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Liftings of Nichols Algebras of Diagonal Type I. Cartan Type A

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After the classification of the finite-dimensional Nichols algebras of diagonal type [17, 18], the determination of its defining relations [6, 7], and the verification of the *generation in degree-s1* conjecture [6], there is still one step missing in the classification of complex finite-dimensional Hopf algebras with abelian group, without restrictions on the order of the latter: the computation of all deformations or liftings. A technique towards solving this question was developed in [5], built on cocycle deformations. In this paper, we elaborate further and present an explicit algorithm to compute liftings. In our main result we classify all liftings of finite-dimensional Nichols algebras of Cartan type A, over a cosemisimple Hopf algebra H. This extends [2], where it was assumed that the parameter is a root of unity of order >3 and that H is a commmutative group algebra. When the parameter is a root of unity of order 2 or 3, new phenomena appear: the quantum Serre relations can be deformed; this allows in turn the power root vectors to be deformed to elements in lower terms of the coradical filtration, but not necessarily in the group algebra. These phenomena are already present in the calculation of the liftings in type A_2 at a parameter of order 2 or 3 over an abelian group [11, 19], done by a different method using a computer program. As a byproduct of our calculations, we present new infinite families of finite-dimensional pointed Hopf algebras.

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1 Introduction

1.1 The general context

This is the first article of a series intended to determine all liftings of finite-dimensional Nichols algebras of diagonal type over an algebraically closed field of characteristic zero $\Bbbk$. The end of this series will also conclude the classification of the finite-dimensional pointed Hopf algebras with abelian group of group-likes, without restrictions on the order of the group. The setting, slightly different than in [5], is the following. We fix:

- A cosemisimple Hopf algebra H.
- A braided vector space of diagonal type (V, c), with a principal realization in ${}^H_H\mathcal{YD}$, such that the Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional.

We place ourselves in this more general context in order to contribute to the classification of Hopf algebras with finite Gelfand–Kirillov dimension, and more precisely to those that are co-Frobenius.

A *lifting* of $V \in {}^H_H\mathcal{YD}$ is a Hopf algebra L such that $\operatorname{gr} L = \mathfrak{B}(V)\#H$, where $\operatorname{gr} L$ is the graded Hopf algebra associated with the coradical filtration. In other words [1, 2.4], L is a lifting of V iff there is an epimorphism of Hopf algebras $\phi : \mathcal{T}(V) := T(V)\#H \to L$ such that $\phi_{|H} = \mathrm{id}_H$ and

$$\phi_{|H\oplus V\#H} : H \oplus V\#H \to L_1 \text{ is an isomorphism of Hopf bimodules.} \tag{1.1}$$

Such ϕ is called a *lifting map*. If emphasis on H is needed, then we say that L is a lifting of V over H; if $H = \Bbbk G$ is the group algebra of the group G, then we also say that L is a lifting of V over G.

The aim of the series is to compute all liftings of every V as Section 1.1. It seems very hard, and probably not feasible, to give a uniform answer to this problem, that is compact formulae valid for all V. We proceed then by a case-by-case analysis of the list in the classification of [18]. Let $r_1, \dots, r_M$ be the defining (homogeneous) relations of $\mathfrak{B}(V)$, computed in [6]; let $n_j = \deg r_j$. If ϕ is a lifting map as Section 1.1, then there exists $p_j \in \bigoplus_{0\le i<n_j} T^i(V)\#H$ such that $\phi(r_j) = \phi(p_j)$, for all j. Our approach is

- ◊ to establish the general form of the p_j's, in terms of the r_j's and some parameters;
- ◊ to define a Hopf algebra $L = \mathcal{T}(V)/\langle r_1 - p_1, \dots, r_M - p_M\rangle$ for each choice of the parameters alluded Section 1.1 and to prove that $\operatorname{gr} L \simeq \mathfrak{B}(V)\#H$; and
- ◊ to show that every lifting can be obtained in this way.

In situations considered in previous work [2, 3] the $\phi(r_j)$'s belong to H and were computed recursively, while the remaining points were dealt with by *ad hoc* manners. In this series we proceed recursively again but following the strategy in [5], inspired by [15, 20]; namely, we compute a sequence of quotients of $\mathcal{T}(V)$ as cocycle deformations of a parallel sequence of quotients of the form $\mathfrak{B}\#H$, describing eventually $L = \mathcal{T}(V)/\langle r_1 - p_1, \dots, r_M - p_M\rangle$ as a cocycle deformation of $\mathfrak{B}(V)\#H = \mathcal{T}(V)/\langle r_1, \dots, r_M\rangle$ (see Section 3).

In this paper we compute all liftings for V of Cartan type A, over a root of unit ξ of order 2 or 3. The case when ξ has order >3 is known for group algebras of finite abelian groups [2, Section 6]; we extend this to a general cosemisimple Hopf algebra (see Theorems 1.6, 1.8, and 1.10). There are three reasons to start with Cartan type A. First, it is the Dynkin diagram of *Her all-embracing Majesty* [24]. Second, formulae for the Nichols algebras of this type are much more explicit than for other types. Third, the experience and results for this type would help to understand and solve the other types.

1.2 The main result

Let $\theta \in \mathbb{N}$ and $\mathbb{I} = \{1, \dots, \theta\}$. Let V be a braided vector space of diagonal type with basis $(x_i)_{i\in\mathbb{I}}$ and braiding matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}}$. Let $(\alpha_i)_{i\in\mathbb{I}}$ be the canonical basis of $\mathbb{Z}^\theta$. The braided Hopf algebra $T(V)$ is $\mathbb{Z}^\theta$-graded by $|x_i| = \alpha_i$, $i \in \mathbb{I}$. Let $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \Bbbk$ be the bilinear form defined by $\chi(\alpha_i, \alpha_j) = q_{ij}$, $i, j \in \mathbb{I}$; set $q_{\alpha\beta} = \chi(\alpha, \beta)$, $\alpha, \beta \in \mathbb{Z}^\theta$. The braided commutator is defined on $\mathbb{Z}^\theta$-homogeneous elements $u, v \in T(V)$ by

$$[u, v]_c = uv - q_{|u||v|} vu.$$

Then $\operatorname{ad}_c x_i(v) := [x_i, v]_c$. Let ξ be a primitive Nth root of unity, $N \geq 2$. We fix a braiding matrix $(q_{ij})_{i,j\in\mathbb{I}}$ such that

$$q_{ii} = \xi, \qquad q_{ij}q_{ji} = \begin{cases} \xi^{-1}, & |i-j| = 1, \\ 1, & |i-j| > 1, \end{cases} \qquad i, j \in \mathbb{I}. \tag{1.2}$$

This is a braided vector space of Cartan type A_θ and the corresponding generalized Dynkin diagram, cf. [18], is $\overset{\xi}{\circ} \overset{\xi^{-1}}{\text{———}} \overset{\xi}{\circ} \overset{\xi^{-1}}{\text{———}} \overset{\xi}{\circ} \cdots\cdots \overset{\xi}{\circ} \overset{\xi^{-1}}{\text{———}} \overset{\xi}{\circ}$. The corresponding Nichols algebra is indeed the multiparametric version of the positive part of the small quantum group, or Frobenius–Lusztig kernel, of type A_θ. For $i \leq j \in \mathbb{I}$, we denote

$(ij) = \sum_{i \le k \le j} \alpha_k \in \mathbb{Z}^\theta$; clearly $\{(ij) : i \le j \in \mathbb{I}\}$ is the set of positive roots of the root system A_θ. The associated Lyndon words are defined recursively by

$$x_{(ij)} = \begin{cases} x_j, & i = j, \\ [x_i, x_{(i+1\,j)}]_c & i+1 \le j, \end{cases}$$

in $T(V)$ or any quotient thereof. We also need the notation $x_{ij} = [x_i, x_j]_c$, $i < j \in \mathbb{I}$. We now state the presentation of $\mathcal{B}(V)$ by generators and relations. Part 1.1 was proved in [2], inspired by [23]; is from [4].

Proposition 1.1.

(1) Assume that $N > 2$. Then $\mathfrak{B}(V)$ is generated by $(x_i)_{i\in\mathbb{I}}$ with relations

$$x_{ij} = 0, \qquad i < j - 1; \tag{1.3}$$

$$(\mathrm{ad}_c\, x_i)^2(x_j) = 0, \qquad |j - i| = 1; \tag{1.4}$$

$$x_{(ij)}^N = 0, \qquad i \le j. \tag{1.5}$$

The distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}(V)$ [8, Definition 1] is generated by $(x_i)_{i\in\mathbb{I}}$ with relations (1.3) and (1.4); this is denoted $\widehat{\mathcal{B}}(V)$ in [2, Section 6.3].

(2) Assume that $N = 2$. Then $\mathfrak{B}(V)$ is generated by $(x_i)_{i\in\mathbb{I}}$ with relations (1.3), (1.5), and

$$[x_{(i-1\,i+1)}, x_i]_c = 0, \qquad 1 < i < \theta. \tag{1.6}$$

The distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}(V)$ [8, Definition 1] is generated by $(x_i)_{i\in\mathbb{I}}$ with relations (1.3), (1.4), and (1.6). □

Remark 1.2. Relations (1.6) hold for $N > 2$, by (1.3) and (1.4).

When $N = 2$, (1.4) becomes

$$x_i^2 x_j + q_{ij}^2 x_j x_i^2 = 0, \qquad |j - i| = 1.$$

Since $x_i^2 = 0$ by (1.5), (1.4) holds in $\mathfrak{B}(V)$; thus $\mathfrak{B}(V)$ is a quotient of $\widetilde{\mathcal{B}}(V)$. □

Remark 1.3. The distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}(V)$ is meant to have the same set of Poincaré Birkhoff Witt (PBW) generators, hence the same root system, as $\mathcal{B}(V)$.

By this reason, the choice of the defining relations is performed so as to guarantee this property. In particular, one needs relations to reorder any pair of PBW generators.

Assume V is of Cartan type A. If $N > 3$, then this is automatically attained provided that the quantum Serre relations (1.3) and (1.4) hold (see [2, Lemmas 6.4 and 6.7]). When $N = 2$, then quantum Serre relations (1.3) and (1.4) are not enough, as we cannot reorder the PBW generators $x_{(i-1\,i+1)}$ and x_i, $1 < i < \theta$; hence the need of (1.6). Now, this enlarged set of relations suffices, as it is shown in Lemma 4.1. □

Assume that $N > 3$. Then, all liftings of V (over a finite abelian group) are classified in [2, Theorem 6.25]. In this paper, we classify all liftings of V when $N = 2$ or 3. To present our main results, we need more notation. Let $(g_i, \chi_i)_{i\in\mathbb{I}}$ be a principal realization of V over H (see Section 2.2); let

$$\Gamma = \langle g_1, \dots, g_\theta \rangle.$$

For $i_1, \dots, i_k \in \mathbb{I}$ distinct, $k \in \mathbb{N}$, set

$$\begin{aligned} g_{i_1,\dots,i_k} &:= g_{i_1} \cdots g_{i_k}, & \chi_{i_1,\dots,i_k} &:= \chi_{i_1} \cdots \chi_{i_k}, & x_{i_1,\dots,i_k} &:= [x_{i_1}, [x_{i_2 \dots, i_k}]_c]_c; \\ g_{(ij)} &:= g_{i,i+1,\dots,j}, & \chi_{(ij)} &:= \chi_{i,i+1,\dots,j}, & i &\le j \in \mathbb{I}. \end{aligned}$$

Also, if $i < j \in \mathbb{I}$, then let us fix $g_{(j\,i)} := 1$, $\chi_{(j\,i)} := \epsilon$.

1.2.1 Component in Γ

Here $N \geq 2$ is arbitrary. For $i \leq j \in \mathbb{I}$, we set

$$C_p = C_{ip}^j = (1 - q^{-1})^N \chi_{(i\,p)}(g_{(p+1\,j)})^{N(N-1)/2}. \tag{1.7}$$

If the quantum Serre relations (1.3) and (1.4) are not deformed, then the lifting problem is equivalent to the following question, which amounts to solving an equation in $\Bbbk\Gamma$ (see [2, (6-36)], [4, Section 3]):

- Find all families $(u_{(ij)})_{i\leq j\in\mathbb{I}}$ of elements in $\Bbbk\Gamma$, such that

$$\Delta(u_{(ij)}) = u_{(ij)} \otimes 1 + g_{(ij)}^N \otimes u_{(ij)} + \sum_{i\leq p<j} C_{ip}^j u_{(i\,p)} g_{(p+1\,j)}^N \otimes u_{(p+1\,j)}. \tag{1.8}$$

The solutions to 1.8 are given in [2, Theorem 6.18]. These are defined recursively on $j - i \geq 0$ [2, 6-40] as elements $u_{(ij)}(\boldsymbol{\gamma})$, for each family (The parameters γ_{ij} are called μ_{ij}

(more precisely μ_α, α a root) in [3]. This is the notation we shall adopt in this article.) of scalars $\boldsymbol{\gamma} = (\gamma_{ij})_{i\le j\in\mathbb{I}}$, by

$$u_{(ij)}(\boldsymbol{\gamma}) = \gamma_{ij}(1 - g^N_{(ij)}) + \sum_{i\le p<j} C^j_{ip}\gamma_{ip}\, u_{(p+1\, j)}(\boldsymbol{\gamma}). \tag{1.9}$$

If $(u_{(ij)}(\boldsymbol{\gamma}))_{i\le j\in\mathbb{I}}$ is a solution, then the quotient of $T(V)\#H$ by the ideal generated by

$$\begin{aligned} r &= 0, && r \text{ (generalized) quantum Serre relation;} \\ a^N_{(ij)} &= u_{(ij)}(\boldsymbol{\gamma}), && i \le j \in \mathbb{I}, \end{aligned} \tag{1.10}$$

is a lifting of V, by Theorems 1.6, 1.8, and 1.10. It was shown in [2, 6.25] that all liftings arise like this if H is a commutative group algebra and $N > 3$. In Theorem 1.10, we extend this to any cosemisimple H. We also compute all liftings when $N \ge 3$.

A key difference in the case $N \le 3$ is that solutions to 1.8 are a *part* of the general solution, see 1.13 below. In particular, we show that the deformations do not necessarily restrict to the coradical (see, e.g., the concrete Examples 4.21 and 5.25).

Remark 1.4. In the present article, we use an equivalent version of 1.9. Namely, we consider families of scalars $\boldsymbol{\mu} = (\mu_{(k\,l)})_{k\le l\in\mathbb{I}}$ subject to

$$\mu_{(k\,l)} = 0, \quad \text{if } \chi^N_{(kl)} \neq \epsilon, \quad \text{or } g^N_{(kl)} = 1. \tag{1.11}$$

We define recursively $\mathrm{u}_{(j\,k)} = \mathrm{u}_{(j\,k)}(\boldsymbol{\mu}) \in \Bbbk\Gamma$, $j \le k \in \mathbb{I}$, by $\mathrm{u}_{(jj)} = 0$, and

$$\mathrm{u}_{(j\,k)} = -\sum_{j\le p<k} C_p \mu_{(p+1\,k)} \left(\mathrm{u}_{(j\,p)} + \mu_{(j\,p)}\left(1 - g^N_{(j\,p)}\right)\right) g^N_{(p+1\,k)}. \tag{1.12}$$

The comparison with the previous solution is as follows: define $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\boldsymbol{\mu})$, $\boldsymbol{\gamma} = (\gamma_{ij})_{i\le j\in I}$, by $\gamma_{ij} = \mu_{(ij)} - \sum_{i\le p<j} C_p \gamma_{ip}\mu_{(p+1\,j)}$, $i \le j$. Then,

$$u_{(jk)}(\boldsymbol{\gamma}) = \mathrm{u}_{(j\,k)}(\boldsymbol{\mu}) + \mu_{(j\,k)}(1 - g^N_{(j,k)}). \qquad \square$$

1.2.2 The shape of the liftings

In the general case $N \ge 2$, we show that the lifting problem is equivalent to solving an algorithm, described synthetically in Section 3.3. An equation similar to 1.9 must be solved recursively, this time with solutions in the previous term of the coradical filtration. We show for type A_θ in Theorems 1.6, 1.8, and 1.10 that any lifting of V is given by

(i) a solution $(u_{(ij)}(\boldsymbol{\mu}))_{i\leq j\in\mathbb{I}}$;
(ii) elements $v_r(\boldsymbol{\lambda}) \in \Bbbk\Gamma$, one for each (generalized) quantum Serre relation and associated with scalars $\boldsymbol{\lambda} = (\lambda_r)_r$ (see (1.18), (1.19), (1.26)); and
(iii) elements $\sigma_{(ij)}(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in T(V)\#H$, computed algorithmically.

The corresponding lifting is the quotient of $T(V)\#H$ by

$$\begin{aligned} r &= v_r(\boldsymbol{\lambda}), \quad r \text{ (generalized) quantum Serre relation;} \\ a_{(ij)}^N &= u_{(ij)}(\boldsymbol{\mu}) + \sigma_{(ij)}(\boldsymbol{\lambda}, \boldsymbol{\mu}), \quad i \leq j \in \mathbb{I}. \end{aligned} \tag{1.13}$$

Compare with 1.10. When $N > 3$, $\lambda_r = 0$ for all r and thus $v_r(\boldsymbol{\lambda}) = 0$, also $\sigma_{(ij)}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = 0$. When $N = 3$, $\lambda_r \neq 0$ only for r of type (1.4) as relations (1.3) remain unchanged.

The case $N = 2$ is actually a bit more involved, as the deformation of the generalized quantum Serre relations (1.6) depends on the deformation of the powers of the simple root vector relations (see (1.19)). Also, in Theorem 1.6, the last line of 1.13 is expressed as $\zeta_{(ij)}^2 = u_{(ij)}$, $i \leq j \in \mathbb{I}$, as $\zeta_{(ij)}^2 = a_{(ij)}^2 +$ terms $\sigma(\boldsymbol{\lambda}, \boldsymbol{\mu})$ (see Remark 4.16). The family $\boldsymbol{\nu} = (\nu_i)_{i\in\mathbb{I}}$ controls the deformations of the generalized quantum Serre relations.

1.2.3 The main result, $N = 2$

Here $\xi = -1$. We fix a family of scalars $\boldsymbol{\mu} = (\mu_{(kl)})_{k\leq l\in\mathbb{I}}$ subject to the constraints and normalizations 1.11. We consider two more families of scalars

$$\boldsymbol{\lambda} = (\lambda_{ij})_{i<j-1\in\mathbb{I}}, \qquad \boldsymbol{\nu} = (\nu_i)_{1<i<\theta}$$

subject to the constraints and normalizations

$$\begin{aligned} \lambda_{ij} &= 0, \quad \text{if } \chi_{ij} \neq \epsilon, \quad \text{or } g_{ij} = 1; \\ \nu_i &= 0, \quad \text{if } \chi_{i-1,i,i,i+1} \neq \epsilon \quad \text{or } g_{i-1,i,i,i+1} = 1. \end{aligned} \tag{1.14}$$

We define families of elements in $\mathcal{T}(V)$ attached to these parameters in the following way. To distinguish from the sequence of pre-Nichols algebras, we denote now by $(a_i)_{i\in\mathbb{I}}$ the generators of $T(V)$; correspondingly, we denote a_{ij}, $a_{(ij)}$, $a_{i_1,\dots,i_k}$, instead of x_{ij}, $x_{(ij)}$, $x_{i_1,\dots,i_k}$.

Let $i,j \in \mathbb{I}$, $|i-j| \geq 2$. We define recursively scalars $d_{ij}(s), b_{ij}(s)$, $s \in \mathbb{N}_0$, as follows: $d_{ij}(0) = 2\lambda_{ij}$, $b_{ij}(0) = -2\chi_j(g_{(ij)})\lambda_{ij}$, and for $s > 0$,

$$d_{ij}(s) = q_{ij} \sum_{0\leq t<s} d_{ij+1}(t) d_{jj+2t+2}(s-t-1), \tag{1.15}$$

$$b_{ij}(s) = \sum_{0\leq t<s} b_{i+1j}(t) d_{ii+2t+2}(s-t-1). \tag{1.16}$$

We define recursively $\zeta_{(j\,k)} \in \mathcal{T}(V)$ as follows: $\zeta_{(j\,j)} = a_j$ and for $j < k$

$$\begin{aligned} \zeta_{(j\,k)} = {} & [a_j, \zeta_{(j+1\,k)}]_c + d_{jk}(0)\chi_{(j\,k)}(g_j)\,\zeta_{(j+1\,k-1)}g_{jk} \\ & + 2\sum_{1\leq t\leq (k-j-1)/2} d_{jk-2t}(t)\chi_{(j+1\,k-2t-1)}(g_j)\zeta_{(j+1\,k-2t-1)}g_j g_{(k-2t\,k)}. \end{aligned} \tag{1.17}$$

Let $\mathfrak{u}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$ be the quotient of $\mathcal{T}(V)$ by the relations

$$a_{ij} = \lambda_{ij}(1 - g_i g_j); \tag{1.18}$$

$$\begin{aligned} [a_{(i-1\,i+1)}, a_i]_c = {} & \nu_i(1 - g_i^2 g_{i-1}g_{i+1}) \\ & - 4\chi_i(g_{i-1})\mu_{(i)}\lambda_{i-1\,i+1}g_{i-1}g_{i+1}(1-g_i^2); \end{aligned} \tag{1.19}$$

$$\zeta_{(j\,k)}^2 = \mu_{(j\,k)}(1 - g_{(j\,k)}^2) + \mathfrak{u}_{(j\,k)}, \tag{1.20}$$

for $\mathfrak{u}_{(j\,k)} = \mathfrak{u}_{(j\,k)}(\boldsymbol{\mu})$ as in 1.12.

The relations (1.18) are deformations of (1.3), while (1.20) are deformations of (1.5), and (1.19) are deformations of (1.6).

Remark 1.5. The quotient $\widetilde{\mathfrak{u}}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$ of $\mathcal{T}(V)$ by the relations (1.18)–(1.20) for $j = k$ is a cocycle deformation of $\widetilde{\mathcal{B}}(V)\#H$. □

Recall that that V is of type A_θ at $\xi = -1$.

Theorem 1.6. The algebra $\mathfrak{u}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$ is a Hopf algebra quotient of $\mathcal{T}(V)$ and is a lifting of V. Reciprocally every lifting of V over H is isomorphic to $\mathfrak{u}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$ for some family of scalars $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ as in (1.14). In particular, every lifting is a cocycle deformation of $\mathcal{B}(V)\#H$. □

Proof. We follow the strategy in Section 3: If $\mathcal{H} = \mathcal{B}(V)\#H$, then $\mathfrak{u} = \mathfrak{u}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$ arises as $L(\mathcal{A}, \mathcal{H})$ for a given $\mathcal{A} = \mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) \in \operatorname{Cleft}\mathcal{H}$ such that $\operatorname{gr}\mathfrak{u} \simeq \mathcal{H}$. The corresponding *stratification*, cf. Section 3.1 of the set of generators of the ideal defining $\mathcal{B}(V)$ is given by $\mathcal{G}_0 = \{(1.3), (1.6), x_i^2, i \in \mathbb{I}\}$, $\mathcal{G}_1 = \{(1.5)\}$. The converse follows by Theorem 3.5.

The cleft objects $\mathcal{A}$ are obtained in Theorem 4.7, while the algebras $\mathfrak{u}$ are described in Theorem 4.17. ■

1.2.4 The main result, $N = 3$

Here $\xi^3 = 1$, $\xi \neq 1$. We fix a family of scalars $\boldsymbol{\mu} = (\mu_{(k\,l)})_{k \leq l \in \mathbb{I}}$ subject to the constraints and normalizations 1.11. Pick an extra family of scalars $\boldsymbol{\lambda} = (\lambda_{iij})_{i,j \in \mathbb{I}, |i-j|=1}$ subject to the constraints and normalizations

$$\lambda_{iij} = 0 \qquad \text{if } \chi_{iij} \neq \epsilon, \quad \text{or } g_{iij} = 1. \tag{1.21}$$

We define families of elements in $\mathcal{T}(V)$ attached to these parameters. As in Section 1.2.3, we denote now by $(a_i)_{i \in \mathbb{I}}$ the generators of $T(V)$; and correspondingly a_{ij}, $a_{(ij)}$, $a_{i_1,\dots,i_k}$. Let us fix $i \leq p < l \in \mathbb{I}$ and set $q := p+1$, $r := p+2$.

First, we define $h_{il}(\boldsymbol{\lambda}) \in \Gamma$ via

$$h_{il}(\boldsymbol{\lambda}) = -9\mu_{(i+2\,l)}\lambda_{ii+1i+1}\lambda_{iii+1}(1 - g_{iii+1})g_{ii+1i+1}g^3_{(i+2\,l)}. \tag{1.22}$$

Next, we consider the following elements in $T(V)\#H$:

$$\varsigma^p(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \lambda_{qrr}\Big(\xi^2 a_{(i\,p)}a_{(i\,q)}a_{(i\,r)} + \chi_{p+2}(g_{(1\,p)})a_{(i\,p)}a_{(i\,r)}a_{(i\,q)} + a_{(i\,r)}a_{(i\,p)}a_{(i\,q)}\Big).$$

Now, we fix $s_p = -3(1-\xi^2)$, $p < l-2$, $s_{l-2} = 1$, and set

$$d_{il}(p) = \chi_{(i\,q)}(g_{(q\,l)}g_{(r+1\,l)})\chi_{(i\,p)}(g_{(r+1\,l)}).$$

We set

$$\varsigma_{il}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = -3\xi^2 \sum_{i \leq p < l} \mu_{(p+3\,l)}\chi_r(g_{(p+3\,l)})d_{il}(p)\varsigma^p(\boldsymbol{\lambda}, \boldsymbol{\mu})g_{qrr}g^3_{(p+3\,l)}, \tag{1.23}$$

cf. Remark 1.9 below for a more complete description. Finally, we set

$$\sigma_{(i\,l)}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = h_{il}(\boldsymbol{\lambda}) + \varsigma_{il}(\boldsymbol{\lambda}, \boldsymbol{\mu}). \tag{1.24}$$

Let $\mathfrak{u}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ be the quotient of $\mathcal{T}(V)$ by the relations

$$a_{ij} = 0, \qquad i < j-1; \tag{1.25}$$

$$a_{iij} = \lambda_{iij}(1 - g_{iij}), \qquad |j - i| = 1; \tag{1.26}$$

$$a_{(il)}^3 = \mu_{(il)}(1 - g_{(il)}^3) + \mathrm{u}_{(il)} + \sigma_{(il)}, \qquad i \le l \in \mathbb{I}. \tag{1.27}$$

for $\mathrm{u}_{(il)} = \mathrm{u}_{(il)}(\boldsymbol{\mu})$ as in 1.12 and $\sigma_{(il)} = \sigma_{(il)}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ as in 1.24.

The relations (1.25) are deformations of (1.3), while (1.26) are deformations of (1.4) and (1.27) are deformations of (1.5).

Remark 1.7. The quotient $\widetilde{\mathfrak{u}}(\boldsymbol{\lambda})$ of $\mathcal{T}(V)$ by the relations (1.25) and (1.26) is a cocycle deformation of $\widetilde{\mathcal{B}}(V)\#H$. □

Theorem 1.8. The algebra $\mathfrak{u}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a Hopf algebra quotient of $\mathcal{T}(V)$ and is a lifting of V. Reciprocally, every lifting of V is isomorphic to $\mathfrak{u}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ for some families $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ as in (1.21). In particular, every lifting is a cocycle deformation of $\mathcal{B}(V)\#H$. □

Proof. Similar to the proof of Theorem 1.6, following the strategy in Section 3. The corresponding stratification of the set of defining relations for $\mathcal{B}(V)$ is given by $\mathcal{G}_0 = \{(1.3), (1.4)\}$, $\mathcal{G}_1 = \{(1.5)\}$. The converse follows by Theorem 3.5. In this case, cleft objects $\mathcal{A} = \mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are obtained in Theorem 5.15, while the algebras $\mathfrak{u}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = L(\mathcal{A}, \mathcal{B}(V)\#H)$ are described in Theorem 5.23. ■

Remark 1.9. We give an explicit description of $\varsigma_{(il)}$ in terms of the PBW basis (see Corollary 5.27). To ease up the notation, we fix

$$j := i + 1, \qquad k := i + 2, \qquad q := p + 1, \qquad r := p + 2.$$

Let the symmetric group $\mathbb{S}_3$ act on $\{r, q, p\}$ via $(12)(r) = q$, $(23)(q) = p$. If $p = i, j$, then $\varsigma_i^p(\boldsymbol{\lambda}, \boldsymbol{\mu}) = 0$. When $p > i + 2$,

$$\begin{aligned} \varsigma_i^p(\boldsymbol{\lambda}, \boldsymbol{\mu}) = {} & -3\lambda_{qrr}\lambda_{qqr}\chi_{(ip)}(g_q)a_{(ip)}^3 g_{qqr} \\ & - 3\lambda_{qrr}\lambda_{iij}\sum_{\sigma \in \mathbb{S}_3}(-1)^{|\sigma|}h_{\sigma,i}a_{(k\,\sigma(p))}a_{(j\,\sigma(q))}a_{(i\,\sigma(r))}, \end{aligned} \tag{1.28}$$

for $h_{\sigma,i} \in \Bbbk$, $\sigma \in \mathbb{S}_3$, given by

$$h_{\mathrm{id},i} = \xi\,\chi_{qqr}(g_{(ip)})\chi_{(ir)}(g_{(jq)}), \qquad h_{(12),i} = (\xi^2 - 1)\chi_{qqr}(g_{(ip)})\chi_i(g_{(kq)}),$$

$$h_{(23),i} = \xi\,\chi_r(g_i)\chi_i(g_{(jp)}), \qquad h_{(13),i} = \xi(\xi - 2)\chi_{(kp)}(g_{ij}),$$

$$h_{(123),i} = 2\chi_r(g_{(ip)})\chi_i(g_{(kp)}), \qquad h_{(132),i} = \xi^2\chi_{(kq)}(g_{(ir)})\chi_{(jp)}(g_r).$$ □

1.2.5 The main result, $N > 3$

Here, ξ is a root of unity of order $N > 3$. We fix a family of scalars $\boldsymbol{\mu} = (\mu_{(k\,l)})_{k\le l\in\mathbb{I}}$ subject to the constraints and normalizations 1.11.

Let $\mathfrak{u}(\boldsymbol{\mu})$ be the quotient of $\mathcal{T}(V)$ by the relations

$$a_{ij} = 0, \qquad i < j-1;$$
$$a_{iij} = 0, \qquad |j-i| = 1;$$
$$a_{(i\,l)}^N = \mu_{(i\,l)}(1 - g_{(i\,l)}^N) + \mathfrak{u}_{(i\,l)}(\boldsymbol{\mu}), \qquad i \le l \in \mathbb{I},$$

for $\mathfrak{u}_{(i\,l)} = \mathfrak{u}_{(i\,l)}(\boldsymbol{\mu})$ as in 1.12.

Theorem 1.10. The algebra $\mathfrak{u}(\boldsymbol{\mu})$ is a Hopf algebra quotient of $\mathcal{T}(V)$ and is a lifting of V. Reciprocally, every lifting of V is isomorphic to $\mathfrak{u}(\boldsymbol{\mu})$ for some $\boldsymbol{\mu}$ as in 1.11. Hence, every lifting is a cocycle deformation of $\mathcal{B}(V)\#H$. □

Proof. As in the case $N = 3$, the stratification of the set of defining relations for $\mathcal{B}(V)$ is given by $\mathcal{G}_0 = \{(1.3), (1.4)\}$, $\mathcal{G}_1 = \{(1.5)\}$. We set $\mathcal{H} = \mathfrak{B}(V)\#H$, $\widetilde{\mathcal{H}} = \mathcal{T}(V)/\langle\mathcal{G}_0\rangle$.

In this case, the relations in $\mathcal{G}_0$ cannot be deformed (cf. [2, Theorem 5.6]). As a result, $\operatorname{Cleft}'\widetilde{\mathcal{H}} = \{\widetilde{\mathcal{H}}\}$ and thus the corresponding deformation $\mathcal{L}_1 = L(\cdot, \widetilde{\mathcal{H}}) \simeq \widetilde{\mathcal{H}}$. This shows 3.12 for $j = 0$ trivially. Let us denote by $\widetilde{\mathcal{A}} = \widetilde{\mathcal{H}}$ as $(\mathcal{L}_1, \widetilde{\mathcal{H}})$-bicleft object.

Pick $\boldsymbol{\mu}$ as in 1.11; set $\mathcal{A}(\boldsymbol{\mu})$ the quotient of $\widetilde{\mathcal{H}}$ by the ideal generated by

$$y_{(i\,l)}^N = \mu_{(i\,l)}, \qquad i \le l.$$

It follows from [8] that ${}^{\mathrm{co}\,\mathcal{H}}\widetilde{\mathcal{H}} \le \widetilde{\mathcal{H}}$ is a normal coideal subalgebra and thus [16, Theorem 4], see also [5, Theorem 3.1], yields:

$$\operatorname{Cleft}'(\mathcal{H}) = \{\mathcal{A}(\boldsymbol{\mu}) | \boldsymbol{\mu} \text{ as in } (1.11)\}.$$

In particular, 3.12 holds for $j = 1$. Now, we use 3.7 recursively, as in p. 37 and 62. More precisely, let $\delta : \widetilde{\mathcal{A}} \to \mathcal{L}_1 \otimes \widetilde{\mathcal{A}}$ denote the left coaction. Assume $i = 1$ to simplify the notation and set

$$A = a_{(1\,l)} \otimes 1, \qquad B = g_{(1\,l)} \otimes y_{(1\,l)}, \qquad X_p = a_{(1\,p)} g_{(p+1\,l)} \otimes y_{(p+1\,l)}, \quad 1 \le p < l,$$

so $\delta(y_{(1\,l)}) = A + B + (1-\xi^{-1})\sum_{1\le p<l} X_p$. Set also C_p as in 1.7 and $r = y_{(1\,l)}^N$. By [2, Remark 6.10] we have

$$\delta(r) = a_{(1\,l)}^N \otimes 1 + g_{(1\,l)}^N \otimes y_{(1\,l)}^N + \sum_{1\le p<l} C_p a_{(1\,p)}^N g_{(p+1\,l)}^N \otimes y_{(p+1\,l)}^N.$$

We apply the deformation procedure following [5, Corollary 5.12], that is we assume recursively $y_{(p+1\,l)}^N = \mu_{(p+1\,l)}$, and thus we get cf. 3.7:

$$\tilde{r} = -\sum_{1\le p<l} C_p \mu_{(p+1\,l)} \left(\mathrm{u}_p + \mu_{(1\,p)}\left(1 - g_{(1\,p)}^N\right)\right) g_{(p+1\,l)}^N. \tag{1.29}$$

Hence, $L(\boldsymbol{\mu}) = L(\mathcal{A}(\boldsymbol{\mu}), \mathcal{H}) \simeq \mathfrak{u}(\boldsymbol{\mu})$, by Proposition 3.3 (c).

The converse follows from Theorem 3.5. ■

1.2.6 Applications

The classification of all finite-dimensional pointed Hopf algebras over a group algebra $H = \Bbbk G$ whose infinitesimal braiding V is a principal realization of a braided vector space with braiding matrix (1.2) follows from our main results because such Hopf algebras are generated in degree 1 [9]. When $\operatorname{ord}\xi > 3$, the classification was obtained in [2, Theorem 6.25] assuming that G is abelian; the methods in Section 3 show that this hypothesis is not necessary. We extend this classification to the case in which H is any cosemisimple Hopf algebra.

As a byproduct, new examples of Hopf algebras are defined, as deformations of intermediate pre-Nichols algebras (see Propositions 4.8 and 5.17). Also, new examples of co-Frobenius Hopf algebras arise (see Section 1.2.7).

1.2.7 New examples of co-Frobenius Hopf algebras

Let $\mathbf{G}$ be an algebraic group and let $H = \mathcal{O}(\mathbf{G})$ be its function algebra; thus $\operatorname{Alg}(\mathcal{O}(\mathbf{G}), \Bbbk) \simeq \mathbf{G}$. An YD-pair for H, cf. Section 2.2, is (g, x), where $g \in \operatorname{Hom}_{\text{alg gp}}(\mathbf{G}, \Bbbk^\times)$, $x \in Z(\mathbf{G})$.

Let $\mathbf{G} = \mathrm{GL}_n(\Bbbk)$. As $Z(\mathbf{G}) = \Bbbk^\times \operatorname{Id}$ and $\operatorname{Hom}_{\text{alg gp}}(\mathbf{G}, \Bbbk^\times) = \langle \det \rangle$, an YD-pair (g, x) as above identifies with $(h, t) \in \mathbb{Z} \times \Bbbk^\times$ via $g = \det^h$, $x = t \operatorname{Id}$.

