is obtained by using Lagrange multipliers.

Thus, the extended functional will be:

$$\overline{\Im}_{MN} = \left\| w''_{MN} + \lambda^2 \overline{\overline{w}}_{MN} \right\|^2 - \Omega^2 \left\| w_{MN} \right\|^2 - \sum_{k=0}^{P-1} \lambda_k \, w_{MN}\left(x_k, y_k\right) \tag{23}$$

By vanishing the first variation we obtain:

$$\sum_{i=1}^{M}\sum_{j=1}^{N} \delta A_{ij} \left\{ \frac{A_{ij}\Delta_{ij}}{4} - \sum_{k=0}^{P-1} \lambda_k L_{kij} \right\} - \sum_{k=0}^{P-1} \delta\lambda_k \left\{ \sum_{i=1}^{M}\sum_{j=1}^{N} A_{ij} L_{kij} \right\} = 0 \tag{24}$$

being:

$$\Delta_{ij} = \left(\alpha_i^2 + \lambda^2 \alpha_j^2\right)^2 - \Omega^2 \tag{25}$$

and

$$L_{kij} = \operatorname{sen}\left(\alpha_i x_k\right) \operatorname{sen}\left(\alpha_j y_k\right) \tag{26}$$

For equation (16) to be satisfied for any arbitrary variations of δA_{ij} and $\delta \lambda_k$ the following equations must be verified:

$$\begin{cases} \dfrac{A_{ij}\Delta_{ij}}{4} - \displaystyle\sum_{k=0}^{P-1} \lambda_k L_{kij} = 0 \quad \begin{pmatrix} i = 1,2,...,M \\ j = 1,2,...,N \end{pmatrix} & (a) \\ \displaystyle\sum_{i=1}^{M}\sum_{j=1}^{N} A_{ij} L_{kij} = 0 \quad (k = 0,1,2,...,(P-1)) & (b) \end{cases} \tag{27}$$

Isolating A_{ij} from (27)(a) and replacing it in (27)(b) we have

$$\sum_{i=1}^{M}\sum_{j=1}^{N} \left(\frac{4}{\Delta_{ij}} \sum_{k=0}^{P-1} \lambda_k L_{kij} \right) L_{qij} = 0 \qquad (q = 0,1,2,...,(P-1)) \tag{28}$$

Defining:

$$L^*_{kq} \equiv \sum_{i=1}^{M}\sum_{j=1}^{N} \frac{L_{kij}\, L_{qij}}{\Delta_{ij}} \qquad (k, q = 0,1,2,...,P-1) \tag{29}$$

equation (20) is reduced to

$$\sum_{k=0}^{P-1} \lambda_k \, L^*_{kq} = 0 \qquad (q = 0,1,2,...,P-1) \tag{30}$$

Equation (30) represents a homogeneous linear system with order P from which P frequencies $\Omega_k \quad (k = 1,2,...,P)$ can be obtained.

B. Partially Clamped Edges

The same ideas exposed before to deal with intermediate supports can be applied to model total or partial clamped edges. In analogous way, we have both a continuous and discrete model as well.

For brevity we show only the discrete model. The partially clamped edges are dealt with the same way as the intermediate supports. The partially clamped edge is replaced by Q equidistantly spaced points, to which normal rotation must be restrained. That is, the first normal derivative is equal to zero:

$$w_n(x_q, y_q) = 0 \qquad \forall q = 0,1,2,..,Q-1 \tag{31}$$

being

$$w_n(x_q, y_q) = \nabla w(x_q, y_q) \cdot \breve{\mathbf{n}} \,,$$

∇ is the operator gradient and $\breve{\mathbf{n}} = (n_x, n_y)$ is the unit normal vector (director cosines) on the clamped edge. Then, per each partial clamped edge we need to add the terms to put them in evidence in the functional. *i.e.*

$$\sum_{q=0}^{Q-1} \eta_q w_n(x_q, y_q) = 0 \tag{32}$$

where η_q is the Lagrange multiplier.

