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Integrated Planning and Scheduling with Due Dates in the Corrugated Board Boxes Industry

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Supporting Information

ABSTRACT: An integrated approach that solves the cutting stock problem and scheduling is considered in this article. The main challenging characteristics of this problem are given by its combinatorial nature, as well as the nonconvexities appearing in the formulation. Scheduling decisions are directly affected by cutting patterns and present changeover times that are sequence-dependent. Given the problem complexity, in general, cutting stock optimization has been considered independently from the scheduling problem. However, a cutting plan that defines the sequence in which patterns are processed is essential in order to obtain a solution in the context of corrugated board boxes industry. In order to obtain a global solution, the approach that has been developed uses a disjunctive technique to transform the original nonconvex formulation into a mixed-integer linear programming (MILP) model; whereas a continuous time representation is assumed to model sequencing decisions by applying general and immediate precedence constraints. Both scheduling models are presented and compared in three examples, showing efficient solutions for the integrated problem.

1. INTRODUCTION

Cutting stock problems (CSPs) have been intensively studied over the past decade, and the number of publications has increased considerably during this period. Generally speaking, a CSP consists of cutting large pieces of material into smaller ones; this process is executed by means of patterns where the same cut is repeated several times. It has many industrial applications, such as the processing of wood, metal, paper, glasses, leather, etc.; a complete characterization of this problem can be found in the work of Wascher et al.[1] According to this typology, the problem is a two-dimensional one, with input minimization, since all small items must be assigned to large items. In addition, there are many items of relatively few different shapes (width, length, and board type) and there are also several large different objects (raw paper rolls). In conclusion, this problem can be classified as a *multiple stock size of cutting stock problem*.

CSP complexity is given by its combinatorial nature and the presence of nonconvex constraints having terms with the product of two variables (bilinear terms). Several objective functions can be applied in order to obtain an optimal solution. For those reasons, the research work is focused on the proposition of new models and algorithms[2] to reach an optimal solution in a reasonable execution time.

The very first relevant works were developed by Gilmore and Gomory,[3−5] where patterns are pregenerated and considered as known vectors in a mixed-integer programming (MIP) optimization model that decides the number of them to produce. Given the NP-hard nature of the problem, many articles have proposed heuristic procedures to solve the CSPs.[6−9] However, some works also present optimization models to solve the problem in different industrial applications, such as cutting pieces of wood,[10] stainless steel,[11] paper rolls[12] and glass industry.[13] Erjavec et al.[14] considered the CSP to define raw material stock size.

Given the problem complexity, in general, cutting stock optimization is considered independently from the scheduling problem. The short-term scheduling of a cutting process consists of deciding the order in which cutting patterns must be processed to guarantee compromised due dates, among other constraints. However, cutting decisions could greatly affect schedule feasibility, so an integrated approach is crucial to offer an optimal result.

In the case of wood cutting application, Yanasse[15] proposed a pattern sequencing method. The main challenge is given by a limited storage space around the saw machine. During the process, stacks of panels being cut can be removed only after their orders are completed. The problem consists of sequencing the patterns in order to minimize the number of stacks opened. One assumption of this approach is that patterns are already defined and the main focus is given by the sequencing problem. Examples of four and six patterns are considered to test the approach, which are solved with a branched-and-bound algorithm.

Westerlund and Isaksson[16] considered the problem of producing paper reels from larger ones, such that certain specifications are satisfied. Cutting patterns are calculated in advance, so the optimization model selects the patterns to cut. Although patterns are not sequenced in the time horizon, they include a constraint that guarantees that the total time consumed to produce the pattern is lower than the time to deliver minus the occupation time for cutting machine. Changeover times are not sequence-dependent. An example with eight orders is used to illustrate the approach proposed.

Giannelos and Georgiadis[17] also presented the scheduling of cutting-stock processes assuming that feasible patterns are determined a priori. They use a relax-and-fix heuristic to solve the problem where multiple identical parallel machines are considered.

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An example with eight product types, three cutting machines, three raw material types, and four time slots for each machine is presented. They also present an integrated optimization approach in the cutting stock problem of paper rolls considering due dates in customer orders,[18] analyzing three industrial cases studies.

Johnston and Sadinlija[19] integrated the optimal cutting plan and sequencing decisions, assuming that the order of the pattern also corresponded to the position of the pattern in the manufacturing sequence. This idea shows some correspondence with the time slot representation of traditional scheduling problems, because the *j*th pattern is processed right before the *j*th + 1 pattern. There is no limitation regarding the possible combinations between orders or in the type of raw material to use in each pattern. With the proposed approach, they solve several examples disregarding sequencing constraints and one example of seven orders in which due dates are taken into account.

In the work presented by Yanasse,[20] an integer linear programming formulation is presented for the integrated problem, and they proposed a solution procedure based on Lagrangian relaxation to decompose the problem into two subproblems. They used a discrete time representation to address the scheduling problem. They also proposed a heuristic procedure to provide feasible solutions to the integrated problem.

In order to ensure regularity of downstream material flow, choosing patterns and run lengths is not enough.[21] A cutting plan that defines the sequence in which patterns are processed is essential in this sense. The problem that was presented by Arbib, Mairnelli, and Pezzella[21] included finding a pattern sequence that fulfilled demand and reduced trim-loss and the number of stacks maintained open during the process. They proposed a heuristic procedure based on tabu search to define a set of feasible cutting patterns and schedule.

Haessler and Talbot[22] developed a 0−1 model to solve the corrugators trim problem. The complexity of the cutting stock and trim-loss problem in the context of the corrugated board boxes industry is described in detail, showing the several tradeoffs involved. A limited number of patterns are pregenerated by applying a set of economical rules. The most convenient patterns are selected by the optimization model. Only some types of patterns are allowed to restrict the problem combinatory; sequencing decisions or due dates are not included in the model. Also, it is assumed that raw materials of different sizes (widths) are available but no consideration regarding different paper types is taken into account. These authors point out that the industrial context of this cutting stock and trim-loss problem present particular characteristics that deserve special attention and study in order to reach an optimal solution.

In our approach, the integration of the optimal cutting plan and scheduling through a mathematical optimization is considered. In general, most articles assume that cutting patterns are already defined when scheduling or sequencing decisions are analyzed or apply heuristic techniques at some point of the execution process to come up with a result. In the proposed work, the CSP and scheduling are considered simultaneously in one optimization model, avoiding the use of heuristic strategies and taking into account the special characteristics of the production process under analysis.

The production of board boxes is even more complex than cutting large raw material into smaller pieces, since additional considerations must be taken into account. In order to solve the problem to global optimality in one step, cutting patterns are not known a priori, because they become part of the decision model. Their definition needs the assignment of papers of specific width and type to the different layers of the board, which becomes an important part of the model. As usual, several customer orders are assigned to cutting patterns where sheets of board of different sizes must be cut. On the other hand, scheduling must consider that processing times are variables that are dependent on the determination of the cutting patterns. Excessively long patterns, for instance, might violate due date commitments with customers. Different changeover times are involved in this problem; some of them are fixed, while others are influenced by the sequence. Only an integrated approach can guarantee an optimal, and even feasible, solution to the problem.

Since the original formulation is nonlinear due to bilinear terms from the cutting pattern decisions, we propose a disjunctive technique to obtain a linear formulation. In addition, scheduling decisions are modeled applying binary precedence variables over a continuous time representation. Three examples will be analyzed in order to test the efficiency of the model.

This article is organized as follows: in section 2, the cutting process in the context of corrugated board boxes is presented; section 3 introduces some modeling issues to take into account in order to formulate the optimization model; the next part (section 4) shows the formulation of the problem as a mixed-integer non-linear program (MINLP), which is then transformed into a linear disjunctive problem and finally relaxed as a mixed-integer linear programming (MILP) formulation; results are presented in section 5, which is finally followed by a discussion and conclusion section.

2. CUTTING PROCESS CHARACTERIZATION

It is a general practice that the manufacturing of corrugated board boxes is implemented as a pull process: board boxes are produced as customers' orders are received. In this scheme, the production process is planned in order to fulfill customer requirements in terms of boxes dimensions, board type, amount of units, and due dates. This characteristic supports the idea that cutting plan and scheduling decisions should be considered in an integrated approach, such that the optimal cutting plan does not compromise scheduling feasibility.