Let V be a braided vector space of type A_2, with parameter ξ. Then there is a principal YD-realization $V \in {}^H_H\mathcal{YD}$ if and only if there are $(h_1, h_2) \in \mathbb{Z}^2$ and $(t_1, t_2) \in \mathbb{C}^2$ such that if $u_i := t_i^n$, then

$$\xi = u_1^{h_1} = u_2^{h_2}; \qquad\qquad \xi^{-1} = u_1^{h_2} u_2^{h_1}.$$

Each solution yields a realization $V \in {}^H_H\mathcal{YD}$ and as a consequence of our main results (Theorems 1.6, 1.8 or 1.10), we obtain new families of co-Frobenius Hopf algebras over $\mathcal{O}(\mathbf{G})$. Examples of solutions are given by:

- $N = 3$, $(u_1, u_2) = (\xi, \xi)$ and $(h_1, h_2) = (1, 1)$.
- $N = 7$, $(u_1, u_2) = (\xi, \xi^4)$ and $(h_1, h_2) = (1, 2)$.

More examples arise considering $\mathbf{G} = \mathrm{GL}_{n_1}(\Bbbk) \times \mathrm{GL}_{n_2}(\Bbbk) \times \cdots \times \mathrm{GL}_{n_s}(\Bbbk)$.

2 Preliminaries

2.1 Conventions

If $n \in \mathbb{N}$, we set $\mathbb{I}_n = \{1, \dots, n\}$; we omit the subscript when it is clear from the context. We denote by $\mathbb{S}_n$ the symmetric group in n letters. Also, $\mathbb{G}_n$ denotes the group of nth roots of 1, and $\mathbb{G}'_n$ is the subset of primitive nth roots.

Let H be a Hopf algebra; we always assume that its antipode is bijective. We use the Heynemann–Sweedler notation for the comultiplication and coaction. We denote by $G(H)$ the group of group-like elements of H and by ${}^H_H\mathcal{YD}$, respectively $\mathcal{YD}^H_H$ the category of left, respectively right, Yetter–Drinfeld modules over H. If A is an algebra and $S \subset A$, then $\langle S \rangle$ denotes the two-sided ideal generated by S.

If H' is a Hopf algebra, we denote by $\mathrm{Isom}(H, H')$ the set of Hopf algebra isomorphisms $\varphi : H \to H'$. If A, A' are right H-comodule algebras, then $\mathrm{Alg}^H(A, A')$ is the set of comodule algebra morphisms between them. We shall denote by $\mathrm{Alg}^H_H(A, B)$ the set of algebra morphisms between two algebras $A, B \in \mathcal{YD}^H_H$. When $H = \Bbbk$, we omit any reference as $\mathrm{Alg}(A, B) = \mathrm{Alg}^H_H(A, B)$ becomes the set of $\Bbbk$-algebra maps $A \to B$.

2.2 Principal realizations

Let H be a Hopf algebra. Let (g, χ) be an *YD-pair* [5], that is $g \in G(H)$ and $\chi \in \mathrm{Alg}(H, \Bbbk)$ satisfy

$$\chi(h)\, g = \chi(h_{(2)}) h_{(1)}\, g\, \mathcal{S}(h_{(3)})$$

for all $h \in H$; this implies that $g \in Z(G(H))$. Then, $\Bbbk^\chi_g := \Bbbk$ with coaction given by g and action given by χ is an object in ${}^H_H\mathcal{YD}$.

Let V be a braided vector space of *diagonal type*, that is, there are a basis $(x_i)_{i\in\mathbb{I}}$ of V and a matrix $\mathbf{q} = \left(q_{ij}\right)_{i,j\in\mathbb{I}}$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$. A *principal realization* of

V over H is a family $((g_i, \chi_i))_{i\in\mathbb{I}}$ of YD-pairs such that $\chi_j(g_i) = q_{ij}$, $i, j \in \mathbb{I}$; so that $V \in {}^H_H\mathcal{YD}$ up to identifying $\Bbbk x_i \simeq \Bbbk_{g_i}^{\chi_i}$, and the braiding c is the categorical one from ${}^H_H\mathcal{YD}$. Clearly

$$\Gamma = \langle g_1, \dots, g_\theta \rangle \leq Z(G(H)) \tag{2.1}$$

and we can realize V as an object in ${}^\Gamma_\Gamma\mathcal{YD} := {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$.

Example 2.1. There are $V \in {}^H_H\mathcal{YD}$ with diagonal braiding but not from a principal realization. Let $H = \Bbbk\mathtt{H}(\mathtt{p})$ where $\mathtt{H}(\mathtt{p})$ is the finite Heisenberg group of upper triangular matrices $\begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix}$ with coefficients in the finite field $\mathbb{F}_p$, p a prime. The conjugacy classes in $\mathtt{H}(\mathtt{p})$ are

$$\mathcal{O}_c = \left\{ \begin{pmatrix} 1 & 0 & c \\ & 1 & 0 \\ & & 1 \end{pmatrix} \right\}, \qquad \mathcal{O}_{(a,b)} = \left\{ \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} : c \in \mathbb{F}_p \right\},$$

for all $c \in \mathbb{F}_p$, $(a, b) \in \mathbb{F}_p^2 - 0$. Then we have the following:

- ◇ If $\rho \in \operatorname{Irr}\mathtt{H}(\mathtt{p})$, then the $M(\mathcal{O}_c, \rho) \in {}^H_H\mathcal{YD}$ is of diagonal type, but does not arise from a principal realization unless $\dim \rho = 1$.
- ◇ If $(a, b) \in \mathbb{F}_p^2 - 0$ and $x \in \mathcal{O}_{(a,b)}$, then the isotropy group $\mathtt{H}(\mathtt{p})^x \simeq \mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathcal{O}_{(a,b)}$ is an abelian rack. Hence $M(\mathcal{O}_{(a,b)}, \rho) \in {}^H_H\mathcal{YD}$ is of diagonal type, but does not arise from a principal realization, for all $\rho \in \operatorname{Irr}\mathtt{H}(\mathtt{p})^x$. □

2.3 Nichols and pre-Nichols algebras

Let H and V be as in Section 1.1. As usual, we denote by $\mathfrak{B}(V)$ the *Nichols algebra* of V and by $\mathcal{J}(V) \subset T(V)$ its defining ideal: $\mathfrak{B}(V) = T(V)/\mathcal{J}(V)$ (see [2]). A *pre-Nichols algebra* is a Hopf algebra $\mathfrak{R} = T(V)/\mathcal{J} \in {}^H_H\mathcal{YD}$ with $\mathcal{J} \subset \mathcal{J}(V)$ a graded Hopf ideal. Every pre-Nichols algebra $\mathfrak{R}$ is a $\mathbb{Z}^\theta$-graded semisimple object in ${}^H_H\mathcal{YD}$. The following identities are well-known. If $x, y, z \in \mathfrak{R}$ are $\mathbb{Z}^\theta$-homogeneous, then

$$[[x, y]_c, z]_c = [x, [y, z]_c]_c + q_{|y||z|}[x, z]_c y - q_{|x||y|} y[x, z]_c \quad \text{(q-Jacobi)}, \tag{2.2}$$

$$\begin{aligned} [xy, z]_c &= x[y, z]_c + q_{|y||z|}[x, z]_c y, \\ [x, yz]_c &= [x, y]_c z + q_{|x||y|} y[x, z]_c. \end{aligned} \tag{2.3}$$

Assume that the generalized Dynkin diagram of V is connected. The generators of the ideal $\mathcal{J}(V)$ were computed theoretically in [7] and concretely, case-by-case in the list of [18], in [6]. These relations can be informally organized into two types:

- ◇ Quantum Serre relations and generalizations—that sometimes involve more than two simple roots (see e.g., (1.6)).
- ◇ Powers of root vectors.

Now there are some special roots called *Cartan roots* [8, (20)]. There is a distinguished pre-Nichols algebra of V with favourable properties, denoted by $\widetilde{\mathcal{B}}(V)$ (cf. [8]). The defining ideal $\mathcal{I}(V)$ of $\widetilde{\mathcal{B}}(V)$ is generated by the same relations as for $\mathcal{B}(V)$, but excluding the powers of Cartan root vectors, and possibly adding some quantum Serre relations redundant for $\mathcal{J}(V)$. We set

$$\mathcal{T}(V) = T(V)\#H, \qquad \mathcal{H} = \mathcal{B}(V)\#H, \qquad \widetilde{\mathcal{H}} = \widetilde{\mathcal{B}}(V)\#H,$$

and $\pi : \mathcal{T}(V) \to \mathcal{H}$, $\widetilde{\pi} : \mathcal{T}(V) \to \widetilde{\mathcal{H}}$ the natural projections.

2.4 Cleft objects and two-cocycles

A (normalized) Hopf two-cocycle is a convolution invertible linear map $\sigma : H \otimes H \to \Bbbk$ such that, for $x, y, z \in H$:

$$\sigma(x, 1) = \sigma(1, x) = \epsilon(x),$$

$$\sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)}).$$

If σ is a Hopf two-cocycle, then it is possible to perturb the multiplication $m(x \otimes y) = xy$ on H on several ways, obtaining new associative products on the vector space H. First, we may consider $m_{(\sigma)}, m_{(\sigma^{-1})} : H \otimes H \to H$ as

$$m_{(\sigma)}(x \otimes y) = \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}, \qquad \text{respectively ,}$$

$$m_{(\sigma^{-1})}(x \otimes y) = \sigma^{-1}(x_{(2)}, y_{(2)})x_{(1)}y_{(1)}.$$

The corresponding algebras will be denoted by $H_{(\sigma)}$, respectively, $H_{(\sigma^{-1})}$. The comultiplications $\Delta : H_{(\sigma)} \to H_{(\sigma)} \otimes H$, $\Delta : H_{(\sigma^{-1})} \to H \otimes H_{(\sigma^{-1})}$, remain algebra maps and hence $H_{(\sigma)}$, respectively $H_{(\sigma^{-1})}$, is a right, respectively left, H-comodule algebra. Yet another associative multiplication m_σ is defined:

$$m_\sigma(x \otimes y) = \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}\sigma^{-1}(x_{(3)}, y_{(3)}), \quad x, y \in H.$$

The corresponding algebra, denoted by H_σ is actually a Hopf algebra with comultiplication Δ—see [14] for the explicit form of the antipode $\mathcal{S}_\sigma$. This Hopf algebra H_σ is referred to as a *cocycle deformation* of H.

2.4.1 Cleft objects

A (right) H-comodule algebra A with trivial coinvariants, *that is* $A^{\operatorname{co} H} = \Bbbk$, is a cleft object of H when there exists an H-colinear convolution-invertible map $\gamma : H \to A$. This map can be assumed to satisfy $\gamma(1) = 1$, in which case it is called a *section*. Left, respectively bi-, cleft objects are defined accordingly.

We shall denote by $\operatorname{Cleft} H$ the set of (isomorphism classes of) right cleft objects of H. If $A \in \operatorname{Cleft} H$, then A is an algebra in ${}^H_H\mathcal{YD}$ via the *Miyashita–Ulbrich action* [13].

For every cleft object A, there is a Hopf two-cocycle $\sigma : H \otimes H \to \Bbbk$ such that $A \simeq H_{(\sigma)}$. Indeed, a section $\gamma : H \to A$ determines σ by

$$\sigma(x, y) = \gamma(x_{(1)})\gamma(y_{(1)})\gamma^{-1}(x_{(2)}y_{(2)}), \quad x, y \in H.$$

If $A \in \operatorname{Cleft} H$, then there is an associated Hopf algebra $L = L(A, H)$ [22] in such a way that A becomes (H, L)-bicleft. Moreover, if $A = H_{(\sigma)}$, then $L \simeq H_\sigma$. Hence, L is a cocycle deformation of H and every cocycle deformation can be obtained in this way (see *loc.cit.*)

3 The Strategy

Let H be a cosemisimple Hopf algebra and V as in Section 1.1. We recall and expand here the strategy developed in [5] to compute the cocycle deformations of $\mathfrak{B}(V)\#H$. Accordingly, let Γ be the abelian group as in 2.1.

Remark 3.1. In [5, Section 1.1] H is assumed to be finite-dimensional. This assumption, however, can be omitted. Indeed, it is only used in [5, Lemma 5.7] and in [5, Section 5.9, Question]. These two instances are independent of the strategy and both of them deal with the evidence of an "intermediate Gunther's Theorem" to simplify the recursive step.

On the other hand, a dimension argument is used to prove exhaustion in the examples (see [5, Theorem 5.20]). We provide an alternative argument in Theorem 3.5, valid in general. □

3.1 The main idea

We explain how to compute all Hopf algebras L which are cocycle deformations of $\mathcal{H} := \mathfrak{B}(V)\#H$ and satisfy $\operatorname{gr} L \simeq \mathcal{H}$. These Hopf algebras arise as $L(\mathcal{A}, \mathcal{H})$ for suitable $\mathcal{A} \in \operatorname{Cleft} \mathcal{H}$ (cf. Section 2.4); in turn, we compute the cleft extensions $\mathcal{A}$ recursively by a method from [16].

Let $\mathcal{G}$ be the set of generators of the ideal $\mathcal{J}(V)$ described in [6] for each connected component, union the q-commutators of vertices in different components. Notice that every $r \in \mathcal{G}$ belongs to $T(V)^{\chi_r}_{g_r}$ for some $g_r \in \Gamma$, $\chi_r \in \operatorname{Alg}(H, \Bbbk)$. We decompose $\mathcal{G}$ as a disjoint union $\mathcal{G} = \mathcal{G}_0 \sqcup \cdots \sqcup \mathcal{G}_\ell$. Let

$$\mathfrak{B}_i := \begin{cases} T(V), & i = 0; \\ T(V)/\langle \mathcal{G}_0 \cup \cdots \cup \mathcal{G}_{i-1} \rangle, & i > 0; \end{cases} \qquad \mathcal{H}_i = \mathfrak{B}_i \# H. \tag{3.1}$$

In particular, $\mathcal{H}_{\ell+1} = \mathcal{H}$. We choose this decomposition in such a way that

$$\text{the elements in (the image of) } \mathcal{G}_i,\ i < \ell, \text{ are primitive in } \mathfrak{B}_i; \tag{3.2}$$

$$\mathcal{G}_\ell \text{ consists of powers of Cartan root vectors.} \tag{3.3}$$

In plain words, the strategy is to deform the relations in $\mathcal{G}$ step by step, that is first those in $\mathcal{G}_0$, then those in $\mathcal{G}_1$, and so on. By (3.2), the form of the deformed relations is particularly simple in the steps 0 to ℓ and depends on a suitable parameter. To check that the proposed deformation has the right properties, we proceed indirectly by defining first a cleft extension for each possible parameter; then the proposed deformation appears as the corresponding cocycle deformation (cf. Section 2.4.1). In the last step, the deformations of the powers of Cartan root vectors require a delicate combinatorial analysis; but the definition of the cleft extensions is facilitated because the algebra of coinvariants ${}^{\mathcal{H}_{\ell+1}}\mathcal{H}_\ell$ is a q-polynomial algebra [8, Theorem 4.10]. To organize the information we pack all the cleft extensions arising in the ith step in a subset $\operatorname{Cleft}' \mathcal{H}_i$ of $\operatorname{Cleft} \mathcal{H}_i$.

Concretely, the inductive procedure starts with

- the Hopf algebra $\mathcal{H}_0$;
- the trivial $\mathcal{A}_0 = \mathcal{H}_0 \in \operatorname{Cleft}(\mathcal{H}_0)$, where the section $\gamma : \mathcal{H}_0 \to \mathcal{A}_0$ is the identity map; and
- and the corresponding Hopf algebra $\mathcal{L}_0 = L(\mathcal{A}_0, \mathcal{H}_0) \simeq \mathcal{H}_0$.

We now define recursively a subset $\operatorname{Cleft}' \mathcal{H}_i$ of $\operatorname{Cleft} \mathcal{H}_i$, $0 \le i \le \ell+1$, see [5, Section 5.2] for more details. First, we clearly have

$$\operatorname{Cleft}' \mathcal{H}_0 := \{\mathcal{A}_0\}.$$

Given $i \ge 0$, $\operatorname{Cleft}' \mathcal{H}_{i+1}$ consists of quotients of each $\mathcal{A} \in \operatorname{Cleft}' \mathcal{H}_i$. To explain this, we fix $\mathcal{A} \in \operatorname{Cleft}' \mathcal{H}_i$; it comes equipped with

- ◇ a section $\gamma : \mathcal{H}_i \to \mathcal{A}$ such that the restriction $\gamma_{|H} : H \to \mathcal{A}$ is an algebra map—see [5, Proposition 6.2 (b)]; and
- ◇ an algebra $\mathcal{E} \in {}^H_H\mathcal{YD}$ such that $\mathcal{A} = \mathcal{E}\#H$ [5, Proposition 5.8 (d)]; actually $\mathcal{E}$ is the image of $T(V)$ under the projection $\mathcal{A}_0 = T(V)\#H \twoheadrightarrow \mathcal{A}$.

Then, we collect in $\operatorname{Cleft}' \mathcal{H}_{i+1}$ all $\mathcal{A}'$ given either as

$$\mathcal{A}' = \mathcal{A}/\mathcal{A}\psi(X_i^+), \qquad \text{where} \qquad X_i := {}^{\operatorname{co}\mathcal{H}_{i+1}}\mathcal{H}_i, \qquad \psi \in \operatorname{Alg}^{\mathcal{H}_i}_{\mathcal{H}_i}(X_i, \mathcal{A}); \tag{3.4}$$

or else as

$$\mathcal{A}' = \mathcal{A}/\langle\varphi(Y_i^+)\rangle, \qquad \text{where} \qquad Y_i := \Bbbk\langle\mathcal{S}(\mathcal{G}_i)\rangle, \qquad \varphi \in \widetilde{\operatorname{Alg}}^{\mathcal{H}_i}(Y_i, \mathcal{A}); \tag{3.5}$$

here, $\widetilde{\operatorname{Alg}}^{\mathcal{H}_i}(Y_i, \mathcal{A}) := \{\varphi \in \operatorname{Alg}^{\mathcal{H}_i}(Y_i, \mathcal{A}) | \langle\varphi(Y_i^+)\rangle \neq \mathcal{A}\}$.

Remark 3.2. The subalgebra X_i is the normalizer of Y_i [5, Remark 5.4]. If $\psi \in \operatorname{Alg}^{\mathcal{H}_i}_{\mathcal{H}_i}(X_i, \mathcal{A})$, then $\psi_{|Y_i} =: \varphi \in \widetilde{\operatorname{Alg}}^{\mathcal{H}_i}(Y_i, \mathcal{A})$ and $\langle\varphi(Y_i^+)\rangle = \mathcal{A}\psi(X_i^+)$. □

Proof. On one hand, $\langle\varphi(Y_i^+)\rangle \subseteq \langle\psi(X_i^+)\rangle = \mathcal{A}\psi(X_i^+)$, the last equality by [16, Theorem 4]. Hence $\langle\varphi(Y_i^+)\rangle \neq \mathcal{A}$, cf. *loc.cit.* The other inclusion follows because $X_i = N(Y_i)$. ■

More explicitly, given a family of scalars $\Lambda_i := (\lambda_r)_{r\in\mathcal{G}_i}$ we define

$$\mathcal{E}(\Lambda_i) = \mathcal{E}/\langle\gamma(r) - \lambda_r : r \in \mathcal{G}_i\rangle, \qquad \mathcal{A}' = \mathcal{A}(\Lambda_i) = \mathcal{E}(\Lambda_i)\#H. \tag{3.6}$$

Set $\mathcal{L} = L(\mathcal{A}, \mathcal{H}_i)$. Recall that for $r \in \mathcal{G}_i$, there are $g_r \in \Gamma$, $\chi_r \in \operatorname{Alg}(H, \Bbbk)$ such that $r \in T(V)^{\chi_r}_{g_r}$. By [5, Corollary 5.12],

$$\nabla(r) := \gamma(r)_{(-1)} \otimes \gamma(r)_{(0)} - g_r \otimes \gamma(r) \in \mathcal{L} \otimes 1, \qquad \text{for all } r \in \mathcal{G}_i. \tag{3.7}$$

Thus, $\nabla(r) = \tilde{r} \otimes 1$ and by *loc.cit.* $\tilde{r}$ is $(g_r, 1)$-primitive in $\mathcal{L}$. Set

$$\mathcal{L}' = \mathcal{L}(\Lambda_i) := \mathcal{L}/\langle\tilde{r} - \lambda_r(1 - g_r) : r \in \mathcal{G}_i\rangle. \tag{3.8}$$

The following proposition is a summary of [5, Section 5.6]; we add a short proof since in *loc.cit.* this is stated for a single element in $\mathcal{G}_i$.

Proposition 3.3. Let $\mathcal{A}' = \mathcal{A}(\Lambda_i)$, $\Lambda_i \in \Bbbk^{\mathcal{G}_i}$.

(a) If $\mathcal{A}' \neq 0$, then $\mathcal{A}' \in \operatorname{Cleft}' \mathcal{H}_{i+1}$.
(b) If $\chi_r \neq \epsilon$ and $\lambda_r \neq 0$ for some $r \in \mathcal{G}_i$, then $\mathcal{A}' = 0$.
(c) $L(\mathcal{A}', \mathcal{H}_{i+1}) \simeq \mathcal{L}(\Lambda_i)$.
(d) If $i = \ell$, then $\operatorname{gr} \mathcal{L}(\Lambda_\ell) \simeq \mathcal{B}(V)\#H$, that is $\mathcal{L}(\Lambda_\ell)$ is a lifting of V. □

Proof. (a) Assume that $i < \ell$. Let us fix a numeration $r_1, \dots, r_s$ of $\mathcal{G}_i$. Let $\mathcal{B}_i^{(0)} := \mathcal{B}_i$, $\mathcal{B}_i^{(t)} := \mathcal{B}_i / \langle r_1, \dots, r_t \rangle$, $t \in \mathbb{I}_s$, so $\mathcal{B}_i^{(s)} = \mathcal{B}_{i+1}$. By abuse of notation, the image of r_j is denoted by r_j throughout. Set, as well,

$$\mathcal{E}^{(t)} := \mathcal{E} / \langle \gamma(r_j) - \lambda_{r_j} : j \in \mathbb{I}_t \rangle, \qquad \mathcal{A}^{(t)} := \mathcal{E}^{(t)}\#H, \qquad \pi^{(t)} : \mathcal{A} \to \mathcal{A}^{(t)}$$

the natural projection. Notice that $\mathcal{A}^{(1)} \neq 0$ since it projects on to $\mathcal{A}'$ and thus $\mathcal{A}^{(1)} \in \operatorname{Cleft}' \mathcal{B}_i^{(1)}\#H$ by [5, Remark 5.11]. Let $\gamma^{(1)} : \mathcal{B}_i^{(1)}\#H \to \mathcal{A}^{(1)}$ be the section. Observe that $\gamma^{(1)}(r_2) = \pi^{(1)}(\gamma(r_2))$. Indeed, the coaction $\rho : \mathcal{A}^{(1)} \to \mathcal{A}^{(1)} \otimes \mathcal{B}_i^{(1)}\#H$ satisfies

$$\pi^{(1)}\left(\gamma^{(1)}(r_2)\right) \xmapsto{\rho} \pi^{(1)}(\gamma^{(1)}(r_2)) \otimes 1 + g_{r_2} \otimes r_2$$

as $\pi^{(1)}$ is a comodule algebra projection that preserves H cf. the *snapshot* in [5, p. 696]. Hence we may iterate the argument and conclude that $\mathcal{A}^{(t)} \in \operatorname{Cleft}' \mathcal{B}_i^{(t)}\#H$, $t \in \mathbb{I}_s$, and $\mathcal{A}' = \mathcal{A}^{(s)} \in \operatorname{Cleft}' \mathcal{H}_{i+1}$.

Next, we consider the case $i = \ell$. In this step, we allow the subset $\mathcal{G}_\ell$ to contain non-primitive elements. However, the previous analysis extends to this case. To see this, we decompose, in turn,

$$\mathcal{G}_\ell = \mathcal{G}_\ell^{(0)} \sqcup \dots \sqcup \mathcal{G}_\ell^{(r)}$$

as a disjoint union of sets satisfying that $\mathcal{G}_\ell^{(0)}$ contains primitive elements in $\mathcal{B}_\ell$ and that (the image of) $\mathcal{G}_\ell^{(i)}$, $i > 1$ is composed of primitive elements in

$$\mathcal{B}_\ell^{(i)} = \mathcal{B}_\ell / \langle \mathcal{G}_\ell^{(0)} \cup \dots \cup \mathcal{G}_\ell^{(i-1)} \rangle.$$

We decompose, accordingly, $\Lambda_\ell = \Lambda_\ell^{(0)} \times \dots \times \Lambda_\ell^{(r)}$ and proceed as before.

(b) Follows by conjugating $\gamma(r) = \lambda_r$ by $g \in G(H)$ with $\chi_r(g) \neq 1$.

(c) Follows by a iterative application of [5, Corollary 5.12], as we proceed element-by-element as in *(a)*.

(d) For each $\mathcal{A}' \in \mathrm{Cleft}'\,\mathcal{H}_{i+1}$, the section $\gamma' : \mathcal{H}_{i+1} \to \mathcal{A}'$ is such that the restriction $\gamma'_{|H} : H \to \mathcal{A}'$ is an algebra map [5, Proposition 6.2 (b)]. Hence, $\mathrm{gr}\, L(\mathcal{A}', \mathcal{H}_{i+1}) \simeq \mathcal{H}_{i+1}$ by [5, Proposition 4.14 (c)]. ■

If $\Lambda = (\lambda_r)_{r\in\mathcal{G}} \in \Bbbk^{\mathcal{G}}$ and $0 \le i \le \ell$, then we set $\Lambda_i = (\lambda_r)_{r\in\mathcal{G}_i} \in \Bbbk^{\mathcal{G}_i}$. Set $\mathcal{A}_0 = T(V)\#H$ and define—using the assignment $\mathcal{A}_i \rightsquigarrow \mathcal{A}_{i+1} := \mathcal{A}_i(\Lambda_i)$ cf. 3.6—the set of deformation parameters

$$\mathcal{R} = \{\Lambda = (\lambda_r)_{r\in\mathcal{G}} \in \Bbbk^{\mathcal{G}} \mid \mathcal{A}_i(\Lambda_i) \neq 0,\ \forall i \text{ and } \lambda_r = 0 \text{ if } g_r = 1\}. \tag{3.9}$$

By Proposition 3.3 we have

Corollary 3.4. For each $\Lambda \in \mathcal{R}$, we obtain a chain of Hopf algebra quotients

$$\mathcal{L}_0 := T(V)\#H \twoheadrightarrow \mathcal{L}_1 := \mathcal{L}_0(\Lambda_0) \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{L}_{\ell+1} := \mathcal{L}_\ell(\Lambda_\ell) \tag{3.10}$$

such that $\mathcal{L}_i$ is a cocycle deformation of $\mathcal{B}_i\#H$. □

For $\Lambda \in \mathcal{R}$, we set $\mathcal{L}(\Lambda) := \mathcal{L}_{\ell+1}$. In this way, we obtain a family $\mathcal{L}(\Lambda)$, $\Lambda \in \mathcal{R}$, of cocycle deformations of $\mathfrak{B}(V)\#H$ that are liftings of V. Next, we check when this family is exhaustive. We consider the following condition on $V \in {}^H_H\mathcal{YD}$: for $\Lambda = (\lambda_r)_{r\in\mathcal{G}} \in \Bbbk^{\mathcal{G}}$,

$$\Lambda \in \mathcal{R} \ \text{ if and only if } \lambda_r = 0, \text{ when } \chi_r \neq \epsilon. \tag{3.11}$$

Observe that the “only if” implication always holds, by Proposition 3.3 (b). Actually, we need a recursive version of 3.11:
Suppose we are given $0 \le j \le \ell$, and families $\Lambda_i = (\lambda_r)_{r\in\mathcal{G}_i} \in \Bbbk^{\mathcal{G}_i}$ for $i \le j$ such that $\lambda_r = 0$, when $\chi_r \neq \epsilon$. Define recursively $\mathcal{A}_0 = \mathcal{T}(V)$, $\mathcal{A}_1 = \mathcal{A}_0(\Lambda_0)$, $\mathcal{A}_i = \mathcal{A}_{i-1}(\Lambda_{i-1})$. The recursive version of 3.11 is

$$\mathcal{A}_j \neq 0. \tag{3.12}$$

Theorem 3.5. Assume that 3.12 holds for all $j \ge 0$. If L is a lifting of V, then there is $\Lambda \in \mathcal{R}$ such that $L \simeq \mathcal{L}(\Lambda)$. In particular, L is a cocycle deformation of $\mathfrak{B}(V)\#H$. □

Proof. Let $\phi : \mathcal{L}_0 = T(V)\#H \to L$ be a lifting map. We shall attach to ϕ a family $(\lambda_r)_{r\in\mathcal{G}} \in \Bbbk^{\mathcal{G}}$ such that $\lambda_r = 0$ if either $g_r = 1$ or else $\chi_r \neq \epsilon$. Let $\mathfrak{S}$ be the set of simple subcoalgebras of H. A direct computation shows that $V\#H \subset \sum_{i\in\mathbb{I}, C\in\mathfrak{S}} g_iC \wedge C$. Since ϕ is a lifting map,

$$L_1 \overset{(1.1)}{=} \phi(H \oplus V\#H) = \sum_{C\in\mathfrak{S}} C + \sum_{i\in\mathbb{I}, C\in\mathfrak{S}} g_iC \wedge C.$$

If $r \in \mathcal{G}_0$, then r is $(g_r, 1)$-primitive in $\mathcal{L}_0$, hence so is $\phi(r) \in L$. That is, $\phi(r) \in \Bbbk g_r \wedge \Bbbk \subset L_1$. Then either $\phi(r) \in H$ or $\phi(r) \in g_iC \wedge C$ for some $i \in \mathbb{I}$, $C \in \mathfrak{S}$. In the former case, $\phi(r) = \lambda_r(1 - g_r)$ for some $\lambda_r \in \Bbbk$. As $g_r \in \Gamma < Z(H) \cap G(H)$, conjugation by $h \in H$ determines that $\lambda_r = 0$ whenever $\chi_r \neq \epsilon$. In the latter, $g_iC = \Bbbk g_r$ and $C = \Bbbk$, thus $g_r = g_i$ and

$$\phi(r) = \lambda_r(1 - g_r) + \sum_{j\in\mathbb{I}: g_j = g_r} \mu_j x_j,$$

for some $\lambda_r, \mu_j \in \Bbbk$. Conjugation by $h \in H$ shows that

$$\lambda_r\chi_r = \lambda_r\epsilon, \qquad \mu_j\chi_r = \mu_j\chi_j, \qquad \text{for } g_j = g_r.$$

Now, by [10, Proposition 6.2] the pair (χ_r, g_r) is different from (χ_i, g_i), $i \in \mathbb{I}$. Thus $\mu_j = 0$ for all such j, hence $\phi(r) = \lambda_r(1 - g_r)$ and $\lambda_r = 0$ whenever $\chi_r \neq \epsilon$. In either case, we can normalize $\lambda_r = 0$ when $g_r = 1$. Set $\Lambda_0 = (\lambda_r)_{r\in\mathcal{G}_0} \in \Bbbk^{\mathcal{G}_0}$; by 3.12 for $j = 0$, $\mathcal{L}_1 := \mathcal{L}'_0(\Lambda_0)$ is a well-defined cocycle deformation of $\mathcal{H}_1$, and clearly ϕ factorizes through $\mathcal{L}_1$.

We proceed inductively: let $i > 0$ and assume that ϕ factorizes through $\mathcal{L}_i := \mathcal{L}'_{i-1}(\Lambda_{i-1})$, $\Lambda_{i-1} \in \Bbbk^{\mathcal{G}_{i-1}}$. Observe that for each $r \in \mathcal{G}_i$, the corresponding image $\tilde{r} \in \mathcal{L}_i$ is $(g_r, 1)$-primitive (cf. 3.7). Arguing as in the previous paragraph, we conclude that $\phi(\tilde{r}) = \lambda_r(1 - g_r)$ and $\lambda_r = 0$ whenever $\chi_r \neq \epsilon$ or $g_r = 1$. Hence there is Λ_i such that ϕ factorizes through $\mathcal{L}_{i+1} = \mathcal{L}'_i(\Lambda_i)$, which is a well-defined cocycle deformation of $\mathcal{H}_{i+1}$ by 3.12 for $j = i$. In the final step ℓ we proceed in the same way, splitting $\mathcal{G}_\ell$ as in the proof of Proposition 3.3. We conclude that there exists $\Lambda \in \mathcal{R}$ such that ϕ factorizes through $\mathcal{L}(\Lambda)$.

Now, the lifting map ϕ is injective when restricted to $V\#H$ by definition, and so is the factorization $\phi : \mathcal{L}(\Lambda) \twoheadrightarrow L$, that is ϕ is injective when restricted to $\mathcal{L}(\Lambda)_1$. Then ϕ is injective [21, Theorem 5.3.1] and thus $L \simeq \mathcal{L}(\Lambda)$. ■

3.2 Isomorphism classes

Let (H, V) be as in Section 1.1, with braiding matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}}$, and $((g_i, \chi_i))_{i\in\mathbb{I}}$ a principal realization. We assume that the generalized Dynkin diagram of V is connected. Let $\Lambda \in \mathcal{R}$ and $\mathcal{L}(\Lambda)$ be as in Theorem 3.5.

3.2.1 The block group

Let

$$\mathbb{I}(i) = \{j \in \mathbb{I} | g_j = g_i \text{ and } \chi_j = \chi_i\} \subseteq \mathbb{I}, \quad i \in \mathbb{I}.$$

Remark 3.6. Either of the following holds:
(1) $|\mathbb{I}(i)| = 1$ for all $i \in \mathbb{I}$.
(2) There exists $i \neq j$ such that $j \in \mathbb{I}(i)$ is not adjacent to i. Then the generalized Dynkin diagram is one of the following:

(a) Type A_3 with $q = -1$ [18, Table 2, Row 1] and matrix $\begin{pmatrix} -1 & x & -1 \\ -x^{-1} & -1 & -x^{-1} \\ -1 & x & -1 \end{pmatrix}$, where $x \in \Bbbk^{\times}$.
(b) [18, Table 2, Row 8, diagram 3 (&4)] with matrix $\begin{pmatrix} -1 & x & -1 \\ (qx)^{-1} & q & (qx)^{-1} \\ -1 & x & -1 \end{pmatrix}$ (one diagram is obtained from the other by $q \mapsto q^{-1}$), where $x \in \Bbbk^{\times}$ and $q \in \mathbb{G}_n$ for some $n \in \mathbb{N}$.
(c) [18, Table, 2, Row 15, diagram 2, respectively 3] with matrix $\begin{pmatrix} -1 & x & -1 \\ \xi x^{-1} & -1 & \xi x^{-1} \\ -1 & x & -1 \end{pmatrix}$ respectively $\begin{pmatrix} -1 & x & -1 \\ \xi x^{-1} & -\xi^2 & \xi x^{-1} \\ -1 & x & -1 \end{pmatrix}$, where $\xi \in \mathbb{G}_3'$ and $x \in \Bbbk^{\times}$.
(d) Type D_θ, $\theta \geq 4$, with $q = -1$ [18, Table 3, Row 5 & Table 4, Row 8].
(e) [18, Table 3, Row 18, diagrams 5 & 6] (rank 4).

(3) There exists $\xi \in \mathbb{G}_3'$ such that the generalized Dynkin diagram is one of the following:

(a) Type A_2 with $q = \xi$, [18, Table 1, Row 1], and matrix $\begin{pmatrix} \xi & \xi \\ \xi & \xi \end{pmatrix}$.
(b) [18, Table 2, Row 15, diagram 4] and matrix $\begin{pmatrix} \xi & \xi & x \\ \xi & \xi & x \\ \xi^2 x^{-1} & \xi^2 x^{-1} & -1 \end{pmatrix}$, where $x \in \Bbbk^{\times}$. □

Proof. First, observe that if $j \in \mathbb{I}(i)$, then every vertex not adjacent to i cannot be adjacent to j as $1 = \chi_k(g_i)\chi_i(g_k) = \chi_k(g_j)\chi_j(g_k)$. Next, if $j \in \mathbb{I}(i)$, $j \neq i$, is not adjacent to i, then $\chi_j(g_j) = \chi_i(g_i) = -1$ as $1 = \chi_j(g_i)\chi_i(g_j) = \chi_j(g_j)^2$. Then, (2) and (3) follow by inspection in [18]. ■

Let us denote by

$$\mathbf{L} := \{s \in \mathrm{GL}_\theta(\Bbbk) | s_{ij} = 0 \text{ if } j \notin \mathbb{I}(i)\}.$$

Observe that $\mathbf{L} \simeq \{(s_i)_{i\in\mathbb{I}} \in \Bbbk^{\times\theta}\}$ if the generalized Dynkin diagram is as in Remark 3.6 (1). If the diagram is as in (3)(a), then $\mathbf{L} = \mathrm{GL}_2(\Bbbk)$.