C. Convergence Study

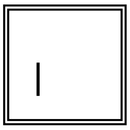
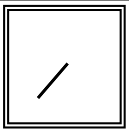
To obtain the eigenvalues with the desired precision, the convergence study must be done adopting at first, a fixed value for *P*. This means adopting a rigid model of *P* points for the intermediate support. Then, the *M* and *N* values are increased until the desired precision is attained. Next, a larger value for *P* is adopted and the calculations are repeated increasing again the *M* and *N* values. The eigenvalue obtained in this step will have a higher or equal value than the obtained in the previous step because this model is more rigid.

III. NUMERICAL RESULTS

In order to illustrate the accuracy and utility of the above-described approach, numerical results are presented for several examples and comparison is made with previously published results when possible.

In Table 1 natural frequencies of a rectangular simply supported plate are given for two different cases of intermediate linear support. Results are obtained for both, continuous and discrete model and they are contrasted with the results obtained by the Finite Element Method.

Table 1: Frequencies Parameters. A: Continuous model, $M = N$ =300, P = 50, B: Discrete model, C: FEM (ALGOR), Mod. Model C: $M = N$ = 300, P = 30; MEF (ALGOR) Mesh: 1024 elements. $P_0=(x_0, y_0)$, $P_1=(x_1,y_1)$.

MODEL	INTERNAL SUPPORT	Ω			
			1	2	3
	P_0 = (0.25,0.25) P_1 = (0.25,0.50)	A	30.91	55.59	76.67
		B	30.92	55.62	76.75
		C	30.93	55.62	76.74
	P_0 = (0.25,025) P_1 = (0.50,0.50)	A	49.34	52.64	72.67
		B	49.34	52.77	72.69
		C	49.35	52.72	72.71

Next, natural frequencies of the square plate with various boundary conditions and an intermediate support along its diagonal are given in Table 2.

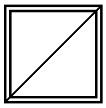
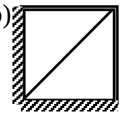
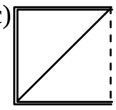
It should be noted that as long as the plate under study has partial or totally free edges, the energetic functional used by this methodology includes the terms corresponding to the Gaussian curvature.

The results are contrasted with the values obtained by the Finite Element Method (FEM) by using the ALGOR® software and with those obtained by Kim (1995) by the Rayleigh-Ritz Method using a polynomial orthogonal set as trial functions.

The case of a square plate with various boundary conditions and a partial intermediate support is shown in Table 3. The results are compared only with the values obtained by the FEM (ALGOR®), since according to the authors, no others previous results have been published in the open literature, which deal with partial intermediate supports using other methodologies.

The numerical results corresponding to the case of a square plate simply supported and a partial intermediate curved support are also shown in Table 4. In this example, the internal support is a circumferential arc with radius r = 0.25 and center coincident with that of the plate.

Table 2: Natural frequencies of a plate with various boundary conditions and an intermediate support along its diagonal. WEM: $M = N$ = 300, P = 30, Q = 90, υ = 0.3. FEM (ALGOR): mesh of 1600 elements. (*) Kim (1995)

Model	Method	Eigenvalues		
		Ω_1	Ω_2	Ω_3
a) 	WEM	49.34	65.79	98.69
	KIM (*)	49.34	65.80	98.69
	FEM	49.36	69.85	98.74
b) 	WEM	60.53	78.95	114.55
	KIM (*)	60.54	79.03	114.60
	FEM	60.57	78.98	114.62
c) 	WEM	26.41	55.38	65.76
	KIM (*)	26.43	55.41	65.86
	FEM	26.40	55.39	65.71

Finally, the numerical results for two examples of rectangular plates with multiple intermediate supports are given in Table 5. The frequency parameters are compared again with the values obtained by FEM using the software ALGOR since there are not any known values obtained with other methodologies.

Table 3: Natural frequencies of a square plate with varied boundary conditions and a partial intermediate support. WEM: $M = N$ = 300, P = 30, Q = 15, υ = 0.3. FEM (ALGOR): mesh of 1600 elements.

