

## Dynamics of the Entanglement between Two Oscillators in the Same Environment

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We provide a complete characterization of the evolution of entanglement between two resonant oscillators coupled to a common environment. We identify three phases with different qualitative long time behavior: There is a phase where entanglement undergoes a sudden death. Another phase (sudden death and revival) is characterized by an infinite sequence of events of sudden death and revival of entanglement. In the third phase (no sudden death) there is no sudden death of entanglement, which persists for a long time. The phase diagram is described and analytic expressions for the boundary between phases are obtained. These results are applicable to a large variety of non-Markovian environments. The case of nonresonant oscillators is also numerically investigated.

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Entanglement is responsible for the most counterintuitive aspects of quantum mechanics. It is not only a source of surprises but also a useful physical resource: it is the key ingredient for teleportation and is believed to be the origin of the power of quantum computers. Disentanglement due to the interaction with external reservoirs was extensively studied for systems of qubits (with common or independent environments) [1–4]. In such a context, a peculiar phenomenon was noticed in [5]: Entanglement can disappear in a finite time (“sudden death” SD of entanglement). Here we study the evolution of the entanglement between two oscillators interacting with a common environment. Entanglement dynamics for such kinds of systems was analyzed before and a variety of results is available. For example: It was shown that the environment may completely disentangle an initial entangled state (inducing a sudden death of entanglement) [6–8]. The evolution of the same type of initially entangled state was analyzed when the two oscillators interact with a common bath under a simplifying assumption: the Markovian approximation [9–11]. In that case, a condition for the existence of an asymptotic entangled state was deduced [9]. If such a condition is not satisfied (see below) sudden death takes place. Also, it was shown that the interaction with a common environment opens the door to the creation of an entangled state from an initial separable state [11]. Most of these works were done under some sort of Markovian approximation. More recently, the non-Markovian regime was analyzed [12–15]. Under special conditions a simple result emerged: The final state remains entangled but half of the initial entanglement is lost [14].

Our work provides a unified picture to understand the origin of the different qualitative behaviors (“phases”) that characterize the evolution of entanglement for long times. We will fully characterize these phases of the evolution and obtain equations describing the boundary between them. For this purpose we will use a well-known tool: the exact master equation for quantum Brownian motion [16]. We consider two harmonic oscillators ( $x_1$  and  $x_2$ ) bilinearly

coupled between them and with a collection of harmonic oscillators [16–19]. The total Hamiltonian is  $H = H_S + H_{\text{int}} + H_{\text{env}}$ , where

$$H_S = \frac{p_1^2 + p_2^2}{2m} + \frac{1}{2}m(\omega_1^2 x_1^2 + \omega_2^2 x_2^2) + mc_{12}x_1x_2,$$

$$H_{\text{env}} = \sum_{n=1}^N \left( \frac{\pi_n^2}{2m_n} + \frac{m_n}{2}w_n^2 q_n^2 \right), \quad (1)$$

$$H_{\text{int}} = (x_1 + x_2) \sum_{n=1}^N c_n q_n.$$

Using coordinates  $x_{\pm} = (x_1 \pm x_2)/\sqrt{2}$ , the Hamiltonian transforms into  $H_S = (p_+^2 + p_-^2)/2m + m(\omega_{\pm}^2 x_{\pm}^2 + \omega_{\mp}^2 x_{\mp}^2)/2 + mc_{+-}x_+x_-$ , where  $\omega_{\pm}^2 = (\omega_1^2 + \omega_2^2)/2 \pm c_{12}$  and  $c_{+-} = (\omega_1^2 - \omega_2^2)/2$ . This model can be exactly solved assuming that the initial state of the environment is thermal with initial temperature  $T$  [16]. The resonant case ( $\omega_1 = \omega_2$ ), where the  $x_{\pm}$  oscillators are decoupled, is the simplest: The exact master equation for the reduced density matrix of the two oscillators  $\rho$  is ( $\hbar = 1$ ) [16,19]:

$$\dot{\rho} = \frac{1}{i}[H_R, \rho] - i\gamma(t)[x_+, \{p_+, \rho\}] - D(t)[x_+, [x_+, \rho]] - f(t)[x_+, [p_+, \rho]]. \quad (2)$$