In order to satisfy customer orders, the company handles a stock of paper reels provided by paper converting mills. These reels are differentiated according to paper type (gramage or color) and reel width. The use of different sizes (widths) allows an efficient use of the raw material, because the width can be selected in order to minimize the trim loss in the cutting process. Different paper widths can be assigned to produce the board and also multiple pattern configurations can be used to cut board sheets, leading a huge number of combinations to define cutting patterns and satisfy customer orders.

The corrugated board is produced using several paper layers. There are two main types of layers: liner and fluted. The board structure is mainly defined by the number of layers assigned and the paper type in each layer. The single wall is a typical board used in the industry which is a rigid structure, with two external liner layers and one fluted in the middle of the board, as it is shown in Figure 1. According to the box dimensions, the size of the board sheet to cut is determined (see Figure 2), which is produced in a special machine that forms board and cuts the sheets (this is called a corrugator).

The production of corrugated sheets is a continuous process. All activities are performed simultaneously in the same corrugator machine, and the order in which these activities must be performed is fixed. The initial step is to place the paper reels in the machine. Since all paper reels are liner, the fluted layer is

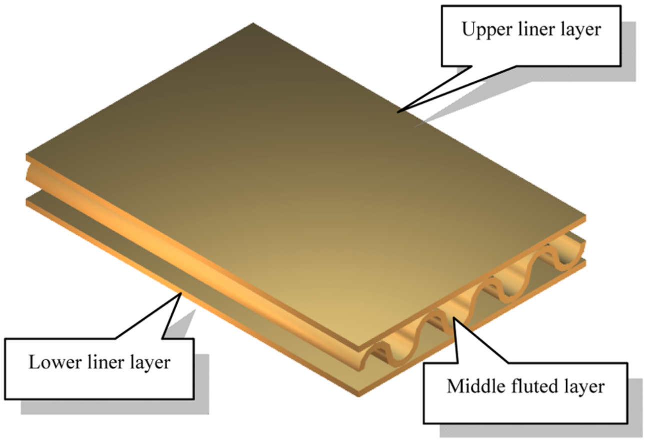

Figure 1. Corrugated board sheet.

obtained in the first stage of the corrugation and cutting process. Adhesive is then added to glue the liner layers to the fluted one. Once corrugated board is formed, it goes through the cutting section of the same machine, where board sheets are finally obtained.

The cutting machine has longitudinal slitters and transversal guillotines to cut the boards. The board is first cut into strips and then chopped to sheets according to the lengths required. The number of longitudinal slitters limits the number of sheets to cut across the width of the pattern, while the guillotines determine the number of orders of different lengths. For example, if the corrugator has five knives, seven pieces of board can be obtained in the pattern width. However, the two external ones are discarded because the layers are not perfectly glued which represents a lower bound to the trim-loss. In general, the corrugator has two guillotines, allowing two different lengths to cut. Figure 3 shows an example of the cutting pattern process.

During the cutting process, some trim-loss is produced since the widths of the papers assigned to form the board are wider than the width used to cut the sheets. This is a very flexible process, since different paper widths can be assigned to define the board and also multiple sheets (of the same board type but

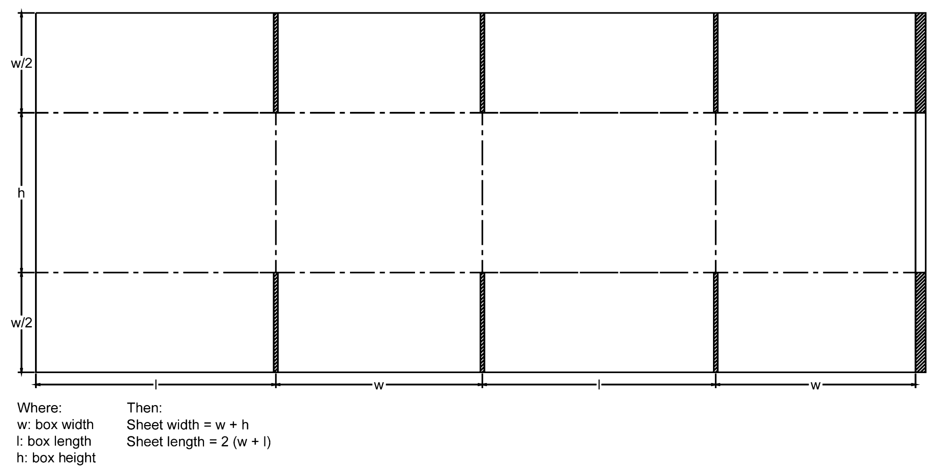

Figure 2. Box and sheet dimensions.

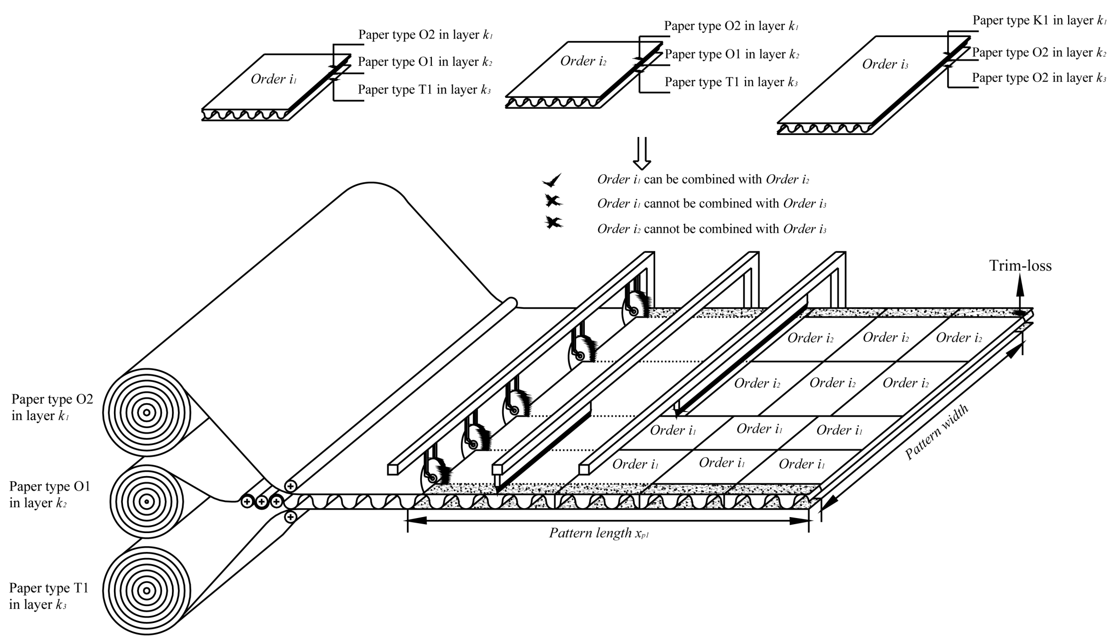

Figure 3. Production process stages.

different sizes) can be used to form a pattern. So one main objective is to find a set of cutting patterns that minimizes the cost of raw material lost during the cutting process.

Finally, whenever a pattern is finished, some additional activities are required before the following pattern starts the production process. These activities determine the changeover times that must be considered in the scheduling decisions. They include:

- changing the position of knives due to the quantity, location, and dimensions of the sheets;
- changing the paper width in some layer(s) of the board; and
- changing the paper type in some layer(s) of the board.

The first activity requires 1 min, whereas the last two activities can take 15−45 min in some cases, depending on the number of layers to implement the change. As a result, there is a minimum setup time, determined by the positioning of the guillotines and slitters, which is fixed and an additional changeover time, depending on pattern configuration and sequence (changing paper width and type). Since all these activities are performed simultaneously, only the greatest time must be considered. All these considerations must be included when modeling scheduling decisions in order to also satisfy the due dates of customers.

3. MODELING ISSUES

Even though the flexibility of the cutting process is a positive characteristic that makes it possible to minimize trim-loss, it also requires an efficient modeling approach to solve the problem. As a result, the assignment of paper width to cut patterns according to the paper stock available and the highly combinatory nature of the problem are the main challenges to finding an optimal solution in practical terms.