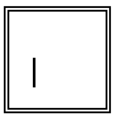
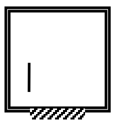
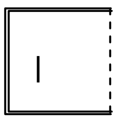
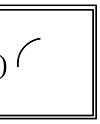
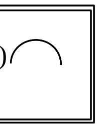
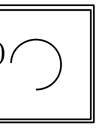
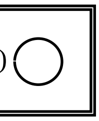
Model	Method	Eigenvalues		
		Ω_1	Ω_2	Ω_3
	WEM	30.926	55.620	76.751
	FEM	30.931	55.627	76.745
	WEM	32.72	60.33	78.52
	FEM	32.74	60.37	78.71
	WEM	14.29	42.46	44.62
	FEM	14.31	42.46	44.63

Table 4: Natural frequencies of a fully simple square plate with an intermediate circumferential arc support. Center (0.5,0.5), r = 0.25. WEM: $M = N = 400$, a) P = 20, b) P = 30, c) P = 35, P = 40. FEM (ALGOR) Mesh of 1600 elements. (*) Young and Dickinson (1993).

Model	Central angle		Eigenvalues		
			WEM	FEM	(*)
a) 	$\pi / 4$	Ω_1 Ω_2 Ω_3	37.05 67.00 84.64	37.04 67.00 84.75	
b) 	$\pi / 2$	Ω_1 Ω_2 Ω_3	52.88 87.62 98.75	53.00 87.94 99.18	
c) 	$3\pi / 2$	Ω_1 Ω_2 Ω_3	95.39 98.89 126.49	96.18 99.50 126.72	
d) 	2π	Ω_1 Ω_2 Ω_3	98.89 126.50 132.48		98.91 126.61 132.64

The plate is simply supported on its edges and the three lowest eigenvalues are given.

V. CONCLUSIONS

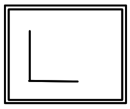
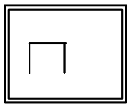
In this work a methodology to determine natural frequencies of thin rectangular plates, with arbitrary precision using the Whole Element Method (WEM) was shown. This is a direct variational method previously founded and developed for boundary value problems, as well as those with initial conditions and others governed by partial differential equations, uni, bi or three-dimensional domains, conservative or not, linear or not. It starts from an *ad-hoc* functional (in this problem, the energetic functional) and proposing an extremizing sequence of functions belonging to a complete set in L_2. It should be noted that the previous study of the mode shapes is not necessary.

The boundary conditions must be satisfied by the complete sequence and eventual non-satisfied conditions are taken into account by Lagrange multipliers. The sequences used are extended trigonometric series, which are systematically generated and formally identical in all cases.

Two models (discrete and continuous) were proposed to consider the intermediate supports. In both cases, the efficiency is similar, however the discrete model is more practical and simpler to implement by computational algorithms.

Although we limited our attention to the vibration of rectangular thin plates with linear intermediate supports, the methodology can be used to analyze plates with complex and irregular internal support conditions as well.

Table 5: Natural frequencies parameters of rectangular plates with multiple partial intermediate supports. λ: aspect ratio. WEM: P = 20, $M = N = 300$, FEM: Mesh of 5120 elements.

Model	Intermediate Supports	Ω	WEM	FEM
	$\lambda = 1.25$ $x = 0.2$ $(0.25 \le y \le 0.75)$ $y = 0.25$ $(0.2 \le x \le 0.6)$	Ω_1 Ω_2 Ω_3	50.53 92.36 121.94	50.57 92.50 121.60
	$\lambda = 1.25$ $x = 0.2$ $(0.25 \le y \le 0.5)$ $y = 0.25$ $(0.2 \le x \le 0.5)$ $x = 0.5$ $(0.25 \le y \le 0.5)$	Ω_1 Ω_2 Ω_3	63.57 93.66 109.79	63.50 93.71 109.86

In the examples we have analyzed, a discrete model for the clamped edges was adopted. Then the effect of the clamped segment on the edges on the vibration is taken into account by replacing the clamped segment on the edges with rows of equidistantly spaced points. These, in turn, are restrained from either transverse deflection or normal rotation respectively.

The generality of the model studied is one of the most important advantages, since a unique computational algorithm allows determining the natural frequencies of thin plates with various and non-classical boundary conditions and with multiple internal linear or irregular supports.

It has been shown that the methodology proposed may be used successfully. Good agreement is observed for all cases and the results can be used for comparative studies by future researchers.

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Received: June 7, 2002.
Accepted: February 16, 2004.
Recommended by Subject Editor E. Dvorkin.