3.2.2 Isomorphisms

We fix a new pair (H', V') as in Section 1.1. Set $\theta' = \dim V'$, $\mathbb{I}' = \mathbb{I}_{\theta'}$. Fix a principal realization $((g'_i, \chi'_i))_{i\in\mathbb{I}'}$ of V' in ${}^{H'}_{H'}\mathcal{YD}$ and let $\Gamma' = \langle g'_i \mid i \in \mathbb{I}'\rangle \leq H'$ be as in 2.1.

Let $\mathcal{G}'$ be the set of generators of the ideal defining $\mathcal{B}(V')$ and $\mathcal{R}' \subseteq \Bbbk^{\mathcal{G}'}$ as in 3.9. Pick $\Lambda' \in \mathcal{R}'$ and consider the Hopf algebra $\mathcal{L}(\Lambda')$. Let

$$\mathbb{S}_{\mathfrak{q}} = \{\sigma \in \mathbb{S}_\theta | q_{ij} = q_{\sigma(i)\sigma(j)} \ \forall i, j \in \mathbb{I}\}.$$

Lemma 3.7. Let $\psi : \mathcal{L}(\Lambda) \to \mathcal{L}(\Lambda')$ be a Hopf algebra isomorphism. Then $\varphi = \psi_{|H} : H \to H'$ is a Hopf algebra isomorphism and $T = \psi_{|V} : V \to V'$ is an isomorphism of braided vector spaces. In particular $\theta = \theta'$. Moreover,

(i) there is $\sigma \in \mathbb{S}_{\mathfrak{q}}$ such that $\varphi(g_j) = g'_{\sigma(j)}$ and $\chi'_{\sigma(j)} \circ \varphi = \chi_j$, $j \in \mathbb{I}$;

(ii) there is $s = (s_{ij}) \in \mathbf{L}$ such that $T(a_i) = \sum_{j\in\mathbb{I}(\sigma(i))} s_{ij} a'_j$, $i \in \mathbb{I}$. □

Proof. Follows since the map ψ preserves both the comultiplication and the coradical filtration, as well as the adjoint action. ■

Remark 3.8. When $|\mathbb{I}(i)| = 1$, $i \in \mathbb{I}$, Lemma 3.7 (ii) reads

(ii′) there are scalars $\{s_i\}_{i\in\mathbb{I}}$ such that $T(a_i) = s_i a'_{\sigma(i)}$. □

Assume that $\theta = \theta'$, $H' \simeq H$. We fix $\varphi \in \mathrm{Isom}(H, H')$, $\sigma \in \mathbb{S}_\theta$, and $s \in \mathbf{L}$. We say that a triple $(\varphi, \sigma, s) : (H, V, \Lambda) \to (H', V', \Lambda')$ is a *lifting data isomorphism* if

- $\sigma \in \mathbb{S}_{\mathfrak{q}}$.
- $g'_i = \varphi(g_{\sigma(i)})$ and $\chi'_i = \chi_{\sigma(i)} \circ \varphi$, $i \in \mathbb{I}$; and
- $\Lambda' = s \cdot \Lambda^\sigma$ (cf. Lemmas 3.10 and 3.11).

Set $\mathrm{Isom}(\Lambda, \Lambda') = \{\text{lifting data isomorphisms} : (H, V, \Lambda) \to (H', V', \Lambda')\}$.

Theorem 3.9. $\mathrm{Isom}(\mathcal{L}(\Lambda), \mathcal{L}(\Lambda')) \simeq \mathrm{Isom}(\Lambda, \Lambda')$. □

Proof. By Lemma 3.7, any $\psi \in \mathrm{Isom}(\mathcal{L}(\Lambda), \mathcal{L}(\Lambda'))$ univocally determines a triple $(\varphi, \sigma, s) \in \mathrm{Isom}(\Lambda, \Lambda')$.

Conversely, let $(\varphi, \sigma, s) \in \mathrm{Isom}(\Lambda, \Lambda')$. In particular, $\mathcal{L}(\Lambda')$ is an H-module via φ. Consider the linear map $T = T_s^\sigma : V \to \mathcal{L}(\Lambda')$ given by $T_s^\sigma(a_i) = \sum_{j \in \mathbb{I}(\sigma(i))} s_{ij} a'_j$, $i \in \mathbb{I}$. By assumption, T is H-linear and hence it defines an algebra epimorphism $F : T(V)\#H \to \mathcal{L}(\Lambda')$ with $F_{|H} = \varphi$ and $F(a_i) = T(a_i)$, $i \in \mathbb{I}$. By a combination of Lemmas 3.10 and 3.11, the map F induces an isomorphism $\widetilde{F} \in \mathrm{Isom}(\mathcal{L}(\Lambda), \mathcal{L}(\Lambda'))$. The assignment $(\varphi, \sigma, s) \mapsto \widetilde{F}$ is injective, as each triple determines a Hopf algebra map in the first term of the coradical filtration, hence in the whole algebra.

These constructions are inverse to each other and define a bijective correspondence $\mathrm{Isom}(\mathcal{L}(\Lambda), \mathcal{L}(\Lambda')) \simeq \mathrm{Isom}(\Lambda, \Lambda')$. ■

We set $\mathcal{H}_i = \mathcal{B}_i(V)\#H$, $i \geq 0$, see 3.1. If $\Lambda \in \mathcal{R}$, then we set, cf. 3.6: $\mathcal{A}_0(\Lambda) := \mathcal{T}(V) \in \mathrm{Cleft}(\mathcal{H}_0)$, $\mathcal{A}_{i+1}(\Lambda) := \mathcal{A}_i(\Lambda_i) \in \mathrm{Cleft}(\mathcal{H}_{i+1})$. Let

$$\rho_i : \mathcal{A}_i \to \mathcal{A}_i \otimes \mathcal{H}_i, \qquad \gamma_i : \mathcal{H}_i \to \mathcal{A}_i$$

denote the coaction and section. Also we set $\mathcal{L}_i(\Lambda) := \mathcal{L}_i$ as in 3.10.

Lemma 3.10. There is a well-defined action $\mathbf{L} \times \mathcal{R} \to \mathcal{R}$ so that if $s \in \mathbf{L}$, $\Lambda \in \mathcal{R}$, then $\mathcal{L}_i(s \cdot \Lambda) \simeq \mathcal{L}_i(\Lambda)$ as Hopf algebras. □

Proof. We fix $\Lambda \in \mathcal{R}$, $s \in \mathbf{L}$. We shall assume for simplicity that each stratum $\mathcal{G}_i$ of $\mathcal{G}$ (cf. 3.1) contains *all* primitive elements of $\mathcal{B}_i(V)$. The general case follows analogously.

We define $s \cdot \Lambda \in \mathcal{R}$. That is, we define for each $i \geq 0$ a family of scalars $s \cdot \Lambda_i \in \mathbb{k}^{\mathcal{G}_i}$ such that the algebras defined recursively as $\mathcal{A}_0^{(s)} = \mathcal{A}_0$ and $\mathcal{A}_{i+1}^{(s)} = \mathcal{A}_i^{(s)}(s \cdot \Lambda_i)$, cf. 3.6, are non zero. Hence $s \cdot \Lambda := (s \cdot \Lambda_i)_{i \geq 0} \in \mathcal{R}$. Moreover, we show that $\mathcal{A}_i^{(s)}(s \cdot \Lambda_i) \simeq \mathcal{A}_i(\Lambda_i)$ as cleft objects, all i. As a result, $\mathcal{L}_i(s \cdot \Lambda) \simeq \mathcal{L}_i(\Lambda)$ as Hopf algebras.

Let V_s be the vector space with basis $\{F_s(x_k)\}_{k \in \mathbb{I}}$. Then V_s is braided, with the braiding from V by assumption on $s \in \mathbf{L}$. Set $\mathcal{H}_{s \cdot i} = \mathfrak{B}_i(V_s)\#H$. Let $F_0 : \mathcal{H}_0 \to \mathcal{H}_{s \cdot 0}$ be the unique algebra automorphism with

$$F_{0|H} = \mathrm{id} \quad \text{and} \quad F_0(x_k) = F_s(x_k), \qquad k \in \mathbb{I}.$$

By assumption, $F_0(\mathbb{k}\mathcal{G}_0) = \mathbb{k}\mathcal{G}_0 \subset T(V_s)$ and thus it induces an algebra automorphism $F_1 : \mathcal{H}_1 \to \mathcal{H}_{s \cdot 1}$. Similarly, $F_1(\mathbb{k}\mathcal{G}_1) = \mathbb{k}\mathcal{G}_1$ and, in general, there is an induced automorphism $F_i : \mathcal{H}_i \to \mathcal{H}_{s \cdot i}$, $i \geq 0$.

Claim 3.1. There is $\mathcal{A}_{s\cdot i} \in \mathrm{Cleft}(\mathcal{H}_{s\cdot i})$ together with an algebra automorphism $f_i : \mathcal{A}_i \to \mathcal{A}_{s\cdot i}$ such that

$$\rho_{s\cdot i} \circ f_i = (f_i \otimes F_i) \circ \rho_i, \qquad f_i \circ \gamma_i(r) = \gamma_{s\cdot i}(F_i(r)),\ r \in \mathcal{G}_i. \tag{3.13}$$

□

This is clear when $i = 0$, for $f_0 = F_0$, $\mathcal{A}_{s\cdot 0} := \mathcal{H}_{s\cdot 0}$, $\rho_{s\cdot 0} = \Delta$, $\gamma_{s\cdot 0} = \mathrm{id}$.

Assume that, for a given $i \geq 0$, we have defined $\mathcal{A}_{s\cdot i}$ so that 3.13 holds. If $r \in \mathcal{G}_i$, then $x = \gamma_{s\cdot i}(F_s(r)) \in \mathcal{A}_{s\cdot i}$ is unique such that

$$\rho_{s\cdot i}(x) = x \otimes 1 + g_r \otimes F_s(r) \in \mathcal{A}_{s\cdot i} \otimes \mathcal{H}_{s\cdot i}.$$

This is satisfied by $x = f_i \circ \gamma_i(r)$ and hence f_i descends as to an isomorphism

$$f_{i+1} : \mathcal{A}_{i+1} \to \mathcal{A}_{s\cdot(i+1)} := \mathcal{A}_{s\cdot i}/\langle \gamma_{s\cdot i}(F_i(r)) - \lambda_r : r \in \mathcal{G}_i\rangle,$$

and 3.13 defines a structure $\mathcal{A}_{s\cdot(i+1)} \in \mathrm{Cleft}(\mathcal{H}_{s\cdot(i+1)})$.

Now, the composition $\mathcal{A}_0 \twoheadrightarrow \mathcal{A}_i(\Lambda_i) \xrightarrow{f_i} \mathcal{A}_{s\cdot i}$ defines a family of scalars $s\cdot\Lambda_i \in \Bbbk^{\mathcal{G}_i}$ with $\mathcal{A}_i^{(s)}(s\cdot\Lambda_i) \simeq \mathcal{A}_i(\Lambda_i)$, $i \geq 0$. Hence $s\cdot\Lambda \in \mathcal{R}$. ■

We consider the action of $\mathbb{S}_{\mathrm{q}}$ on $T(V)$ by permutations of the generators. If $\Lambda = (\lambda_r)_{r\in\mathcal{G}} \in \mathcal{R}$, then we set $\Lambda^\sigma := (\lambda_{\sigma\cdot r})_{r\in\mathcal{G}} \in \Bbbk^{\mathcal{G}}$, $\sigma \in \mathbb{S}_{\mathrm{q}}$.

Lemma 3.11. There is a well-defined action $\mathbb{S}_{\mathrm{q}} \times \mathcal{R} \to \mathcal{R}$ so that if $\sigma \in \mathbb{S}_{\mathrm{q}}$, $\Lambda \in \mathcal{R}$, then $\mathcal{L}_i(\Lambda^\sigma) \simeq \mathcal{L}_i(\Lambda)$ as Hopf algebras. □

Proof. Proceed as in Lemma 3.10, *mutatis mutandis*. ■

We give examples of the action $\mathbf{L} \times \mathcal{R} \to \mathcal{R}$ from Lemma 3.10.

Example 3.12. (1) Assume $|\mathbb{I}(i)| = 1$, $i \in \mathbb{I}$; hence $\mathbf{L} \simeq \Bbbk^{\times\theta}$. If $s = (s_i)_{i\in\mathbb{I}} \in \mathbf{L}$ and $r \in T(V)$ is a $\mathbb{Z}^\theta$-homogeneous element with $\deg r = (d_1, \dots, d_\theta)$, then we set $s_r := s_1^{d_1} \cdots s_\theta^{d_\theta} \in \Bbbk^\times$. If $\Lambda = (\lambda_r)_{r\in\mathcal{G}} \in \mathcal{R}$ and $s \in \mathbf{L}$, then $s\cdot\Lambda := (s_r\lambda_r)_{r\in\mathcal{G}}$.

(2) Assume V is as in Remark 3.6 (3)(a), so $\mathbf{L} = \mathrm{GL}_2(\Bbbk)$. In this case, $\Lambda = (\lambda_{112}, \lambda_{122}, \mu_1, \mu_2, \mu_{12}) \in \Bbbk^5$ by Theorem 1.8. Let $s = \left(\begin{smallmatrix} s_{11} & s_{12} \\ s_{21} & s_{22}\end{smallmatrix}\right) \in \mathbf{L}$ and denote $s\cdot\Lambda := (\lambda^s_{112}, \lambda^s_{122}, \mu^s_1, \mu^s_2, \mu^s_{12})$. Then

$$\mu^s_1 = s^3_{11}\mu_1 + s^3_{12}\mu_2 + s^2_{11}s_{12}\lambda_{112} + s_{11}s^2_{12}\lambda_{122},$$

$$\begin{aligned}
\mu_2^s &= s_{21}^3\mu_1 + s_{22}^3\mu_2 + s_{21}^2 s_{22}\lambda_{112} + s_{21}s_{22}^2\lambda_{122},\\
\lambda_{112}^s &= 3s_{11}^2 s_{21}\mu_1 + 3s_{12}^2 s_{22}\mu_2 + (s_{11}^2 s_{22} + 2s_{11}s_{12}s_{21})\lambda_{112}\\
&\quad + (2s_{11}s_{12}s_{22} + s_{12}^2 s_{21})\lambda_{122},\\
\lambda_{122}^s &= 3s_{11}s_{21}^2\mu_1 + 3s_{12}s_{22}^2\mu_2 + (2s_{11}s_{21}s_{22} + s_{12}s_{21}^2)\lambda_{112}\\
&\quad + (s_{11}s_{22}^2 + 2s_{12}s_{21}s_{22})\lambda_{122},\\
\mu_{12}^s &= (s_{11}s_{22} - s_{12}s_{21})^3\mu_{12}.
\end{aligned}$$ □

3.3 The algorithm

Our strategy reduces the lifting problem to an algorithm, that we describe next.

Let H, V be as in Section 1.1, Γ as in 2.1. Let $\mathcal{G}$ be the set of generators of the ideal $\mathcal{J}(V)$ defining $\mathfrak{B}(V)$ as described in [6] for each connected component, union the q-commutators of vertices in different components. Decompose it as $\mathcal{G} = \mathcal{G}_0 \sqcup \cdots \sqcup \mathcal{G}_\ell$ so that (3.2) and (3.3) hold.

The algorithm involves $\ell+1$ recursive steps. At each Step **i**, the *input* are two Hopf algebras $\mathcal{H}_i$ and $\mathcal{L}_i$, a $(\mathcal{L}_i, \mathcal{H}_i)$-bicleft object $\mathcal{A}_i$, with coactions ρ_i, δ_i and a choice of scalars $\Lambda_i = (\lambda_r)_{r\in\mathcal{G}_i} \in \Bbbk^{\mathcal{G}_i}$ such that

$$\lambda_r = 0, \quad \text{if } \chi_r \neq \epsilon, \quad \text{or } g_r = 1.$$

The *output* is a new triple $(\mathcal{H}_{i+1}, \mathcal{A}_{i+1}, \mathcal{L}_{i+1})$, as quotient of the input data. Step **0** starts with $\mathcal{H}_0 = \mathcal{L}_0 = T(V)\#H$ and $\mathcal{A}_0 = \mathcal{H}_0$, with $\rho_0 = \delta_0 = \Delta$.

The *final outcome* of the algorithm is a list of liftings of V in terms of families $\Lambda \in \Bbbk^{\mathcal{G}}$. All of them are cocycle deformations of $\mathfrak{B}(V)\#H$. If no step produces a zero object, then this list is exhaustive.

The recursive step is the following:

Step i.

(1) Compute $r' \in \mathcal{A}_i$, $r \in \mathcal{G}_i$. These elements are defined by the equation:

$$\rho_i(r') = r' \otimes 1 + g_r \otimes r, \quad r \in \mathcal{G}_i.$$

(2) Set $\mathcal{A}_{i+1} := \mathcal{A}_i(\Lambda_i) = \mathcal{A}_i/\langle r' - \lambda_r : r \in \mathcal{G}_i\rangle$ and check $\mathcal{A}_{i+1} \neq 0$.

(3) Compute $\tilde{r} \in \mathcal{L}_i$, $r \in \mathcal{G}_i$. These elements are defined by the equation:

$$\delta_i(r') = \tilde{r} \otimes 1 + g_r \otimes r', \quad r \in \mathcal{G}_i.$$

(4) Set $\mathcal{H}_{i+1} := \mathcal{H}_i/\langle \mathcal{G}_i \rangle$, $\mathcal{L}_{i+1} := \mathcal{L}_i/\langle \tilde{r} - \lambda_r(1 - g_r) : r \in \mathcal{G}_i \rangle$. □

Remark 3.13. We make some comments regarding the recursive step.

(1) At Step **0**, $r' = r$, for each $r \in \mathcal{G}_0$.

(2) At Step $\boldsymbol{\ell}$, $\mathcal{A}_{\ell+1} \neq 0$ automatically.

(3) At Step **i**, $1 \leq i \leq \ell$, the verification of (2) is facilitated by the fact that $\mathcal{A}_{i+1} = \mathcal{E}_{i+1}\#H$, for $\mathcal{E}_{i+1} \in {}^H_H\mathcal{YD}$, $i \geq 0$, the algebra defined recursively by

$$\mathcal{E}_0 = T(V), \qquad \mathcal{E}_{i+1} = \mathcal{E}_i/\langle r' - \lambda_r : r \in \mathcal{G}_i \rangle.$$ □

4 The Case $N = 2$

Let H, V as in Section 1.1, Γ as in 2.1. Assume moreover that V is of type A_θ, $\theta \in \mathbb{N}$, associated with $\xi = -1$. Let $\mathfrak{B}(V)$ be the corresponding Nichols algebra. In this section, we compute the liftings of V. We show that all of them arise as cocycle deformations of $\mathfrak{B}(V)\#H$.

Recall the definition of the distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}(V)$, see Proposition 1.1 (2). Set $\widetilde{\mathcal{H}} = \widetilde{\mathcal{B}}(V)\#H$.

Lemma 4.1. Let $i \leq j \leq k \leq l$. The following relations hold in $\widetilde{\mathcal{H}}$:

$$[x_{(ij)}, x_{(ik)}]_c = 0, \qquad [x_{(ik)}, x_{(jk)}]_c = 0, \tag{4.1}$$

$$[x_{(il)}, x_{(jk)}]_c = 0, \qquad [x_{(ik)}, x_{(jl)}]_c = 2\chi_{(jk)}(g_{(ik)})x_{(jk)}x_{(il)}. \tag{4.2}$$

The coproduct of $\widetilde{\mathcal{H}}$ satisfies

$$\Delta(x_{(ij)}) = x_{(ij)} \otimes 1 + g_{(ij)} \otimes x_{(ij)} + 2\sum_{k=i}^{j-1} x_{(ik)}g_{(k+1j)} \otimes x_{(k+1j)},$$

$$\Delta(x^2_{(ij)}) = x^2_{(ij)} \otimes 1 + g^2_{(ij)} \otimes x^2_{(ij)} + 4\sum_{k=i}^{j-1} \chi_{(ik)}(g_{(k+1j)})x^2_{(ik)}g^2_{(k+1j)} \otimes x^2_{(k+1j)}.$$ □

Proof. It follows as in [2, Section 6], see also [4, Section 3], by induction. The key point to show 4.2 is to use (1.6) as the initial step. On the other hand, relations 4.1 follow from (1.3) and (1.4). The formula for the coproduct now follows. ■

In the remaining part of this section, we deal with a quotient of $\widetilde{\mathcal{B}}(V)$, namely we fix the algebra $\widehat{\mathcal{B}}(V)$ generated by $x_1, \dots, x_\theta$ with relations

$$x_{ij} = 0,\ i < j-1; \qquad [x_{(i-1 i+1)}, x_i]_c = 0,\ 2 \le i < \theta; \qquad x_k^2 = 0,\ k \in \mathbb{I}_\theta. \tag{4.3}$$

This algebra is an intermediate quotient between $\widetilde{\mathcal{B}}(V)$ and $\mathcal{B}(V)$, see Proposition 1.1 and Remark 1.2. We prefer the quotient (4.3) as it is more suitable for our computations. We set $\widehat{\mathcal{H}} = \widehat{\mathcal{B}}(V)\#H$. Observe that Lemma 4.1 holds for $\widehat{\mathcal{H}}$.

Recall also that the Nichols algebra $\mathcal{B}(V)$ is generated by $x_1, \dots, x_\theta$ with the previous defining relations and also $x_{(ij)}^2 = 0$ for $i < j$. We set $\mathcal{H} = \mathcal{B}(V)\#H$. Let $\pi : \widehat{\mathcal{H}} \twoheadrightarrow \mathcal{H}$ be the canonical Hopf algebra map. Recall that $\widehat{\mathcal{H}}^{\operatorname{co}\pi}$ is the subalgebra generated by $x_{(ij)}^2$, $i < j$, which is a polynomial algebra with these elements as generators.

4.1 Cleft objects

Let $\boldsymbol{\lambda} = (\lambda_{ij})_{1 \le i < j-1 < \theta}$, $\boldsymbol{\mu} = (\mu_{(k\,l)})_{1 \le k \le l \le \theta}$, $\boldsymbol{\nu} = (\nu_i)_{1 < i < \theta}$ be families of scalars such that

$$\begin{aligned} &\lambda_{ij} = 0 \text{ if } \chi_i\chi_j \neq \epsilon, \qquad \mu_{(k\,l)} = 0 \text{ if } \chi_{(kl)}^2 \neq \epsilon, \\ &\nu_i = 0 \text{ if } \chi_i^2\chi_{i-1}\chi_{i+1} \neq \epsilon. \end{aligned} \tag{4.4}$$

Let us set, following Proposition 3.3, $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}(\boldsymbol{\lambda})$ the quotient of $T(V)\#H$ by the relations

$$\begin{aligned} &y_{ij} = \lambda_{ij},\ i < j-1; \qquad [y_{(i-1 i+1)}, y_i]_c = \nu_i,\ 2 \le i < \theta; \\ &y_k^2 = \mu_{(k)},\ 1 \le k \le \theta. \end{aligned} \tag{4.5}$$

Here, we have renamed the basis $\{x_1, \dots, x_\theta\}$ of V by $\{y_1, \dots, y_\theta\}$.

Proposition 4.2. The algebras $\widehat{\mathcal{A}}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$ are cleft objects for $\widehat{\mathcal{H}}$. Hence

$$\operatorname{Cleft}' \widehat{\mathcal{H}} = \{\widehat{\mathcal{A}}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) | \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu} \text{ as in } (4.5)\}. \qquad \square$$

In particular, this shows that 3.12 holds for $j = 0$.

Proof. Set $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}(\boldsymbol{\lambda})$ and $\widehat{\mathcal{E}}$ the quotient of $T(V)$ by the ideal I generated by (4.5). Observe that $\widehat{\mathcal{A}} \simeq \widehat{\mathcal{E}}\#H$, as I is an object in ${}^H_H\mathcal{YD}$. Hence we need to show that $\widehat{\mathcal{E}} \neq 0$.

For this we use Diamond Lemma [12, Theorem 1.2]. We introduce a notation close to the one in *loc. cit.* Let $\Xi_{ij} = (w_{ij}, f_{ij})$ be the pair associated to the relation $y_{ij} - \lambda_{ij}$; we choose $w_{ij} = y_i y_j$, so $f_{ij} = q_{ij} y_j y_i + \lambda_{ij}$. Similarly we set $\Xi_i = (y_i^2, \mu_{(i)})$ for $1 \leq i \leq \theta$, and $\Xi_i' = (y_{i-1} y_i y_{i+1} y_i, f_i')$, $2 \leq i \leq \theta - 1$, for the relation $[y_{(i-1i+1)}, y_i]_c - \nu_i$, where for $i = 2$,

$$f_2' = q_{12}^2 q_{13} y_2 y_3 y_2 y_1 - q_{12} q_{13} q_{23} y_3 y_2 y_1 y_2 - q_{12} q_{32} y_2 y_1 y_2 y_3 + 2q_{23} \lambda_{13} \mu_{(2)} + \nu_2.$$

There are no *inclusion ambiguities*. There are eight *overlap ambiguities*:

(1) $(\Xi_{ij}, \Xi_{jk}, y_i, y_j, y_k)$. Both $y_i f_{jk}$ and $f_{ij} y_k$ reduce to

$$q_{ij} q_{ik} q_{jk} y_k y_j y_i + \lambda_{ij} y_k + q_{jk} \lambda_{ik} y_j + \lambda_{jk} y_i,$$

since $\chi_k(g_i g_j)\lambda_{ij} = \chi_{ij}^{-1}(g_k)\lambda_{ij} = \lambda_{ij}$.

(2) $(\Xi_{ij}, \Xi_j, y_i, y_j, y_j)$. As $q_{ij}^2 \mu_j = \mu_j$ and $\lambda_{ij}(1 + q_{ij}) = 0$, both $y_i f_j$ and $f_{ij} y_j$ reduce to $\mu_j y_i$.

(3) $(\Xi_i, \Xi_{ij}, y_i, y_i, y_j)$. Analogous to the previous case.

(4) $(\Xi_i, \Xi_{i+1}', y_i, y_i, y_{i+1} y_{i+2} y_{i+1})$. For simplicity set $i = 1$. To prove that $y_1 f_2'$ reduces to $\mu_{(1)} y_2 y_3 y_2$ we use the identities $[y_1, y_{123}]_c = 0$ obtained from Ξ_1, and $[y_{12}, y_{123}]_c = 0$, which is obtained from Ξ_2' and the previous relation.

(5) $(\Xi_{i+1}', \Xi_{i+1}, y_i y_{i+1} y_{i+2}, y_{i+1}, y_{i+1})$. Again set $i = 1$. Then, $f_2' y_2$ reduces to $\mu_{(2)} y_1 y_2 y_3$ up to reduce by Ξ_2'.

(6) $(\Xi_{i+1}', \Xi_{i+1j}, y_i y_{i+1} y_{i+2}, y_{i+1}, y_j)$. Again set $i = 1$. If $j > 4$, then both $f_2' y_j$ and $y_1 y_2 y_3 f_{2j}$ reduce to

$$\begin{aligned} &q_{1j} q_{2j}^2 q_{3j} \Big(q_{12}^2 q_{13} y_j y_2 y_3 y_2 y_1 - q_{12} q_{13} q_{23} y_j y_3 y_2 y_1 y_2 - q_{12} q_{32} y_j y_2 y_1 y_2 y_3 \\ &\quad + 2q_{23} \lambda_{13} \mu_{(2)} y_j + \nu_2 y_j \Big) + \lambda_{1j} q_{2j}^2 q_{3j} y_2 y_3 y_2 + \lambda_{2j} \lambda_{13} q_{2j} q_{3j} y_2 \\ &\quad + \lambda_{2j} q_{13} q_{3j} y_3 y_1 y_2 + \lambda_{3j} \mu_{(2)} q_{2j} y_1 + \lambda_{2j} y_1 y_2 y_3 \end{aligned}$$

by direct computation. If $j = 4$, then use the relation $[y_{(14)}, y_2]_c = 0$ obtained from f_2' and f_2 to reduce the word $y_1 y_2 y_3 y_4 y_2$ and obtain the same reduction for both $f_2' y_4$ and $y_1 y_2 y_3 f_{24}$.

(7) $(\Xi_{ij}, \Xi_{j+1}', y_i, y_j, y_{j+1} y_{j+2} y_{j+1})$. Fix $j = 1, j = 3$ to simplify the notation. By direct computation we reduce both expressions to

$$\begin{aligned} &\lambda_{13} y_4 y_5 y_4 + \lambda_{14} q_{13} q_{35} y_5 y_3 y_4 + \lambda_{14} q_{13} q_{14} q_{15} y_3 y_4 y_5 + \lambda_{15} \mu_{(4)} q_{13} q_{14} y_3 \\ &\quad + q_{13} q_{14}^2 q_{15} \Big(q_{34}^4 q_{35} y_4 y_5 y_4 y_3 - q_{34} q_{35} q_{45} y_5 y_4 y_3 y_4 - q_{34} q_{54} y_4 y_3 y_4 y_5 \Big) y_1 \\ &\quad + \lambda_{14} \lambda_{35} q_{13} y_4 + 4q_{45} \lambda_{35} \mu_{(4)} y_1 + \nu_4 y_1. \end{aligned}$$

(8) $(\Xi'_{i+1}, \Xi'_{i+2}, y_i y_{i+1} y_{i+2}, y_{i+1}, y_{i+2} y_{i+3} y_{i+2})$. Again assume that $i = 1$. Both $y_1 y_2 y_3 f'_3$ and $f'_2 y_3 y_4 y_3$ reduce to

$$\begin{aligned}
&q_{12}q_{43}\lambda_{13}\mu_{(2)}\mu_{(3)}y_4 - 2q_{12}q_{23}q_{24}\lambda_{14}\mu_{(2)}\mu_{(3)}y_3 + q_{23}\lambda_{13}\mu_{(2)}y_3y_4y_3\\
&\quad + \nu_3 y_1y_2y_3 + \nu_2 y_3y_4y_3 - q_{12}q_{32}q_{13}\mu_{(3)}\lambda_{24}y_2y_3y_1 + q_{12}^2q_{13}^2\lambda_{14}\mu_{(3)}y_2y_3y_2\\
&\quad + q_{12}q_{13}q_{23}^3q_{24}^2\lambda_{13}y_3y_4y_2y_3y_2 - q_{12}q_{32}q_{14}q_{24}^2\mu_{(3)}y_4y_2y_1y_2y_3\\
&\quad + q_{12}q_{13}^2q_{14}q_{23}^3q_{24}\lambda_{13}y_3y_2y_3y_4y_2 - q_{12}q_{13}q_{23}^2q_{24}q_{34}\lambda_{14}y_3y_2y_3y_2y_3\\
&\quad + q_{12}^2q_{13}q_{24}\lambda_{13}y_2y_3y_4y_2y_1 + q_{12}^2q_{13}^2q_{14}\lambda_{13}y_2y_3y_2y_3y_4\\
&\quad + q_{12}q_{13}^3q_{14}q_{23}^3q_{24}y_3y_2y_3y_4y_3y_1y_2 - q_{12}q_{13}^2q_{14}q_{23}^2q_{24}^2q_{34}y_3y_4y_2y_3y_1y_2y_3\\
&\quad + q_{34}\lambda_{24}\mu_{(3)}y_1y_2y_3 - q_{12}q_{13}^2q_{23}^2q_{43}y_3y_2y_3y_1y_2y_3y_4.
\end{aligned}$$

Note that $\lambda_{13}\lambda_{24} = 0$ since $\chi_{13}\chi_{24}(g_{(1\,4)}) = -1$, so either $\chi_{13} \neq \epsilon$ or else $\chi_{24} \neq \epsilon$.

The proposition now follows from Proposition 3.3. ■

Lemma 4.3. For all $j < k$,

$$\rho(y_{(j\,k)}) = y_{(j\,k)} \otimes 1 + g_{(j\,k)} \otimes x_{(j\,k)} + 2\sum_{l=j+1}^{k-1} y_{(j\,l)}g_{(l+1\,k)} \otimes x_{(l+1\,k)}. \qquad \square$$

Proof. By induction on $k - j$. If $k = j + 1$, then

$$\rho(y_{(j\,j+1)}) = y_{(j\,j+1)} \otimes 1 + g_{(j\,j+1)} \otimes x_{(j\,j+1)} + 2y_j g_{j+1} \otimes x_{j+1}$$

by direct computation. If it holds for $k - j$, then

$$\begin{aligned}
\rho(y_{(j-1\,k)}) &= \rho(y_{j-1})\rho(y_{(j\,k)}) - \chi_{(j\,k)}(g_{j-1})\rho(y_{(j\,k)})\rho(y_{j-1})\\
&= y_{(j-1\,k)} \otimes 1 + \big(1 - \chi_{(j\,k)}(g_{j-1})\chi_{j-1}(g_{(j\,k)})\big)\, y_{j-1}g_{(j\,k)} \otimes x_{(j\,k)}\\
&\quad + 2\sum_{l=j+1}^{k-1}\big(y_{j-1}y_{(j\,l)} - \chi_{(j\,k)}(g_{j-1})\chi_{j-1}(g_{(l+1\,k)})y_{(j\,l)}y_{j-1}\big)g_{(l+1\,k)} \otimes x_{(l+1\,k)}\\
&\quad + g_{(j\,k)} \otimes x_{(j\,k)} + 2\sum_{l=j+1}^{k-1}\chi_{(j\,l)}(g_{j-1})y_{(j\,l)}g_{j-1}g_{(l+1\,k)} \otimes [x_{j-1}, x_{(l+1\,k)}]_c.
\end{aligned}$$

Notice that $1-\chi_{(j\,k)}(g_{j-1})\chi_{j-1}(g_{(j\,k)})=2$. For each $j+1\le l\le k-1$,

$$y_{j-1}y_{(j\,l)}-\chi_{(j\,k)}(g_{j-1})\chi_{j-1}(g_{(l+1\,k)})y_{(j\,l)}y_{j-1}=y_{(j-1\,l)},$$

$[x_{j-1},x_{(l+1\,k)}]_c=0$ by Lemma 4.1, and the inductive step follows. ■

Lemma 4.4. For all $j\le k<l$, $y_{(j\,k)}y_{(j\,l)}=\chi_{(j\,l)}(g_{(j\,k)})y_{(j\,l)}y_{(j\,k)}$. □

Proof. Set $\mathrm{y}_{j,k,l}=y_{(j\,k)}y_{(j\,l)}-\chi_{(j\,l)}(g_{(j\,k)})y_{(j\,l)}y_{(j\,k)}$. If $k=j$, $l=j+1$, then

$$\begin{aligned}\mathrm{y}_{j,j,j+1}&=y_jy_{(j\,j+1)}-q_{jj}q_{(j\,j+1)}y_{(j\,j+1)}y_j\\&=(\mu_jy_{j+1}-q_{(j\,j+1)}y_jy_{j+1}y_j)+q_{(j\,j+1)}(-q_{(j\,j+1)}\mu_jy_{j+1}+y_jy_{j+1}y_j)\\&=(1-\mu_jq_{(j\,j+1)}^2)y_{j+1}=(1-\mu_jq_{(j\,j+1)}^2\chi_j^2(g_{j+1}))y_{j+1}=0.\end{aligned}$$

Assume it holds for all k',l' such that $k'+l'<k+l$. Then

$$\rho(\mathrm{y}_{j,k,l})=\mathrm{y}_{j,k,l}\otimes 1+\text{(i)}+\text{(ii)}+\text{(iii)}+\text{(iv)}+\text{(v)}+\text{(vi)}+\text{(vii)}.$$

We compute now the other seven summands. We use repeatedly Lemma 4.1 and inductive hypothesis.

(i) $=(1-\chi_{(j\,k)}(g_{(j\,l)})\chi_{(j\,l)}(g_{(j\,k)}))y_{(j\,k)}g_{(j\,l)}\otimes x_{(j\,l)}=2y_{(j\,k)}g_{(j\,l)}\otimes x_{(j\,l)}$.

(ii) $=2\sum_{t=1}^{l-1}\big(y_{(j\,k)}y_{(j\,t)}-\chi_{(j\,l)}(g_{(j\,k)})\chi_{(j\,k)}(g_{(t+1\,l)})y_{(j\,t)}y_{(j\,k)}\big)g_{(t+1\,l)}\otimes x_{(t+1\,l)}$. If $t>k$, then $\chi_{(j\,l)}(g_{(j\,k)})\chi_{(j\,k)}(g_{(t+1\,l)})=\chi_{(j\,t)}(g_{(j\,k)})$ so the summand is zero by Lemma 4.1. For $t=k$, $\chi_{(j\,l)}(g_{(j\,k)})\chi_{(j\,k)}(g_{(k+1\,l)})=1$, so the summand is also zero. If $t<k$, then $y_{(j\,t)}y_{(j\,k)}=\chi_{(j\,k)}(g_{(j\,t)})y_{(j\,k)}y_{(j\,t)}$, so we have that

(ii) $=4\sum_{t=1}^{k-1}y_{(j\,k)}y_{(j\,t)}g_{(t+1\,l)}\otimes x_{(t+1\,l)}$.