Nevertheless, cutting decisions also affect model convexity. Bilinear terms appear in the assignment constraints and in the product of length and width of a cutting pattern in order to calculate the number of units produced with it. For that reason, transformation techniques must be applied to reformulate the bilinear terms. In addition, this problem is highly combinatory, since there are many possible cutting patterns that can be used to satisfy customer orders.

Bilinear terms appear in the formulation in paper assignment and pattern area calculation, as well as stock and demand constraints.

There are several methods to transform bilinear terms into linear ones; different alternatives have been studied by Li and Lu,[23] Li et al.,[24] and Rodriguez and Vecchietti.[25] In order to obtain a linear reformulation, Rodriguez and Vecchietti[26] analyzed two different approaches. In the first place, pattern pregeneration is considered in order to define all feasible patterns. Then, a MILP formulation is developed to select the optimal cutting pattern and length to produce in order to satisfy demand requirements and minimize the trim-loss costs. A second strategy that reformulates the entire problem as MILP and solves it in one step is also considered and compared. This approach was previously applied by Harjunkoski et al.[27] to the trim-loss problem in the paper mill. The comparison carried out shows that the one-step strategy is much more inefficient than the first one, because of both a relaxation of the original problem and an increased number of variables and constraints. In this article, a disjunctive reformulation is used to reformulate cutting bilinearities, which has been successfully proposed for several applications by Rodriguez and Vecchietti.[25]

The scheduling consists of defining the order in which cutting patterns must be produced. Since, in this formulation, patterns are decision variables, then their processing times are variables that are dependent on pattern length (decision variable of the cutting plan) and the machine velocity (parameter). Furthermore, the width of papers assigned to each layer of the patterns affects setup times. Suppose that all patterns are of the same width, the setup time between two consecutive patterns will be given by the change in knife positions and paper types. These relationships between cutting and sequencing problems show the importance of an integrated approach.

Customer orders, considered as board sheets in the cutting and corrugation process, are not sequenced directly because they are assigned to patterns. They can be satisfied using more than one cutting pattern so the ending time of an order must take into account the patterns in which it is assigned. For that reason, the order is finished when the last pattern that includes this order ends.

Since processing times are not known a priori, a continuous time representation is more appropriate for the scheduling problem. In this case, general and immediate precedence strategies are used and compared. Castro and Grossmann[28] proposed a disjunctive approach to better understand and derive general precedence constraints in a scheduling problem. Immediate precedence variables allow handling changeover in a direct manner while general precedence includes a lesser number of binary variables.[29] One limiting characteristic of the general precedence approach is that it might overestimate the final time of the patterns when setup times are involved. We show in section 4.2 that there is a minimum pattern processing time that allows the use and comparison of both models.

4. PROBLEM FORMULATION

4.1. The Original MINLP Model.

4.1.1. Cutting Stock Problem (CSP).

$$\text{Min} \sum_p \sum_k cp_{pk} + \sum_p CY \cdot yr_p \tag{1}$$

As previously mentioned, when the board is cut into smaller pieces to produce the sheets, some waste of material always occurs, because of the differences between the pattern widths and the paper widths. Since raw material cost represents a relevant part of product cost, the objective function is to minimize the cost of paper trim loss in the cutting process. Variable cp_{pk} represents the trim loss cost of pattern p in each layer k, which is dependent on the widths of papers assigned to each layer, the dimensions of the orders assigned, the pattern length and paper cost of each layer. In addition, the changeover cost CY is included whenever a new pattern p is produced. This equation is shown in eq 1.

$$\sum_i n_{ip} \le Nlong - 1 \qquad \forall\, p \in P \tag{2}$$

Equation 2 defines that the maximum number of sheets to cut per pattern width must be lower than or equal to the number of longitudinal knives $Nlong$ minus one, where n_{ip} is an integer variable that indicates the number of units of the order i assigned to pattern p.

$$\sum_i y_{ip} \le Ntrans \qquad \forall\, p \in P \tag{3}$$

Equation 3 defines that the number of different orders assigned to each pattern could be, at most, *Ntrans*, which represents the number of transversal guillotines in the cutting stage and y_{ip} is a binary variable that indicates the assignment of order i to pattern p.

$$ta_{pk} \geq \sum_{ap} w_{ap\,kp} \cdot AP_{ap} \cdot x_p \qquad \forall\ p \in P, \forall\ k \in K \tag{4}$$

Equation 4 determines the total area of pattern p in layer k, ta_{pk}, where $w_{ap\,k\,p}$ is a binary variable indicating the selection of certain paper width ap for layer k of pattern p, while AP_{ap} is a parameter corresponding to the paper width, and x_p is a positive variable that indicates the pattern length. Note that this inequality presents bilinear terms in which variable x_p is multiplied by binary variable $w_{ap\,k\,p}$.

$$ua_{pk} \geq \sum_{i} n_{ip} \cdot WI_i \cdot x_p \qquad \forall\ p \in P, \forall\ k \in K \tag{5}$$

Equation 5 calculates the area of pattern p, ua_{pk}, considering the number of order sheets i included in pattern p (n_{ip}), the width of each sheet i (WI_i), and the length of the pattern (x_p). In this case, there is a product of the integer variable n_{ip} multiplied by the continuous variable x_p.

$$cp_{pk} \geq CO_{pk} \cdot (ta_{pk} - ua_{pk}) \qquad \forall\ p \in P, \forall\ k \in K \tag{6}$$

Equation 6 defines the pattern trim-loss cost for each layer k, given by the continuous variable cp_{pk}, where CO_{pk} is a parameter that indicates the paper cost in layer k of pattern p. The trim-loss area in each layer k of pattern p is calculated as the difference between variables ta_{pk} and ua_{pk}, which are defined in eqs 4 and 5, respectively.

$$y_{ip} + y_{i'p} \leq 1 \qquad \forall\ p \in P, \forall\ i \neq i' \in Not\_comb_{i\,i'} \tag{7}$$

Equation 7 establishes that orders i and i' cannot be combined in the same pattern if they have different type of board. This relationship is given by set $Not\_comb_{i\,i'}$.

$$\sum_{ap} w_{ap\,k\,p} \cdot AP_{ap} - \sum_{i} n_{ip} \cdot WI_i \geq Minloss \cdot yr_p \qquad \forall\ p \in P \tag{8}$$

$$\sum_{ap} w_{ap\,k\,p} \cdot AP_{ap} - \sum_{i} n_{ip} \cdot WI_i \leq Maxloss \cdot yr_p \qquad \forall\ p \in P \tag{9}$$

Constraints (8) and (9) define a minimum and maximum trim-loss per width of pattern p, respectively. This loss is calculated on the left-hand side of these equations, considering the width AP_{ap} assigned to each layer of pattern p, according to binary variable $w_{ap\,k\,p}$ minus the number of orders n_{ip} assigned multiplied by the order widths WI_i. Note that only one width ap can be selected for each layer k of pattern p, so $\sum_{ap} w_{ap\,k\,p}$ is, at most, 1. In fact, if no width ap is assigned to pattern p, it means that the pattern p is not used and consequently, the width will be also 0.

$$\sum_{ap} w_{ap\,k\,p} \leq 1 \qquad \forall\ p \in P \tag{10}$$

Equation 10 constrains that only one paper width ap can be assigned to each layer k of pattern p.

$$\sum_{ap} w_{ap\,k\,p} \geq yr_p \qquad \forall\ p \in P \tag{11}$$

Equation 11 establishes that if pattern p exists ($yr_p = 1$), then at least one paper width ap must be assigned to each layer k of the pattern p. Note that even yr_p is not a binary variable, it only takes the values of 0 or 1, because of eqs 20−22.

$$\sum_{p} \sum_{k \in Rel_{pktp}} x_p \cdot \alpha_k \cdot w_{ap\,k\,p} \leq S_{tp\,ap} \qquad \forall\ ap \in AP, \forall\ tp \in TP \tag{12}$$

Equation 12 establishes that paper consumption cannot be greater than the amount in stock, where parameter $S_{tp\,ap}$ corresponds to the amount of raw material in stock, while $w_{ap\,k\,p}$ is a binary variable that selects a paper width ap of layer k in pattern p. If the width ap is assigned to layer k of pattern p, the length used in all layers of all patterns can be, at most, $S_{tp\,ap}$. It should be noted that α_k is a parameter representing the paper consumption of layer k (fluted layers consumes more than one meter to produce one meter length of board). The set $Rel_{p\,k\,tp}$ determines the paper type tp associated to each layer k of pattern p.