Here, the renormalized Hamiltonian is  $H_R = H_S + m\delta\omega^2(t)x_{\pm}^2/2$ . The coefficients  $\delta\omega^2(t)$ ,  $\gamma(t)$ ,  $D(t)$  and  $f(t)$  depend on the spectral density of the environment, defined as  $J(\omega) = \sum_n c_n^2 \delta(\omega - w_n)/2m_n w_n$ , and also on its initial temperature. The explicit form of these coefficients is rather cumbersome and was studied in detail elsewhere [16,20]. We will focus on the Ohmic environment with a spectral density  $J(\omega) = 2m\gamma_0\omega\theta(\omega - \Lambda)/\pi$ . The high frequency cutoff  $\Lambda$  defines a characteristic time scale  $\Lambda^{-1}$  over which the coefficients  $\gamma(t)$  and  $\delta\omega^2(t)$  vary. For times  $t \gg \Lambda^{-1}$  these two temperature independent coefficients settle into asymptotic values:  $\gamma(t) \rightarrow \gamma = 2\gamma_0$  and  $\delta\omega^2(t) \rightarrow -4m\Lambda\gamma/\pi$ . The time dependent fre-

quencies  $\Omega_{1,2}^2(t) = \omega_{1,2}^2 + \delta\omega^2(t)/2$  approach cutoff independent values only if the bare frequencies  $\omega_{1,2}$  have an appropriate dependence on the cutoff. The coupling constant  $c_{1,2}$  must also be renormalized in the same way so that the time dependent coupling  $C_{12}(t) = c_{12} + \delta\omega^2(t)/2$  approaches a finite cutoff independent value. The behavior of the diffusion coefficients  $D(t)$  and  $f(t)$  is more complicated and depends on the initial temperature. For the moment we just need to mention here that for realistic environments these coefficients approach asymptotic values after a temperature-dependent time (which equals  $\Lambda^{-1}$  only in the high temperature regime). A word on notation: upper case letters will be used to denote renormalized quantities. The time label will be omitted when referring to the asymptotic value of the corresponding function (i.e.,  $\Omega_{1,2}$  denotes the asymptotic value of the renormalized frequency of the oscillators, etc.).

From the master equation (2) we can derive simple evolution equations for the covariance matrix  $V_{ij}(t) = \text{Tr}(\rho(t)\{r_i, r_j\})/2 - \text{Tr}(\rho(t)r_i)\text{Tr}(\rho(t)r_j)$ , where  $i, j = 1, \dots, 4$ , and  $\vec{r} = (x_-, p_-, x_+, p_+)$ . Some of these equations are particularly illuminating: In fact, equations for the covariances split into two blocks of  $2 \times 2$ . The evolution of the first block formed with the second moments of  $x_-$  and  $p_-$  corresponds to a free oscillator with frequency  $\Omega_-(t)$ . The evolution of the second block, formed with the second moments of  $x_+$  and  $p_+$ , satisfies

$$\frac{d}{dt} \left( \frac{\langle p_+^2 \rangle}{2m} + \frac{m}{2} \Omega^2(t) \langle x_+^2 \rangle \right) = -\frac{2\gamma(t)}{m} \langle p_+^2 \rangle + \frac{D(t)}{m}, \quad (3)$$

$$\frac{1}{2} \frac{d^2 \langle x_+^2 \rangle}{dt^2} + \gamma(t) \frac{d \langle x_+^2 \rangle}{dt} + \Omega^2(t) \langle x_+^2 \rangle = \frac{\langle p_+^2 \rangle}{m^2} - \frac{f(t)}{m}. \quad (4)$$

These equations contain most of the necessary information to analyze the evolution of the entanglement between initial Gaussian states. To solve them we need the time-dependent coefficients that appear in the master equation. But, remarkably, we can use the above equations to understand the qualitative behavior of entanglement. For this, we only need to assume that the time dependent coefficients in (2) approach asymptotic values. In such case, there is a stable stationary solution where the dispersions  $\Delta^2 x_+ = \langle x_+^2 \rangle$  and  $\Delta^2 p_+ = \langle p_+^2 \rangle$  are

$$\Delta p_+ = \sqrt{\frac{D}{2\gamma}}; \quad \Omega \Delta x_+ = \sqrt{\frac{D}{2m^2\gamma} - \frac{f}{m}}. \quad (5)$$