(iii) $=g_{(j\,k)}g_{(j\,l)}\otimes[x_{(j\,k)},x_{(j\,l)}]_c=0$.

$$\begin{aligned}\text{(iv)}&=2\sum_{t=1}^{l-1}\chi_{(j\,t)}(g_{(j\,k)})y_{(j\,t)}g_{(j\,k)}g_{(t+1\,l)}\otimes[x_{(j\,k)},x_{(t+1\,l)}]_c\\&=-4\sum_{t=1}^{k-1}\chi_{(s+1\,t)}(g_{(k+1\,s)})y_{(j\,t)}g_{(j\,k)}g_{(t+1\,l)}\otimes x_{(j\,l)}x_{(t+1\,k)}\\&\quad-2y_{(j\,k)}g_{(j\,l)}\otimes x_{(j\,l)}.\end{aligned}$$

$$\text{(v)}=2\sum_{s=1}^{k-1}\chi_{(j\,l)}(g_{(s+1\,k)})[y_{(j\,s)},y_{(j\,l)}]_cg_{(s+1\,k)}\otimes x_{(s+1\,k)}=0.$$

$$\text{(vi)} = 2\sum_{s=1}^{k-1} y_{(j\,s)}g_{(s+1\,k)}g_{(j\,l)}$$

$$\otimes \left(x_{(j+1\,s)}x_{(j\,l)} - \chi_{(j\,l)}(g_{(j\,k)})\chi_{(j\,s)}(g_{(j\,l)})x_{(j\,l)}x_{(s+1\,k)}\right)$$

$$= 4\sum_{t=1}^{k-1} \chi_{(k+1\,l)}(g_{(t+1\,k)})y_{(j\,t)}g_{(j\,k)}g_{(t+1\,l)} \otimes x_{(j\,l)}x_{(t+1\,k)} = -\text{(iv)} - \text{(i)}.$$

$$\text{(vii)} = 4\sum_{t=1}^{l-1}\sum_{s=1}^{k-1} \chi_{(j\,t)}(g_{(s+1\,k)})y_{(j\,s)}y_{(j\,t)}g_{(s+1\,k)}g_{(t+1\,l)} \otimes x_{(s+1\,k)}x_{(t+1\,l)}$$

$$- \chi_{(j\,l)}(g_{(j\,k)})\chi_{(j\,s)}(g_{(t+1\,l)})y_{(j\,t)}y_{(j\,s)}g_{(t+1\,l)}g_{(s+1\,k)} \otimes x_{(t+1\,l)}x_{(s+1\,k)}.$$

For (vii) there are three subcases:

- If $t > k$, then $y_{(j\,s)}y_{(j\,t)} = \chi_{(j\,t)}(g_{(j\,s)})y_{(j\,t)}y_{(j\,s)}$ and $x_{(s+1\,k)}x_{(t+1\,l)} = \chi_{(t+1\,l)}(g_{(s+1\,k)})x_{(t+1\,l)}x_{(s+1\,k)}$. Hence, these summands are 0.
- If $t = k$, $x_{(s+1\,k)}x_{(k+1\,l)} = x_{(s+1\,l)} + \chi_{(k+1\,l)}(g_{(s+1\,k)})x_{(k+1\,l)}x_{(s+1\,k)}$, and $y_{(j\,s)}y_{(j\,k)} = \chi_{(j\,k)}(g_{(j\,s)})y_{(j\,k)}y_{(j\,s)}$, so the summand is $-$(ii).
- For $t < k$, the summands cancel between themselves.

Thus $\rho(\mathrm{y}_{j,k,l}) = \mathrm{y}_{j,k,l} \otimes 1$, so $\mathrm{y}_{j,k,l} \in \Bbbk$. Also, $\mathrm{y}_{j,k,l} \in \widehat{\mathcal{A}}_{\chi_{(j\,k)}\chi_{(j\,l)}}$. As $\chi_{(j\,k)}\chi_{(j\,l)}(g_{(j\,k)}g_{(j\,l)}) = -1$, we have that $\widehat{\mathcal{A}}_{\chi_{(j\,k)}\chi_{(j\,l)}} \cap \Bbbk = 0$ so $\mathrm{y}_{j,k,l} = 0$. ■

Lemma 4.5. For all $j < k$,

$$\rho(y_{(j\,k)}^2) = y_{(j\,k)}^2 \otimes 1 + g_{(j\,k)}^2 \otimes x_{(j\,k)}^2 + 4\sum_{s=j+1}^{k-1} \chi_{(j\,s)}(g_{(s+1\,k)})y_{(j\,s)}^2 g_{(s+1\,k)}^2 \otimes x_{(s+1\,k)}^2. \qquad \square$$

Proof. As ρ is an algebra map,

$$\rho(y_{(j\,k)}^2) = \left(y_{(j\,k)} \otimes 1 + g_{(j\,k)} \otimes x_{(j\,k)} + 2\sum_{s=j+1}^{k-1} y_{(j\,s)}g_{(s+1\,k)} \otimes x_{(s+1\,k)}\right)^2.$$

By Lemmas 4.1 and 4.4 all the summands q-commute. ■

Lemma 4.6. For all $j < k$ and all i, $y_{(j\,k)}^2 y_i = \chi_i(g_{(j\,k)}^2)y_i y_{(j\,k)}^2$. □

Proof. By induction on $k - j$. If $k = j + 1$, then

$$\rho(y_{(j\,j+1)}^2 y_i - \chi_i(g_{(j\,j+1)}^2)y_i y_{(j\,j+1)}^2) = (y_{(j\,j+1)}^2 y_i - \chi_i(g_{(j\,j+1)}^2)y_i y_{(j\,j+1)}^2) \otimes 1$$

since $x^2_{(jj+1)}x_i = \chi_i(g^2_{(jj+1)})x_i x^2_{(jj+1)}$. But $y^2_{(jj+1)}y_i - \chi_i(g^2_{(jj+1)})y_i y^2_{(jj+1)} \in \widehat{\mathcal{A}}_{\chi_i\chi^2_{(jj+1)}}$ and $\chi_i\chi^2_{(jj+1)}(g_i g^2_{(jj+1)}) = -1$, so $y^2_{(jj+1)}y_i = \chi_i(g^2_{(jj+1)})y_i y^2_{(jj+1)}$.

A similar proof follows for the inductive step since for all $j < k$ and all i, $x^2_{(jk)}x_i = \chi_i(g^2_{(jk)})x_i x^2_{(jk)}$ (see [8, Proposition 4.1]). ■

The following theorem shows that 3.12 holds for $j = 1$, hence 3.11 holds in general.

Theorem 4.7. Let $\mathcal{A} = \mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$ be the quotient of $\widehat{\mathcal{A}}$ by the relations

$$y^2_{(ij)} = \mu_{(ij)}, \ 1 \le i < j \le \theta. \tag{4.6}$$

Then $\mathcal{A} \in \operatorname{Cleft}\mathcal{H}$. As a result,

$$\operatorname{Cleft}'\mathcal{H} = \{\mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) | \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu} \text{ as in } (4.5)\}.$$ □

Proof. Indeed these algebras are obtained following [16, Theorem 4]. As in *loc. cit.* we need to describe the $\widehat{\mathcal{H}}$-linear and colinear algebra maps ${}^{\mathrm{co}\,\pi}\widehat{\mathcal{H}} \to \widehat{\mathcal{A}}$. As ${}^{\mathrm{co}\,\pi}\widehat{\mathcal{H}}$ is a polynomial ring in the variables $x^2_{(ij)}g^{-2}_{(jk)}$, it is enough to determine the value on $x^2_{(ij)}g^{-2}_{(jk)}$. Set $f(x^2_{(ij)}g^{-2}_{(jk)}) = y^2_{(ij)}g^{-2}_{(jk)} - \mu_{(ij)}g^{-2}_{(jk)}$. Then f is $\widehat{\mathcal{H}}$-colinear by Lemmas 4.1 and 4.5. We claim that f is also $\widehat{\mathcal{H}}$-linear. Indeed, for all $g \in H$ and all $1 \le k \le \theta$,

$$f(g \cdot x^2_{(ij)}g^{-2}_{(jk)}) = \chi^2_{ij}(g)f(x^2_{(ij)}g^{-2}_{(jk)}) = g \cdot f(x^2_{(ij)}g^{-2}_{(jk)}),$$
$$f(x_k \cdot x^2_{(ij)}g^{-2}_{(jk)}) = 0 = x_k \cdot f(x^2_{(ij)}g^{-2}_{(jk)}),$$

where the first equality holds by (4.4) and the second by Lemma 4.6 and [8, Proposition 4.1]. The claim follows since $\widehat{\mathcal{H}}$ is generated by H and the x_k's as an algebra. Then $\widehat{\mathcal{A}}/\widehat{\mathcal{A}}f(({}^{\mathrm{co}\,\pi}\widehat{\mathcal{H}})^+) = \widehat{\mathcal{A}}/\widehat{\mathcal{A}}f(({}^{\mathrm{co}\,\pi}\widehat{\mathcal{H}})^+)\widehat{\mathcal{A}} = \mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$ is a cleft object of $\mathcal{H}$ by Proposition 3.3. ■

4.2 Liftings

In this subsection, we give a presentation for the Hopf algebras $L(\widehat{\mathcal{A}}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}), \widehat{\mathcal{H}})$ and $L(\mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}), \mathcal{H})$. We apply Proposition 3.3 together with formula 3.7.

Proposition 4.8. The Hopf algebra $\widehat{\mathcal{L}}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) = L(\widehat{\mathcal{A}}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}), \widehat{\mathcal{H}})$ is the quotient of $\mathcal{T}(V)$ by relations

$$a_{ij} = \lambda_{ij}(1 - g_i g_j); \tag{4.7}$$

$$a_k^2 = \mu_{(k)}(1 - g_k^2); \tag{4.8}$$

$$[a_{(i-1\,i+1)}, a_i]_c = \nu_i(1 - g_i^2 g_{i-1} g_{i+1}) \tag{4.9}$$
$$- 4\chi_i(g_{i-1})\mu_{(i)}\lambda_{i-1\,i+1} g_{i-1} g_{i+1}(1 - g_i^2).$$

In particular, it is a cocycle deformation of $\widehat{\mathcal{H}}$ with gr $\widehat{\mathcal{L}}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) \simeq \widehat{\mathcal{H}}$. □

Proof. We follow Proposition 3.3 (c): the x_{ij}'s and the x_k^2's are skew-primitive elements in $T(V)\#H$, so we quotient $T(V)\#H$ by relations (4.7) and (4.8) to obtain the corresponding lifting. Again, Proposition 3.3 (c), see also [5, Corollary 5.12], applies for the relation $[x_{(i-1 i+1)}, x_i]_c$ since it is primitive, and $\tilde{u} = [a_{(i-1\,i+1)}, a_i]_c + 4\chi_i(g_{i-1})\mu_{(i)}\lambda_{i-1\,i+1} g_{i-1} g_{i+1}(1 - g_i^2)$ is the corresponding skew-primitive element (see 3.7). ■

Let $i \neq j \in \mathbb{I}$. If $|i-j| \geq 2$, then we define recursively scalars $d_{ij}(s)$, $b_{ij}(s)$, $s \geq 0$, as: $d_{ij}(0) = 2\lambda_{ij}$, $b_{ij}(0) = -2\chi_j(g_{(ij)})\lambda_{ij}$, and for $s > 0$,

$$d_{ij}(s) = q_{ij} \sum_{l=0}^{s-1} d_{i\,j+1}(l) d_{j\,j+2l+2}(s - l - 1), \tag{4.10}$$

$$b_{ij}(s) = \sum_{l=0}^{s-1} b_{i+1\,j}(l) d_{i\,i+2l+2}(s - l - 1). \tag{4.11}$$

If $|i-j| = 1$, then we set $d_{ij}(s) = b_{ij}(s) = 0$, for $s \geq 0$. In what follows $y_{(k+1\,k)} := 1$, to simplify the summation formulas.

Remark 4.9. Notice that $d_{ij}(s) = 0$ if $\chi_i \chi_{(j\,j+2s)} \neq \epsilon$. □

Lemma 4.10. Let $j < k$, $i \notin \{j-1, j, \dots, k+1\}$. Then

$$[y_i, y_{(j\,k)}]_c = \sum_{s=0}^{\frac{k-j}{2}} d_{ij}(s)\, y_{(j+2s+1\,k)}. \tag{4.12}$$

□

Proof. By induction on $j - k$. If $k = j + 1$, then

$$\begin{aligned}
[y_i, y_{(j\,j+1)}]_c &= \lambda_{ij}(1 - \chi_{j+1}(g_i g_j)) y_{j+1} + \lambda_{i\,j+1}(\chi_j(g_i) - \chi_{j+1}(g_j)) y_j \\
&= \lambda_{ij}(1 - \chi_{j+1}(g_i g_j)\chi_i\chi_j(g_{j+1})) y_{j+1} + \lambda_{i\,j+1}(\chi_j(g_i) - \chi_i^{-1}(g_j)) y_j \\
&= 2\lambda_{ij} y_{j+1}.
\end{aligned}$$

The inductive step follows from the following formula:

$$\begin{aligned}[y_i, y_{(j-1\,k)}]_c &= 2\lambda_{ij-1} y_{(j\,k)} + q_{ij-1}\big[y_{j-1}, [y_i, y_{(j\,k)}]_c\big]_c \\ &= d_{ij-1}(0) y_{(j\,k)} + q_{ij-1} \sum_{s=0}^{\frac{k-j}{2}} d_{ij}(s) \big[y_{j-1}, y_{(j+2s+1\,k)}\big]_c \\ &= d_{ij-1}(0) y_{(j\,k)} + q_{ij-1} \sum_{s=0}^{\frac{k-j}{2}} d_{ij}(s) \sum_{t=0}^{\frac{k-j-1-2s}{2}} d_{j-1\,j+2s+1}(t)\, y_{(j+2(s+t+1)\,k)}.\end{aligned}$$

Here, we have applied the inductive hypothesis twice. ■

Lemma 4.11. Let $j < k$. Then

$$[y_{(j\,k)}, y_{k+1}]_c = y_{(j\,k+1)} - \sum_{s=1}^{\frac{k-j}{2}} b_{j\,k+1}(s)\, y_{(j+2s+1\,k)}, \tag{4.13}$$

$$[y_{(j\,k)}, y_k]_c = \sum_{s=0}^{\frac{k-j-1}{2}} b_{j\,k}(s)\, y_{(j+2s+1\,k)}. \tag{4.14}$$

□

Proof. First, we prove (4.13) by induction on $k-j$. For $k = j+1$, we have

$$\begin{aligned}[y_{(j\,j+1)}, y_{j+2}]_c &= [[y_j, y_{j+1}]_c, y_{j+2}]_c \\ &= y_{(j\,j+2)} - \chi_{j+1}(g_j) y_{j+1} [y_j, y_{j+2}]_c + \chi_{j+2}(g_{j+1}) [y_j, y_{j+2}]_c y_{j+1} \\ &= y_{(j\,j+2)} + \chi_{j+2}(g_{j+1})(1 + \chi_{j+1}(g_{j\,j+2}))\lambda_{jj+2} y_{j+1} \\ &= y_{(j\,j+2)} - b_{jj+2}(0) y_{j+1}.\end{aligned}$$

Now assume it holds for j', k' such that $k' - j' < k - j$. Then by inductive hypothesis, Lemma 4.10 and using $\chi_{k+1}^{-1} = \chi_j$ if $\lambda_{jk+1} \neq 0$ we obtain

$$\begin{aligned}[y_{(j\,k)}, y_{k+1}]_c &= [[y_j, y_{(j+1\,k)}]_c, y_{k+1}]_c \\ &= [y_j, [y_{(j+1\,k)}, y_{k+1}]_c]_c + \big(\chi_{k+1}(g_{(j+1\,k)}) - \chi_{(j+1\,k)}(g_j)\big)\lambda_{jk+1} y_{(j+1\,k)} \\ &= \left[y_j, y_{(j+1\,k+1)} - \sum_{s=1}^{\frac{k-j}{2}} b_{j+1k+1}(s)\, y_{(j+2s+1\,k+1)} \right]_c \\ &\quad + \chi_{k+1}(g_{(j+1\,k)})\big(1 - \chi_j(g_{(j+1\,k)})\chi_{(j+1\,k)}(g_j)\big)\lambda_{jk+1} y_{(j+1\,k)}\end{aligned}$$

$$= y_{(j\,k+1)} - \sum_{s=1}^{\frac{k-j}{2}} b_{j+1k+1}(s) \sum_{t=0}^{\frac{k-2s-j}{2}} d_{jj+2s+2t+2}\, y_{(j+2s+2t+2\,k+1)} - b_{jk+1}(0) y_{(j+1\,k)}.$$

Now we prove (4.14). For $k = j+1, j+2$ we have

$$\begin{aligned}
[y_{(j\,j+1)}, y_{j+1}]_c &= \mu_{(j+1)}(1 - q^2_{(j\,j+1)}) y_j = 0,\\
[y_{jj+2}, y_{j+2}]_c &= [[y_j, y_{j+1j+2}]_c, y_{k+2}]_c\\
&= \lambda_{jj+2}(\chi_{j+2}(g_{j+1j+2}) - \chi_{j+1j+2}(g_j)) y_{j+1j+2}\\
&= -2\chi_{j+2}(g_{j+1})\lambda_{jj+2} y_{j+1j+2} = b_{jj+2}(0) y_{j+1j+2}.
\end{aligned}$$

Then we argue by induction in $k - j$ as for (4.13). ■

We define recursively $\zeta_{(j\,k)} \in \widehat{\mathcal{L}}$ as follows: $\zeta_{(j\,j)} = a_j$ and for $j < k$

$$\zeta_{(j\,k)} = [a_j, \zeta_{(j+1\,k)}]_c + d_{jk}(0)\chi_{(j\,k)}(g_j)\zeta_{(j+1\,k-1)} g_{jk} + 2\sum_{t=1}^{\frac{k-j-1}{2}} d_{jk-2t}(t)\chi_{(j+1\,k-2t-1)}(g_j)\zeta_{(j+1\,k-2t-1)} g_j g_{(k-2t\,k)}. \quad (4.15)$$

Remark 4.12. If $s \neq t$, then $d_{jk-2t}(t) d_{jk-2s}(s) = 0$ by Remark 4.9. Thus, there is at most one non-trivial summand besides $[a_j, \zeta_{(j+1\,k)}]_c$ in (4.15). □

Lemma 4.13. The $\widehat{\mathcal{L}}$-coaction δ of $\widehat{\mathcal{A}}$ satisfies:

$$\delta(y_{(j\,k)}) = \zeta_{(j\,k)} \otimes 1 + g_{(j\,k)} \otimes y_{(j\,k)} + 2\sum_{s=j}^{k-1} \zeta_{(j\,s)} g_{(s+1\,k)} \otimes y_{(s+1\,k)}.$$ □

Proof. Again by induction: the case $k = j+1$ is direct. Now assume that it holds for $k - j$. Then, we compute

$$\begin{aligned}
\delta(y_{(j-1\,k)}) &= \delta(y_{j-1})\delta(y_{(j\,k)}) - \chi_{(j\,k)}(g_{j-1})\delta(y_{(j\,k)})\delta(y_{j-1})\\
&= [a_{j-1}, \zeta_{(j\,k)}]_c \otimes 1 + 2a_{j-1} g_{(j\,k)} \otimes y_{(j\,k)}\\
&\quad + 2\sum_{s=j}^{k-1} [a_{j-1}, \zeta_{(j\,s)}]_c g_{(s+1\,k)} \otimes y_{(s+1\,k)} + g_{(j-1\,k)} \otimes y_{(j-1\,k)}
\end{aligned}$$

$$+\ 2\sum_{s=j}^{k-1}\chi_{(j\,s)}(g_{j-1})\zeta_{(j\,s)}g_{j-1}g_{(s+1\,k)}\otimes[y_{j-1},y_{(s+1\,k)}]_c$$

$$=[a_{j-1},\zeta_{(j\,k)}]_c\otimes 1+2a_{j-1}g_{(j\,k)}\otimes y_{(j\,k)}$$

$$+\ 2\sum_{s=j}^{k-1}[a_{j-1},\zeta_{(j\,s)}]_c g_{(s+1\,k)}\otimes y_{(s+1\,k)}+g_{(j-1\,k)}\otimes y_{(j-1\,k)}$$

$$+\ 2\sum_{s=j}^{k-2}\chi_{(j\,s)}(g_{j-1})\zeta_{(j\,s)}g_{j-1}g_{(s+1\,k)}\otimes\sum_{t=0}^{\frac{k-s-1}{2}}d_{j-1s+1}(t)\,y_{(s+2t+2\,k)}$$

$$+\ 2\chi_{(j\,k-1)}(g_{j-1})\zeta_{(j\,k-1)}g_{j-1k}\otimes\lambda_{j-1k},$$

by Lemma 4.10. The proof follows by reordering the summands. ■

Now, for each $m\geq 1$, consider the m-adic approximation $\widehat{\mathfrak{B}}_m(V)$. This is the quotient of $T(V)$ by relations (4.3) and

$$x_{(k\,l)}^2,\qquad 1\leq l-k<m.$$

Thus, we obtain a family of cleft objects $\mathcal{A}_m(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu})$ for $\mathcal{H}_m=\widehat{\mathfrak{B}}_m(V)\#H$ given by the quotient of $\mathcal{T}(V)$ by relations (4.5) together with

$$y_{(k\,l)}^2-\mu_{(k\,l)},\qquad 1\leq l-k<m. \tag{4.16}$$

Let $\mathcal{L}_m(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}):=L(\mathcal{A}_m,\mathcal{H}_m)$. Notice that $\mathcal{L}_0=\widehat{\mathcal{L}}$. We keep the name $\delta:\mathcal{A}_m\to\mathcal{L}_m\otimes\mathcal{A}_m$ for the coaction at each level.

For the next two lemmas we consider a fixed m.

Lemma 4.14. Let $i<j<k$ be such that $k-j<m$. Then, there exist $c_{ijk}(s,t)\in\Bbbk$ such that

$$[y_{(i\,k)},y_{(j\,k)}]_c=\sum_{i<s<t\leq k+1}c_{ijk}(s,t)\,y_{(t\,k)}y_{(s\,k)}. \tag{4.17}$$

□

Proof. By induction on $j-i$. If $j=i+1$, then

$$\begin{aligned}[y_{(j-1\,k)},y_{(j\,k)}]_c&=\big(y_{j-1}y_{(j\,k)}-\chi_{(j\,k)}(g_{j-1})y_{(j\,k)}y_{j-1}\big)y_{(j\,k)}\\&\quad-\ \chi_{(j\,k)}(g_{(j-1\,k)})y_{(j\,k)}\big(y_{j-1}y_{(j\,k)}-\chi_{(j\,k)}(g_{j-1})y_{(j\,k)}y_{j-1}\big)\\&=\mu_{(j\,k)}(1-\chi_{(j\,k)}^2(g_{j-1}))y_{j-1}=0,\end{aligned}$$

since $\chi_{(j\,k)}(g_{(j-1\,k)}) = \chi_{(j\,k)}(g_{j-1})\chi_{(j\,k)}(g_{(j\,k)}) = -\chi_{(j\,k)}(g_{j-1})$.

Now assume that it holds for all $i' < j'$ such that $j - i > j' - i'$. Then,

$$[y_{(i-1\,k)}, y_{(j\,k)}]_c = [y_{i-1}, [y_{(i\,k)}, y_{(j\,k)}]_c]_c - \chi_{(i\,k)}(g_{i-1})y_{(i\,k)}[y_{i-1}, y_{(j\,k)}]_c$$
$$+ \chi_{(j\,k)}(g_{(i\,k)})[y_{i-1}, y_{(j\,k)}]_c y_{(i\,k)} = \sum_{i<s<t\le k+1} c_{ijk}(s,t)[y_{i-1}, y_{(t\,k)}y_{(s\,k)}]_c$$
$$+ \sum_{r=0}^{\frac{k-j}{2}} d_{i-1j}(s)\big(2\chi_{(j\,k)}(g_{(i\,k)})y_{(j+2s+1\,k)}y_{(i\,k)} - \chi_{(j\,k)}(g_{i-1})[y_{(i\,k)}, y_{(j+2s+1\,k)}]_c\big).$$

We apply the inductive step to express $[y_{(i\,k)}, y_{(j+2s+1\,k)}]_c$ as a linear combination of products $y_{(t\,k)}y_{(s\,k)}$. Also,

$$[y_{i-1}, y_{(t\,k)}y_{(s\,k)}]_c = [y_{i-1}, y_{(t\,k)}]_c y_{(s\,k)} + \chi_{(t\,k)}(g_{i-1})y_{(t\,k)}[y_{i-1}, y_{(s\,k)}]_c.$$

We apply Lemma 4.10 and the inductive step to obtain a linear combination of elements $y_{(t\,k)}y_{(s\,k)}$ for $k+1 \ge t \ge s > i$. Assume that some $y_{(t\,k)}^2$ appears with non-zero coefficient. Then $\chi_{(i\,k)}\chi_{(j\,k)} = \chi_{(t\,k)}^2$, so $\chi_{(i\,t-1)}\chi_{(j\,t-1)} = \epsilon$, which contradicts $\chi_{(i\,t+1)}\chi_{(j\,t+1)}(g_{(i\,t+1)}g_{(j\,t+1)}) = -1$. ■

Lemma 4.15. Let $j \le k$. There exist $z_{(j\,k)}(s,t) \in \widehat{\mathcal{L}}$ such that

$$\delta(y_{(j\,k)}^2) = \zeta_{(j\,k)}^2 \otimes 1 + 4\sum_{s=j}^{k-1} \chi_{(j\,s)}(g_{(s+1\,k)})\zeta_{(j\,s)}^2 g_{(s+1\,k)}^2 \otimes y_{(s+1\,k)}^2$$
$$+ g_{(j\,k)}^2 \otimes y_{(j\,k)}^2 + \sum_{i<s<t\le k+1} z_{(j\,k)}(s,t) \otimes y_{(t\,k)}y_{(s\,k)}. \tag{4.18}$$

□

Proof. As δ is an algebra map,

$$\delta(y_{(j\,k)}^2) = \left(\zeta_{(j\,k)} \otimes 1 + g_{(j\,k)} \otimes y_{(j\,k)} + 2\sum_{s=j}^{k-1} \zeta_{(j\,s)}g_{(s+1\,k)} \otimes y_{(s+1\,k)}\right)^2.$$

Then, we apply Lemma 4.14 to write the right-hand side as a linear combination of elements $y_{(t\,k)}y_{(s\,k)}$ (remember that $y_{(k+1\,k)} = 1$). ■

Notice that $\mathcal{H}_{m+1} = \mathcal{H}_m/I_{m+1}$ is such that I_{m+1} is generated by skew-primitive elements [2, Remark 6.10]. According to Proposition 3.3 cf. 3.7, to describe $\mathcal{L}_{m+1}$ as a quotient of $\mathcal{L}_m$ we need the *deforming elements* $\mathrm{u}_{(j\,k)}$ defined by the equation:

$$\zeta_{(j\,k)}^2 \otimes 1 - \delta(y_{(j\,k)})^2 = \mathrm{u}_{(j\,k)} \otimes 1. \tag{4.19}$$

Recall the definition of $\zeta_{(j\,k)}$ in (4.15).

Remark 4.16. As in the case of $x_{(jk)}$, $y_{(jk)}$, we define recursively $a_{(jj)} = a_j$, $a_{(jk)} = [a_j, a_{(j+1\,k)}]_c$. By induction we see that

$$\zeta_{(jk)} = a_{(jk)} + \text{ other terms with factors } a_{(st)},\ t-s < k-j.$$

For example,

$$\zeta_{(12)} = a_{(12)}, \qquad \zeta_{(13)} = a_{(13)} + 2\lambda_{13} q_{12} a_2 g_1 g_3.$$ □

Theorem 4.17. The algebra $\mathcal{L}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) := L(\mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}), \mathcal{H})$ is the quotient of $\widehat{\mathcal{L}}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$ by

$$\zeta_{(j\,k)}^2 = \mu_{(j\,k)}(1 - g_{(j\,k)}^2) + \mathrm{u}_{(j\,k)}, \tag{4.20}$$

where $\mathrm{u}_{(j\,k)}$ is defined recursively as: $\mathrm{u}_{(kk)} = 0$, $k \in \mathbb{I}$, and for $k > j$

$$\mathrm{u}_{(j\,k)} = -4 \sum_{j \le p < k} \chi_{p+1,k}(g_{j,p}) \mu_{(p+1\,l)} \left(\mathrm{u}_{(j\,p)} + \mu_{(j\,p)} \left(1 - g_{(j\,p)}^2\right)\right) g_{(p+1\,k)}^2.$$ □

Proof. We prove the statement by induction in $m = k - j$. We work over $\mathcal{H}_m$, $\mathcal{A}_m$, $\mathcal{L}_m$. Then $x_{(j\,k)}^2$ is primitive and $\gamma(x_{(j\,k)}^2) = y_{(j\,k)}^2$. Set $\pi_m : \widehat{\mathcal{L}} \twoheadrightarrow \mathcal{L}_m$ the canonical projection. By (4.18),

$$\begin{aligned}\delta(y_{(j\,k)}^2) = {} & \left(\zeta_{(j\,k)}^2 - \mathrm{u}_{(j\,k)}\right) \otimes 1 + g_{(j\,k)}^2 \otimes y_{(j\,k)}^2 \\ & + \sum_{i<s<t\le k+1} \pi_m\left(\mathrm{z}_{(j\,k)}(s,t)\right) \otimes y_{(t\,k)} y_{(s\,k)}.\end{aligned}$$

By Proposition 3.3, $\pi_m\left(\mathrm{z}_{(j\,k)}(s,t)\right) = 0$ and the theorem follows. ■

Example 4.18. For $\theta = 2, 3$ we have the following relations

$$\begin{aligned}\zeta_{(12)}^2 & - \mu_{(12)}(1 - g_{12}^2) - \mathrm{u}_{(12)} \\ & = a_{(12)}^2 - \mu_{(12)}(1 - g_{12}^2) - 4q_{21}\mu_{(1)}\mu_{(2)}(1 - g_1^2)g_2^2, \\ \zeta_{(13)}^2 & - \mu_{(13)}(1 - g_{(1\,3)}^2) - \mathrm{u}_{(13)}\end{aligned}$$

$$
\begin{aligned}
&= (a_{(13)} + 2\lambda_{13}q_{12}a_2g_{13})^2 - \mu_{(13)}(1 - g^2_{(1\,3)}) - \mathrm{u}_{(13)}\\
&= a^2_{(13)} + 2q_{12}\lambda_{13}\nu_2(1 - g_{(13)}g_2)g_{13} + 4q^2_{12}\lambda^2_{13}\mu_{(2)}(1 - g^2_2)g^2_{13}\\
&\quad - \mu_{(13)}(1 - g^2_{(1\,3)}) - 4q_{31}q_{21}\mu_{(23)}\mu_{(1)}(1 - g^2_1)g^2_{23}\\
&\quad - 4q_{31}q_{32}\mu_{(3)}\Big(\mu_{(12)}(1 - g^2_{12}) + 4q_{21}\mu_{(1)}\mu_{(2)}(1 - g^2_1)g^2_2\Big)g^2_3. \qquad \square
\end{aligned}
$$

Remark 4.19. We set $G = (\mathbb{Z}/2n\mathbb{Z})^3$ for some $n \geq 2$, g_i, $i = 1, 2, 3$, the generators of each cyclic factor. Set $H = \Bbbk G$. Given the matrix $\mathbf{q} = \left(\begin{smallmatrix} -1 & 1 & -1 \\ -1 & -1 & -1 \\ -1 & 1 & -1 \end{smallmatrix}\right)$, there exist $\chi_i \in \widehat{\Gamma}$, $i = 1, 2, 3$, such that $\chi_j(g_i) = q_{ij}$, so V is realized in ${}^H_H\mathcal{YD}$. Notice that $\chi_i^2 = \epsilon$, $i = 1, 2, 3$, $\chi_1\chi_3 = \epsilon$, so the scalars $\mu_{(i)}$, λ_{13} can be simultaneously non-zero by (4.4) for the Yetter–Drinfeld module V with basis $v_i \in V^{\chi_i}_{g_i}$.

But for $\theta \geq 4$ we have that $\lambda_{13}\lambda_{24} = 0$. Indeed $\chi_{(14)}(g_{(14)}) = -1$, so either $\chi_1\chi_3 \neq \epsilon$ or else $\chi_2\chi_4 \neq \epsilon$.

Also $\nu_2\nu_3 = 0$. Otherwise $\chi_1\chi_2^3\chi_3 = \chi_2\chi_3^2\chi_4$, which implies that $\chi_{12} = \chi_{34}$, so $\chi_{(14)}(g_{(14)}) = \chi^2_{12}(g_{12})\chi^2_{34}(g_{34}) = 1$, a contradiction. $\square$

Remark 4.20. The computation of $\zeta^2_{(jk)}$, $j < k$, involves the computation of $[a_{(rs)}, a_{(r's')}]_c$ for $r \leq s$, $r' \leq s'$. A general abstract formula involves all the scalars μ_{rs}, ν_s, λ_{st}. However not all of them can be non-zero simultaneously (see Remark 4.19). For example,

$$
\begin{aligned}
[a_1, a_{(35)}]_c &= 2\lambda_{13}a_{45} - 2\chi_{34}(g_1)\lambda_{15}a_{34}g_{15} + 4\chi_3(g_1)\lambda_{14}\lambda_{35}(1 - g_{35})\\
[a_{(14)}, a_3]_c &= 2\lambda_{13}\chi_3(g_{(24)})a_{(24)} - 2\nu_3a_1g_{2343} + 8\lambda_{24}\mu_{(3)}\chi_3(g_2)a_1g_{24}(g^2_3 - 1),\\
[a_{(14)}, a_2]_c &= 2\lambda_{24}a_{(13)}g_{24} + 2\chi_2(g_4)\nu_2a_4 + 4\lambda_{24}\chi_3(g_{12})a_3a_{12}g_{24}\\
&\quad + 4\lambda_{24}\chi_{23}(g_1)a_{23}a_1g_{24} - 8\lambda_{24}\chi_{23}(g_{12})a_3a_2a_1g_{24}.
\end{aligned}
$$

In the first identity, we necessarily have $\lambda_{13}\lambda_{14} = \lambda_{15}\lambda_{14} = 0$. Similar conditions follow for the other two identities. $\square$

Example 4.21. Set $\theta = 5$ and consider the braiding matrix

$$
\begin{pmatrix} -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \end{pmatrix}.
$$

Thus χ_{13}, χ_{15}, χ_{24}, $\chi_{(13)}\chi_2$, $\chi_{(24)}\chi_3 \neq \epsilon$. We may assume that there are g_i, χ_j are such that

$$
\chi_{14} = \chi_{35} = \chi_{(35)}\chi_4 = \chi_i^2 = \epsilon, \qquad 1 \leq i \leq 5.
$$

Notice that $\zeta_{(13)} = a_{(13)}$, $\zeta_{(24)} = a_{(24)}$, but

$$\zeta_{(14)} = a_{(14)} - 4\lambda_{14}a_{23}g_{14}, \qquad \zeta_{(15)} = a_{(15)} + 4\lambda_{14}\lambda_{35}a_2g_{35} + 4\lambda_{35}a_4a_{12}g_{35},$$
$$\zeta_{(25)} = a_{(25)} + 4\lambda_{35}a_4a_2g_{35}.$$

We compute the relations involving $\zeta^2_{(i,i+1)}$, $\zeta^2_{(i,i+2)}$ as in Example 4.18. For the other cases we need some auxiliary computations as

$$[a_{(14)}, a_{23}]_c = [a_2, a_{(25)}]_c = [a_{(25)}, a_4]_c = 0,$$
$$[a_{(15)}, a_2]_c = 8\lambda_{14}\lambda_{35}\mu_{(2)}(1 - g_2^2)(1 + g_{14})g_{35},$$
$$[a_{12}, a_{(15)}]_c = [a_1, [a_2, a_{(15)}]_c]_c = 0,$$
$$[a_{(15)}, a_4]_c = 8\lambda_{35}\mu_{(4)}a_{12}g_{35}(g_4^2 - 1) - 2\lambda_{14}a_{(25)}g_{14} - 2\nu_4a_{12}g_{(35)}g_4.$$

We have that

$$\begin{aligned}
\zeta^2_{(14)} &= a^2_{(14)} + 16\lambda^2_{14}\big(\mu_{(23)}(1 - g^2_{23}) - 2\mu_{(2)}\mu_{(3)}(1 - g_2^2)g_3^2\big)g^2_{14},\\
\zeta^2_{(25)} &= a^2_{(25)} + 16\lambda^2_{35}\mu_{(4)}\mu_{(2)}(1 - g_4^2)(1 - g_2^2)g^2_{35},\\
\zeta^2_{(15)} &= a^2_{(15)} - 16\lambda^2_{14}\lambda^2_{35}\mu_{(2)}(1 - g_2^2)g^2_{35}\\
&\quad - 16\lambda^2_{35}\mu_{(4)}(1 - g_4^2)\big(\mu_{(12)}(1 - g^2_{12}) + 2\mu_{(1)}\mu_{(2)}(1 - g_1^2)g_2^2\big)g^2_{35}\\
&\quad + 8\lambda_{35}a_4a_{(15)}a_{12}g_{35} + 32\lambda^2_{14}\lambda^2_{35}\mu_{(2)}(1 - g_2^2)(1 + g_{14})g^2_{35}\\
&\quad - 8\nu_4\lambda_{35}\big(\mu_{(12)}(1 - g^2_{12}) + 2\mu_{(1)}\mu_{(2)}(1 - g_1^2)g_2^2\big)g^2_{(35)}\\
&\quad - 8\lambda_{14}\lambda_{35}a_{(25)}a_{12}g_{14}g_{35} - 32\lambda_{14}\lambda^2_{35}a_4a_2a_{12}g^2_{35},
\end{aligned}$$

so last relation can be deformed in higher strata of the coradical filtration. □

5 The Case $N = 3$

Let H, V as in Section 1.1, Γ as in 2.1. Assume that V is of type A_θ, $\theta \in \mathbb{N}$, associated with (primitive) cubic root of unity ξ. Let $\mathfrak{B}(V)$ be the corresponding Nichols algebra. In this section we compute the liftings of V. We show that all of them arise as cocycle deformations of $\mathfrak{B}(V)\#H$.