Another bilinear term also appears in eq 12. In this inequality, the bilinear product is formed by the continuous variable x_p and the binary variable $w_{ap\,k\,p}$.

$$\sum_{p} n_{ip} \cdot \frac{x_p}{L_i} \geq D_i \qquad \forall\ i \in I \tag{13}$$

$$\sum_{p} n_{ip} \cdot \frac{x_p}{L_i} \leq (1 + \delta_i) D_i \qquad \forall\ i \in I \tag{14}$$

Equations 13 and 14 are the demand constraints. Equation 13 defines that the number of sheets produced for one order i in all patterns p must satisfy the demand D_i. Equation 14 establishes an overproduction upper bound δ_i, which gives flexibility to the cutting plan.

$$x_p \geq CRmin_p \cdot yr_p \qquad \forall\ p \in P \tag{15}$$

$$x_p \leq CRmax_p \cdot yr_p \qquad \forall\ p \in P \tag{16}$$

Equations 15 and 16 give a minimum and maximum run length for pattern p given by parameters $CRmin_p$ and $CRmax_p$, respectively. Both constraints are related to the production time limits. The minimum run length is especially important for scheduling constraints, because it guarantees that processing times are greater than the setup times and general precedence decisions can be applied.

$$\sum_{p} y_{ip} \geq 1 \qquad \forall\ i \in I \tag{17}$$

Equation 17 determines that every order i must be assigned to some pattern p.

$$y_{ip} - n_{ip} \leq 0 \qquad \forall\ i \in I, \forall\ p \in P \tag{18}$$

$$n_{ip} - Nlong \cdot y_{ip} \leq 0 \qquad \forall\ i \in I, \forall\ p \in P \tag{19}$$

Equations 18 and 19 are logical constraints relating variables n_{ip} and y_{ip}. The first one determines that if $n_{ip} = 0$, then $y_{ip} = 0$. The second one defines that if $n_{ip} > 0$, y_{ip} must be 1.

$$yr_p \geq y_{ip} \qquad \forall\ i \in I, \forall\ p \in P \tag{20}$$

$$yr_p \leq 1 \qquad \forall\ p \in P \tag{21}$$

$$yr_p \leq \sum_i y_{ip} \qquad \forall\ p \in P \tag{22}$$

Equations 20–22 represents lower and upper bounds to the variable yr_p, ensuring that it would take values of 0 or 1, even if it is declared as a positive variable. Equation 20 is a lower bound and establishes that if some order i is assigned to pattern p, then yr_p must be at least 1. The upper bound in eq 21 limits yr_p to 1. Finally, eq 22 establishes that yr_p must be 0 if no order i is assigned to pattern p.

4.1.2. Scheduling. The cutting problem is integrated into the scheduling model in order to obtain a global optimal solution. Scheduling constraints are given as follows:

$$pt_p = \frac{x_p}{VEL} \qquad \forall\ p \in P \tag{23}$$

Equation 23 calculates the processing time of pattern p, according to the pattern length x_p and the cutting machine velocity. This is one of the most important relationships between the cutting and scheduling problems.

$$wk_{pp'} \geq w_{ap\,kp} + w_{ap'\,kp'} - 1 \qquad \forall\ k \in K,\ \forall\ (ap, ap') \in APap \neq ap', \forall\ (p, p') \in Pp \neq p' \tag{24}$$

As it was mentioned, whenever a paper width is changed, some time is required between pattern executions. This time depends on the selection of width for each pattern, which is a decision of the cutting process ($w_{ap\ k\ p}$). So eq 24 considers that a change in the paper width will occur in any layer k of patterns p and p', given by binary variable $wk_{p\ p'}$, if two different widths are assigned to the same layer k of patterns p and p'.

$$wk_{pp'} \leq 2 - (w_{ap\,kp} + w_{ap\,kp'}) \qquad \forall\ (p, p') \in P,\ p \neq p' \tag{25}$$

Equation 25 determines that, if the same width ap is assigned to layer k of patterns p and p', then no width change is considered.

$$ct_{pp'} \geq CW \cdot wk_{pp'} \qquad \forall\ (p, p') \in P,\ p \neq p' \tag{26}$$

$$ct_{pp'} \geq CB_{pp'} \qquad \forall\ (p, p') \in P,\ p \neq p' \tag{27}$$

$$ct_{pp'} \geq CN \qquad \forall\ (p, p') \in P,\ p \neq p' \tag{28}$$

Equations 26–28 define lower bounds for variable $ct_{p\ p'}$ modeling the changeover time between patterns p and p'. Equation 26 considers the setup time due to changes in the widths of the patterns, while eq 27 determines the minimum time due to board change. In addition, eq 28 calculates the time required for changing the position of knifes in the cutting machine whenever a new pattern is produced, which is independent of the sequence. Equation 26 applies if there is a change of widths between pattern p and p'. If this is the case, the changeover time is CW and the binary variable $wk_{p\ p'}$ is 1. Parameter $CB_{p\ p'}$ can be determined a priori, since the board type is considered to be known for each pattern. Therefore, if two patterns have the same paper widths but different paper types in some layers, a paper change must be considered. If, in contrast, both patterns are of the same board type but their widths are different, a changeover time is also required. The minimum time to change knives is not dependent on the patterns definition or sequencing decisions; this is a fixed time given by parameter CN in eq 28. Since all these activities are performed simultaneously, only the longest time determines the total setup time.

$$ft_p \geq pt_p \qquad \forall\ p \in P \tag{29}$$

Equation 29 determines that the final time for pattern p must be at least equal to its processing time. This inequality will be active in the case of the first scheduled pattern.

$$to_i \geq ft_p + M\cdot(1 - y_{ip}) \qquad \forall\ i \in I,\ \forall\ p \in P \tag{30}$$

$$to_i \leq DD_i \qquad \forall\ i \in I \tag{31}$$

Equations 30 and 31 constrain variable to_i, which is the final time of order i. First, eq 30 defines that the final time of an order must be greater than or equal to the final time of pattern p if that order has been assigned to this pattern (y_{ip}). On the other hand, by eq 31, the final time of order i cannot be greater than the due date compromised, given by parameter DD_i.

4.1.3. Immediate and General Precedence. Considering the advantages and drawbacks of immediate and general precedence, both formulations are presented in this section and computational results are compared.

Immediate precedence is given by eqs 32–35, while general precedence constraints are presented in eqs 36 and 37.

$$ft_p \geq pt_p + ft_{p'} + ct_{p'p} - M\cdot(1 - y^{pred}_{p'p}) \qquad \forall\ (p, p') \in P,\ p \neq p' \tag{32}$$

Equation 32 is a big-M constraint to determine a lower bound for the final time for a pattern. The ending time of pattern p must be greater than or equal to the processing time of the pattern plus the ending time of pattern p' and the changeover time from pattern p to pattern p if pattern p' is a predecessor of pattern p. Note that variable $y^{pred}_{p'p}$ is 1 if pattern p' is sequenced before pattern p, and is 0 otherwise.

Some additional constraints are required in order to guarantee the correct sequencing process when immediate precedence is considered.

$$\sum_{\substack{p' \\ p \neq p'}} y^{pred}_{pp'} \leq yr_p \qquad \forall\ p \in P \tag{33}$$

$$\sum_{\substack{p \\ p \neq p'}} y^{pred}_{pp'} \leq yr_{p'} \qquad \forall\ p' \in P \tag{34}$$

$$\sum_{\substack{p \\ p \neq p'}} \sum_{p'} y^{pred}_{pp'} \geq \sum_p yr_p - 1 \tag{35}$$

Equation 33 determines that pattern p can be a predecessor of only one pattern if this pattern is active ($yr_p = 1$). Similarly, eq 34 defines that pattern p' can be an immediate successor of only one pattern if p' is selected ($yr_{p'} = 1$). Finally, eq 35 determines the lower bound for the number of active sequencing variables.