Entanglement for Gaussian states is entirely determined by the properties of the covariance matrix  $V_{ij}$ . In fact, a good measure of entanglement for such states is the logarithmic negativity  $E_{\mathcal{N}}$  [21–24]:

$$E_{\mathcal{N}} = \max\{0, -\ln(2\nu_{\min})\}, \quad (6)$$

where  $\nu_{\min}$  is the smallest symplectic eigenvalue of the partially transposed covariance matrix. There are known

expressions for  $E_{\mathcal{N}}$  for Gaussian states which we used as initial conditions. Thus, it is useful to mention them here: The two-mode squeezed state, obtained from the vacuum as  $\exp(-r(a_1^\dagger a_2^\dagger - a_1 a_2))|0\rangle$ , has  $E_{\mathcal{N}} = 2|r|$ . This is a minimum uncertainty state with dispersions satisfying  $\delta x_+ \delta p_+ = \delta x_- \delta p_- = 1/2$ . The squeezing  $r$  determines the ratio between variances:  $m\Omega \delta x_+ / \delta p_+ = \delta p_- / (m\Omega \delta x_-) = \exp(2r)$ . As  $r \rightarrow \infty$  the state localizes in the  $p_+$  and  $x_-$  variables approaching an ideal EPR state [25]. Another initial state we will consider is a separable squeezed state for which  $m\Omega \delta x_{1,2} / \delta p_{1,2} = \exp(2r)$ .

The evolution of entanglement for general initial Gaussian states can then be simply analyzed by using the previous results. For the resonant case, the  $x_-$  virtual oscillator decouples exactly. Using the Eqs. (3) and (4) we see that  $\Delta x_+$  and  $\Delta p_+$  approach asymptotic values after a time scale  $1/\gamma$ , fixed by the dissipation rate. After that time, the covariance matrix in the  $(x_+, x_-)$  bases has a  $2 \times 2$  block with oscillatory functions (corresponding to the  $x_-$  oscillator) and another  $2 \times 2$  block, corresponding to the  $x_+$  virtual oscillator, which is diagonal. Using this form for the covariance matrix (and changing basis to obtain covariances of the original  $x_{1,2}$  oscillators) it is simple to compute the following logarithmic negativity

$$E_{\mathcal{N}}(t) \rightarrow \max\{0, \tilde{E}_{\mathcal{N}} + \Delta E_{\mathcal{N}} G(t)\}. \quad (7)$$

Where  $G(t)$  is a function with period  $\pi/\Omega_-$  in  $\{-1, 1\}$ . The mean value  $\tilde{E}_{\mathcal{N}}$  [26] and the amplitude  $\Delta E_{\mathcal{N}}$  are

$$\tilde{E}_{\mathcal{N}} = \max\{r, r_{\text{crit}}\} - \frac{1}{2} \ln(2\Delta x_+ \Delta p_+),$$

$$\Delta E_{\mathcal{N}} = \min\{r, r_{\text{crit}}\}.$$

Here  $r$  is the initial squeezing factor defined as  $r = |\frac{1}{2} \times \ln(m\Omega_- \frac{\delta x_-}{\delta p_-})|$ ,  $r_{\text{crit}}$  is related to the squeezing factor of the equilibrium state for the  $x_+$ -oscillator defined as  $r_{\text{crit}} = |\frac{1}{2} \ln(m\Omega_- \frac{\Delta x_+}{\Delta p_+})|$ . The dispersions  $\Delta x_+$  and  $\Delta p_+$  are the asymptotic values (5).

These simple results will enable us to draw general conclusions about the dynamics of entanglement for long times. Three qualitatively different scenarios emerge: First, entanglement may persist for arbitrary long times. This phase, which we call “NSD” (for no-sudden death), is realized when the initial state is such that  $\tilde{E}_{\mathcal{N}} - \Delta E_{\mathcal{N}} > 0$ , which translates into  $|r - r_{\text{crit}}| > \ln(2\Delta x_+ \Delta p_+)/2 \equiv S_r$ . Then, there is a phase where entanglement undergoes an infinite sequence of events of “sudden death and revival” (SDR) [27,28]. This SDR phase occurs if the initial state is such that  $|E_c| \leq r \leq -E_c + 2r_{\text{crit}}$ , where  $E_c \equiv r_{\text{crit}} - S_r$ . Finally, a third phase characterized by a final event of “sudden death” (SD) of entanglement may be realized. The condition for the SD phase is  $r \leq -E_c$ .