5.1 Cleft objects

Let us set, following Proposition 3.3, $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}(\boldsymbol{\lambda})$ the quotient of $T(V)\#H$ by the relations

$$y_{ij} = 0, \ i < j-1; \qquad\qquad y_{iij} = \lambda_{iij}, \ |j-i| = 1 \tag{5.1}$$

for some family of scalars $\boldsymbol{\lambda} = (\lambda_{iij})$ satisfying

$$\lambda_{iij} = 0 \quad \text{ if } \chi_{iij} \neq \epsilon. \tag{5.2}$$

Here, we have renamed the basis $\{x_1, \ldots, x_\theta\}$ of V by $\{y_1, \ldots, y_\theta\}$.

If $\widetilde{\mathcal{A}} \neq 0$, then these algebras are cleft objects for $\widetilde{\mathcal{H}}$, by Proposition 3.3. The coaction $\rho : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{H}}$ given by

$$\rho(y_i) = y_i \otimes 1 + g_i \otimes y_i, \qquad i \in \mathbb{I}.$$

We will show that $\widetilde{\mathcal{A}}(\boldsymbol{\lambda}) \neq 0$ for every $\boldsymbol{\lambda}$ satisfying 5.2. In particular, this will show that 3.12 holds for $j = 0$. First, we develop some technicalities about these scalars.

Lemma 5.1. (1) Let $1 \le i \neq j < \theta$, $|\, i - j \,| = 1$. If $\chi_{iij} = \epsilon$, then $q_{ij} = q_{ji} = \xi$.

(2) Let $1 \le i < \theta - 2$, $j = i+1$, $k = i+2$. If $\chi_{iij} = \epsilon$ or $\chi_{ijj} = \epsilon$, then $\chi_{jjk} \neq \epsilon$ and $\chi_{jkk} \neq \epsilon$.

(3) Let $1 \le i < \theta - 3$, $j = i+1$, $k = i+2$, $l = i+3$. If $\chi_{iij} = \epsilon$ or $\chi_{ijj} = \epsilon$, then $\chi_{kkl} \neq \epsilon$ and $\chi_{kll} \neq \epsilon$. □

Proof. We set $i = 1$ to simplify the notation. For (1), observe that $\chi_{112} = \epsilon$ gives $1 = \chi_{112}(g_1) = \xi^2 q_{12}$ and $1 = \chi_{112}(g_2) = \xi^2 q_{21}$. Hence $q_{21} = q_{12} = \xi$. Idem for $\chi_{122} = \epsilon$. For (2), we have

$$\chi_{112}(g_{223})\chi_{223}(g_{112}) = (q_{31}q_{13})^2 (q_{21}q_{12})^4 (q_{23}q_{32}) q_{22}^4 = \xi^2.$$

Hence, if $\chi_{112} = \epsilon$, then $\chi_{223} \neq \epsilon$. The other combinations follow analogously. For (3), it follows that $\chi_{112}(g_{334})\chi_{334}(g_{112}) = \xi$. Thus if $\chi_{112} = \epsilon$, then $\chi_{334} \neq \epsilon$. A similar computation yields the other combinations. ■

Proposition 5.2. Let $\boldsymbol{\lambda} = (\lambda_{iij})$ satisfy 5.2. Then $\widetilde{\mathcal{A}}(\boldsymbol{\lambda}) \neq 0$. Hence $\widetilde{\mathcal{A}}(\boldsymbol{\lambda}) \in \operatorname{Cleft} \widetilde{\mathcal{H}}$ and, in particular,

$$\operatorname{Cleft}' \widetilde{\mathcal{H}} = \{\widetilde{\mathcal{A}}(\boldsymbol{\lambda}) | \boldsymbol{\lambda} \text{ as in } (5.2)\}.$$

□

Proof. Set $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}(\boldsymbol{\lambda})$. Observe that $\widetilde{\mathcal{A}} \simeq \widetilde{\mathcal{E}}_\theta \# H$, for

$$\widetilde{\mathcal{E}}_\theta = \Bbbk\langle y_1, \dots, y_\theta \mid y_{ij},\ i < j-1;\ y_{iij} - \lambda_{iij},\ |j-i| = 1\rangle$$

as the ideal of $T(V)$ generated by (5.1) is an object in ${}^H_H\mathcal{YD}$.

Hence we need to show that $\widetilde{\mathcal{E}}_\theta \neq 0$. We see this by induction on $\theta \geq 2$. For $\theta = 2$, we distinguish three cases, namely

(i) $\lambda_{112} = \lambda_{122} = 0$; (ii) $\lambda_{112} \neq 0, \lambda_{122} = 0$; (iii) $\lambda_{112}\lambda_{122} \neq 0$.

Case (i) is clear, as $\widetilde{\mathcal{E}}_2$ is the distinguished pre-Nichols algebra of type A_2 (cf. p. 4). For both cases (ii) and (iii) notice that, as $q_{12} = q_{21} = \xi$ by Lemma 5.1, the defining relations become:

$$\lambda_{112} = y_1^2 y_2 + y_1 y_2 y_1 + y_2 y_1^2, \qquad \lambda_{122} = y_2^2 y_1 + y_2 y_1 y_2 + y_1 y_2^2.$$

Now case (iii) follows by observing that $\alpha : \widetilde{\mathcal{E}}_2 \to \Bbbk$ given by

$$\alpha(y_1) = \left(\frac{\lambda_{112}^2}{3\lambda_{122}}\right)^{\frac{1}{3}}, \qquad \alpha(y_2) = \left(\frac{\lambda_{122}^2}{3\lambda_{112}}\right)^{\frac{1}{3}} \tag{5.3}$$

is a well-defined one-dimensional representation.

For case (ii), we have the representation $\alpha : \widetilde{\mathcal{E}}_2 \to \Bbbk^{3\times 3}$:

$$\alpha(y_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \alpha(y_2) = \begin{pmatrix} -\lambda_{112} & 0 & -\lambda_{112} \\ 0 & 0 & 0 \\ \lambda_{112} & 0 & \lambda_{112} \end{pmatrix}. \tag{5.4}$$

We turn now to the general case $\theta \geq 3$. If $\lambda_{112} = \lambda_{122} = 0$, then we have an algebra isomorphism

$$\widetilde{\mathcal{E}}_\theta / \langle y_1 \rangle \simeq \widetilde{\mathcal{E}}_{\theta-1}, \qquad \bar{y}_i \mapsto a_{i-1}, \quad 2 \leq i \leq \theta,$$

where a_i, $1 \leq i \leq \theta - 1$ stand for the generators of $\widetilde{\mathcal{E}}_{\theta-1}$. Hence we may assume $\lambda_{112} \neq 0$. Thus it follows that $\lambda_{223} = \lambda_{233} = 0$ and (if $\theta \geq 4$) $\lambda_{334} = \lambda_{344} = 0$, by Lemma 5.1. In particular, if $\theta = 3$, there is an isomorphism

$$\widetilde{\mathcal{E}}_3 / \langle y_3 \rangle \simeq \widetilde{\mathcal{E}}_2, \qquad \overline{y_i} \mapsto a_i, \quad 1 \leq i \leq 2$$

which shows that $\widetilde{\mathcal{E}}_3 \neq 0$. Let then $\theta \geq 4$ and assume $\widetilde{\mathcal{E}}_\vartheta \neq 0$ for every $\vartheta < \theta$. For $i \in \mathbb{N}$, we set $i^* = i + 3$. Let $\mathcal{Q} = (q_{ij})_{1 \leq i,j \leq \theta}$ be the braiding matrix and consider the submatrix $\mathcal{Q}' = (q'_{ij})_{1\le i,j\le\theta-3}$, $q'_{ij} = q_{i^*j^*}$ and the subfamily $\boldsymbol{\lambda}' = (\lambda'_{iij})$, $\lambda'_{iij} = \lambda_{i^*i^*j^*}$. Consider the corresponding algebra $\widetilde{\mathcal{E}}_{\theta-3}(\boldsymbol{\lambda}')$, with generators $a_1, \dots, a_{\theta-3}$. We denote this algebra by B and rename the generators $w_{i+3} := a_i$, $1 \le i \le \theta - 3$. That is, B is the algebra generated by $w_4, \dots, w_\theta$ with relations defined by a subset of the relations in (5.1). Let A be the algebra $\widetilde{\mathcal{E}}_2(\lambda_{112}, \lambda_{122})$, with generators s_1, s_2. Let us also set $E = \widetilde{\mathcal{E}}_\theta/\langle y_3\rangle$. We will show that $E \ne 0$, hence $\widetilde{\mathcal{E}}_\theta \ne 0$. Let us denote by $y_1, y_2, y_4, \dots, y_\theta \in E$ the images of the corresponding generators of $\widetilde{\mathcal{E}}_\theta$.

Let $\alpha : A \to k^{m\times m}$ be the representation defined in 5.3 if $\lambda_{122} \ne 0$ or the representation in 5.4 if $\lambda_{122} = 0$, with $m = 1$ or $m = 3$, respectively. Set, accordingly, $M_1 = \alpha(s_1), M_2 = \alpha(s_2)$. Let us consider the vector space $W = B \otimes V$. We define $\varrho : E \to \operatorname{End}(W)$ given by

$$\varrho(y_j)(b\otimes v) = \begin{cases} g_j \cdot b \otimes M_j\, v, & j < 3 \\ w_j b \otimes v, & j > 3. \end{cases}$$

This is a well-defined representation. For instance:

$$\varrho(y_{112} - \lambda_{112})(w_k \otimes v) = (\chi_k(g_{112})\lambda_{112} - \lambda_{112}) w_k \otimes v = 0,$$
$$\varrho(y_1 y_j - q_{jk} y_j y_1)(w_k \otimes v) = (q_{1j} q_{jk} - q_{jk} q_{1j}) w_j w_k \otimes M_1\, v = 0, \quad j > 3,$$
$$\varrho(y_{jjl} - \lambda_{jjl})(w_k \otimes v) = w_{jjl} w_k \otimes v - \lambda_{jjl} w_k \otimes v = 0, \quad 3 < j = l - 1.$$

Hence the proposition follows. ■

We need to compute $\rho(y_{(kl)}^3) = \rho(y_{(kl)})^3$, $1 \le k < l \le \theta$. We have:

$$\rho(y_i)^3 = y_i^3 \otimes 1 + g_i^3 \otimes x_i^3, \quad i \in \mathbb{I} \tag{5.5}$$

as these elements are skew-primitives for the coaction. Now, for every k, l,

$$\rho(y_{(kl)}) = y_{(kl)} \otimes 1 + g_{(kl)} \otimes x_{(kl)} + (1 - \xi^2) \sum_{k\le p<l} y_{(kp)} g_{(p+1l)} \otimes x_{(p+1l)},$$

again as relation $y_{ij} = 0$, $i < j - 1$ holds in $\widetilde{\mathcal{A}}$.

Consider a family of indeterminate variables $\mathbf{t} = (\mathtt{t}_{iij})$ and let $R = \Bbbk[\mathbf{t}]$ be the polynomial ring on those variables. Let us denote by $\widetilde{\mathcal{H}}_R$ and $\widetilde{\mathcal{A}}_R$ the R-algebras defined by the same relations as $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{A}}$. Given a family $\boldsymbol{\lambda}$ as above, we consider the evaluation

$t_{iij} \mapsto \lambda_{iij}$. Thus, we have $\widetilde{\mathcal{A}}_R \otimes_R \Bbbk = \widetilde{\mathcal{A}}$, $\widetilde{\mathcal{H}}_R \otimes_R \Bbbk = \widetilde{\mathcal{H}}$. Let R^+ be the ideal generated by $\mathbf{t}$. By (5.1) we have:

$$y_{(kr)}y_{(ks)} = \chi_{(ks)}(g_{(kr)})y_{(ks)}y_{(kr)} + R^+\widetilde{\mathcal{A}}_R.$$

If $\lambda_{iij} = 0$ for every $1 \le i,j \le \theta$, then $\widetilde{\mathcal{A}} \simeq \widetilde{\mathcal{H}}$ and thus

$$\rho(y^3_{(k\,l)}) = y^3_{(k\,l)} \otimes 1 + g^3_{(k\,l)} \otimes x^3_{(k\,l)} + \sum_{k\le p<l} Cpy^3_{(k\,p)}g^3_{(p+1\,l)} \otimes x^3_{(p+1\,l)},$$

for B_p as in 1.7. Hence,

$$\rho(y^3_{(k\,l)}) = y^3_{(k\,l)} \otimes 1 + g^3_{(k\,l)} \otimes x^3_{(k\,l)} + \sum_{k\le p<l} C_p y^3_{(k\,p)}g^3_{(p+1\,l)} \otimes x^3_{(p+1\,l)} + R^+\widetilde{\mathcal{A}}_R \otimes \widetilde{\mathcal{H}}_R.$$

Hence, in the computation of $\rho(y^3_{(k\,l)})$ in the general case, we need to focus on the terms in which a scalar λ_{***} may appear. See Example 5.3 for $\theta = 2$. This example shows the philosophy behind our calculations. Also, it introduces the notation $\rightsquigarrow$ in 5.6 to keep only the terms with a factor λ_{***}.

Example 5.3. We will show that

$$\rho(y^3_{12}) = y^3_{12} \otimes 1 + g^3_{12} \otimes x^3_{12} + (1-\xi^2)^3\chi_1(g_2)^3 y^3_1 g^3_2 \otimes x^3_2.$$

Hence, we can take a section $\gamma : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{A}}$ such that $\gamma(x^3_{12}) = y^3_{12}$.

Let us compute $\rho(y_{12})^3$. Set

$$A = y_{12} \otimes 1, \qquad B = g_{12} \otimes x_{12}, \qquad C = y_1 g_2 \otimes x_2.$$

Hence $\rho(y_{12}) = A + B + (1-\xi^2)C$. As said, we need to focus on the terms in which a factor λ_{***} may appear. These are related with the fact that y_1 appears to the left of y_{12} and are precisely:

$$CAB, \quad CBA, \quad BCA, \quad CAA, \quad ACA, \quad CAC, \quad CCA.$$

Now, for instance

$$\begin{aligned} CAB &= y_1 g_2 y_{12} g_{12} \otimes x_2 x_{12} = \chi_{12}(g_2) y_1 y_{12} g_{122} \otimes x_2 x_{12} \\ &= \lambda_{112}\chi_{12}(g_2) g_{122} \otimes x_2 x_{12} + \chi_{12}(g_2)\chi_{12}(g_1) y_{12} y_1 g_{122} \otimes x_2 x_{12}. \end{aligned}$$

We only need to keep the term involving λ_{112}. Hence, we write

$$CAB \rightsquigarrow \lambda_{112}\chi_{12}(g_2)g_{122} \otimes x_2x_{12} = \lambda_{112}\xi^2 g_{122} \otimes x_2x_{12} \tag{5.6}$$

as $\lambda_{112}\chi_{12}(g_2) = \lambda_{112}\xi^2$. We will do this for every term. We need the following equalities:

$$\begin{aligned}
y_1y_{12} &= \lambda_{112} + \chi_{12}(g_1)y_{12}y_1 \rightsquigarrow \lambda_{112};\\
y_1y_{12}^2 &= \lambda_{112}(1+\xi^2)y_{12} + \chi_{12}(g_1)^2y_{12}^2y_1 \rightsquigarrow \lambda_{112}(1+\xi^2)y_{12};\\
y_{12}y_1y_{12} &= \lambda_{112}y_{12} + \chi_{12}(g_1)y_{12}^2y_1 \rightsquigarrow \lambda_{112}y_{12};\\
y_1y_{12}y_1 &= \lambda_{112}y_1 + \chi_{12}(g_1)y_{12}y_1^2 \rightsquigarrow \lambda_{112}y_1;\\
y_1^2y_{12} &= \lambda_{112}(1+\xi^2)y_1 + \chi_{12}(g_1)^2y_{12}y_1^2 \rightsquigarrow \lambda_{112}(1+\xi^2)y_1.
\end{aligned}$$

We have

$$CAB \rightsquigarrow \lambda_{112}\xi^2 g_{122} \otimes x_2x_{12}; \qquad CBA \rightsquigarrow \lambda_{112}g_{122} \otimes x_2x_{12};$$
$$BCA \rightsquigarrow \lambda_{112}\xi g_{122} \otimes x_2x_{12}.$$

Thus $CAB + CBA + BCA \rightsquigarrow 0$.

$$CAA \rightsquigarrow \lambda_{112}\xi(1+\xi^2)y_{12}g_2 \otimes x_2; \qquad ACA \rightsquigarrow \lambda_{112}\xi^2 y_{12}g_2 \otimes x_2.$$

Thus $CAA + ACA \rightsquigarrow 0$.

$$CAC \rightsquigarrow \lambda_{112}y_1g_2^2 \otimes x_2^2; \qquad CCA \rightsquigarrow \lambda_{112}\xi^2(1+\xi^2)y_1g_2^2 \otimes x_2^2.$$

Thus $CAC + CCA \rightsquigarrow 0$. Therefore,

$$\rho(y_{12})^3 = y_{12}^3 \otimes 1 + g_{12}^3 \otimes x_{12}^3 + (1-\xi^2)^3\chi_1(g_2)^3y_1^3g_2^3 \otimes x_2^3. \qquad \square$$

From now on we consider the case $\theta \geq 3$. We will collect some technical identities needed to compute $\rho(y_{(kl)})^3$ in a series of general lemmas.

Lemma 5.4. The following identities hold in $\widetilde{\mathcal{A}}$:

$$[y_{(1\,l)}, y_2]_c = \lambda_{122}(1-\xi^2)\chi_2(g_{(3\,l)})y_{(3\,l)}, \qquad l \geq 3. \qquad \square$$

Proof. Assume first $l = 3$. We have two cases, namely $\chi_{122} = \epsilon$ or not, (in which case it is possible to have $\chi_{223} = \epsilon$). We proceed as in [2, Lemma 1.11]. In the first case, we have the lemma. In the second, we get $[y_{(13)}, y_2]_c = \lambda_{223}(1 - \chi_{223}(g_1))x_1 = 0$, hence the lemma also holds (as $\lambda_{122} = 0$). For the general case, we get

$$\begin{aligned}
[y_{(1\,l)}, y_2]_c &= [[y_{(1\,3)}, y_{(4\,l)}]_c, y_2]_c \\
&= \chi_2(g_{(4\,l)})[y_{(1\,3)}, y_2]_c y_{(4\,l)} - \chi_{(4\,l)}(g_{(1\,3)})y_{(4\,l)}[y_{(1\,3)}, y_2]_c \\
&= \lambda_{122} q_{32}(1 - \xi^2)\chi_2(g_{(4\,l)})\Big(y_3 y_{(4\,l)} - \chi_{(4\,l)}(g_3)\chi_{(4\,l)}(g_{122})y_{(4\,l)}y_3\Big) \\
&= \lambda_{122}(1 - \xi^2)\chi_2(g_{(3\,l)})y_{(3\,l)},
\end{aligned}$$

as $\lambda_{122}\chi_{(5\,l)}(g_{122}) = \lambda_{122}$, $1 = \chi_2(g_{(4\,l)})\chi_{(4\,l)}(g_2)$ plus q-Jacobi (2.2). ■

Lemma 5.5. The following identities hold in $\widetilde{\mathcal{A}}$, for $1 \le p \le l$:

$$[y_p, y_{(1\,l)}]_c = \begin{cases} \lambda_{112}(1 - \xi^2)y_{(3\,l)}, & p = 1; \\ \lambda_{122}(1 - \xi)y_{(3\,l)}, & p = 2; \\ 0, & 2 < p < l; \\ (1 - \xi)y_l y_{(1\,l)}, & p = l. \end{cases}, \qquad l \ge 3. \qquad \square$$

Proof. Using that $\lambda_{112}\chi_{112} = \lambda_{112}\epsilon$,

$$\begin{aligned}
[y_1, y_{(1\,l)}]_c &= \lambda_{112}(1 - \chi_{(3\,l)}(g_{112}))y_{(3\,l)} \\
&= \lambda_{112}(1 - \chi_{(3\,l)}(g_{112})\chi_{112}(g_{(3\,l)}))y_{(3\,l)} = \lambda_{112}(1 - \xi^2)y_{(3\,l)}.
\end{aligned}$$

If $p = 2$, it follows by Lemma 5.4 that $[y_2, y_{(1\,l)}]_c = \lambda_{122}(1 - \xi)y_{(3\,l)}$.

Assume $2 < p < l$. Then

$$\begin{aligned}
[y_p, y_{(1\,l)}]_c &= [y_p, [y_{(1\,p-2)}, y_{(p-1\,l)}]_c]_c = -\chi_{(p-1\,l)}(g_{(1\,p-2)})[y_p, y_{(p-1\,l)}]_c y_{(1\,p-2)} \\
&\quad + \chi_{(1\,p-2)}(g_p)y_{(1\,p-2)}[y_p, y_{(p-1\,l)}]_c \\
&\overset{\text{case } p=2}{=} -\chi_{(p-1\,l)}(g_{(1\,p-2)})\lambda_{p-1pp}(1 - \xi)y_{(p+1\,l)}y_{(1\,p-2)} \\
&\quad + \chi_{(1\,p-2)}(g_p)\lambda_{p-1pp}(1 - \xi)y_{(1\,p-2)}y_{(p+1\,l)} \\
&= \lambda_{p-1pp}(1 - \xi)\chi_{(1\,p-2)}(g_p)\Big(y_{(1\,p-2)}y_{(p+1\,l)} \\
&\quad - \chi_{(p+1\,l)}(g_{(1\,p-2)})\chi_{p-1pp}(g_{(1\,p-2)})y_{(p+1\,l)}y_{(1\,p-2)}\Big) \\
&= \lambda_{p-1pp}(1 - \xi)\chi_{(1\,p-2)}(g_p)[y_{(1\,p-2)}, y_{(p+1\,l)}]_c = 0.
\end{aligned}$$

Finally, if $p = l$, using that $[y_l, y_{l-1\,l}]_c = (1-\xi)y_l y_{l-1\,l} - \lambda_{l-1ll}\xi^2$ and q-Jacobi (2.2) we arrive to

$$\begin{aligned}
[y_l, y_{(1\,l)}]_c &= -\chi_{l-1l}(g_{(1\,l-2)})\Big((1-\xi)y_l y_{l-1\,l} - \lambda_{l-1ll}\xi^2\Big)y_{(1\,l-2)} \\
&\quad + \chi_{(1\,l-2)}(g_l)y_{(1\,l-2)}\Big((1-\xi)y_l y_{l-1\,l} - \lambda_{l-1ll}\xi^2\Big) \\
&= (1-\xi)y_l y_{(1\,l)} \\
&\quad + \lambda_{l-1ll}\xi^2\chi_{l-1l}(g_{(1\,l-2)})(1 - \chi_{(1\,l-2)}(g_l)\chi_l(g_{(1\,l-2)}))y_{(1\,l-2)},
\end{aligned}$$

and the lemma follows using that $\lambda_{l-1ll}\chi_{l-1l}(g_{(1\,l-2)})^{-1} = \lambda_{l-1ll}\chi_l(g_{(1\,l-2)})$ and $\chi_l(g_{(1\,l-2)})\chi_{(1\,l-2)}(g_l) = 1$. ■

Remark 5.6. If $l = 2$, then $[y_1, y_{(1\,l)}]_c = \lambda_{112}$. □

Remark 5.7. If $2 < p \leq l$, then

$$[y_{(1\,l)}, y_p]_c = 0.$$

Indeed, if $p = l$,

$$\begin{aligned}
[y_{(1\,l)}, y_l]_c &= (1 - \chi_l(g_{(1\,l)})\chi_{(1\,l)}(g_l))y_{(1\,l)}y_l - \chi_l(g_{(1\,l)})[y_l, y_{(1\,l)}]_c \\
&= (1-\xi)y_{(1\,l)}y_l - \chi_l(g_{(1\,l)})(1-\xi)y_l y_{(1\,l)} \\
&= (1-\xi)[y_{(1\,l)}, y_l]_c.
\end{aligned}$$

If $1 < p < l$ this follows from

$$[y_{(1\,l)}, y_p]_c = (1 - \chi_p(g_{(1\,l)})\chi_{(1\,l)}(g_p))y_{(1\,l)}y_p - \chi_p(g_{(1\,l)})[y_p, y_{(1\,l)}]_c = 0.$$ □

Remark 5.8. Let $1 \leq p \leq l-2$. Then,

$$[y_{(1\,p+1)}, y_{(1\,l)}]_c = \chi_{(1\,l)}(g_{p+1})[[y_{(1\,p)}, y_{(1\,l)}]_c, y_{p+1}]_c. \tag{5.7}$$

Hence,

$$[y_{(1\,p)}, y_{(1\,l)}]_c = \chi_{(1\,l)}(g_{(2\,p)})[\dots[y_1, y_{(1\,l)}]_c, y_2]_c, \dots, y_{p-1}]_c, y_p]_c.$$

Indeed, using q-Jacobi (2.2) we see that 5.7 holds:

$$[y_{(1\,p+1)}, y_{(1\,l)}]_c = [y_{(1\,p)}, [y_{p+1}, y_{(1\,l)}]_c]_c + \chi_{(1\,l)}(g_{p+1})[y_{(1\,p)}, y_{(1\,l)}]_c y_{p+1}$$

$$-\chi_{p+1}(g_{(1\,p)})y_{p+1}[y_{(1\,p)}, y_{(1\,l)}]_c = \chi_{(1\,l)}(g_{p+1})[[y_{(1\,p)}, y_{(1\,l)}]_c, y_{p+1}]_c,$$

as $[y_{(1\,p)}, [y_{p+1}, y_{(1\,l)}]_c]_c = 0$. This last equality is clear if $p \geq 2$ by Lemma 5.5 that also yields $[y_1, [y_2, y_{(1\,l)}]_c]_c = \lambda_{122}(1-\xi)(y_1 y_{(3\,l)} - \chi_{122}(g_1)\chi_{(3\,l)}(g_1)) = \lambda_{122}(1-\xi)[y_1, y_{(3\,l)}]_c = 0$. □

Lemma 5.9. The following identities hold in $\widetilde{\mathcal{A}}$.

(1) $[y_{(1\,l)}, y_{(k\,p)}]_c = 0$, for $3 \leq k \leq p \leq l$.
(2) $[y_{(1\,p)}, y_{(3\,l)}]_c = \chi_{(3\,p)}(g_{(1\,p)})(1-\xi^2)y_{(3\,p)}y_{(1\,l)}$, for $3 \leq p < l$. □

Proof. (1) Fix k. Recall that $[y_{(1\,l)}, y_j]_c = 0$, for $3 \leq j \leq l$, by Remark 5.7. In particular, $[y_{(1\,l)}, y_k]_c = 0$. Now, using induction on p and q-Jacobi (2.2),

$$[y_{(1\,l)}, y_{(k\,p)}]_c = -\chi_p(g_{(k\,p-1)})[y_{(1\,l)}, y_p]_c y_{(k\,p-1)} + \chi_{(k\,p-1)}(g_{(1\,l)})y_{(k\,p-1)}[y_{(1\,l)}, y_p]_c = 0.$$

(2) We have, using q-Jacobi (2.2) and Item (1) for $k = 3$:

$$\begin{aligned}[y_{(1\,p)}, y_{(3\,l)}]_c &= [y_{(1\,p)}, [y_{(3\,p)}, y_{(p+1\,l)}]_c]_c \\ &= -\chi_{(p+1\,l)}(g_{(3\,p)})y_{(1\,l)}y_{(3\,p)} + \chi_{(3\,p)}(g_{(1\,p)})y_{(3\,p)}y_{(1\,l)} \\ &= \chi_{(3\,p)}(g_{(1\,p)})(1 - \chi_{(p+1\,l)}(g_{(3\,p)})\chi_{(3\,p)}(g_{(p+1\,l)}))y_{(3\,p)}y_{(1\,l)}\end{aligned}$$

and (2) follows as $\chi_{(p+1\,l)}(g_{(3\,p)})\chi_{(3\,p)}(g_{(p+1\,l)}) = \xi^2$. ■

Lemma 5.10. The following identities hold in $\widetilde{\mathcal{A}}$:

$$[y_{(1\,l)}, y_{(2\,l)}]_c = -3\lambda_{122}\chi_2(g_{(1\,l)})y_{(3\,l)}^2, \qquad l \geq 3.$$ □

Proof. Follows using Lemma 5.4 combined with Lemma 5.9 (1):

$$\begin{aligned}[y_{(1\,l)}, y_{(2\,l)}]_c &= [y_{(1\,l)}, [y_2, y_{(3\,l)}]_c]_c = [[y_{(1\,l)}, y_2]_c, y_{(3\,l)}]_c \\ &= \lambda_{122}(1-\xi^2)\chi_2(g_{(3\,l)})(1 - \chi_{(3\,l)}(g_{122}g_{(3\,l)}))y_{(3\,l)}^2 \\ &= \lambda_{122}(1-\xi^2)\chi_2(g_{(3\,l)})(1-\xi^2)y_{(3\,l)}^2 = -3\lambda_{122}\xi^2\chi_2(g_{(3\,l)})y_{(3\,l)}^2,\end{aligned}$$

and thus the lemma follows as $\lambda_{122}\chi_2(g_{(1\,l)}) = \lambda_{122}\xi^2\chi_2(g_{(3\,l)})$. ■

Lemma 5.11. The following identities hold in $\widetilde{\mathcal{A}}$:

(1) $[y_1, y_{(1l)}]_c = \lambda_{112}(1-\xi^2)y_{(3l)}$, $l \geq 3$.

(2) $[y_{(12)}, y_{(1l)}]_c = -3\xi^2\lambda_{112}\chi_{(1l)}(g_2)y_{(3l)}y_2 + \lambda_{112}(1-\xi)y_{(2l)}$, $l \geq 3$.

(3) For $3 \leq p < l$:

$$[y_{(1p)}, y_{(1l)}]_c = -3\xi^2\lambda_{112}\chi_{(1l)}(g_{(2p)})y_{(3l)}y_{(2p)} + 3\lambda_{112}\chi_1(g_{(3p)})y_{(3p)}y_{(2l)}. \qquad \square$$

Proof. (1) is Lemma 5.5 for $p = 1$. For (2) we use 5.7 and (1) to get

$$[y_{12}, y_{(1l)}]_c = \chi_{(1l)}(g_2)[[y_1, y_{(1l)}]_c, y_2]_c = \lambda_{112}(1-\xi^2)\chi_{(1l)}(g_2)[y_{(3l)}, y_2]_c.$$

Now $[y_{(3l)}, y_2]_c = (1-\xi^2)y_{(3l)}y_2 - \chi_2(g_{(3l)})y_{(2l)}$ and (2) follows using the equality $\lambda_{112}\chi_2(g_{(3l)})\chi_{(1l)}(g_2) = \lambda_{112}\xi$.

For (3), we use q-Jacobi (2.2) and Lemma 5.9 to get

$$\begin{aligned}
[y_{(1p)}, y_{(1l)}]_c &= \chi_{(1l)}(g_{(3p)})[y_{12}, y_{(1l)}]_c y_{(3p)} - \chi_{(3p)}(g_{12})y_{(3p)}[y_{12}, y_{(1l)}]_c \\
&= -3\xi^2\lambda_{112}\chi_{(1l)}(g_2)\Big(\chi_{(1l)}(g_{(3p)})y_{(3l)}y_2y_{(3p)} - \chi_{(3p)}(g_{12})y_{(3p)}y_{(3l)}y_2\Big) \\
&\quad + \lambda_{112}(1-\xi)\Big(\chi_{(1l)}(g_{(3p)})y_{(2l)}y_{(3p)} - \chi_{(3p)}(g_{12})y_{(3p)}y_{(2l)}\Big) \\
&= -3\xi^2\lambda_{112}\chi_{(1l)}(g_2)\chi_{(1l)}(g_{(3p)})y_{(3l)}y_{(2p)} \\
&\quad + 3\xi^2\lambda_{112}\chi_{(1l)}(g_2)\chi_{(3p)}(g_{12})[y_{(3p)}, y_{(3l)}]_c y_2 \\
&\quad + \lambda_{112}(1-\xi)\chi_{(1l)}(g_{(3p)})[y_{(2l)}, y_{(3p)}]_c \\
&\quad + \lambda_{112}(1-\xi)\chi_{(1l)}(g_{(3p)})\chi_{(3p)}(g_{(2l)})(1-\xi^2)y_{(3p)}y_{(2l)}.
\end{aligned}$$

We use this equality and Lemma 5.1 to deduce $\lambda_{112}[y_{(3p)}, y_{(3l)}]_c = 0$. We use this fact together with Lemma 5.9 (1) and Lemma 5.4 to get

$$\begin{aligned}
[y_{(1l)}, y_{(2p)}]_c &= [[y_{(1l)}, y_2]_c, y_{(3p)}]_c \\
&= \lambda_{122}(1-\xi^2)\chi_2(g_{(3l)})(1-\chi_{(3p)}(g_{122}g_{(3l)})\chi_{(3l)}(g_{(3p)}))y_{(3l)}y_{(3p)} \\
&\quad - \lambda_{122}(1-\xi^2)\chi_2(g_{(3l)})\chi_{(3p)}(g_{(3l)})\xi^2[y_{(3p)}, y_{(3l)}]_c \\
&= -3\xi^2\lambda_{122}\chi_2(g_{(3l)})y_{(3l)}y_{(3p)}.
\end{aligned} \tag{5.8}$$

In particular, $\lambda_{112}[y_{(2l)}, y_{(3p)}]_c = 0$ by Lemma 5.1. Hence (3) follows. ∎

Remark 5.12. We have, for $2 \le p \le l$:

$$[y_{(1\,l)}, y_{(2\,p)}]_c = \begin{cases} \lambda_{122}(1-\xi^2)\chi_2(g_{(3\,l)})y_{(3\,l)}, & p=2 \\ \lambda_{122}(1-\xi^2)^2\chi_2(g_{(3\,l)})y_{(3\,l)}y_{(3\,p)}, & p>2. \end{cases}$$

For $p = 2$ this is Lemma 5.4. For $p > 2$ this is 5.8. □

Proposition 5.13. For C_p as in 1.7 we have

$$\rho(y_{(k\,l)})^3 = y_{(k\,l)}^3 \otimes 1 + g_{(k\,l)}^3 \otimes x_{(k\,l)}^3 + \sum_{k\le p<l} C_p y_{(k\,p)}^3 g_{(p+1\,l)}^3 \otimes x_{(p+1\,l)}^3. \qquad \square$$

Proof. Let us set $k = 1 < l$ to simplify the notation. We may assume $l \ge 3$ as case $l = 1$ is 5.5 and case $l = 2$ is Example 5.3. Set

$$A = y_{(1\,l)} \otimes 1, \qquad B = g_{(1\,l)} \otimes x_{(1\,l)}, \qquad X_p = y_{(1\,p)}g_{(p+1\,l)} \otimes x_{(p+1\,l)},\, 1 \le p < l.$$

In what respects to commutation rules, we may set, without lack of rigour, $X_l := A$, as with the convention $g_{(l+1\,l)} = x_{(l+1\,l)} = 1$ it becomes

$$X_l = y_{(1\,l)}g_{(l+1\,l)} \otimes x_{(l+1\,l)} = y_{(1\,l)} \otimes 1.$$

As in Example 5.3, we need to focus on the terms of $(A + B + (1-\xi^2)\sum X_p)^3$ involving a factor λ_{***}. These are divided into three big groups, namely:

- (G1) For every pair $p < q$, terms XYZ involving $X, Y, Z \in \{B, X_p, X_q\}$, all different, X_p to the left of X_q.
- (G2) For every pair $p < q$, terms XYZ involving $X, Y, Z \in \{X_p, X_q\}$, not all equal and with a factor X_p to the left of X_q.
- (G3) For every triple $p < q < r$, terms XYZ from *distinct* $X, Y, Z \in \{X_p, X_q, X_r\}$ and with X_p to the left of X_q or X_r or with X_q to the left of X_r.