If general precedence is chosen, eqs 36 and 37 are required instead of eq 32.

$$ft_p \geq pt_p + ft_{p'} + ct_{p'p} - M\cdot(1 - y^{pred}_{p'p}) \qquad \forall\ (p, p') \in P,\ p < p' \tag{36}$$

$$ft_{p'} \geq pt_{p'} + ft_p + ct_{pp'} - M\cdot y^{pred}_{p'p} \qquad \forall\ (p, p') \in P,\ p < p' \tag{37}$$

It is worth mentioning that, for both approaches, parameter M can be determined as follows:

$$M = \max_i\{DD_i\} + \max\{CW;\ CN;\ CB_{pp'}\}$$

Note that, in the case of eq 37, the only change in the value of M is given by using $CB_{p'p}$ instead of $CB_{pp'}$.

One clear advantage of the general precedence method is that it requires half binary variables. If pattern p' is a global predecessor of pattern p, then eq 36 is applied. In contrast, if p is a global predecessor of pattern p', meaning that $y^{pred}_{p'p} = 0$, eq 37 is activated. However, using this approach, setup times are not considered directly, so the ending time can be overestimated. This situation occurs if the changeover time between two patterns that are not immediately sequenced is greater than the processing time of the patterns assigned between them. Figure 4 shows an example of this situation.

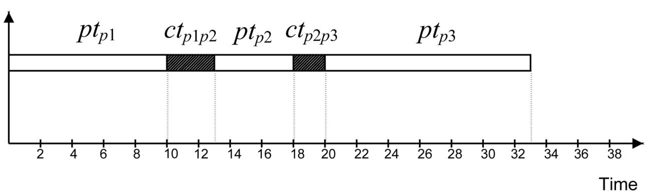

Figure 4. Example of completion time with sequence-dependent changeover time.

Let consider an example with three patterns—$p1$, $p2$, and $p3$—to be sequenced, having the following symmetric setup times:

$$ct_{p1p2} = 3$$

$$ct_{p1p3} = 15$$

$$ct_{p2p3} = 2$$

and the following processing times:

$$pt_{p1} = 10$$

$$pt_{p2} = 5$$

$$pt_{p3} = 13$$

For this case, the optimal sequence is given by $p1 \rightarrow p2 \rightarrow p3$; then, the precedence variables are

$$y^{pred}_{p1p2} = 1$$

$$y^{pred}_{p1p3} = 1$$

$$y^{pred}_{p2p3} = 1$$

The optimal completion time fp_p of each pattern is given by

$$fp_{p1} = pt_{p1} = 10$$

$$\begin{aligned} fp_{p2} &= pt_{p1} + ct_{p1p2} + pt_{p2} \\ &= 10 + 3 + 5 \\ &= 18 \end{aligned}$$

$$\begin{aligned} fp_{p3} &= pt_{p1} + ct_{p1p2} + pt_{p2} + ct_{p2p3} + pt_{p3} \\ &= 10 + 3 + 5 + 2 + 13 \\ &= 33 \end{aligned}$$

Then, the optimal final completion time for all patterns is 33, which also is shown in Figure 4.

However, since $y^{pred}_{p1\,p3} = 1$ in the general precedence model, eq 36 applies when precedence between p_1 and p_3 is considered:

$$\begin{aligned} fp_{p3} &\geq fp_{p1} + ct_{p1p3} + pt_{p3} \\ &= 10 + 15 + 13 \\ &= 38 \end{aligned}$$

Figure 5 shows that the ending time for pattern $p3$ will be calculated as 38, according to eq 36, meaning that the total completion time is overestimated.

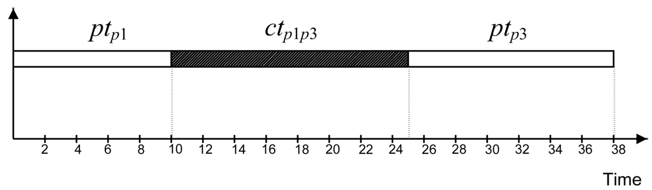

Figure 5. Example of general precedence with sequence-dependent changeover time.

The previous example shows that when setup times are involved, general precedence might overestimate the final time of tasks. This limitation is overcome by the immediate or direct precedence variables, where setup times are considered in a direct manner. In order to guarantee that immediate and general precedence models are equivalent, the following condition must hold:[30]

$$ct_{pp'} \leq ct_{pp''} + pt_{p''} + ct_{p''p'} \qquad \forall\ (p, p', p'') \in P \tag{38}$$

In order to analyze this statement, suppose that, in the worst case:

$$ct_{pp''} = 0$$

$$ct_{p''p'} = 0$$

Then, condition (38) is transformed to (39):

$$ct_{pp'} \leq pt_{p''} \qquad \forall\ (p, p', p'') \in P \tag{39}$$

In our case, setup times vary from 1 min to 45 min, according to the activities involved (changing knifes positions or changing paper reels in 1−3 layers of the board). In addition, the processing time $pt_{p''}$ is a variable that is dependent on the pattern length $x_{p''}$, as shown in eq 23. However, given the minimum run length condition in eq 15, the company wants to prevent the use of short patterns such that the setup times are never greater than the processing times. This condition is given as follows:

$$ct_{pp'} \leq \frac{CRmin_{p''}}{VEL} \qquad \forall\ (p, p', p'') \in P \tag{40}$$

Then, general and direct precedence models are equivalent and comparable.

4.2. Disjunctive Reformulation. As mentioned, some bilinear terms appear in the formulation, because of cutting constraints. The first bilinear term obtained in eqs 4 and 12 is used to calculate the total area of a cutting pattern in the first case and the material stock consumption in the second case. This is a product of a binary multiplied by a continuous variable. The other bilinear term is formed by a discrete variable and a continuous one, which appears in eq 5, to determine the area in each pattern and in eqs 13 and 14, to satisfy demand constraints. Both types of bilinear terms can be transformed to a linear formulation applying a disjunctive approach, which takes advantage of the discrete nature of one of the variable involved. This alternative was proposed by Rodriguez and Vecchietti.[25]

The reformulation of the first bilinear term given by $w_{ap\,k\,p}\cdot x_p$ is straightforward. The existent binary variable $w_{ap\,k\,p}$ is now redefined as a Boolean variable, $\omega_{ap\,k\,p}$. A new slack variable, $l_{ap\,k\,p}$, is used in eq 41 to represent the nonlinear term presented in eqs 4 and 12.

$$\bigvee_{ap\in AP}\begin{bmatrix}\omega_{ap\,k\,p}\\ l_{ap\,k\,p}=x_p\end{bmatrix}\qquad \forall\ p\in P,\ \forall\ k\in K \tag{41}$$

Variable $l_{ap\,k\,p}$ can now replace the bilinear term in the model. Note that if Boolean variable $\omega_{ap\,k\,p}$ is true, then $l_{ap\,k\,p}$ is equal to x_p, otherwise it takes a value of zero.

For the second term, the disjunctive procedure applies a 2-based formulation given in eq 42. New variables are again introduced to redefine bilinear terms. Now, the discrete variable n_{ip} is calculated outside the disjunctions as the summation of J terms, defined by the positive variable nj_{ipj}, given in each disjunction (i,p,j). In this case, if the Boolean variable β_{ipj} is true, a positive value is assigned to nj_{ipj}; otherwise, $nj_{ipj}=0$. This new formulation is given by eqs 42 and 43.

$$\begin{bmatrix}\beta_{ipj}\\ nxj_{ipj}=2^{j-1}\cdot x_p\\ nj_{ipj}=2^{j-1}\end{bmatrix}\vee\begin{bmatrix}\neg\beta_{ipj}\\ nxj_{ipj}=0\\ nj_{ipj}=0\end{bmatrix}$$
$$\forall\ i\in I,\ \forall\ p\in P,\ \forall\ j\in J \tag{42}$$

Variables nj_{ipj} and nxj_{ipj} are added to the original model to represent the integer variable nj_{ipj} and the bilinear term, respectively. The definition of n_{ip} is now written as shown in eq 43.

$$n_{ip}=\sum_j nj_{ipj}\qquad \forall\ i\in I,\ \forall\ p\in P \tag{43}$$

Given these transformations, eqs 4, 5, and 12−14 are replaced by eqs 44−48, respectively.

$$ta_{pk}\geq\sum_{ap} l_{apkp}\cdot AP_{ap}\qquad \forall\ p\in P,\ \forall\ k\in K \tag{44}$$

$$ua_{pk}\geq\sum_i\sum_j nxj_{ipj}\cdot Wi_i\qquad \forall\ p\in P,\ \forall\ k\in K \tag{45}$$

$$\sum_p\ \sum_{k\in Rel_{pktp}}\alpha_k\cdot l_{ap\,k\,p}\leq S_{tp\,ap}\qquad \forall\ ap\in AP,\ \forall\ tp\in TP \tag{46}$$

$$\sum_p\sum_j\frac{nxj_{ipj}}{L_i}\geq D_i\qquad \forall\ i\in I \tag{47}$$

$$\sum_p\sum_j\frac{nxj_{ipj}}{L_i}\leq(1+\delta_i)D_i\qquad \forall\ i\in I \tag{48}$$

4.3. Model Relaxation. In order to implement this model as a MILP, this disjunctive representation is reformulated using convex hull relaxation.