Depending on the properties of the environment (initial temperature, damping rate, etc) an initial state with squeezing  $r$  will belong to one of the three phases. For the Ohmic

environment we used exact expressions for the coefficients  $D$  and  $f$  [20] to obtain the phase diagram shown in Fig. 1. Areas corresponding to each of the three phases are displayed and, as a reference, we show the temperature dependence of  $S_r$  and  $r_{\text{crit}}$ .

It is worth mentioning some features of the phase diagram, that provide complete information about the asymptotic behavior of entanglement. The low temperature NSD phase is purely non-Markovian and nonperturbative. Its area shrinks as the damping rate decreases. This phase includes the coherent states, that can become entangled at very low temperature. The zero temperature line is also interesting: It contains states in the NSD phase for small and large squeezings. However, for an intermediate range of squeezings centered about  $r_{\text{crit}}$  states belong to the SDR phase. To the contrary, the high temperature region of the diagram is quite different. Thus, for high temperatures we have  $E_c < 0$  (which implies that coherent states do not get entangled) and also  $r_{\text{crit}} \rightarrow 0$  (which implies that the region covered by the SDR phase becomes narrower). Thus, for high temperatures initial states with large squeezing ( $r > \ln(2\Delta x_+ \Delta p_+)/2 = S_r$ ) retain some of their entanglement while those with squeezing factors smaller than the critical value  $S_r$  suffer from sudden death. This is almost the same condition obtained previously for persistence of entanglement in the Markovian regime [9]. However, our analysis shows that the boundary between SD and NSD phases is rather subtle: for any finite temperature the two phases are separated by a very narrow portion of SDR phase (the amplitude of the oscillations,  $r_{\text{crit}}$ , decreases as temperature grows). This is not the only interesting non-Markovian effect identified by our analysis. Thus, it is remarkable the relevant role played by the anomalous diffusion in

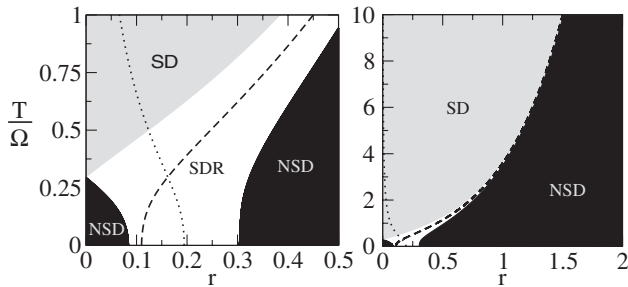


FIG. 1. Phase diagram for Ohmic environment ( $\Omega = 1$ ,  $\gamma_0 = 0.15$ ,  $\Lambda = 20$ ,  $m = 1$ ,  $C_{12} = 0$ ). The SD, NSD, and SDR phases describe the three different qualitative long time behaviors for the entanglement between two oscillators. For temperatures above the one for which  $S_r = r_{\text{crit}}$  the SDR phase is centered about the dashed line  $S_r$  and has a width given by the dotted line  $r_{\text{crit}}$ . Below this temperature the role of  $S_r$  and  $r_{\text{crit}}$  are interchanged. SDR separates the SD and NSD phases. The low temperature NSD island is non-Markovian and nonperturbative.  $\tilde{E}_{\mathcal{N}}$  in the NSD phase is the distance to the dashed line for  $r > r_{\text{crit}}$ , and the distance between the dashed and dotted lines for  $r \leq r_{\text{crit}}$ .

the master equation: The SDR phase, the entanglement oscillations in the NSD phase and the possibility of entangling initial coherent states are effects that disappear when  $f/D \rightarrow 0$  (for example, in the high temperature limit). Thus, the final entanglement of initial coherent states is a constant  $E_c$  that decreases when  $f \rightarrow 0$  and grows with  $\gamma_0$  (as the final state localizes more in the  $x_+$  coordinate). A final comment on the phase diagram: The NSD phase is characterized by a nonvanishing asymptotic entanglement that can be quantified from the phase diagram itself. The average value of the logarithmic negativity is simply the distance to the dashed line (which signals the midpoint of the SDR phase) or just the distance between the dashed and dotted lines for  $r \leq r_{\text{crit}}$ .