Since our aim is to show that there is no term involving a factor λ_{***}, we may further restrict these groups, as the other resulting combinations provide equivalent terms. For instance, we have

$$\begin{aligned} X_pX_l &= y_{(1\,p)}g_{(p+1\,l)}y_{(1\,l)}g_{(l+1\,l)} \otimes x_{(p+1\,l)}x_{(l+1\,l)} \\ &= y_{(1\,p)}g_{(p+1\,l)}y_{(1\,l)} \otimes x_{(p+1\,l)} \end{aligned}$$

$$= \chi_{(1\,l)}(g_{(p+1\,l)})y_{(1\,p)}y_{(1\,l)}g_{(p+1\,l)} \otimes x_{(p+1\,l)}.$$

$$X_pX_q = y_{(1\,p)}g_{(p+1\,l)}y_{(1\,q)}g_{(q+1\,l)} \otimes x_{(p+1\,l)}x_{(q+1\,l)}$$

$$= \chi_{(1\,q)}(g_{(p+1\,l)})\chi_{(q+1\,l)}(g_{(p+1\,l)})y_{(1\,p)}y_{(1\,l)}g_{(p+1\,l)}g_{(q+1\,l)} \otimes x_{(q+1\,l)}x_{(p+1\,l)}$$

$$= \chi_{(1\,l)}(g_{(p+1\,l)})y_{(1\,p)}y_{(1\,l)}g_{(p+1\,l)}g_{(q+1\,l)} \otimes x_{(q+1\,l)}x_{(p+1\,l)}.$$

Hence, we restrict to the following subgroups:

(G1′) For every $p < l$, terms XYZ involving $X, Y, Z \in \{B, X_p, A\}$, all different, X_p to the left of A.

(G2′) For every $p < l$, terms XYZ involving $X, Y, Z \in \{X_p, A\}$, not all equal and with a factor X_p to the left of A.

(G3′) For every pair $p < q < l$, terms XYZ arising from *distinct* $X, Y, Z \in \{X_p, X_q, A\}$ and with X_p to the left of X_q or A or with X_q to the left of A.

We start with group (G1′): notice that, for any p:

$$X_pAB + X_pBA + BX_pA$$
$$= (1+\xi+\xi^2)\chi_{(1\,l)}(g_{p+1\,l})y_{(1\,p)}y_{(1\,l)}g_{(1\,l)}g_{(p+1\,l)} \otimes x_{(p+1\,l)}x_{(1\,l)} = 0.$$

We now proceed to group (G2′), that is . terms of the form X_pAX_p, X_pX_pA and X_pAA, AX_pA. We further divide this group into

(G2′.1) factors arising from $\{A, X_1\}$,

(G2′.2) factors arising from $\{A, X_2\}$, and

(G2′.3) factors arising from $\{A, X_p\}$, $p \geq 3$.

The computations for item (G2′.1) are analogous to the ones in Example 5.3, and we get that the factor involving λ_{***} is zero. For (G2′.2), we need the following computations:

$$y_{12}y_{(1\,l)}^2 \rightsquigarrow -3\lambda_{112}(1+\xi)\chi_{(3\,l)}(g_2)y_{(3\,l)}y_2y_{(1\,l)} + \lambda_{112}(\xi^2-\xi)y_{(2\,l)}y_{(1\,l)};$$

$$y_{(1\,l)}y_{12}y_{(1\,l)} \rightsquigarrow -3\lambda_{112}\chi_{(3\,l)}(g_{12})y_{(3\,l)}y_2y_{(1\,l)} + \lambda_{112}(\xi-\xi^2)\chi_{(2\,l)}(g_1)y_{(2\,l)}y_{(1\,l)};$$

$$y_{12}y_{(1\,l)}y_{12} \rightsquigarrow -3\xi^2\lambda_{112}\chi_{(1\,l)}(g_2)y_{(3\,l)}y_2y_{12} + \lambda_{112}(1-\xi)y_{(2\,l)}y_{12};$$

$$y_{12}^2y_{(1\,l)} \rightsquigarrow 3\lambda_{112}\xi\,\chi_{(2\,l)}(g_{122})y_{(3\,l)}y_2y_{12} + \lambda_{112}\chi_{(2\,l)}(g_{12})(\xi^2-\xi)y_{(2\,l)}y_{12}.$$

We have $X_2AA + AX_2A \rightsquigarrow 0$:

$$\begin{aligned} X_2AA &= \chi_{(1\,l)}(g_{(3\,l)})^2 y_{12} y_{(1\,l)}^2 g_{(3\,l)} \otimes x_{(3\,l)} \\ &\rightsquigarrow \lambda_{112}\Big(3y_{(3\,l)}y_2 y_{(1\,l)} g_{(3\,l)} - (1-\xi)\chi_2(g_{(3\,l)}) y_{(2\,l)} y_{(1\,l)} g_{(3\,l)}\Big) \otimes x_{(3\,l)}; \\ AX_2A &= \chi_{(1\,l)}(g_{(3\,l)}) y_{(1\,l)} y_{12} y_{(1\,l)} g_{(3\,l)} \otimes x_{(3\,l)} \\ &\rightsquigarrow \lambda_{112}\Big(-3y_{(3\,l)}y_2 y_{(1\,l)} g_{(3\,l)} + (1-\xi)\chi_2(g_{(3\,l)}) y_{(2\,l)} y_{(1\,l)} g_{(3\,l)}\Big) \otimes x_{(3\,l)}. \end{aligned}$$

Also, $X_2AX_2 + X_2X_2A \rightsquigarrow 0$:

$$\begin{aligned} X_2AX_2 &= \chi_{12}(g_{(3\,l)})\chi_{(1\,l)}(g_{(3\,l)}) y_{12} y_{(1\,l)} y_{12} g_{(3\,l)}^2 \otimes x_{(3\,l)}^2 \\ &\rightsquigarrow \lambda_{112}\Big(-3\xi y_{(3\,l)} y_2 y_{12} g_{(3\,l)}^2 + (\xi-\xi^2)\chi_2(g_{(3\,l)}) y_{(2\,l)} y_{12} g_{(3\,l)}^2\Big) \otimes x_{(3\,l)}^2; \\ X_2X_2A &= \chi_{12}(g_{(3\,l)})\chi_{(1\,l)}(g_{(3\,l)})^2 y_{12}^2 y_{(1\,l)} g_{(3\,l)}^2 \otimes x_{(3\,l)}^2 \\ &\rightsquigarrow \lambda_{112}\Big(3\xi y_{(3\,l)} y_2 y_{12} g_{(3\,l)}^2 + \chi_2(g_{(3\,l)})(\xi^2-\xi) y_{(2\,l)} y_{12} g_{(3\,l)}^2\Big) \otimes x_{(3\,l)}^2. \end{aligned}$$

We move on to (G2′.3). We need

$$\begin{aligned} y_{(1\,p)} y_{(1\,l)}^2 &\rightsquigarrow \chi_{(1\,l)}(g_{(1\,p)}) y_{(1\,l)} y_{(1\,p)} y_{(1\,l)} - 3\xi^2 \lambda_{112} \chi_{(1\,l)}(g_{(2\,p)}) y_{(3\,l)} y_{(2\,p)} y_{(1\,l)} \\ &\quad + 3\lambda_{112}(1-\xi^2)\chi_{(1\,l)}(g_{(2\,p)})\chi_{(3\,p)}(g_2) y_{(3\,l)} y_{(3\,p)} y_2 y_{(1\,l)}. \\ y_{(1\,l)} y_{(1\,p)} y_{(1\,l)} &\rightsquigarrow -3\xi^2 \lambda_{112} \chi_{(3\,l)}(g_{(1\,l)}) y_{(3\,l)} y_{(2\,p)} y_{(1\,l)} \\ &\quad + 3\lambda_{112}(1-\xi^2)\chi_{(3\,p)}(g_2)\chi_{(3\,l)}(g_{(1\,l)}) y_{(3\,l)} y_{(3\,p)} y_2 y_{(1\,l)}. \\ y_{(1\,p)} y_{(1\,l)} y_{(1\,p)} &\rightsquigarrow -3\xi^2 \lambda_{112} \chi_{(1\,l)}(g_{(2\,p)}) y_{(3\,l)} y_{(2\,p)} y_{(1\,p)} \\ &\quad + 3\lambda_{112}(1-\xi^2)\chi_{(1\,l)}(g_{(2\,p)})\chi_{(3\,p)}(g_2) y_{(3\,l)} y_{(3\,p)} y_2 y_{(1\,p)}. \\ y_{(1\,p)}^2 y_{(1\,l)} &\rightsquigarrow \chi_{(1\,l)}(g_{(1\,p)}) y_{(1\,p)} y_{(1\,l)} y_{(1\,p)} \\ &\quad - 3\xi^2 \lambda_{112} \chi_{(1\,l)}(g_{(2\,p)})\chi_{(3\,l)}(g_{(1\,p)})\chi_{(2\,p)}(g_{(1\,p)}) y_{(3\,l)} y_{(2\,p)} y_{(1\,p)} \\ &\quad - 3\lambda_{112}(1-\xi)\chi_{(1\,l)}(g_{(2\,p)})\chi_{(3\,l)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(1\,p)}) y_{(3\,l)} y_{(3\,p)} y_2 y_{(1\,p)}. \end{aligned}$$

Now, $X_pAA + AX_pA$ equals

$$\chi_{(1\,l)}(g_{(p+1\,l)})\Big(\chi_{(1\,l)}(g_{(p+1\,l)}) y_{(1\,p)} y_{(1\,l)}^2 + y_{(1\,l)} y_{(1\,p)} y_{(1\,l)}\Big) g_{(p+1\,l)} \otimes x_{(p+1\,l)}$$

and the factor between brackets, according to the equalities above is

$$\begin{aligned}
&\rightsquigarrow (1+\xi)y_{(1\,l)}y_{(1\,p)}y_{(1\,l)} - 3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,l)})y_{(3\,l)}y_{(2\,p)}y_{(1\,l)}\\
&\quad + 3\lambda_{112}(1-\xi^2)\chi_{(1\,l)}(g_{(2\,l)})\chi_{(3\,p)}(g_2)y_{(3\,l)}y_{(3\,p)}y_2y_{(1\,l)}\\
&\rightsquigarrow -3(1+\xi^2)\lambda_{112}\chi_{(3\,l)}(g_{(1\,l)})y_{(3\,l)}y_{(2\,p)}y_{(1\,l)}\\
&\quad + 3\lambda_{112}(\xi-\xi^2)\chi_{(3\,p)}(g_2)\chi_{(3\,l)}(g_{(1\,l)})y_{(3\,l)}y_{(3\,p)}y_2y_{(1\,l)}\\
&\quad - 3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,l)})y_{(3\,l)}y_{(2\,p)}y_{(1\,l)}\\
&\quad + 3\lambda_{112}(1-\xi^2)\chi_{(1\,l)}(g_{(2\,l)})\chi_{(3\,p)}(g_2)y_{(3\,l)}y_{(3\,p)}y_2y_{(1\,l)}\\
&= -3\lambda_{112}\chi_{(1\,l)}(g_{(2\,l)})(\xi^2+(1+\xi^2)\chi_{(3\,l)}(g_1)\chi_1(g_{(2\,l)}))y_{(3\,l)}y_{(2\,p)}y_{(1\,l)}\\
&\quad + 3\lambda_{112}\chi_{(1\,l)}(g_{(2\,l)})\chi_{(3\,p)}(g_2)(1-\xi^2+(\xi-\xi^2)\chi_{(3\,l)}(g_1)\chi_1(g_{(2\,l)}))\\
&\quad \times\ y_{(3\,l)}y_{(3\,p)}y_2y_{(1\,l)}\\
&= -3\lambda_{112}\chi_{(1\,l)}(g_{(2\,l)})(1+\xi+\xi^2)y_{(3\,l)}y_{(2\,p)}y_{(1\,l)}\\
&\quad + 3\lambda_{112}\chi_{(1\,l)}(g_{(2\,l)})\chi_{(3\,p)}(g_2)(1-\xi^2+(\xi-\xi^2)\xi)y_{(3\,l)}y_{(3\,p)}y_2y_{(1\,l)} = 0.
\end{aligned}$$

Here we use that $\lambda_{112}\chi_{12}^{-1} = \lambda_{112}\chi_1$ and $\lambda_{112}\chi_1(g_2) = \xi$. Analogously, if $\alpha = \chi_{(1\,p)}(g_{(p+1\,l)})\chi_{(1\,l)}(g_{(p+1\,l)})$, then $X_pAX_p + X_pX_pA$ is

$$\alpha\Big(\chi_{(1\,l)}(g_{(p+1\,l)})y_{(1\,p)}^2y_{(1\,l)} + y_{(1\,p)}y_{(1\,l)}y_{(1\,p)}\Big)g_{(p+1\,l)}^2\otimes x_{(p+1\,l)}^2$$

and the factor between brackets is now

$$\begin{aligned}
&\rightsquigarrow -3(1+\xi^2)\lambda_{112}\chi_{(1\,l)}(g_{(2\,p)})y_{(3\,l)}y_{(2\,p)}y_{(1\,p)}\\
&\quad + 3\lambda_{112}(\xi-\xi^2)\chi_{(1\,l)}(g_{(2\,p)})\chi_{(3\,p)}(g_2)y_{(3\,l)}y_{(3\,p)}y_2y_{(1\,p)}\\
&\quad - 3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,l)})\chi_{(3\,l)}(g_{(1\,p)})\chi_{(2\,p)}(g_{(1\,p)})y_{(3\,l)}y_{(2\,p)}y_{(1\,p)}\\
&\quad - 3\lambda_{112}(1-\xi)\chi_{(1\,l)}(g_{(2\,l)})\chi_{(3\,l)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(1\,p)})y_{(3\,l)}y_{(3\,p)}y_2y_{(1\,p)}\\
&= -3\lambda_{112}\chi_{(1\,l)}(g_{(2\,p)})(1+\xi+\xi^2)y_{(3\,l)}y_{(2\,p)}y_{(1\,p)}\\
&\quad + 3\lambda_{112}\chi_{(1\,l)}(g_{(2\,p)})\chi_{(3\,p)}(g_2)(1-\xi)(\xi-\xi^4)y_{(3\,l)}y_{(3\,p)}y_2y_{(1\,p)} = 0.
\end{aligned}$$

We are left with group (G3′). Again, we subdivide it

(G3′.1) Case $p = 1, q = 2$;
(G3′.2) Case $p = 1, q \geq 3$;

(G3′.3) Case $p = 2$, $q \geq 3$; and

(G3′.4) Case $p \geq 3$.

For (G3′.1), we have

$$X_1X_2X_l + X_1X_lX_2 + X_lX_1X_2 + X_2X_lX_1 + X_2X_1X_l = \mathbf{Y}_1 g_{(2\,l)} g_{(3\,l)} \otimes x_{(3\,l)} x_{(2\,l)},$$

$$\begin{aligned}
\mathbf{Y}_1 =\ & \chi_{(1\,l)}(g_{(2\,l)}g_{(3\,l)})\chi_{12}(g_{(2\,l)})\chi_{(3\,l)}(g_{(2\,l)})y_1y_{12}y_{(1\,l)} \\
& + \chi_{(1\,l)}(g_{(2\,l)})\chi_{12}(g_{(2\,l)})\chi_{(3\,l)}(g_{(2\,l)})y_1y_{(1\,l)}y_{12} \\
& + \chi_{12}(g_{(2\,l)})\chi_{(3\,l)}(g_{(2\,l)})y_{(1\,l)}y_1y_{12} + \chi_{(1\,l)}(g_{(3\,l)})\chi_1(g_{(3\,l)})y_{12}y_{(1\,l)}y_1 \\
& + \chi_{(1\,l)}(g_{(2\,l)}g_{(3\,l)})\chi_1(g_{(3\,l)})y_{12}y_1y_{(1\,l)} \\
=\ & \xi^2\chi_1(g_{(3\,l)})\chi_{112}(g_{(2\,l)})y_1y_{12}y_{(1\,l)} + \xi^2\chi_{112}(g_{(2\,l)})\chi_{(3\,l)}(g_2)y_1y_{(1\,l)}y_{12} \\
& + \xi\chi_1(g_{(2\,l)})y_{(1\,l)}y_1y_{12} + \xi\chi_{112}(g_{(3\,l)})y_{12}y_{(1\,l)}y_1 \\
& + \xi^2\chi_1(g_{(2\,l)})\chi_{112}(g_{(3\,l)})y_{12}y_1y_{(1\,l)}.
\end{aligned}$$

Hence, we need

$$\begin{aligned}
& y_1y_{12}y_{(1\,l)} \rightsquigarrow \chi_{12}(g_1)y_{12}y_1y_{(1\,l)} + \lambda_{112}y_{(1\,l)}; \\
& y_1y_{(1\,l)}y_{12} \rightsquigarrow \lambda_{112}\chi_{(1\,l)}(g_1)y_{(1\,l)} + \lambda_{112}(1-\xi^2)y_{(3\,l)}y_{12}; \\
& y_{(1\,l)}y_1y_{12} \rightsquigarrow \lambda_{112}y_{(1\,l)}; \\
& y_{12}y_{(1\,l)}y_1 \rightsquigarrow -3\xi^2\lambda_{112}\chi_{(1\,l)}(g_2)y_{(3\,l)}y_2y_1 + \lambda_{112}(1-\xi)y_{(2\,l)}y_1; \\
& y_{12}y_1y_{(1\,l)} \rightsquigarrow -3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{12})y_{(3\,l)}y_2y_1 + \lambda_{112}(1-\xi^2)y_{(1\,l)} \\
& \qquad + \lambda_{112}(1-\xi)\chi_{(1\,l)}(g_1)y_{(2\,l)}y_1 + \lambda_{112}(1-\xi^2)\chi_{(3\,l)}(g_{12})y_{(3\,l)}y_{12}.
\end{aligned}$$

Using the above identities,

$$\begin{aligned}
\mathbf{Y}_1 \rightsquigarrow\ & \Big(\xi^2\chi_1(g_{(3\,l)})\chi_{112}(g_{(2\,l)})\chi_{12}(g_1) + \xi^2\chi_1(g_{(2\,l)})\chi_{112}(g_{(3\,l)})\Big)y_{12}y_1y_{(1\,l)} \\
& + \lambda_{112}(\xi^2-\xi)\chi_1(g_{(3\,l)})y_{(1\,l)} + \lambda_{112}(\xi^2-\xi)\chi_{(3\,l)}(g_2)y_{(3\,l)}y_{12} \\
& - 3\xi^2\lambda_{112}\chi_{(3\,l)}(g_2)y_{(3\,l)}y_2y_1 + \lambda_{112}(\xi-\xi^2)y_{(2\,l)}y_1 \\
\rightsquigarrow\ & -3(1+\xi)\lambda_{112}\chi_{(3\,l)}(g_2)y_{(3\,l)}y_2y_1 + \lambda_{112}(\xi-\xi^2)\chi_1(g_{(3\,l)})y_{(1\,l)} \\
& + \lambda_{112}(\xi^2-\xi)y_{(2\,l)}y_1 + \lambda_{112}(\xi-\xi^2)\chi_{(3\,l)}(g_2)y_{(3\,l)}y_{12} \\
& + \lambda_{112}(\xi^2-\xi)\chi_1(g_{(3\,l)})y_{(1\,l)} + \lambda_{112}(\xi^2-\xi)\chi_{(3\,l)}(g_2)y_{(3\,l)}y_{12}
\end{aligned}$$

$$
\begin{aligned}
&- 3\xi^2\lambda_{112}\chi_{(3\,l)}(g_2)y_{(3\,l)}y_2y_1 + \lambda_{112}(\xi - \xi^2)y_{(2\,l)}y_1 \\
=& -3(1+\xi+\xi^2)\lambda_{112}\chi_{(3\,l)}(g_2)y_{(3\,l)}y_2y_1 \\
&+ \lambda_{112}((\xi^2-\xi)+(\xi-\xi^2))\chi_{(3\,l)}(g_2)y_{(3\,l)}y_{12} \\
&+ \lambda_{112}((\xi^2-\xi)+(\xi-\xi^2))y_{(2\,l)}y_1 \\
&+ \lambda_{112}\chi_1(g_{(3\,l)})((\xi-\xi^2)+(\xi^2-\xi))y_{(1\,l)} = 0.
\end{aligned}
$$

Now we turn to (G3′.2): we have, for $q \geq 3$,

$$
X_1X_qX_l + X_1X_lX_q + X_lX_1X_q + X_qX_lX_1 + X_qX_1X_l = \mathbf{Y}_2 g_{(2\,l)}g_{(q+1\,l)} \otimes x_{(q+1\,l)}x_{(2\,l)},
$$

for

$$
\begin{aligned}
\mathbf{Y}_2 = &\ \xi^2\chi_1(g_{(2\,l)})^2\chi_{(1\,l)}(g_{(q+1\,l)})y_1y_{(1\,q)}y_{(1\,l)} + \xi^2\chi_1(g_{(2\,l)})^2y_1y_{(1\,l)}y_{(1\,q)} \\
&+ \xi\chi_1(g_{(2\,l)})y_{(1\,l)}y_1y_{(1\,q)} + \chi_{112}(g_{(q+1\,l)})\chi_{(3\,l)}(g_{(q+1\,l)})y_{(1\,q)}y_{(1\,l)}y_1 \\
&+ \xi\chi_1(g_{(2\,l)})\chi_{112}(g_{(q+1\,l)})\chi_{(3\,l)}(g_{(q+1\,l)})y_{(1\,q)}y_1y_{(1\,l)}
\end{aligned}
$$

and thus we need

$$
\begin{aligned}
y_1y_{(1\,q)}y_{(1\,l)} &\rightsquigarrow \chi_{(1\,q)}(g_1)y_{(1\,q)}y_1y_{(1\,l)} + \lambda_{112}(1-\xi^2)y_{(3\,q)}y_{(1\,l)}; \\
y_1y_{(1\,l)}y_{(1\,q)} &\rightsquigarrow \lambda_{112}(1-\xi^2)\chi_{(1\,l)}(g_1)\chi_{(3\,q)}(g_{(1\,l)})y_{(3\,q)}y_{(1\,l)} \\
&\qquad + \lambda_{112}(1-\xi^2)y_{(3\,l)}y_{(1\,q)}; \\
y_{(1\,l)}y_1y_{(1\,q)} &\rightsquigarrow \lambda_{112}(1-\xi^2)\chi_{(3\,q)}(g_{(1\,l)})y_{(3\,q)}y_{(1\,l)}; \\
y_{(1\,q)}y_{(1\,l)}y_1 &\rightsquigarrow -3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,q)})y_{(3\,l)}y_{(2\,q)}y_1 + 3\lambda_{112}\chi_1(g_{(3\,q)})y_{(3\,q)}y_{(2\,l)}y_1; \\
y_{(1\,q)}y_1y_{(1\,l)} &\rightsquigarrow -3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(1\,q)})y_{(3\,l)}y_{(2\,q)}y_1 - 3\lambda_{112}\chi_{(3\,q)}(g_{12})y_{(3\,q)}y_{(1\,l)} \\
&\qquad + 3\lambda_{112}\chi_{(1\,l)}(g_1)\chi_1(g_{(3\,q)})y_{(3\,q)}y_{(2\,l)}y_1 + \lambda_{112}(1-\xi^2)\chi_{(3\,l)}(g_{(1\,q)})y_{(3\,l)}y_{(1\,q)}.
\end{aligned}
$$

That is,

$$
\begin{aligned}
\mathbf{Y}_2 \rightsquigarrow &-3\lambda_{112}(1+\xi+\xi^2)\chi_{(3\,l)}(g_2)\chi_{12}(g_{(3\,q)})y_{(3\,l)}y_{(2\,q)}y_1 \\
&+ 3\lambda_{112}(1+\xi+\xi^2)\xi\chi_{(3\,l)}(g_{(q+1\,l)})\chi_1(g_{(3\,q)})y_{(3\,q)}y_{(2\,l)}y_1 \\
&+ \lambda_{112}(\xi-\xi^2)\chi_{(3\,l)}(g_2)(1-\chi_{112}(g_{(3\,l)}))y_{(3\,l)}y_{(1\,q)} \\
&- 3\lambda_{112}\chi_{(3\,q)}(g_{(1\,l)})\chi_1(g_{(2\,l)})(1+\xi+\xi^2)y_{(3\,q)}y_{(1\,l)} \rightsquigarrow 0.
\end{aligned}
$$

For (G3′.3), we have, for $q \geq 3$,

$$X_2X_qX_l + X_2X_lX_q + X_lX_2X_q + X_qX_lX_2 + X_qX_2X_l = \mathbf{Y}_3 g_{(3\,l)}g_{(q+1\,l)} \otimes x_{(q+1\,l)}x_{(3\,l)},$$

for

$$\begin{aligned}\mathbf{Y}_3 = {}& \chi_{12}(g_{(3\,l)})^2\chi_{(1\,q)}(g_{q+1\,l})y_{12}y_{(1\,q)}y_{(1\,l)} + \xi^2\chi_{12}(g_{(3\,l)})^2 y_{12}y_{(1\,l)}y_{(1\,q)}\\ &+ \xi\chi_{12}(g_{(3\,l)})y_{(1\,l)}y_{12}y_{(1\,q)} + \xi\chi_{112}(g_{(q+1\,l)})\chi_{(2\,q)}(g_{q+1\,l})y_{(1\,q)}y_{(1\,l)}y_{12}\\ &+ \xi^2\chi_{12}(g_{(3\,l)})\chi_{(2\,q)}(g_{q+1\,l})\chi_{112}(g_{(q+1\,l)})y_{(1\,q)}y_{12}y_{(1\,l)}.\end{aligned}$$

Hence, we need

$$\begin{aligned}
y_{12}y_{(1\,q)}y_{(1\,l)} \rightsquigarrow {}& \chi_{(1\,q)}(g_{12})y_{(1\,q)}y_{12}y_{(1\,l)} - 3\lambda_{112}\xi\chi_{(3\,q)}(g_2)y_{(3\,q)}y_2y_{(1\,l)}\\
&+ \lambda_{112}(1-\xi)y_{(2\,q)}y_{(1\,l)};\\
y_{12}y_{(1\,l)}y_{(1\,q)} \rightsquigarrow {}& \chi_{(1\,l)}(g_{12})y_{(1\,l)}y_{12}y_{(1\,q)}\\
&- 3\lambda_{112}\xi\chi_{(3\,l)}(g_2)y_{(3\,l)}y_2y_{(1\,q)}\\
&+ \lambda_{112}(1-\xi)y_{(2\,l)}y_{(1\,q)};\\
y_{(1\,l)}y_{12}y_{(1\,q)} \rightsquigarrow {}& -3\lambda_{112}\xi\chi_{(3\,q)}(g_{122})\chi_{(3\,q)}(g_{(q+1\,l)})\chi_2(g_{(3\,l)})y_{(3\,q)}y_2y_{(1\,l)}\\
&+ \lambda_{112}(1-\xi)\chi_{(2\,q)}(g_{(1\,l)})y_{(2\,q)}y_{(1\,l)};\\
y_{(1\,q)}y_{(1\,l)}y_{12} \rightsquigarrow {}& -3\lambda_{112}\xi^2\chi_{(1\,l)}(g_{(2\,q)})y_{(3\,l)}y_{(2\,q)}y_{12}\\
&+ 3\lambda_{112}\chi_1(g_{(3\,q)})y_{(3\,q)}y_{(2\,l)}y_{12};\\
y_{(1\,q)}y_{12}y_{(1\,l)} \rightsquigarrow {}& -3\lambda_{112}\xi^2\chi_{(1\,l)}(g_{(3\,q)})\chi_{(1\,l)}(g_{122})y_{(3\,l)}y_{(2\,q)}y_{12}\\
&+ 3\lambda_{112}\xi\chi_1(g_{(3\,q)})\chi_{(3\,l)}(g_{12})y_{(3\,q)}y_{(2\,l)}y_{12}\\
&- 3\xi^2\lambda_{112}\chi_{(1\,l)}(g_2)\chi_{(3\,l)}(g_{(1\,q)})\chi_2(g_{(1\,q)})y_{(3\,l)}y_2y_{(1\,q)}\\
&+ \lambda_{112}(1-\xi)\chi_{(2\,l)}(g_{(1\,q)})y_{(2\,l)}y_{(1\,q)}\\
&- 3\xi^2\lambda_{112}\chi_{(1\,l)}(g_2)\chi_{(3\,q)}(g_{(1\,q)})\chi_2(g_{(1\,l)})(1-\xi^2)y_{(3\,q)}y_2y_{(1\,l)}\\
&+ 3\lambda_{112}\xi^2\chi_{(3\,q)}(g_1)y_{(2\,q)}y_{(1\,l)}.
\end{aligned}$$

We have used that, by q-Jacobi (2.2) and Remark 5.12 we have

$$\begin{aligned}[y_{(1\,q)}, y_{(2\,l)}]_c = {}& [y_1, [y_{(1\,q)}, y_{(2\,l)}]_c]_c + \chi_{(2\,l)}(g_{(2\,q)})[y_{(1\,l)}, y_{(2\,q)}]_c\\ &- \chi_{(2\,q)}(g_1)(1-\xi)y_{(2\,q)}y_{(1\,l)}\end{aligned}$$

$$= [y_1, [y_{(1\,q)}, y_{(2\,l)}]_c]_c - 3\lambda_{122}\xi^2\chi_{(2\,l)}(g_{(3\,q)})y_{(3\,l)}y_{(3\,q)}$$
$$- \ \chi_{(2\,q)}(g_1)(1-\xi)y_{(2\,q)}y_{(1\,l)}$$

and thus, combining Lemmas 5.1 and 5.11 (3):

$$\lambda_{112}[y_{(1\,q)}, y_{(2\,l)}]_c = -3\lambda_{112}\lambda_{122}\xi^2\chi_{(2\,l)}(g_{(3\,q)})y_{(3\,l)}y_{(3\,q)}$$
$$- \lambda_{112}\chi_{(2\,q)}(g_1)(1-\xi)y_{(2\,q)}y_{(1\,l)}. \tag{5.9}$$

Hence,

$$\begin{aligned}\mathbf{Y}_3 \rightsquigarrow & -3\lambda_{112}\chi_{122}(g_{(q+1\,l)})\chi_{(3\,q)}(g_{q+1\,l})(1+\xi+\xi^2)y_{(3\,q)}y_2y_{(1\,l)}\\ & + \ 3\lambda_{112}(1+\xi+\xi^2)\chi_2(g_{(3\,l)})\chi_{(1\,q)}(g_{(q+1\,l)})y_{(2\,q)}y_{(1\,l)}\\ & - \ 3\lambda_{112}(1+\xi+\xi^2)y_{(3\,l)}y_2y_{(1\,q)}\\ & + \ \lambda_{112}((1-\xi^2)-(1-\xi^2))\chi_2(g_{(3\,l)})y_{(2\,l)}y_{(1\,q)}\\ & - \ 3\lambda_{112}(1+\xi+\xi^2)\chi_1(g_{(3\,q)})y_{(3\,l)}y_{(2\,q)}y_{12}\\ & + \ 3\lambda_{112}(1+\xi+\xi^2)\chi_1(g_{(3\,q)})\chi_{(2\,q)}(g_{q+1\,l})y_{(3\,q)}y_{(2\,l)}y_{12} \rightsquigarrow 0.\end{aligned}$$

Finally, for (G3′.4), we have, for $3 \le p < q$,

$$X_pX_qX_l + X_pX_lX_q + X_lX_pX_q + X_qX_lX_p + X_qX_pX_l$$
$$= \mathbf{Y}_4 g_{(p+1\,l)}g_{(q+1\,l)} \otimes x_{(q+1\,l)}x_{(p+1\,l)},$$

$$\begin{aligned}\mathbf{Y}_4 = \ & \chi_{(1\,l)}(g_{(p+1\,l)})^2\chi_{(1\,l)}(g_{(q+1\,l)})y_{(1\,p)}y_{(1\,q)}y_{(1\,l)}\\ & + \ \chi_{(1\,l)}(g_{(p+1\,l)})^2y_{(1\,p)}y_{(1\,l)}y_{(1\,q)} + \chi_{(1\,l)}(g_{(p+1\,l)})y_{(1\,l)}y_{(1\,p)}y_{(1\,q)}\\ & + \ \chi_{(1\,l)}(g_{(q+1\,l)})\chi_{(1\,p)}(g_{(q+1\,l)})y_{(1\,q)}y_{(1\,l)}y_{(1\,p)}\\ & + \ \chi_{(1\,l)}(g_{(p+1\,l)})\chi_{(1\,l)}(g_{(q+1\,l)})\chi_{(1\,p)}(g_{(q+1\,l)})y_{(1\,q)}y_{(1\,p)}y_{(1\,l)}.\end{aligned}$$

Hence, we need

$$\begin{aligned} y_{(1\,p)}y_{(1\,q)}y_{(1\,l)} \rightsquigarrow \ & \chi_{(1\,q)}(g_{(1\,p)})y_{(1\,q)}y_{(1\,p)}y_{(1\,l)}\\ & - \ 3\xi^2\lambda_{112}\chi_{(1\,q)}(g_{(2\,p)})y_{(3\,q)}y_{(2\,p)}y_{(1\,l)} + 3\lambda_{112}\chi_1(g_{(3\,p)})y_{(3\,p)}y_{(2\,q)}y_{(1\,l)};\\ y_{(1\,p)}y_{(1\,l)}y_{(1\,q)} \rightsquigarrow \ & \chi_{(1\,l)}(g_{(1\,p)})y_{(1\,l)}y_{(1\,p)}y_{(1\,q)}\\ & - \ 3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,p)})y_{(3\,l)}y_{(2\,p)}y_{(1\,q)} + 3\lambda_{112}\chi_1(g_{(3\,p)})y_{(3\,p)}y_{(2\,l)}y_{(1\,q)};\end{aligned}$$

$$
\begin{aligned}
y_{(1\,l)}y_{(1\,p)}y_{(1\,q)} \rightsquigarrow & -3\xi^2\lambda_{112}\chi_{(1\,q)}(g_{(2\,p)})\chi_{(3\,q)}(g_{(1\,l)})\chi_{(2\,p)}(g_{(1\,l)})y_{(3\,q)}y_{(2\,p)}y_{(1\,l)} \\
& + 3\lambda_{112}\chi_1(g_{(3\,p)})\chi_{(3\,p)}(g_{(1\,l)})\chi_{(2\,q)}(g_{(1\,l)})y_{(3\,p)}y_{(2\,q)}y_{(1\,l)}; \\
y_{(1\,q)}y_{(1\,l)}y_{(1\,p)} \rightsquigarrow & -3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,q)})y_{(3\,l)}y_{(2\,q)}y_{(1\,p)} \\
& + 3\lambda_{112}\chi_1(g_{(3\,q)})y_{(3\,q)}y_{(2\,l)}y_{(1\,p)}; \\
y_{(1\,q)}y_{(1\,p)}y_{(1\,l)} \rightsquigarrow & -3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,q)})\chi_{(1\,l)}(g_{(1\,p)})y_{(3\,l)}y_{(2\,q)}y_{(1\,p)} \\
& + 3\lambda_{112}\chi_1(g_{(3\,q)})\chi_{(1\,l)}(g_{(1\,p)})y_{(3\,q)}y_{(2\,l)}y_{(1\,p)} \\
& - 3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,p)})\chi_{(3\,l)}(g_{(1\,q)})\chi_{(2\,p)}(g_{(1\,q)})y_{(3\,l)}y_{(2\,p)}y_{(1\,q)} \\
& - 3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,p)})\chi_{(3\,q)}(g_{(1\,q)})(1-\xi^2)\chi_{(2\,p)}(g_{(1\,l)})y_{(3\,q)}y_{(2\,p)}y_{(1\,l)} \\
& + 3\lambda_{112}\chi_1(g_{(3\,p)})\chi_{(3\,p)}(g_{(1\,q)})\chi_{(2\,l)}(g_{(1\,q)})y_{(3\,p)}y_{(2\,l)}y_{(1\,q)} \\
& - 3\lambda_{112}\chi_1(g_{(3\,p)})\chi_{(3\,p)}(g_{(1\,q)})\chi_{(2\,q)}(g_1)(1-\xi)y_{(3\,p)}y_{(2\,q)}y_{(1\,l)}.
\end{aligned}
$$