In the case of disjunction (41), the convex hull relaxation is given by eqs 49−53.

$$\sum_{ap} w_{ap\,k\,p}\leq 1\qquad \forall\ p\in P \tag{49}$$

$$l_{ap\,k\,p}=x^*_{ap\,k\,p}\qquad \forall\ p\in P,\ \forall\ ap\in AP,\ \forall\ k\in K \tag{50}$$

$$x_p=\sum_{ap} x^*_{ap\,k\,p}\qquad \forall\ p\in P,\ \forall\ k\in K \tag{51}$$

$$l_{ap\,k\,p}\leq CRmax_p\cdot w_{ap\,k\,p}\qquad \forall\ p\in P,\ \forall\ ap\in AP,\ \forall\ k\in K \tag{52}$$

$$x^*_{ap\,k\,p}\leq CR\max_p\cdot w_{ap\,k\,p}$$
$$\forall\ p\in P,\ \forall\ ap\in AP,\ \forall\ k\in K \tag{53}$$

Note that $x^*_{ap\,k\,p}$ can be eliminated from the previous formulation and is replaced by $l_{ap\,k\,p}$, and eq 49 is already considered in eq 10. Then, eqs 49−53 can be rewritten as eqs 10, 52, and 54.

$$x_p=\sum_{ap} l_{ap\,k\,p}\qquad \forall\ p\in P,\ \forall\ k\in K \tag{54}$$

Regarding the second disjunction (eq 42), the convex hull relaxation is given in eqs 55−59.

$$nj_{ipj}=2^{j-1}\cdot\beta_{ipj}\qquad \forall\ i\in I,\ \forall\ p\in P,\ \forall\ j\in J \tag{55}$$

$$nxj_{ipj}=2^{j-1}\cdot x^1_{ipj}\qquad \forall\ i\in I,\ \forall\ p\in P,\ \forall\ j\in J \tag{56}$$

$$x_p=x^1_{ipj}+x^2_{ipj}\qquad \forall\ i\in I,\ \forall\ p\in P,\ \forall\ j\in J \tag{57}$$

$$x^1_{ipj}\leq CRmax_p\cdot\beta_{ipj}\qquad \forall\ i\in I,\ \forall\ p\in P,\ \forall\ j\in J \tag{58}$$

$$x^2_{ipj}\leq CRmax_p\cdot(1-\beta_{ipj})\qquad \forall\ i\in I,\ \forall\ p\in P,\ \forall\ j\in J \tag{59}$$

Finally, two MILP formulations are obtained, which integrate the cutting process and the scheduling decision. The first one is given by eqs 1−3, 6−11, 15−35, 43−48, 52, and 54−59, and it considers immediate precedence relationships for sequencing decisions. The second one is formulated by eqs 1−3, 6−11, 15−31, 36−37, 43−48, 52, and 54−59, considering general precedence variables in scheduling constraints. Both models are applied in the following section to solve three examples and compare results.

5. RESULTS

The mathematical formulation presented in the previous section was implemented in GAMS 23.7 and solved using CPLEX 12.3. In this section, both MILP models are compared in terms of the number of variables, constraints, and execution time. Even though the examples are not of industrial size, they allow analyzing model performance as well as comparison of the impact of considering scheduling and cutting decisions in an integrated approach.

5.1. Example 1. This example presents nine orders from customers. The number of boxes of each order, as well as the sheets size required from the corrugating process, are given in the Supporting Information.

Solution performance is presented in Table 1 for both models. In this example, the difference in the formulations is also noticed in the execution times. Although general precedence strategy uses 1.5% less constraints than immediate precedence formulation and there is a difference of <1% in the number of positive and binary variables, the impact in the execution time is more than 27%, showing that even small improvements in the formulations can greatly affect solution results.

Table 1. Models Performance for Example 1 with Nine Orders

	constraints	positive variables	binary variables	CPUs	objective function
immediate precedence model	13062	1582	648	36.65	317.44
general precedence model	12870	1573	603	26.82	317.44

Models results are also different in terms of variable values. Table 2 shows the patterns information for immediate precedence strategy, while Table 3 presents the same for the second sequencing method. Note that even though the only difference between both models is given in the precedence variables and sequencing constraints, pattern definition also changes from one model to the other. This is mainly because there are alternative solutions that give the same final solution in the objective function.

Order assignment to patterns and the number of sheets produced are shown in Tables 4 and 5. Note that overproduction is required to minimize trim-loss costs in the objective function only in one case.

Regarding sequencing decisions, Tables 6 and 7 show the ending and processing time for each pattern. The sequence is again different, depending on the strategy used, but the final time is the same in both cases. Note that even when setup times are the same (which is not necessarily the case), the final time for all orders could be different, since processing times are decision variables that are dependent on pattern length.

From a production management perspective, if different sequences are obtained satisfying due date constraints and obtaining the same Objective Function value, there are some additional considerations that can help the planner to select the best cutting and scheduling alternative, including:

- the shortest final time for processing all patterns,
- a lesser number of patterns,
- lowest overproduction,
- average earliness of all orders, and
- delivery considerations such as producing first orders that are sent together in one shipment

In order to show the importance of solving the cutting and scheduling problems in an integrated approach, due dates are changed in this example. Instead of the due dates presented in Table A.1 in the Supporting Information, tighter due dates are considered for orders i_1 and i_6, assuming 26 and 35 h,

Table 2. Pattern Characteristics for Example 1 Using Immediate Precedence

	Patterns									
	*p*1	*p*2	*p*3	*p*4	*p*5	*p*6	*p*7	*p*8	*p*9	*p*10
pattern length, x_p (m)	1050	845	1895	555	1048		1309	200	563	1050
pattern width (mm)	1350	1050	950	1140	1160		1244	1250	1350	1180
paper type										
external layer	O1	O1	B1	K1	K1		O2	B2	B2	B1
fluted layer	O1	O1	T1	O2	O2		O1	O1	O1	T1
internal layer	O1	O1	K1	O2	O2		O2	O2	O2	K1

Table 3. Pattern Characteristics for Example 1 Using General Precedence

	Patterns									
	*p*1	*p*2	*p*3	*p*4	*p*5	*p*6	*p*7	*p*8	*p*9	*p*10
pattern length, x_p (m)	845	1050	1050	1048	555	1309		200	563	1895
pattern width (mm)	1050	1350	1180	1160	1140	1244		1250	1350	950
paper type										
external layer	O1	O1	B1	K1	K1	O2		B2	B2	B1
fluted layer	O1	O1	T1	O2	O2	O1		O1	O1	T1
internal layer	O1	O1	K1	O2	O2	O2		O2	O2	K1

Table 4. Orders Assigned to Patterns in Example 1 with Nine Orders and Immediate Precedence

orders assigned	*p*1	*p*2	*p*3	*p*4	*p*5	*p*6	*p*7	*p*8	*p*9	*p*10	production (units)	overproduction (%)
i_1	1	1									1000	0
i_2	1										1500	0
i_3					2						1800	0
i_4				1							1000	0
i_5							4				2500	0
i_6								3	1		1000	0
i_7								1	3		1662	11
i_8			1								1000	0
i_9										1	1500	0