We verified our predictions by an exact numerical solution (performed by discretizing the environment and numerically solving the equations for all covariances). For an Ohmic environment with  $\gamma_0 = 0.15$ ,  $\Lambda = 20$ ,  $m = 1$ ,  $\Omega = 1$ ,  $C_{12} = 0$  are shown in Fig. 2. All evolutions obtained for  $E_{\mathcal{N}}$  correspond to one of the above phases.

Our analysis can be extended to other types of system-environment coupling: If  $H_{\text{int}} = \sum_{i=1}^N \lambda_n (a_{i+}^\dagger b_n + a_i b_n^\dagger)$  an exact master equation can also be obtained ( $a_{i+}^\dagger$  and  $b_n^\dagger$  are creation operators of the  $x_+$  and the  $q_n$  oscillators). Such an equation can be obtained, for all spectral densities and temperature, by symmetrizing (2) with respect to  $x_+$  and  $p_+$  and appropriately defining new time dependent coefficients (the zero temperature version of such equation, which in the Markovian limit coincides with the

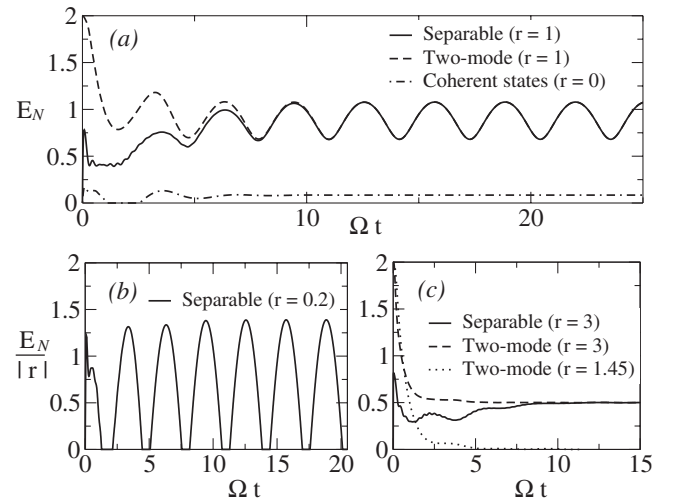


FIG. 2. Logarithmic negativity for resonant oscillators in the same environment. (a) For  $T = 0$  the NSD phase appears both for large and small squeezing. Asymptotic behavior of initially entangled or separable states only depends on  $r$ . The amplitude of oscillations vanishes when  $r \rightarrow 0$ . (b) The SDR phase appears for intermediate values of squeezing at zero temperature. (c)  $T/\Omega = 10$ , the SD phase appears for small  $r$  and NSD phase for large squeezings, oscillations in the steady state are attenuated as the temperature increases.

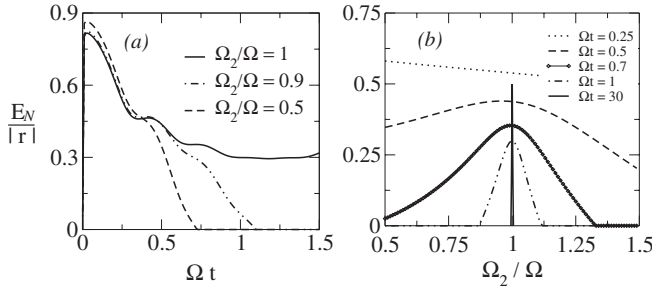


FIG. 3. Entanglement dynamics for nonresonant oscillators initially in a squeezed separable state ( $r = 3$ ) and  $T/\Omega = 10$ . (a) Entanglement is created between nonresonant oscillators but it banishes in finite time. (b)  $E_{\mathcal{N}}$  for different times as a function of the frequency of the second oscillator, the resonant condition is essential for asymptotic entanglement.

quantum optical master equation, was discussed in [14]. Conclusions drawn from this equations differ from the above ones in a simple way: Symmetry implies that the anomalous diffusion term vanishes. Thus, the final state of  $x_+$  is not squeezed, i.e.  $r_{\text{crit}} = 0$ . Also, for  $T = 0$  we have  $S_r = 0$  as the asymptotic state is the vacuum. Therefore, for this type of interaction (or, equivalently, in the RWA) the SDR phase and the low temperature NSD phase do not exist. There are only two phases (SD and NSD), separated by the  $S_r$  line (which in this case starts at the origin of the phase diagram). In this case, entanglement displays no oscillations in the asymptotic regime. At zero temperature entanglement approaches  $r$  (half the initial value in a two-mode squeezed state).