Thus, we get to

$$
\begin{aligned}
\mathbf{Y}_4 \rightsquigarrow & -3\lambda_{112}\xi^2\chi_{(1\,l)}(g^2_{(p+1\,l)}g_{(q+1\,l)})\chi_{(1\,q)}(g_{(2\,p)})(1+\xi+\xi^2)y_{(3\,q)}y_{(2\,p)}y_{(1\,l)} \\
& + 3\lambda_{112}\chi_1(g_{(2\,p)})\chi_{(1\,l)}(g_{(q+1\,l)}g^2_{(p+1\,l)})(1+\xi+\xi^2)y_{(3\,p)}y_{(2\,q)}y_{(1\,l)} \\
& - 3\lambda_{112}\chi_1(g_{(2\,l)})\chi_{(1\,l)}(g_{(p+1\,l)})(1+\xi+\xi^2)y_{(3\,l)}y_{(2\,p)}y_{(1\,q)} \\
& + 3\lambda_{112}\chi_1(g_{(3\,p)})\chi_{(1\,l)}(g_{(p+1\,l)})^2(1+\xi+\xi^2)y_{(3\,p)}y_{(2\,l)}y_{(1\,q)} \\
& - 3\lambda_{112}\chi_1(g_{(2\,l)})\chi_{(1\,p)}(g_{(q+1\,l)})(1+\xi+\xi^2)y_{(3\,l)}y_{(2\,q)}y_{(1\,p)} \\
& + 3\lambda_{112}\chi_{(1\,l)}(g_{(q+1\,l)})\chi_{(1\,p)}(g_{(q+1\,l)})\chi_1(g_{(3\,q)})(1+\xi+\xi^2)y_{(3\,q)}y_{(2\,l)}y_{(1\,p)} \\
\rightsquigarrow & \; 0,
\end{aligned}
$$

which establishes the lemma. ■

Lemma 5.14. We have $[y_i, y^3_{(k\,l)}]_c = 0$ for every $1 \le i \le \theta$, $1 \le k \le l \le \theta$. □

Proof. We show this by induction on $l-k$. If $l-k=0$, it is straightforward that $[y_i, y^3_k]_c = 0$ for $\mid k-i \mid > 1$ by (5.1). This is also clear if $k=i$. If $k=i+1$, say $i=1, k=2$, we get:

$$
\begin{aligned}
[y_1, y^3_2]_c &= y_{12}y^2_2 + \chi_2(g_1)y_2y_{12}y_2 + \chi_2(g_1)^2y^2_2y_{12} \\
&= \chi_2(g_{12})y_2y_{12}y_2 + \lambda_{122}(1+\xi)y_2 + \chi_2(g_1)^2(1+\xi)y^2_2y_{12} \\
&= \lambda_{122}(1+\xi+\xi^2)y_2 + \chi_2(g_1)^2(1+\xi+\xi^2)y^2_2y_{12} = 0.
\end{aligned}
$$

Case $i = k+1$ is analogous. Fix k and assume $[y_i, y^3_{(k\,p)}]_c = 0$ for every $k \le p \le l$, every $1 \le k \le l \le \theta$. It follows from Proposition 5.13 that

$$\rho([y_i, y^3_{(k\,l)}]_c) = [y_i, y^3_{(k\,l)}]_c \otimes 1 + g_i g^3_{(k\,l)} \otimes [x_i, x^3_{(k\,l)}]_c + \sum_{k \le p < l} C_p [y_i, y^3_{(k\,p)}]_c g^3_{(p+1\,l)} \otimes x^3_{(p+1\,l)} + \sum_{k \le p < l} C_p \gamma_p^3 y^3_{(k\,p)} g_i g^3_{(p+1\,l)} \otimes [x_i, x^3_{(p+1\,l)}]_c,$$

for C_p as in 1.7 and $\gamma_p = \chi_{(k\,p)}(g_i)$. By induction, we have $[y_i, y^3_{(k\,p)}]_c = 0$ for every $k \le p < l$ while $[x_i, x^3_{(k\,l)}]_c = [x_i, x^3_{(p+1\,l)}]_c = 0$ for every $k \le p < l$ by [8, Proposition 4.1]. That is,

$$\rho([y_i, y^3_{(k\,l)}]_c) = [y_i, y^3_{(k\,l)}]_c \otimes 1,$$

that is $[y_i, y^3_{(k\,l)}]_c \in \widetilde{\mathcal{A}}^{\operatorname{co} \widetilde{\mathcal{H}}} = \Bbbk$. Set $\Bbbk \ni s := [y_i, y^3_{(k\,l)}]_c$. Now,

$$s = g_i [y_i, y^3_{(k\,l)}]_c g_i^{-1} = \xi\, \chi_{(k\,l)}(g_i)^3 [y_i, y^3_{(k\,l)}]_c.$$

Hence $s = 0$ if $\chi_{(k\,l)}(g_i)^3 \neq \xi^2$. On the other hand,

$$s = g^3_{(k\,l)} [y_i, y^3_{(k\,l)}]_c g^{-3}_{(k\,l)} = \chi_i(g_{(k\,l)})^3 [y_i, y^3_{(k\,l)}]_c$$

and thus $s = 0$ if $\chi_i(g_{(k\,l)})^3 \neq 1$. But we cannot have both $\chi_i(g_{(k\,l)})^3 = 1$ and $\chi_{(k\,l)}(g_i)^3 = \xi^2$ as it contradicts $1 = \chi_{(k\,l)}(g_i)^3 \chi_i(g_{(k\,l)})^3$. Therefore $s = 0$ and the lemma follows. ∎

The following shows 3.12 for $j = 1$ and thus 3.11 in general.

Theorem 5.15. Let $\mathcal{A} = \mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ be the algebra quotient of $\mathcal{T}(V)$ by

$$y_{ij} = 0, \; i < j-1 \in \mathbb{I}; \qquad y_{iij} = \lambda_{iij}, \; i, j \in \mathbb{I}, |j - i| = 1; \tag{5.10}$$

$$y^3_{(k\,l)} = \mu_{(k\,l)}, \; k \le l \in \mathbb{I}, \tag{5.11}$$

for families of scalars $\boldsymbol{\lambda} = (\lambda_{iij})_{i,j}$ and $\boldsymbol{\mu} = (\mu_{(k\,l)})_{k,l}$ satisfying 5.2 and

$$\mu_{(k\,l)} = 0 \quad \text{if } \chi^3_{(k\,l)} \neq \epsilon. \tag{5.12}$$

Then $\mathcal{A} \in \operatorname{Cleft} \mathcal{H}$. In particular,

$$\operatorname{Cleft}' \mathcal{H} = \{\mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu}) | \boldsymbol{\lambda} \text{ as in (5.2)}, \boldsymbol{\mu} \text{ as in (5.12)}\}.$$ □

Proof. By [8], $X = {}^{\mathrm{co}\,\mathcal{H}}\widetilde{\mathcal{H}}$ is the polynomial algebra in the variables

$$\mathrm{x}_{kl} := g_{(k\,l)}^{-3} x_{(k\,l)}^{3}, \qquad 1 \le k \le l \le \theta.$$

We will show that the $\widetilde{\mathcal{H}}$-colinear algebra maps $f : X \to \widetilde{\mathcal{A}}$ generated by

$$\mathrm{x}_{kl} \mapsto \mathrm{y}_{kl} - \mathrm{g}_{kl},$$

for $\mathrm{y}_{kl} := g_{(k\,l)}^{-3} y_{(k\,l)}^{3}$ and $\mathrm{g}_{kl} := \mu_{(k\,l)} g_{(k\,l)}^{-3}$ are also $\widetilde{\mathcal{H}}$-linear, when we consider the right adjoint action $\cdot : X \otimes \widetilde{\mathcal{H}} \to X$ and the Miyashita–Ulbrich action $\leftharpoonup : \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{H}} \to \widetilde{\mathcal{A}}$. We have, for $h \in H$:

$$f(\mathrm{x}_{kl} \cdot h) = \chi_{(k\,l)}(h)^3 (\mathrm{y}_{kl} - \mathrm{g}_{kl}) = f(\mathrm{x}_{kl}) \leftharpoonup h,$$

as $\chi_{(k\,l)}(g_i)^3 \mu_{(k\,l)} = \mu_{(k\,l)}$ by 5.12. Also, by [8, Proposition 4.1]

$$\mathrm{x}_{kl} \cdot x_i = (1 - \chi_{(k\,l)}(g_i)^3 \chi_i(g_{(k\,l)})^3) \mathrm{x}_{kl} x_i = 0.$$

On the other hand, $(\mathrm{y}_{kl} - \mathrm{g}_{kl}) \leftharpoonup x_i = 0$, by Lemma 5.14. Then, the theorem follows by [16, Theorem 8], see also [5, Theorem 3.3]. ■

5.2 Liftings

Let $\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}(\boldsymbol{\lambda})$ be the quotient of $T(V)\#H$ by the relations

$$a_{ij} = 0, \; i < j - 1; \qquad\qquad a_{iij} = \lambda_{iij}(1 - g_{iij}), \; |j - i| = 1 \tag{5.13}$$

for some family of scalars $\boldsymbol{\lambda} = (\lambda_{iij})$ satisfying 5.2 and normalized by

$$\lambda_{iij} = 0 \quad \text{if } g_{iij} = 1. \tag{5.14}$$

Here, we rename the basis $\{x_1, \ldots, x_\theta\}$ of V by $\{a_1, \ldots, a_\theta\}$.

Remark 5.16. Observe that normalization 5.14 is not necessary when $\theta \ge 3$. Take, for simplicity, $i = 1, j = 2$. Then, we have $\xi^2 = \chi_{112}(g_3)\chi_3(g_{112})$. □

Recall the definition of the distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}(V)$ from p. 4 and the bosonization $\widetilde{\mathcal{H}} = \widetilde{\mathcal{B}}(V)\#H$.

Proposition 5.17. Let $\boldsymbol{\lambda} = (\lambda_{iij})$ satisfy 5.2 and 5.14. Then

(1) $\widetilde{\mathcal{L}}(\boldsymbol{\lambda}) = L(\widetilde{\mathcal{A}}(\boldsymbol{\lambda}), \widetilde{\mathcal{A}}(\boldsymbol{\lambda}))$; hence $\widetilde{\mathcal{L}}(\boldsymbol{\lambda})$ is a cocycle deformation of $\widetilde{\mathcal{H}}$.

(2) $\widetilde{\mathcal{L}}(\boldsymbol{\lambda})$ is a pointed Hopf algebra with $\operatorname{gr} \widetilde{\mathcal{L}} = \widetilde{\mathcal{H}}$. □

Proof. Follows directly from Proposition 3.3 (c); see also the case $N = 2$ in Proposition 4.8. ■

Let $\delta : \widetilde{\mathcal{A}}(\boldsymbol{\lambda}) \to \widetilde{\mathcal{L}}(\boldsymbol{\lambda}) \otimes \widetilde{\mathcal{A}}(\boldsymbol{\lambda})$ be the coaction. We have

$$\delta(y_{(kl)}) = a_{(kl)} \otimes 1 + g_{(kl)} \otimes y_{(kl)} + (1 - \xi^2) \sum_{k \le p < l} a_{(kp)} g_{(p+1l)} \otimes y_{(p+1l)}.$$

We proceed to describe the algebra $L(\mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu}), \mathcal{H})$. As $\mathcal{H}$ is obtained from $\widetilde{\mathcal{H}}$ as a quotient of an ideal not generated exclusively by (skew-)primitive elements, we thus need to prepare the setting accordingly, cf. (the proof of) Proposition 3.3 (a).

For each $m \ge 1$, consider the m-adic approximation $\widehat{\mathfrak{B}}_m(V)$ to $\mathfrak{B}(V)$. This is the quotient of $T(V)$ by relations (1.3) and (1.4) together with

$$x_{(k\,l)}^3, \qquad 1 \le l - k < m. \tag{5.15}$$

Thus, we obtain a family of cleft objects $\mathcal{A}_m(\boldsymbol{\lambda}, \boldsymbol{\mu}_m)$ for $\mathcal{H}_m = \widehat{\mathfrak{B}}_m(V)\#H$ given by the quotient of $\mathcal{T}(V)$ by relations (5.1) for each together with

$$y_{(k\,l)}^3 - \mu_{(k\,l)}, \qquad 1 \le l - k < m.$$

Here, $\boldsymbol{\lambda} = (\lambda_{iij})_{i,j}$ satisfies 5.2 and $\boldsymbol{\mu}_m = (\mu_{(k\,l)})_{k \le l}$ satisfies 5.12.

Now, fix $\boldsymbol{\lambda}, \boldsymbol{\mu}_m$ and set $\mathcal{A}_m = \mathcal{A}_m(\boldsymbol{\lambda}, \boldsymbol{\mu}_m)$. Let $\mathcal{L}_m(\boldsymbol{\lambda}, \boldsymbol{\mu}_m) := L(\mathcal{A}_m, \mathcal{H}_m)$. Notice that $\mathcal{L}_0 = \widetilde{\mathcal{L}}$. We keep the name $\delta : \mathcal{A}_m \to \mathcal{L}_m \otimes \mathcal{A}_m$ for the coaction at each level. Thus, $\mathcal{H}_{m+1} = \mathcal{H}_m / I_{m+1}$ is such that I_{m+1} is generated by skew primitive elements [2, Remark 6.10]. Hence, by Proposition 3.3, $\mathcal{L}_{m+1}$ is the quotient of $\mathcal{L}_m$ by the ideal generated by

$$a_{(k\,l)}^3 - \sigma_{(k\,l)} - \mu_{(k\,l)}(1 - g_{(k,l)}^3), \tag{5.16}$$

where, according to 3.7, the *deforming elements* $\sigma_{(k\,l)}$ are defined by:

$$a_{(k\,l)}^3 \otimes 1 - \delta(y_{(k\,l)})^3 = \sigma_{(k\,l)} \otimes 1. \tag{5.17}$$

We give a description of these elements in Proposition 5.22.

In this way, we obtain a description of the full algebra $\mathcal{L} = L(\mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu}), \mathcal{H})$ in the final step of this procedure. We further normalize $\boldsymbol{\mu}$ by

$$\mu_{(k\,l)} = 0 \quad \text{if } g^3_{(k\,l)} = 1. \tag{5.18}$$

We illustrate this situation in the following two examples.

Example 5.18. $\mathcal{L}_1$ is the quotient of $\mathcal{T}(V)$ by relations (5.13) and

$$a_k^3 = \mu_{(k)}(1 - g_k^3).$$

In particular, $\sigma_{(k\,k)} = 0$, $k \in \mathbb{I}$. □

Proof. Let $m = 0$. The elements y_k^3, generating I_1, satisfy:

$$\delta(y_k)^3 = a_k^3 \otimes 1 + g_k^3 \otimes y_k^3.$$

Hence $\mathrm{u}_{(k)} = 0$ and the statement follows. ■

The following example contains the spirit of our computations ahead.

Example 5.19. $\mathcal{L}_2$ is the quotient of $\mathcal{T}(V)$ by relations (5.13) and

$$\begin{aligned} a_k^3 &= \mu_{(k)}(1 - g_k^3), \quad 1 \le k \le \theta; \\ a_{kk+1}^3 &= \mu_{(kk+1)}(1 - g_k^3 g_{k+1}^3) - \mu_{(k+1)}\mu_{(k)}(1-\xi)^3 \chi_k(g_{k+1})^3 (1 - g_k^3) g_{k+1}^3 \\ &\quad - \lambda_{kk+1k+1}\lambda_{kkk+1}\xi^2(1 - g_k^2 g_{k+1}) g_k g_{k+1}^2 \quad 1 \le k < \theta. \end{aligned}$$ □

Proof. Set $m = 1$. We have already described $\mathcal{L}_1$ in Example 5.18. We need to compute the elements $\mathrm{u}_{(k,k+1)}$, $k < \theta$. It will be enough to understand $\delta(y_{12})^3$. Set

$$A = a_{12} \otimes 1, \qquad B = g_{12} \otimes y_{12}, \qquad C = a_1 g_2 \otimes y_2,$$

so that $\delta(y_{12}) = A + B + (1 - \xi^2)C$. As before, we focus on the terms in which a factor λ_{***} may appear. These are related with two possible facts:

(1) Fact **A**: a_1 appears to the left of a_{12}, that is:

$$CAB, \quad CBA, \quad BCA, \quad CAA, \quad ACA, \quad CAC, \quad CCA.$$

(2) Fact **B**: y_{12} appears to the left of y_2, that is:

$$ABC, \qquad BAC, \qquad BCA, \qquad BCB, \qquad BBC, \qquad BCC, \qquad CBC.$$

We have

$$\begin{aligned}
CAB &= a_1 g_2 a_{12} g_{12} \otimes y_2 y_{12} = \chi_{12}(g_2) a_1 a_{12} g_{122} \otimes y_2 y_{12} \\
&\quad \rightsquigarrow \lambda_{112}\xi^2(1 - g_{122}) g_{122} \otimes y_2 y_{12}; \\
CBA &\rightsquigarrow \lambda_{112}(1 - g_{122}) g_{122} \otimes y_2 y_{12}; \\
BCA &= \chi_1(g_2) a_1 a_{12} g_{122} \otimes y_2 y_{12} + \lambda_{122}\xi^2 a_1 a_{12} g_{122} \otimes 1 \\
&\quad \rightsquigarrow \lambda_{112}\xi(1 - g_{112}) g_{122} \otimes y_2 y_{12} + \lambda_{122}\xi a_{12} a_1 g_{122} \otimes 1 \\
&\qquad + \lambda_{122}\lambda_{112}\xi^2(1 - g_{112}) g_{122} \otimes 1; \\
CAA &= \chi_{12}(g_1)^2 a_1 a_{12}^2 g_2 \otimes y_2 \rightsquigarrow \lambda_{112}(1+\xi) a_{12}(1 - g_{112}) \otimes y_2; \\
ACA &= \chi_{12}(g_2) a_{12} a_1 a_{12} g_2 \otimes y_2 \rightsquigarrow \lambda_{112}\xi^2 a_{12}(1 - g_{112}) \otimes y_2; \\
CAC &= \chi_{112}(g_2) a_1 a_{12} a_1 g_2^2 \otimes y_2^2 \rightsquigarrow \lambda_{112} a_1 (1 - g_{112}) g_2^2 \otimes y_2^2; \\
CCA &= \chi_{112}(g_2)\chi_{12}(g_2) a_1^2 a_{12} g_2^2 \otimes y_2^2 \rightsquigarrow \lambda_{112}(\xi + \xi^2) a_1 (1 - g_{112}) g_2^2 \otimes y_2^2.
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
ABC &= \chi_1(g_{12}) a_{12} a_1 g_{122} \otimes y_{12} y_2 \rightsquigarrow \lambda_{122}\xi^2 a_{12} a_1 g_{122} \otimes 1; \\
BAC &\rightsquigarrow \lambda_{122} a_{12} a_1 g_{122} \otimes 1; \\
BCB &= \chi_1(g_{12}) a_1 g_{122} g_{12} \otimes y_{12} y_2 y_{12} \rightsquigarrow \lambda_{122}\xi^2 a_1 g_{122} g_{12} \otimes y_2; \\
BBC &= \chi_1(g_{12})^2 a_1 g_{122} g_{12} \otimes y_{12}^2 y_2 \rightsquigarrow \lambda_{122}(1+\xi) a_1 g_{122} g_{12} \otimes y_{12}; \\
BCC &= \chi_1(g_{122})\chi_1(g_{12}) a_1^2 g_{122} g_2 \otimes y_{12} y_2^2 \rightsquigarrow \lambda_{122}(\xi + \xi^2) a_1^2 g_{122} g_2 \otimes y_2; \\
CBC &= \lambda_{122} a_1^2 g_{122} g_2 \otimes y_2.
\end{aligned}$$

Hence,

$$CAB + CBA + BCA \rightsquigarrow \lambda_{122}\xi a_{12} a_1 g_{122} \otimes 1 + \lambda_{122}\lambda_{112}\xi^2(1 - g_{112}) g_{122} \otimes 1$$

and $CAA + ACA \rightsquigarrow 0$, $CAC + CCA \rightsquigarrow 0$. Also, we have

$$ABC + BAC \rightsquigarrow (1+\xi^2)\lambda_{122}a_{12}a_1g_{122} \otimes 1;$$
$$BCB + BBC \rightsquigarrow 0, \qquad BCC + CBC \rightsquigarrow 0.$$

Therefore,

$$\begin{aligned}\delta(y_{12})^3 &= a_{12}^3 \otimes 1 + g_{12}^3 \otimes y_{12}^3 + (1-\xi)^3\chi_1(g_2)^3 a_1^3 g_2^3 \otimes y_2^3\\ &\quad + \lambda_{122}\lambda_{112}\xi^2(1-g_{112})g_{122} \otimes 1\\ &= a_{12}^3 \otimes 1 + g_{12}^3 \otimes y_{12}^3 + \mu_{(2)}\mu_{(1)}(1-\xi)^3\chi_1(g_2)^3(1-g_1^3)g_2^3 \otimes 1\\ &\quad + \lambda_{122}\lambda_{112}\xi^2(1-g_{112})g_{122} \otimes 1.\end{aligned}$$

In particular, as $\mu_1\chi_1(g_2)^3 = 1$,

$$\mathrm{u}_{(1,2)} = -\mu_{(2)}\mu_{(1)}(1-\xi)^3(1-g_1^3)g_2^3 - \xi^2\lambda_{122}\lambda_{112}(1-g_1^2g_2)g_1g_2^2.$$

The statement follows. ■

Remark 5.20. When $\theta = 2$, then $\mathcal{L}_2$ as in Example 5.19 is a lifting of type A_2. It coincides with the liftings found in [11] for this type. □

5.3 The deforming elements

The expressions for both $\sigma_{(i)} := \sigma_{(i\,i)}$ and $\sigma_{(i\,i+1)}$ follow from Examples 5.18 and 5.19. Namely,

$$\begin{aligned}\sigma_{(i)} &= 0, \quad i \in \mathbb{I},\\ \sigma_{(i\,i+1)} &= -\mu_{(i+1)}\mu_{(i)}(1-\xi)^3\chi_i(g_{i+1})^3(1-g_i^3)g_{i+1}^3\\ &\quad - \lambda_{ii+1i+1}\lambda_{iii+1}\xi^2(1-g_i^2g_{i+1})g_ig_{i+1}^2, \quad i < \theta \in \mathbb{I}.\end{aligned}$$

For the general case of $\sigma_{(i\,l)}$, $i, l \in \mathbb{I}$, we proceed in a similar fashion.

We first define $\mathrm{u}_{(i\,l)}(\boldsymbol{\mu})$ and $h_{il}(\boldsymbol{\lambda})$ in $\Bbbk\Gamma$. We set $\mathrm{u}_{(ii)} = 0$, and, recursively,

$$\mathrm{u}_{(i\,l)}(\boldsymbol{\mu}) = -\sum_{i\le p<l} C_p\mu_{(p+1\,l)}\left(\mathrm{u}_{(i\,p)} + \mu_{(i\,p)}\left(1-g_{(i\,p)}^N\right)\right)g_{(p+1\,l)}^N. \tag{5.19}$$

Now, we set $h_{ii}(\boldsymbol{\lambda}) = h_{ii+1}(\boldsymbol{\lambda}) = 0$ and, for $l \ge i+2$,

$$h_{il}(\boldsymbol{\lambda}) = -9\mu_{(i+2\,l)}\lambda_{ii+1i+1}\lambda_{iii+1}(1-g_{iii+1})g_{ii+1i+1}g_{(i+2\,l)}^3. \tag{5.20}$$

Next, for $i \leq p < l$, we set $q = p+1$, $r = p+2$ and consider the following elements in $T(V)\#H$:

$$\varsigma_i^p(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \lambda_{qrr}\Big(\xi^2 a_{(i\,p)}a_{(i\,q)}a_{(i\,r)} + \chi_r(g_{(1\,p)})a_{(i\,p)}a_{(i\,r)}a_{(i\,q)} + a_{(i\,r)}a_{(i\,p)}a_{(i\,q)}\Big). \tag{5.21}$$

Let us fix $s_p = -3(1-\xi^2)$, $p < l-2$, $s_{l-2} = 1$, and set

$$d_{il}(p) = \chi_{(i\,q)}(g_{(q\,l)}g_{(r+1\,l)})\chi_{(i\,p)}(g_{(r+1\,l)})s_p.$$

Finally, we consider:

$$\varsigma_{il}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = -3\xi^2 \sum_{i \leq p < l} \mu_{(p+3\,l)}\chi_r(g_{(p+3\,l)})d_{il}(p)\varsigma_i^p(\boldsymbol{\lambda}, \boldsymbol{\mu})g_{qrr}g^3_{(p+3\,l)}. \tag{5.22}$$

Recall that $g_{(l+1\,l)} = 1$; also we set $\mu_{(l+1\,l)} := 1$.

Remark 5.21. Observe that nor $\varsigma^p(\boldsymbol{\lambda}, \boldsymbol{\mu})$ neither $\varsigma_{il}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are expressed in the PBW basis. This is an arduous computation that we perform in full generality in Section 5.3, see Corollary 5.27 for a complete answer. □

Proposition 5.22. Let i, l be as above. Then

$$\sigma_{(i\,l)}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \mathrm{u}_{(i\,l)}(\boldsymbol{\mu}) + h_{il}(\boldsymbol{\lambda}) + \varsigma_{il}(\boldsymbol{\lambda}, \boldsymbol{\mu}). \tag{5.23}$$

□

See below for a proof. As a result, we have the following.

Theorem 5.23. The Hopf algebra $L(\mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu}), \mathcal{H}) := \mathcal{L}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is the quotient of $\mathcal{T}(V)$ by relations (5.13) and

$$a^3_{(i\,l)} = \mu_{(i\,l)}(1 - g^3_{(i\,l)}) + \sigma_{(i\,l)}(\boldsymbol{\lambda}, \boldsymbol{\mu}),$$

for $\sigma_{(i\,l)}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ as in 5.23. □

Proof. Follows by Proposition 3.3 (c), see 5.16. ■

See Example 5.25 below for a lifting of a concrete V. Next, we prove Proposition 5.22:

Proof. We take $i = 1$ to ease up the notation, so $l \geq 3$. Set

$$A = a_{(1\,l)} \otimes 1, \qquad B = g_{(1\,l)} \otimes y_{(1\,l)}, \qquad X_p = a_{(1\,p)} g_{(p+1\,l)} \otimes y_{(p+1\,l)}, 1 \leq p < l$$

so $\delta(y_{(1\,l)}) = A + B + (1-\xi^2)\sum_{1\leq p<l} X_p$. We will also denote $X_l := A$, by identifying as usual $g_{(l+1\,l)} := 1$, $y_{(l+1\,l)} := 1$. Finally, set

$$\sigma_p := \sigma_{(1\,p)}, \qquad 1 \leq p \leq l. \tag{5.24}$$

As in Example 5.19, we need to focus on the terms of $(A + B + (1-\xi^2)\sum X_p)^3$ involving a factor λ_{***}, as by [2, Remark 6.10] we have, for C_p as in 1.7:

$$\begin{aligned}\delta(y^3_{(1\,l)}) = a^3_{(1\,l)} \otimes 1 + g^3_{(1\,l)} \otimes y^3_{(1\,l)} \\ + \sum_{1\leq p<l} C_p a^3_{(1\,p)} g^3_{(p+1\,l)} \otimes y^3_{(p+1\,l)} \\ + \text{terms involving a factor } \lambda_{***}.\end{aligned} \tag{5.25}$$

Combining this with the recursive deformation procedure following [5, Corollary 5.12], that is we assume $y^3_{(p+1\,l)} = \mu_{(p+1\,l)}$, we obtain

$$\sigma_l = (5.19) - \text{terms involving a factor } \lambda_{***}. \tag{5.26}$$

As in Proposition 5.13, we consider the cases (here we need to distinguish a factor A from a factor X_p, identified previously):

(L1) For every $p < q$, terms XYZ involving $X, Y, Z \in \{B, X_p, X_q\}$, all different, X_p to the left of X_q.

(L2) For every pair $p < q$, terms XYZ involving $X, Y, Z \in \{X_p, X_q\}$, not all equal and with a factor X_p to the left of X_q.

(L3) For every triple $p < q < r$, terms XYZ involving *distinct* $X, Y, Z \in \{X_p, X_q, X_r\}$ and with X_p to the left of X_q or X_r or with X_q to the left of X_r.

However, as Example 5.19 illustrates, we also need to consider

(L4) terms ABX_p and BAX_p, $1 \leq p < l$;

(L5) terms BX_pB and BBX_p, $1 \leq p < l$; and

(L6) terms BX_qX_p and X_qBX_p, $1 \leq p \leq q < l$.

Remark 5.24. In cases (L1) and (L2) it is enough to consider $q < l$, as a factor $X_l = A$ will not contribute to σ_l. Case (L3) is different, and we will take this difference into account: the main difference lays in the fact that the factors X_l—unlike X_p, $p < l$—are not multiplied by $(1 - \xi^2)$. Hence commutativity computations follow rather smoothly, and we only have to recall this in the final expression. □

Claim 5.1. Cases (L4–L6) do not contribute to σ_l. □

These cases easily follow from Lemmas 5.4 and 5.9. In (L4) we have $BAX_p = \xi ABX_p$ and

$$BAX_p + ABX_p \rightsquigarrow \begin{cases} 0, & p > 1; \\ 3\xi^2 \lambda_{122} \chi_{12}(g_{(1\,l)}) a_{(1\,l)} a_1 g_{(1\,l)} g_{(2\,l)} \otimes y_{(3\,l)}^2, & p = 1. \end{cases}$$

In (L5) we get

$$BX_pB \rightsquigarrow \begin{cases} 0, & p > 1; \\ -3\lambda_{122} \chi_{12}(g_{(1\,l)}) a_1 g_{(2\,l)} g_{(1\,l)}^2 \otimes y_{(3\,l)}^2 y_{(1\,l)}, & p = 1. \end{cases}$$

$$BBX_p \rightsquigarrow \begin{cases} 0, & p > 1; \\ 3\lambda_{122} \chi_{12}(g_{(1\,l)}) a_1 g_{(2\,l)} g_{(1\,l)}^2 \otimes y_{(3\,l)}^2 y_{(1\,l)}, & p = 1. \end{cases}$$

In particular, $BX_pB + BBX_p \rightsquigarrow 0$. For (L6), as $q + 1 \geq 3$,

$$BX_qX_p \rightsquigarrow \begin{cases} 0, & p > 1; \\ -3\xi \lambda_{122} \chi_{12}(g_{(1\,l)}) \chi_1(g_{(q+1\,l)}) a_{(1\,q)} a_1 g_{(1\,l)} g_{(q+1\,l)} g_{(2\,l)} & \\ \qquad \otimes y_{(q+1\,l)} y_{(3\,l)}^2 & p = 1. \end{cases}$$

$$X_qBX_p \rightsquigarrow \begin{cases} 0, & p > 1; \\ -3\lambda_{122} \chi_{12}(g_{(1\,l)}) \chi_1(g_{(q+1\,l)}) a_{(1\,q)} a_1 g_{(q+1\,l)} g_{(1\,l)} g_{(2\,l)} & \\ \qquad \otimes y_{(q+1\,l)} y_{(3\,l)}^2, & p = 1. \end{cases}$$

Hence $X_qBX_p = \xi BX_qX_p$. In particular, they do not contribute to σ_l and the claim follows.

We deal with cases (L1–L3) using the identities developed in Section 5.4. We need to take into account Equation (5.35).

Claim 5.2. Case (L1) contributes to σ_l with 5.20. □

We have to analyse terms BX_pX_q, X_pBX_q, X_pX_qB, $p < q$. Now, if $p > 1$, as $[y_{(1\,l)}, y_{(p+1\,l)}]_c = [y_{(1\,l)}, y_{(q+1\,l)}]_c = 0$ it follows

$$\begin{aligned} B^{p,q} &:= BX_pX_q + X_pBX_q + X_pX_qB \\ &= (1+\xi+\xi^2)\chi_{(1\,q)}(g_{p+1\,l})a_{(1\,p)}a_{(1\,q)}g_{(p+1\,l)}g_{(q+1\,l)}g_{(1\,l)} \\ &\qquad \otimes y_{(p+1\,l)}y_{(q+1\,l)}y_{(1\,l)} = 0. \end{aligned}$$

If $p = 1$, then still $[y_{(1\,l)}, y_{(q+1\,l)}]_c = 0$ as $q + 1 \geq 3$ and using Lemma 5.4 to compute $[y_{(1\,l)}, y_{(2\,l)}]_c$ we get

$$\begin{aligned} B^{1,q} &:= BX_pX_q + X_pBX_q + X_pX_qB \\ &\rightsquigarrow -3\lambda_{122}\chi_{12}(g_{(1\,l)})\chi_{(1\,q)}(g_{(1\,l)})\chi_{(1\,q)}(g_{(2\,l)})a_1a_{(1\,q)}g_{(2\,l)}g_{(q+1\,l)}g_{(1\,l)} \\ &\qquad \otimes y^2_{(3\,l)}y_{(q+1\,l)}. \end{aligned}$$

Hence, as $\lambda_{122}[y_{(3\,l)}, y_{(q+1\,l)}]_c = 0$, $q > 2$:

$$B^{1,q} \rightsquigarrow \begin{cases} -3\xi\lambda_{122}\lambda_{112}(1-g_{112})g_{(1\,l)}g_{(2\,l)}g_{(3\,l)} \otimes y^3_{(3\,l)}, & q = 2; \\ 3(1-\xi)\lambda_{122}\lambda_{112}\chi_{12}(g_{(3\,l)})a_{(3\,q)}g_{(1\,l)}g_{(2\,l)}g_{(q+1\,l)} & \\ \qquad \otimes y_{(q+1\,l)}y^2_{(3\,l)}, & q \geq 3. \end{cases}$$

Notice that in this way $B^{1,2}$ will contribute to σ_l, as by the induction process we have $y^3_{(3\,l)} = \mu_{(3\,l)}$, that is we get a term 5.20.

Claim 5.3. Case (L2) does not contribute to σ_l. □

We have to deal with terms $X_pX_q^2$, $X_qX_pX_q$, $X_pX_qX_p$, and $X_p^2X_q$. According to (5.35), we have to distinguish cases

(L2i) $p+1 < q < l$,
(L2ii) $p+1 = q < l-1$,
(L2iii) $p+1 = q = l-1$.

In case (L2i), we have

$$\begin{aligned} X_pX_q^2 &= \chi_{(1\,q)}(g_{(q+1\,l)}g^2_{(p+1\,l)})a_{(1\,p)}a^2_{(1\,q)}g^2_{(q+1\,l)}g_{(p+1\,l)} \otimes y_{(p+1\,l)}y^2_{(q+1\,l)} \\ &= \chi_{(1\,q)}(g_{(q+1\,l)})\chi_{(1\,l)}(g_{(p+1\,l)})^2a_{(1\,p)}a^2_{(1\,q)}g^2_{(q+1\,l)}g_{(p+1\,l)} \otimes y^2_{(q+1\,l)}y_{(p+1\,l)} \end{aligned}$$

which does not contribute to $\mathfrak{u}_l$. For (L2ii),

$$X_pX_q^2 \rightsquigarrow \chi_{(1\,q)}(g_{(q+1\,l)}g_{(p+1\,l)})\chi_{(1\,l)}(g_{(p+1\,l)})a_{(1\,p)}a_{(1\,q)}^2g_{(q+1\,l)}^2g_{(p+1\,l)}$$
$$\otimes y_{(q+1\,l)}[y_{(p+1\,l)}, y_{(q+1\,l)}]_c$$
$$+ \chi_{(1\,q)}(g_{(q+1\,l)}g_{(p+1\,l)}^2)a_{(1\,p)}a_{(1\,q)}^2g_{(q+1\,l)}^2g_{(p+1\,l)}$$
$$\otimes [y_{(p+1\,l)}, y_{(q+1\,l)}]_c y_{(q+1\,l)}$$
$$= -3\xi^2\lambda_{p+1p+2p+2}\chi_{(p+2\,l)}(g_{(p+3\,l)})\chi_{(1\,p+1)}(g_{(p+2\,l)}g_{(p+1\,l)})\chi_{(1\,l)}(g_{(p+1\,l)})$$
$$a_{(1\,p)}a_{(1\,p+1)}^2g_{(p+2\,l)}^2g_{(p+1\,l)} \otimes y_{(p+2\,l)}y_{(p+3\,l)}^2$$
$$- 3\xi^2\lambda_{p+1p+2p+2}\chi_{(p+2\,l)}(g_{(p+3\,l)})\chi_{(1\,p+1)}(g_{(p+2\,l)}g_{(p+1\,l)}^2)$$
$$a_{(1\,p)}a_{(1\,p+1)}^2g_{(p+2\,l)}^2g_{(p+1\,l)} \otimes y_{(p+3\,l)}^2y_{(p+2\,l)}$$

and we see that this does not contribute to $\mathfrak{u}_l$, using Lemma 5.1 to deduce $\lambda_{p+1p+2p+2}[y_{(p+2\,l)}, y_{(p+3\,l)}]_c = 0$. The same holds for (L2iii), as in this case

$$X_pX_q^2 \rightsquigarrow \lambda_{l-1ll}(1+\xi^2)\chi_{(1\,l-1)}(g_{(l-1\,l)}^2g_l)a_{(1\,l-2)}a_{(1\,l-1)}^2g_l^2g_{(l-1\,l)} \otimes y_l.$$

The same holds for the combinations $X_qX_pX_q$, $X_pX_qX_p$, and $X_p^2X_q$.