Table 5. Orders Assigned to Patterns in Example 1 with Nine Orders and General Precedence

orders assigned	$p1$	$p2$	$p3$	$p4$	$p5$	$p6$	$p7$	$p8$	$p9$	$p10$	production (units)	overproduction (%)
i_1	1	1									1000	0
i_2		1									1500	0
i_3				2							1800	0
i_4					1						1000	0
i_5						4					2500	0
i_6								1	3		1000	0
i_7								3	1		1662	11
i_8										1	1000	0
i_9			1								1500	0

Table 6. Pattern Sequence in Example 1 with Nine Orders and Immediate Precedence

	Pattern Sequence								
	$p2$	$p9$	$p1$	$p8$	$p3$	$p5$	$p10$	$p4$	$p7$
final time (h)	11.27	20.03	34.86	38.78	65.29	80.52	95.77	104.42	140
processing time (h)	11.27	7.51	14	2.67	25.27	13.98	14	7.4	17.46

Table 7. Pattern Sequence in Example 1 with Nine Orders and General Precedence

	Pattern Sequence								
	$p8$	$p10$	$p9$	$p2$	$p1$	$p5$	$p3$	$p4$	$p6$
final time (h)	2.67	29.18	37.94	52.78	66.54	75.19	90.44	106	140
processing time (h)	2.67	25.27	7.51	14	11.27	7.4	14	13.98	17.46

Table 8. Pattern Characteristics for Example 1 Using Tighter Due Dates

	Pattern									
	$p1$	$p2$	$p3$	$p4$	$p5$	$p6$	$p7$	$p8$	$p9$	$p10$
pattern length, x_p (m)	1895	263	1895	1048	555	1309		630	200	1050
pattern width (mm)	1050	1200	950	1160	1140	1244		1350	900	1180
paper type										
external layer	O1	O1	B1	K1	K1	O2		B2	B2	B1
fluted layer	O1	O1	T1	O2	O2	O1		O1	O1	T1
internal layer	O1	O1	K1	O2	O2	O2		O2	O2	K1

Table 9. Orders Assigned to Patterns in Example 1 with Nine Orders with Tighter Due Dates

orders assigned	$p1$	$p2$	$p3$	$p4$	$p5$	$p6$	$p7$	$p8$	$p9$	$p10$	production (units)	overproduction (%)
i_1	1										1000	0
i_2		4									1500	0
i_3				2							1800	0
i_4					1						1000	0
i_5						4					2500	0
i_6								3			1000	0
i_7								1	3		1757	17
i_8			1								1000	0
i_9										1	1500	0

Table 10. Pattern Sequence in Example 1 with Nine Orders with Tighter Due Dates

	Pattern Sequence								
	$p1$	$p8$	$p2$	$p9$	$p4$	$p3$	$p10$	$p5$	$p6$
final time (h)	25.27	34.92	39.67	44.42	59.65	86.16	101.41	110.06	140
processing time (h)	25.27	8.4	3.5	2.67	13.98	25.27	14	7.4	17.46

respectively. One important result is given by the execution times. While the general precedence model is solved in 37.2 s, the immediate precedence approach reaches the integer optimal solution in 200 s with a gap of 24.30%, with respect to the relaxed solution. After 1000 s, the gap is still 20.49%.

The most important result of this study is that the cutting plan is modified in order to satisfy the new due dates. New patterns characteristics and orders assigned are shown in Tables 8 and 9 respectively. Scheduling decision are clearly different, processing first patterns $p1$ and $p8$ in which orders i_1 and i_6 are assigned, respectively (Table 10). The final objective function is

338.54, which is worse than in the previous case, since greater trim-loss is required to satisfy customer dates. One important conclusion is that if this example is considered without taking into account due dates, the optimal cutting plan would be infeasible in practical terms. In industrial practices, this means that the due dates will be violated.

5.2. Example 2. Twelve (12) orders are presented in this example. Table A.2 in the Supporting Information shows the demand level for each order, as well as the sheet length and width.

Immediate and general precedence approaches are also applied to this example. Again, the second one presents better performance, as shown in Table 11, but the difference is 6%.

Table 11. Models Performance for Example 2 with 12 Orders

	constraints	positive variables	binary variables	CPUs	objective function
immediate precedence model	19328	2370	956	49.86	498.07
general precedence model	19050	2358	890	46.86	498.07

The cutting trim-loss cost plus the patterns cost, in both cases, is $498.07.

Pattern characteristics, applying both strategies, are shown in Tables 12 and 13.

Regarding scheduling decisions, the two strategies present alternative solutions: both are feasible but have different final times for all orders. As mentioned in the previous example, this could occur not only due to setup times but also because the processing time is calculated in the model according to the pattern length. In this case, immediate precedence approach offers a scheduling solution that can be executed in 200 h while the other method, considering general precedence, requires a production time of 220 h to execute all the patterns and satisfy customer orders. These results are presented in Tables 14 and 15.

5.3. Example 3. The last example is composed of 15 customer orders with the characteristics presented in Table A.3 in the Supporting Information.

Performances and model sizes are presented in Table 16. In this example, the difference in the execution time is even more important than in the previous cases; the general precedence model reaches the solution in half the time, compared to the immediate precedence formulation.

Table 16. Models Performance for Example 3 with 15 Orders

	constraints	positive variables	binary variables	CPUs	objective function
immediate precedence model	30576	3604	1408	186.84	599.33
general precedence model	30124	3587	1303	90.74	599.33

Table 12. Pattern Characteristics for Example 2 Using Immediate Precedence

	*p*1	*p*2	*p*3	*p*4	*p*5	*p*6	*p*7	*p*8	*p*9	*p*10	*p*11	*p*12
pattern length, x_p (m)	525	1895	1350	699	1710	800	1109	1050	420	1895	490	1440
pattern width (mm)	1400	950	1180	1140	940	1461	1244	970	1340	950	1200	1150
paper type												
external layer	O1	O1	B1	K1	K1	O2	O2	B2	B2	B1	T1	T1
fluted layer	O1	O1	T1	O2	O2	O1	O1	O1	O1	T1	O1	O1
internal layer	O1	O1	K1	O2	O2	O2	O2	K2	K2	K1	O2	O2

Table 13. Pattern Characteristics for Example 2 Using General Precedence

	Pattern											
	*p*1	*p*2	*p*3	*p*4	*p*5	*p*6	*p*7	*p*8	*p*9	*p*10	*p*11	*p*12
pattern length, x_p (m)	1895	525	1895	699	1710	800	1190	1490	200	1350	490	1440
pattern width (mm)	950	1400	950	1140	940	1244	1244	970	1340	1180	1200	1150
paper type												
external layer	O1	O1	B1	K1	K1	O2	O2	B2	B2	B1	T1	T1
fluted layer	O1	O1	T1	O2	O2	O1	O1	O1	O1	T1	O1	O1
internal layer	O1	O1	K1	O2	O2	O2	O2	O2	O2	K1	O2	O2

Table 14. Pattern Sequence in Example 2 with 12 Orders and Immediate Precedence

	Pattern Sequence											
	*p*5	*p*11	*p*4	*p*8	*p*3	*p*1	*p*7	*p*6	*p*10	*p*12	*p*2	*p*9
final time (h)	22.8	30.58	41.15	56.40	75.65	83.9	99.95	111.86	138.38	158.83	193.15	200
processing time (h)	22.8	6.53	9.32	14	18	7	14.79	10.67	25.27	19.2	25.27	5.6

Table 15. Pattern Sequence in Example 2 with 12 Orders and General Precedence

	Pattern Sequence											
	*p*6	*p*9	*p*7	*p*10	*p*12	*p*5	*p*11	p_3	*p*8	*p*4	*p*2	*p*1
final time (h)	10.67	14.58	30.63	49.88	70.33	94.38	102.16	128.68	149.79	160.36	193.48	220
processing time (h)	10.67	2.67	14.79	18	19.2	22.8	6.53	25.27	19.87	9.32	7	25.27

Table 17. Pattern Characteristics for Example 3 Using Immediate Precedence

	Pattern														
	*p*1	*p*2	*p*3	*p*4	*p*5	*p*6	*p*7	*p*8	*p*9	*p*10	*p*11	*p*12	*p*13	*p*14	*p*15
pattern length, x_p (m)	525	1895	1895	1710	1149	1600	909	788	551	1050	417	1440	750	293	234
pattern width (mm)	1400	1150	950	1140	1160	1461	1244	1270	1340	1180	1200	1150	1300	1410	1000
paper type															
external layer	O1	O1	B1	K1	K1	O2	O2	B2	B2	B1	T1	T1	T2	T2	T1
fluted layer	O1	O1	T1	O2	O2	O1	O1	O1	O1	T1	O1	O1	O1	O1	O1
internal layer	O1	O1	K1	O2	O2	O2	O2	O2	O2	K1	O2	O2	O2	O2	O2