The above properties are valid if the two oscillators are resonant. If this is not the case the master equation is no longer valid since the  $x_-$  and  $x_+$  virtual oscillators are coupled (i.e.,  $x_-$  is no longer protected from the environment and reaches equilibrium). Figure 3, obtained from the exact numerical solution, shows that when we move away from resonance  $E_{\mathcal{N}}$  decays much faster. Also, the value of  $E_{\mathcal{N}}$  away from resonance shows a resonance peak that becomes sharper as time grows.

We provided a complete picture of the behavior of entanglement for resonant oscillators in a common environment. The analysis applies whenever the coefficients of the master equation approach constant asymptotic values. This analysis may enable us to obtain simple conclusions. For example: for super-Ohmic environments it is known that  $\gamma(t)$  vanishes asymptotically (in the limit of large cutoff). Then, we predict the absence of equilibrium for the  $x_+$  oscillator. In this case, we expect that entanglement will oscillate approximately recovering its initial value after some time. This is confirmed by our numerical simulations but contradicts results reported in [14]. Finally, we remark once again that our results for the asymptotic entanglement are highly dependent on the validity of the

resonance condition. For nonresonant oscillators the generic fate for long times is disentanglement due to the interaction with the environment (similar effects are induced by nonlinearities). This issue will be analyzed in detail elsewhere.

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- [1] D. Braun, Phys. Rev. Lett. **89**, 277901 (2002).
- [2] F. Benatti, R. Floreanini, and M. Piani, Phys. Rev. Lett. **91**, 070402 (2003).
- [3] S. Oh and J. Kim, Phys. Rev. A **73**, 062306 (2006).
- [4] C. Anastopoulos, S. Shresta, and B. L. Hu, arXiv:quant-ph/0610007.
- [5] T. Yu and J. H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004).
- [6] M. G. Paris, J. Opt. B **4**, 442 (2002).
- [7] A. Serafini, F. Illuminati, M. G. A. Paris, and S. De Siena, Phys. Rev. A **69**, 022318 (2004).
- [8] S. Maniscalco, S. Olivares, and M. G. A. Paris, Phys. Rev. A **75**, 062119 (2007).
- [9] J. S. Prauzner-Bechcicki, J. Phys. A **37**, L173 (2004).
- [10] P. J. Dodd and J. J. Halliwell, Phys. Rev. A **69**, 052105 (2004).
- [11] F. Benatti and R. Floreanini, J. Phys. A **39**, 2689 (2006).
- [12] M. Ban, J. Phys. A **39**, 1927 (2006).
- [13] K.-L. Liu and H.-S. Goan, Phys. Rev. A **76**, 022312 (2007).
- [14] J.-H. An and W.-M. Zhang, Phys. Rev. A **76**, 042127 (2007).
- [15] C. Hörhammer and H. Büttner, Phys. Rev. A **77**, 042305 (2008).
- [16] B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D **45**, 2843 (1992).
- [17] R. P. Feynman and F. L. Vernon, Ann. Phys. (Leipzig) **24**, 118 (1963).
- [18] A. O. Caldeira and A. J. Leggett, Physica (Amsterdam) **121A**, 587 (1983).
- [19] C.-H. Chou, T. Yu, and B. L. Hu, Phys. Rev. E **77**, 011112 (2008).
- [20] C. H. Fleming, B. L. Hu, and A. Roura, arXiv:0705.2766.
- [21] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
- [22] J. Eisert, Ph.D. thesis, University of Potsdam, 2001.
- [23] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A **70**, 022318 (2004).
- [24] J. Eisert and M. B. Plenio, Phys. Rev. Lett. **89**, 137902 (2002).
- [25] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
- [26] Here we consider the case where  $\delta x_- \delta p_- = 1/2$ . In general, we can apply the same results replacing  $\Delta x_+ \Delta p_+ \rightarrow 2\Delta x_+ \Delta p_+ \delta x_- \delta p_-$ .
- [27] M. Yöncac, T. Yu, and J. H. Eberly, J. Phys. B **39**, S621 (2006).
- [28] M. Yöncac, T. Yu, and J. H. Eberly, J. Phys. B **40**, S45 (2007).