Table 18. Pattern Characteristics for Example 3 Using General Precedence

	Pattern														
	*p*1	*p*2	*p*3	*p*4	*p*5	*p*6	*p*7	*p*8	*p*9	*p*10	*p*11	*p*12	*p*13	*p*14	*p*15
pattern length, x_p (m)	1895	525	1895	1149	1710	909	1600	200	1490	1050	1440	417	310	880	234
pattern width (mm)	1150	1400	950	1160	1140	1244	1461	1340	970	1180	1150	1200	1300	1120	1000
paper type															
external layer	O1	O1	B1	K1	K1	O2	O2	B2	B2	B1	T1	T1	T2	T2	T1
fluted layer	O1	O1	T1	O2	O2	O1	O1	O1	O1	T1	O1	O1	O1	O1	O1
internal layer	O1	O1	K1	O2	O2	O2	O2	O2	O2	K1	O2	O2	O2	O2	O2

Table 19. Pattern Sequence in Example 3 with 15 Orders and Immediate Precedence

	Pattern Sequence														
	*p*15	*p*7	*p*2	*p*6	*p*11	*p*4	*p*10	*p*9	*p*13	*p*8	*p*14	*p*12	*p*3	*p*5	*p*1
final time (h)	3.12	16.49	43	65.59	72.40	96.45	111.7	120.30	131.13	142.88	148.04	168.49	210	240	260
processing time (h)	3.12	12.13	25.27	21.3	5.55	22.8	14	7.35	10	10.5	3.91	19.2	25.27	13.98	7

Table 20. Patterns Sequence in Example 3 with 15 Orders and General Precedence

	Pattern Sequence														
	*p*15	*p*13	*p*8	*p*10	*p*12	*p*6	*p*7	*p*5	*p*1	*p*11	*p*14	*p*9	*p*3	*p*4	*p*2
final time (h)	3.12	8.5	12	27.25	34.05	46.60	69.18	93.23	119.75	139.78	151.93	173.05	210	240	260
processing time (h)	3.12	4.13	2.67	14	5.55	12.13	21.33	22.8	25.27	19.2	11.73	19.87	25.27	13.98	7

In this case, the immediate precedence model defines cutting patterns shown in Table 17. Similar results are obtained in the general precedence model shown in Table 18.

Finally, sequencing decision for both formulations are given in Tables 19 and 20. Note that, in this case, even though the sequence is not the same, final time for all orders obtained applying the general precedence approach is again equal to the one obtained when the immediate precedence assumption is considered.

6. DISCUSSION AND CONCLUSIONS

In this article, we present an integrated model that has been developed to solve the cutting plan and scheduling decisions in the production of corrugated board boxes. Although the cutting process has been extensively studied, this particular problem has not been addressed by the literature. The integrated approach of cutting plan and scheduling decisions has been analyzed in some articles for different industries, such as the production of paper rolls, wood pieces and glass production, among others. In most cases, heuristic techniques are applied to come up with an efficient solution. Some articles also apply optimization models to define the scheduling of the cutting process, where feasible patterns are considered already known.

Since cutting decisions affect some scheduling variables, an integrated approach is crucial to obtain a global solution. Such decisions are mainly pattern processing and setup times, which are traditionally treated as parameters when sequencing decisions are taken into account.

The cutting process itself is also challenging due to the combinatory complexity, as well as the presence of bilinear terms. A disjunctive approach is used in order to linearize the problem, taking advantage of the discrete nature of one of the variables of the bilinear terms.

Regarding the formulation of the scheduling problem, continuous time representation is considered more appropriate in this context, because processing and setup times are not input data in the model. In contrast, they are decision variables that are defined according to the pattern definition. Precedence decisions define the final time for pattern execution applying big-M constraints. The final times of orders are calculated in the same model, according to the patterns they are assigned.

In addition, two different methods are used to model scheduling decisions. Considering the NP-hard nature of the problem, which is highly combinatory, and the model sizes tested, results show efficient solutions for the integrated problem, regardless of whether the general or immediate precedence approach is used. However, general precedence models offer a more-compact formulation, which is, in all cases, related to a shorter execution time.

Future work could include a parallel cutting machine, in order to analyze orders and pattern assignment in a multiple machine production context. In that case, the problem involves not only pattern definition and sequence but also how to assign patterns to machines together with the sequencing decisions in each machine.

■ ASSOCIATED CONTENT

Supporting Information

This material is available free of charge via the Internet at http://pubs.acs.org.

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Notes

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■ NOMENCLATURE

Sets

I = customer orders
P = cutting patterns
K = board layers
Ap = paper widths
Tp = paper types
$Rel_{p\,k\,tp}$ = set that relates the paper type tp corresponding to layer k of pattern p

Parameters

CY = cost of changing pattern
$Nlong$ = number of longitudinal knifes in the cutting machine
$Ntrans$ = number of transversal knifes in the cutting machine
AP_{ap} = paper width of paper ap
WI_i = width of order i
$CO_{p\,k}$ = cost of the paper type assigned to layer k of pattern p
$Minloss$ = minimum trim-loss in the pattern width
$Maxloss$ = maximum trim-loss in the pattern width
α_k = length factor of layer k (fluted layers consume more than 1 meter per meter of board)
$S_{tp\,ap}$ = material stock available of paper type tp and width ap
L_i = length of order i
D_i = number of board sheets required by order i (demand of order i)
δ_i = overproduction allowed of order i
$CRmin_p$ = minimum length allowed for pattern p
$CRmax_p$ = maximum length allowed for pattern p
VEL = cutting machine velocity
CW = change over time if paper width is changed from one pattern to the next one
$CB_{p\,p'}$ = changeover time between patterns p and p' due to the change in board type
CN = changeover time between patterns due to the repositioning of knives in the corrugator
M = big-M parameter
DD_i = due date of order i

Positive Variables

cp_{pk} = trim-loss cost of pattern p in layer k
yr_p = indicates the use of pattern p in the cutting plan
ta_{pk} = total area assigned to layer k of pattern p
ua_{pk} = used area in layer k of pattern p
x_p = length of pattern p
pt_p = processing time of pattern p
$ct_{p\,p'}$ = change over time between p and p'
ft_p = final time of processing pattern p
to_i = final time of processing order i
$l_{ap\,k\,p}$ = auxiliary variable introduced to reformulate the bilinear term $w_{ap\,k\,p}{\cdot}x_p$
nxj_{ipj} = auxiliary variable used to reformulate the bilinear term $n_{ip}{\cdot}x_p$
nj_{ipj} = auxiliary variable used to reformulate variable n_{ip}
$x^*_{ap\,k\,p}$ = auxiliary variable used in the convex hull relaxation to replace bilinear term $w_{ap\,k\,p}{\cdot}x_p$
x^1_{ipj} = auxiliary variable used in the convex hull relaxation to replace bilinear term $n_{ip}{\cdot}x_p$
x^2_{ipj} = auxiliary variable used in the convex hull relaxation to replace bilinear term $n_{ip}{\cdot}x_p$

Integer Variable

n_{ip} = number of orders i assigned in the width of pattern p

Binary Variables

y_{ip} = has a value of 1 if order i is assigned to pattern p, 0 otherwise
$w_{ap\,k\,p}$ = has a value of 1 if width ap is assigned to layer k of pattern p, 0 otherwise
$wk_{p\,p'}$ = has a value of 1 if there is a change of width in any layer between pattern p and p'
$y^{pred}_{p\,p'}$ = has a value of 1 if p is processed before p' (in the case of eqs 31−34, it means immediate precedence, whereas in eqs 35 and 36, it means general precedence)

Boolean Variables

$\omega_{ap\,k\,p}$ = auxiliary Boolean variable used to reformulate the bilinear term $w_{ap\,k\,p}{\cdot}x_p$
β_{ipj} = auxiliary Boolean variable used to reformulate the bilinear term $n_{ip}{\cdot}x_p$

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