Claim 5.4. If $p < 3$, then case (L3) does not contribute to σ_l.
If $p \geq 3$, then case (L3) contributes to σ_l with 5.21. □

Here we deal with terms $X_pX_qX_r$, $X_pX_rX_q$, $X_qX_rX_p$, $X_rX_pX_q$, $X_qX_pX_r$, $p < q < r \leq l$, which we denote by $C^{x,y,z}$, $x, y, z \in \{p, q, r\}$. By the computations above and the commutation rule (5.35) we see that we will get a factor contributing to σ_l if and only if

$$q = p+1, \qquad r = p+2,$$

and p is on the left of q.

Thus, we are left with cases $C^{p,q,r}$, $C^{p,r,q}$, $C^{r,p,q}$, $q = p+1$, $r = p+2$. Set

$$c_p = \begin{cases} -3\lambda_{p+1p+2p+2}\chi_{p+2}(g_{(p+3\,l)})\mu_{(p+3\,l)}, & p \leq l-3 \\ \lambda_{p+1p+2p+2}, & p = l-2. \end{cases} \tag{5.27}$$

Then, for each of these terms, the corresponding factor in $\Bbbk$ that arises in the second tensorand ($\chi_{(l+1\,l)} := \epsilon$) is

$$C^{p,q,r} \dashrightarrow c_p, \qquad C^{p,r,q} \dashrightarrow c_p\chi_{(r+1\,l)}(g_{(p+1\,l)}), \qquad C^{r,p,q} \dashrightarrow c_p.$$

Set, with the convention, for the case $r = l$, $g_{(l+1\,l)} = 1$, $\chi_{(l+1\,l)} = \epsilon$,

$$\omega_{x,y,z} = \chi_{(1\,z)}(g_{(y+1\,l)}g_{(x+1\,l)})\chi_{(1\,y)}(g_{(x+1\,l)}), \qquad x, y, z \in \{p, q, r\}. \tag{5.28}$$

Set also, $g_{p,q,r} := g_{(p+1\,l)}g_{(p+2\,l)}g_{(p+3\,l)}$ and let us set

$$\begin{aligned} &a_{p,q,r} := a_{(1\,p)}a_{(1\,q)}a_{(1\,r)}, \qquad a_{p,r,q} := a_{(1\,p)}a_{(1\,r)}a_{(1\,q)}, \\ &a_{r,p,q} := a_{(1\,r)}a_{(1\,p)}a_{(1\,q)}. \end{aligned} \tag{5.29}$$

Set (cf. Remark 5.24)

$$\Xi_p = \begin{cases} (1-\xi^2)^3 = 3(\xi - \xi^2), & p < l-2; \\ (1-\xi^2)^2 = -3\xi^2, & p = l-2. \end{cases} \tag{5.30}$$

Hence, the contribution of these terms to σ_l is

$$c_p\,\Xi_p\Big(\omega_{p,q,r}a_{p,q,r} + \omega_{p,r,q}\chi_{(p+3\,l)}(g_{(q\,l)})a_{p,r,q} + \omega_{r,p,q}a_{r,p,q}\Big)g_{p,q,r}. \tag{5.31}$$

Notice that

$$\begin{aligned} c_p\omega_{p,q,r} &= c_p\xi^2\chi_{(1\,q)}(g_{(p+1\,l)})\chi_{(1\,q)}(g_{(r+1\,l)})\chi_{(1\,p)}(g_{(r+1\,l)}), \\ c_p\omega_{p,r,q}\chi_{(p+3\,l)}(g_{(q\,l)}) &= c_p\chi_{(1\,q)}(g_{(p+1\,l)})\chi_{(1\,q)}(g_{(r+1\,l)})\chi_{(1\,p)}(g_{(r+1\,l)})\chi_r(g_{(1\,p)}), \\ c_p\omega_{r,p,q} &= c_p\chi_{(1\,q)}(g_{(p+1\,l)})\chi_{(1\,q)}(g_{(r+1\,l)})\chi_{(1\,p)}(g_{(r+1\,l)}). \end{aligned}$$

Set

$$d_p' = \begin{cases} -3\mu_{(p+3\,l)}\chi_{(1\,q)}(g_{qr})\chi_{qqr}(g_{(r+1\,l)}), & p < l-2; \\ \chi_{(1\,q)}(g_{(q\,l)}), & p = l-2, \end{cases} \tag{5.32}$$

and $d_p = \lambda_{qrr}d_p'$. Observe that

$$c_p\chi_{(1\,q)}(g_{(p+1\,l)})\chi_{(1\,q)}(g_{(r+1\,l)})\chi_{(1\,p)}(g_{(r+1\,l)}) = d_p.$$

To see this, we use the identity

$$\mu_{(p+3\,l)}\lambda_{qrr}\chi_{(1\,q)}(g_{(q\,l)})\chi_{(1\,r)}(g_{(r+1\,l)})\chi_{(1\,p)}(g_{(r+1\,l)}) = \mu_{(p+3\,l)}\lambda_{qrr}\chi_{(1\,p)}(g_{(r+1\,l)})^3\chi_{(1\,q)}(g_{qr})\chi_{qqr}(g_{(r+1\,l)})$$

and $\mu_{(p+3\,l)}\lambda_{qrr}\chi_{(1\,p)}(g_{(r+1\,l)})^3 = \mu_{(p+3\,l)}\lambda_{qrr}\chi_{(r+1\,l)}(g_{(1\,p)})^{-3} = \mu_{(p+3\,l)}$.

Hence 5.31 becomes

$$d_p\,\Xi_p\Big(\xi^2 a_{p,q,r} + \chi_r(g_{(1\,p)})a_{p,r,q} + a_{r,p,q}\Big)g_{p,q,r} = d'_p\,\Xi_p\varsigma^p g_{p,q,r}.$$

Adding all of these terms and reordering the scalars, we get 5.22.

Finally, adding all of these contributions, we obtain $\sigma_{(i\,l)}(\boldsymbol{\lambda},\boldsymbol{\mu})$ as in 5.23 and the proposition follows. ■

Example 5.25. Set $\theta = 5$, so $\mathbb{I} = \mathbb{I}_5$ and consider the braiding matrix

$$\mathfrak{q} = \begin{pmatrix} \xi & \xi & 1 & 1 & 1 \\ \xi & \xi & \xi^2 & 1 & 1 \\ 1 & 1 & \xi & 1 & 1 \\ 1 & 1 & \xi^2 & \xi & \xi \\ 1 & 1 & 1 & \xi & \xi \end{pmatrix}.$$

Set $G = (\mathbb{Z}/3n\mathbb{Z})^5$, $n \geq 2$, so G is an abelian group such that $V \in {}^H_H\mathcal{YD}$, $H = \Bbbk G$. Indeed, let g_i, $i \in \mathbb{I}$, the generators of each cyclic factor. Let $q \in \mathbb{G}'_{3n}$ with $q^n = \xi$. Observe that $\widehat{G}$ is generated by φ_i, $i \in \mathbb{I}$, with $\varphi_i(g_i) = q$ and $\varphi_i(g_j) = 1$, $i \neq j \in \mathbb{I}$. A principal realization is given by $((g_i,\chi_i))_{i\in\mathbb{I}}$, for $\chi_1 = \chi_2 = \varphi_1^n\varphi_2^n$, $\chi_4 = \chi_5 = \varphi_4^n\varphi_5^n$, and $\chi_3 = \varphi_2^{2n}\varphi_3^n\varphi_4^{2n}$. In particular $\chi_{112} = \chi_{455} = \epsilon$.

Let us choose $\boldsymbol{\lambda}$ such that all $\lambda_{iij} = 0$ except λ_{112}, λ_{455}. Choose $\boldsymbol{\mu}$ with $\mu_{(k\,l)} = 0$ for every $1 \leq k \leq l \leq 5$. Then, $\mathcal{L}(\boldsymbol{\lambda},\boldsymbol{\mu})$ is the algebra generated by Γ and $a_1,\dots,a_5$ satisfying

$$a_{ik} = 0, \quad |\, i-k\,| > 1, \qquad a^3_{(k\,l)} = 0, \quad |\, k-l\,| < 4,$$

$$a_{iiij} = \begin{cases} \lambda_{112}(1-g_1^2g_2), & i=1, j=2, \\ \lambda_{455}(1-g_4g_5^2), & i=4, j=5, \\ 0, & \text{else}, \end{cases} \qquad |\, i-j\,| = 1,$$

$$a^3_{(1\,5)} = 9\lambda_{112}\lambda_{455}\sum_{\sigma\in\mathbb{S}_3}(-1)^{|\sigma|}h_{\sigma,1}a_{(3\,\sigma(5))}a_{(2\,\sigma(4))}a_{(1\,\sigma(3))}g_4^2g_5.$$

The scalars $h_{\sigma,1} \in \Bbbk^\times$, $\sigma \in \mathbb{S}_3$, are as in Corollary 5.27 and can be explicitly computed from the matrix $\mathfrak{q}$. □

Remark 5.26. If $\theta \geq 4$, then the relations of the Nichols algebra $\mathfrak{B}(V)$ become deformed in the lifting $\mathcal{L}$ by elements in the group algebra $\Bbbk\Gamma \leq H$, as in the case $\operatorname{ord}(\xi) > 3$ of [2], as Lemma 5.1 only allows a single pair $(\lambda_{kkk+1}, \lambda_{kk+1k+1})$ to have a non-zero entry. If $\theta \geq 5$ the relations may be deformed in higher strata of the coradical filtration, as in Example 5.25. □

Now, we give a full description of $\varsigma_i^p(\boldsymbol{\lambda}, \boldsymbol{\mu})$, cf. 5.21, as a linear combination in the PBW basis. We set $q = p+1$, $r = p+2$ and $j = i+1$, $k = i+2$. We consider the action of $\mathbb{S}_3$ on $\{r, q, p\}$ by

$$(12)(r) = q, \qquad (23)(q) = p.$$

Corollary 5.27. If $p = i, j$, then $\varsigma_i^p(\boldsymbol{\lambda}, \boldsymbol{\mu}) = 0$. When $p > i+2$,

$$\begin{aligned}\varsigma_i^p(\boldsymbol{\lambda}, \boldsymbol{\mu}) = &-3\lambda_{qrr}\lambda_{qqr}\chi_{(i\,p)}(g_q)a_{(i\,p)}^3 g_{qqr}\\ &- 3\lambda_{qrr}\lambda_{iij}\sum_{\sigma\in\mathbb{S}_3}(-1)^{|\sigma|}h_{\sigma,i}a_{(k\,\sigma(p))}a_{(j\,\sigma(q))}a_{(i\,\sigma(r))},\end{aligned} \tag{5.33}$$

for $h_{\sigma,i} \in \Bbbk$, $\sigma \in \mathbb{S}_3$, given by

$$\begin{aligned} h_{\mathrm{id},i} &= \xi\chi_{qqr}(g_{(i\,p)})\chi_{(i\,r)}(g_{(j\,q)}), & h_{(12),i} &= (\xi^2-1)\chi_{qqr}(g_{(i\,p)})\chi_i(g_{(k\,q)}),\\ h_{(23),i} &= \xi\chi_r(g_i)\chi_i(g_{(j\,p)}), & h_{(13),i} &= \xi(\xi-2)\chi_{(k\,p)}(g_{ij}),\\ h_{(123),i} &= 2\chi_r(g_{(i\,p)})\chi_i(g_{(k\,p)}), & h_{(132),i} &= \xi^2\chi_{(k\,q)}(g_{(i\,r)})\chi_{(j\,p)}(g_r).\end{aligned}$$
□

Proof. First, we show that $\varsigma^p(\boldsymbol{\lambda}, \boldsymbol{\mu})$ equals

$$\lambda_{qrr}\xi^2 a_{(i\,p)}[a_{(i\,q)}, a_{(i\,r)}]_c - \lambda_{qrr}\xi^2\chi_r(g_{(i\,p)})[a_{(i\,p)}, a_{(i\,r)}]_c a_{(i\,q)}. \tag{5.34}$$

In particular, by Lemma 5.1 and Corollary 5.33, we have $\lambda_{qrr}\varsigma^p(\boldsymbol{\lambda}, \boldsymbol{\mu}) = 0$ if $p - i < 3$. Hence $\varsigma_{(i\,i)} = \varsigma_{(i\,i+1)} = 0$. Indeed, it follows that

$$\begin{aligned}\varsigma^p(\boldsymbol{\lambda}, \boldsymbol{\mu}) = &\lambda_{qrr}\alpha_p\chi_{(i\,q)}(g_{(i\,p)})\, a_{(i\,r)}a_{(i\,q)}a_{(i\,p)}\\ &+ \lambda_{qrr}\Big(\xi^2\chi_{(i\,r)}(g_{(i\,q)}g_{(i\,p)}) + 1 + \chi_r(g_{(i\,p)})\chi_{(i\,r)}(g_{(i\,p)})\Big)a_{(i\,r)}[a_{(i\,p)}, a_{(i\,q)}]_c\\ &+ \xi^2\lambda_{qrr}a_{(i\,p)}[a_{(i\,q)}, a_{(i\,r)}]_c\\ &+ \lambda_{qrr}\Big(\chi_{p+2}(g_{(i\,p)}) + \xi^2\chi_{(i\,r)}(g_{(i\,q)})\Big)[a_{(i\,p)}, a_{(i\,r)}]_c a_{(i\,q)},\end{aligned}$$

for $\alpha_p = 1+\xi^2\chi_{(ir)}(g_{(iq)}g_{(ip)}) + \chi_{p+2}(g_{(ip)})\chi_{(ir)}(g_{(ip)})$. Notice that $\alpha_p = 1+\xi\,\chi_r(g_q)\chi_{qrr}(g_{(ip)}) + \xi\,\chi_{qrr}(g_{(ip)})$ and thus it follows that $\lambda_{qrr}\alpha_p = 0$ as $\lambda_{qrr}\chi_{qrr} = \lambda_{qrr}\epsilon$, $\lambda_{qrr}\chi_r(g_q) = \lambda_{qrr}\xi$ and $1+\xi+\xi^2 = 0$. On the other hand, we use $\lambda_{qrr}\chi_{(ir)}(g_{(iq)}) = \lambda_{qrr}\xi^2\chi_r(g_{(ip)})$ to simplify the coefficients of third and fourth summands. As for the second, we have

$$\begin{aligned}\lambda_{qrr}\chi_{(ir)}(g_{(iq)}g_{(ip)}) &= \lambda_{qrr}\xi^2\chi_r(g_{(ip)})\chi_{(ir)}(g_{(ip)})\\ &= \lambda_{qrr}\chi_r(g_{(ip)})\chi_{qr}(g_{(ip)}) = \lambda_{qrr}.\end{aligned}$$

Also, $\lambda_{qrr}\chi_r(g_{(ip)})\chi_{(ir)}(g_{(ip)}) = \lambda_{qrr}\xi\,\chi_{qrr}(g_{(ip)}) = \lambda_{qrr}$ and thus the coefficient is $\lambda_{qrr}(1+\xi+\xi^2) = 0$. Hence, we have 5.34.

Next we show 5.33, using Proposition 5.37. We have

$$\begin{aligned}\varsigma^p(\boldsymbol{\lambda},\boldsymbol{\mu}) = {}& \lambda_{qrr}\chi_q(g_{(ip)})(1+\xi^2\chi_{qrr}(g_{(ip)})+\xi)a_{(ir)}a_{(iq)}a_{(ip)}\\ &- 3\xi\lambda_{qrr}\lambda_{qqr}\chi_{(ir)}(g_q)a_{(ip)}^3 g_{qqr}\\ &- 3\xi\lambda_{qrr}\lambda_{iij}\chi_{qqr}(g_{(ip)})\chi_{(ir)}(g_{(jq)})a_{(kr)}a_{(jq)}a_{(ip)}\\ &- 3\xi\lambda_{qrr}\lambda_{iij}\Big(\chi_{qqr}(g_{(ip)})\chi_{(kr)}(g_{(jq)})\chi_{(jp)}(g_{(iq)})\\ &+ \xi\,\chi_{(ir)}(g_{(jp)})\chi_r(g_{(ip)})\Big)a_{(kr)}a_{(jp)}a_{(iq)}\\ &+ 3\xi^2\lambda_{qrr}\lambda_{iij}\chi_{qqr}(g_{(ip)})\chi_i(g_{(kq)})a_{(kq)}a_{(jr)}a_{(ip)}\\ &- 3\xi^2\lambda_{iij}\lambda_{qrr}\Big(\chi_{(kq)}(g_{(ir)})\chi_{(jp)}(g_r)\\ &- 2\chi_{(iq)}(g_{(jp)})\\ &\chi_{(ir)}(g_{(ip)})\chi_{(kq)}(g_{(ir)})\chi_r(g_i)\Big)a_{(kq)}a_{(jp)}a_{(ir)}\\ &+ 3\xi^2\lambda_{qrr}\lambda_{iij}\Big(\chi_{(kp)}(g_{ij})\chi_r(g_{(iq)})\\ &+ \xi\,\chi_i(g_{(kp)})\chi_r(g_{(ip)})\Big)a_{(kp)}a_{(jr)}a_{(iq)}\\ &+ 3\xi^2\lambda_{qrr}\lambda_{iij}\Big((1-\xi^2)\chi_{(kp)}(g_{ij})\\ &+ \xi\,\chi_r(g_{(ip)})\chi_i(g_{(kp)})\chi_{(iq)}(g_r)\\ &+ \xi\,\chi_{(kp)}(g_{(jr)})\chi_{(jq)}(g_{(ir)})\Big)a_{(kp)}a_{(jq)}a_{(ir)}.\end{aligned}$$

First, $\lambda_{qrr}(1+\xi^2\chi_{qrr}(g_{(ip)})+\xi) = \lambda_{qrr}(1+\xi^2+\xi) = 0$. Next, observe that

$$\begin{aligned}&\lambda_{qrr}\lambda_{iij}\Big(\chi_{qqr}(g_{(ip)})\chi_{(kr)}(g_{(jq)})\chi_{(jp)}(g_{(iq)}) + \xi\,\chi_{(ir)}(g_{(jp)})\chi_r(g_{(ip)})\Big)\\ &\quad= \lambda_{qrr}\lambda_{iij}\xi^2\chi_r(g_i)\chi_i(g_{(jp)})\Big(\chi_q(g_{iij})\chi_{(kp)}(g_{iij})+1\Big)\end{aligned}$$

$$= \lambda_{qrr}\lambda_{iij}\xi^2\chi_r(g_i)\chi_i(g_{(j\,p)})(\xi^2+1) = -\lambda_{qrr}\lambda_{iij}\chi_r(g_i)\chi_i(g_{(j\,p)}).$$

Similarly,

$$\lambda_{iij}\lambda_{qrr}\Big(\chi_{(k\,q)}(g_{(i\,r)})\chi_{(j\,p)}(g_r) + \chi_{(i\,r)}(g_{(i\,p)})\chi_{(k\,q)}(g_{(i\,r)})\chi_r(g_i)$$
$$- 2\chi_{(i\,q)}(g_{(j\,p)})\Big) = \lambda_{iij}\lambda_{qrr}\chi_{(k\,q)}(g_{(i\,r)})\chi_{(j\,p)}(g_r)\Big(1+\xi-2\xi\Big)$$
$$= \lambda_{iij}\lambda_{qrr}\chi_{(k\,q)}(g_{(i\,r)})\chi_{(j\,p)}(g_r)(1-\xi).$$

Also, we have

$$\lambda_{qrr}\lambda_{iij}\Big(\chi_{(k\,p)}(g_{ij})\chi_r(g_{(i\,q)}) + \xi\,\chi_i(g_{(k\,p)})\chi_r(g_{(i\,p)})\Big)$$
$$= 2\lambda_{qrr}\lambda_{iij}\xi\,\chi_r(g_{(i\,p)})\chi_i(g_{(k\,p)}).$$

Finally,

$$\lambda_{qrr}\lambda_{iij}\Big((1-\xi^2)\chi_{(k\,p)}(g_{ij}) + \xi\,\chi_r(g_{(i\,p)})\chi_i(g_{(k\,p)})\chi_{(i\,q)}(g_r)$$
$$+\,\xi\,\chi_{(k\,p)}(g_{(j\,r)})\chi_{(j\,q)}(g_{(i\,r)})\Big) = \lambda_{qrr}\lambda_{iij}\chi_{(k\,p)}(g_{ij})(1-2\xi^2).$$

Thus, we have

$$\begin{aligned}\varsigma^p(\boldsymbol{\lambda},\boldsymbol{\mu}) = & -3\xi\lambda_{qrr}\lambda_{qqr}\chi_{(i\,r)}(g_q)a^3_{(i\,p)}g_{qqr} \\ & - 3\xi\lambda_{qrr}\lambda_{iij}\chi_{qqr}(g_{(i\,p)})\chi_{(i\,r)}(g_{(j\,q)})a_{(k\,r)}a_{(j\,q)}a_{(i\,p)} \\ & + 3\xi\lambda_{qrr}\lambda_{iij}\chi_r(g_i)\chi_i(g_{(j\,p)})a_{(k\,r)}a_{(j\,p)}a_{(i\,q)} \\ & - 3(1-\xi^2)\lambda_{qrr}\lambda_{iij}\chi_{qqr}(g_{(i\,p)})\chi_i(g_{(k\,q)})a_{(k\,q)}a_{(j\,r)}a_{(i\,p)} \\ & - 3\xi^2\lambda_{iij}\lambda_{qrr}\chi_{(k\,q)}(g_{(i\,r)})\chi_{(j\,p)}(g_r)a_{(k\,q)}a_{(j\,p)}a_{(i\,r)} \\ & + 6\lambda_{qrr}\lambda_{iij}\chi_r(g_{(i\,p)})\chi_i(g_{(k\,p)})a_{(k\,p)}a_{(j\,r)}a_{(i\,q)} \\ & - 3\xi(2-\xi)\lambda_{qrr}\lambda_{iij}\chi_{(k\,p)}(g_{ij})a_{(k\,p)}a_{(j\,q)}a_{(i\,r)}.\end{aligned}$$

Hence the lemma follows by defining the scalars $h_{\sigma,i}$ appropriately. ■

5.4 Technical identities

To compute the elements $\varsigma_{(i\,l)}$ in 5.22 in the PBW basis, we need a large series of technical identities involving commutators. This is the content of this section.

Lemma 5.28. The following identities hold in $\widetilde{\mathcal{L}}$.

(1) $[a_{(1\,l)}, a_2]_c = \begin{cases} \lambda_{122}(1-\xi^2)\chi_2(g_3)a_3 - \lambda_{223}(1-\xi^2)a_1 g_{223}, & l=3; \\ \lambda_{122}(1-\xi^2)\chi_2(g_{(3\,l)})a_{(3\,l)}, & l \geq 4. \end{cases}$

(2) $[a_{(1\,l)}, a_p]_c = 0$, $3 \leq p < l-1$.

(3) $[a_{(1\,l)}, a_{(p\,k)}]_c = 0$, $3 \leq p \leq k < l-1$.

(4) $[a_{(1\,l)}, a_{l-1}]_c = -\lambda_{l-1l-1l}(1-\xi^2)a_{(1\,l-2)}g_{l-1l-1l}$.

(5) $[a_{(1\,l)}, a_l]_c = -\lambda_{l-1ll}(1-\xi^2)a_{(1\,l-2)}g_{l-1ll}$. □

Proof. (1) Case $l=3$ follows once again mimicking [2, Lemma 1.11] as in Lemma 5.4. The general case $l \geq 4$ follows as in Lemma 5.4: in this situation, if $\lambda_{223} \neq 0$, then $\lambda_{122} = 0$ by Lemma 5.1 and

$$[a_{(1\,l)}, a_2]_c = -\lambda_{223}(1-\xi^2)\chi_{(4\,l)}(g_{23})[a_1, a_{(4\,l)}]_c g_{223} = 0,$$

using q-Jacobi (2.2). For (2), first we have that

$$[a_{(1\,l)}, a_p]_c = [[a_{(1\,p+1)}, a_{(p+2\,l)}]_c, a_p]_c = \chi_p(g_{(p+2\,l)})[a_{(1\,p+1)}, a_p]_c a_{(p+2\,l)} \\ - \chi_{(p+2\,l)}(g_{(1\,p+1)})a_{(p+2\,l)}[a_{(1\,p+1)}, a_p]_c.$$

Now, by (1), $\lambda_{ppp+1}\chi_{ppp+1} = \lambda_{ppp+1}\epsilon$ and $[a_{(1\,p-2)}, a_p]_c = 0$,

$$\begin{aligned}[a_{(1\,p+1)}, a_p]_c &= [a_{(1\,p-2)}, [a_{(p-1\,p+1)}, a_p]_c]_c \\ &= -\lambda_{ppp+1}(1-\xi^2)[a_{(1\,p-2)}, a_{p-1}g_{ppp+1}]_c \\ &= -\lambda_{ppp+1}(1-\xi^2)\Big(a_{(1\,p-2)}a_{p-1}g_{ppp+1} \\ &\quad - \chi_{p-1}(g_{(1\,p-2)})a_{p-1}g_{ppp+1}a_{(1\,p-2)}\Big) \\ &= -\lambda_{ppp+1}(1-\xi^2)a_{(1\,p-1)}g_{ppp+1},\end{aligned}$$

as $\lambda_{ppp+1}\chi_{(1\,p-2)}(g_{ppp+1}) = \lambda_{ppp+1}$. In particular, this shows (4) for $p = l-1$. Now, if $p < l-1$ we get

$$\begin{aligned}[a_{(1\,l)}, a_p]_c &= -\lambda_{ppp+1}(1-\xi^2)\chi_p(g_{(p+2\,l)})\big(a_{(1\,p-1)}g_{ppp+1}a_{(p+2\,l)} \\ &\quad - \chi_{(p+2\,l)}(g_p)\chi_{(p+2\,l)}(g_{(1\,p+1)})a_{(p+2\,l)}a_{(1\,p-1)}g_{ppp+1}\big) \\ &= -\lambda_{ppp+1}(1-\xi^2)\chi_p(g_{(p+2\,l)})\chi_{(p+2\,l)}(g_{ppp+1}) \\ &\qquad [a_{(1\,p-1)}, a_{(p+2\,l)}]_c g_{ppp+1} = 0.\end{aligned}$$

(3) follows from (2) by induction. For (5), we get, as $\lambda_{l-1ll}\chi_{l-1ll} = \lambda_{l-1ll}\epsilon$,

$$[a_{(1\,l)}, a_l]_c = [[a_{(1\,l-2)}, a_{l-1l}]_c, a_l]_c = \lambda_{l-1ll}(\chi_{(1\,l-2)}(g_{l-1ll}) - 1)a_{1\,l-2}g_{l-1ll}$$

and since

$$\lambda_{l-1ll}\chi_{1\,l-2}(g_{l-1ll}) = \lambda_{l-1ll}\chi_{(1\,l-2)}(g_{l-1ll})\chi_{l-1ll}(g_{(1\,l-2)}) = \lambda_{l-1ll}\xi^2,$$

the lemma follows. ∎

Remark 5.29. As $[a_2, a_{(1\,3)}]_c = -\chi_{(1\,3)}(g_2)[a_{(1\,3)}, a_2]_c$, we get

$$[a_2, a_{(1\,3)}]_c = \lambda_{122}(1-\xi)a_3 - \lambda_{223}\chi_1(g_2)(\xi^2 - \xi)a_1 g_{223}.$$ □

Lemma 5.30. The following identities hold in $\widetilde{\mathcal{L}}$.

(1) $[a_{(1\,l)}, a_{(3\,p)}]_c = [a_{(3\,p)}, a_{(1\,l)}]_c = 0,\ 3 \le p < l-1.$

(2) $[a_{(1\,l)}, a_{(3\,l)}]_c = -\lambda_{l-1ll}(1-\xi)a_{(1\,l-1)}g_{l-1ll} + 3\lambda_{l-1ll}\chi_{l-1}(g_{(1\,l-2)})a_{l-1}a_{(1\,l-2)}g_{l-1ll}.$

(3) $$[a_{(1\,l)}, a_{(3\,l-1)}]_c = \begin{cases} -\lambda_{334}(1-\xi^2)a_{12}g_{334}, & l = 4, \\ 3\xi^2\lambda_{l-1l-1l}\chi_{(3\,l-2)}(g_{(1\,l)}) \\ \qquad a_{(3\,l-2)}a_{(1\,l-2)}g_{l-1l-1l}, & l \ge 5. \end{cases}$$ □

Proof. (1) follows by induction on p and using q-Jacobi (2.2), case $p = 3$ being Lemma 5.28 (2).

(2) Using q-Jacobi (2.2) and Lemma 5.28 (4–5), we have

$$\begin{aligned}[a_{(1\,l)}, a_{(3\,l)}]_c &= [[a_{(1\,l)}, a_{l-1}]_c, a_l]_c + \chi_{l-1}(g_{(1\,l)})a_{l-1}[a_{(1\,l)}, a_l]_c \\ &- \chi_l(g_{l-1})[a_{(1\,l)}, a_l]_c a_{l-1} = -\lambda_{l-1l-1l}(1-\xi^2)[a_{(1\,l-2)}, a_l]_c g_{l-1l-1l} \\ &- \lambda_{l-1ll}(1-\xi)a_{(1\,l-1)}g_{l-1ll} + 3\lambda_{l-1ll}\chi_{l-1}(g_{(1\,l-2)})a_{l-1}a_{(1\,l-2)}g_{l-1ll}.\end{aligned}$$

(3) Case $l = 4$ is Lemma 5.28 (1), using q-Jacobi (2.2).

Now, if $l \ge 5$, using q-Jacobi (2.2), item (1) and Lemma 5.28 (4) we get

$$\begin{aligned}[a_{(1\,l)}, a_{(3\,l-1)}]_c &= -\lambda_{l-1l-1l}(1-\xi)\chi_{l-1}(g_{(3\,l-2)})[a_{(1\,l-2)}, a_{(3\,l-2)}]_c g_{l-1l-1l} \\ &- \lambda_{l-1l-1l}(1-\xi^2)^2\chi_{(3\,l-2)}(g_{(1\,l)})a_{(3\,l-2)}a_{(1\,l-2)}g_{l-1l-1l}.\end{aligned}$$

Hence (3) follows using (2) and $\lambda_{l-1l-1l}\lambda_{l-3l-2l-2} = 0$. Also, we use the fact that $\lambda_{l-1l-1l}\chi_{l-1}(g_{(3\,l-2)}) = \lambda_{l-1l-1l}\xi\,\chi_{(3\,l-2)}(g_{l-1l})$. ■

Lemma 5.31. The following identities hold in $\widetilde{\mathcal{L}}$.

(1) $[a_1, a_{12}]_c = \lambda_{112}(1 - g_{112})$.

(2) $[a_1, a_{(1\,l)}]_c = \lambda_{112}(1 - \xi^2)a_{(3\,l)}$, $l \geq 3$.

(3) $[a_{12}, a_{(1\,3)}]_c = -3\xi^2\lambda_{112}\chi_{(1\,3)}(g_2)a_3a_2 + \lambda_{112}(1 - \xi)a_{23} - 3\lambda_{223}\xi\,\chi_1(g_2)a_1^2 g_{223}$.

(4) $[a_{(12)}, a_{(1\,l)}]_c = -3\xi^2\lambda_{112}\chi_{(1\,l)}(g_2)a_{(3\,l)}a_2 + \lambda_{112}(1 - \xi)a_{(2\,l)}$, $l \geq 4$. □

Proof. (1) is by definition. For (2), we have

$$[a_1, a_{(1\,l)}]_c = [a_1, [a_{12}, a_{(3\,l)}]_c]_c = \lambda_{112}(1 - \xi^2)a_{(3\,l)} - \lambda_{112}\left(g_{112}a_{(3\,l)} - \chi_{(3\,l)}(g_{112})a_{(3\,l)}g_{112}\right) = \lambda_{112}(1 - \xi^2)a_{(3\,l)}.$$

(3) and (4) follow as in Lemma 5.11. In this case, when $l = 3$ and extra term involving $a_1^2 g_{223}$ arises, which gets killed for bigger l. ■

Proposition 5.32. The following identities hold in $\widetilde{\mathcal{L}}$.

(1) For $3 \leq p < l - 1$:

$$[a_{(1\,p)}, a_{(1\,l)}]_c = -3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,p)})a_{(3\,l)}a_{(2\,p)} + 3\lambda_{112}\chi_1(g_{(3\,p)})a_{(3\,p)}a_{(2\,l)}.$$

(2) For $l \geq 5$,

$$[a_{(1\,l-1)}, a_{(1\,l)}]_c = -3\xi^2\chi_{(1\,l)}(g_{l-1})\lambda_{l-1l-1l}a_{(1\,l-2)}^2 g_{l-1l-1l} - 3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,l-1)})a_{(3\,l)}a_{(2\,l-1)} + 3\lambda_{112}\chi_1(g_{(3\,l-1)})a_{(3\,l-1)}a_{(2\,l)}.$$

□

Proof. (1) We use q-Jacobi (2.2) and Lemma 5.30 (1) to get

$$[a_{(1\,p)}, a_{(1\,l)}]_c = \chi_{(1\,l)}(g_{(3\,p)})[a_{(12)}, a_{(1\,l)}]_c a_{(3\,p)} - \chi_{(3\,p)}(g_{12})a_{(3\,p)}[a_{(12)}, a_{(1\,l)}]_c.$$

$$\begin{aligned}[a_{(12)}, a_{(1\,l)}]_c a_{(3\,p)} &= -3\xi^2\lambda_{112}\chi_{(1\,l)}(g_2)a_{(3\,l)}a_2a_{(3\,p)} + \lambda_{112}(1 - \xi)a_{(2\,l)}a_{(3\,p)} \\ &= -3\xi^2\lambda_{112}\chi_{(1\,l)}(g_2)\Big(a_{(3\,l)}a_{(2\,p)} + \chi_{(3\,p)}(g_2)a_{(3\,l)}a_{(3\,p)}a_2\Big) \\ &\quad + \lambda_{112}(1 - \xi)\chi_{(3\,p)}(g_{(2\,l)})a_{(3\,p)}a_{(2\,l)},\end{aligned}$$

as $\lambda_{112}\lambda_{223} = 0$. We arrive to (1) using $\lambda_{112}\lambda_{334} = 0$:

$$a_{(3\,p)}[a_{(12)}, a_{(1\,l)}]_c = -3\xi\lambda_{112}\chi_{(3\,l)}(g_{(2\,p)})a_{(3\,l)}a_{(3\,p)}a_2 + \lambda_{112}(1-\xi)a_{(3\,p)}a_{(2\,l)}.$$

(2) We have, using q-Jacobi (2.2),

$$\begin{aligned}[a_{(1\,l-1)}, a_{(1\,l)}]_c &= [[a_{(1\,l-2)}, a_{l-1}]_c, a_{(1\,l)}]_c = [a_{(1\,l-2)}, [a_{l-1}, a_{(1\,l)}]_c]_c \\ &\quad + \chi_{(1\,l)}(g_{l-1})\big([a_{(1\,l-2)}, a_{(1\,l)}]_c a_{l-1} - \chi_{l-1}(g_{(1\,l)}g_{(1\,l-2)})a_{l-1}[a_{(1\,l-2)}, a_{(1\,l)}]_c\big).\end{aligned}$$

Now, by Lemma 5.28 (4),

$$\begin{aligned}[a_{l-1}, a_{(1\,l)}]_c &= -\chi_{(1\,l)}(g_{l-1})[a_{(1\,l)}, a_{l-1}]_c \\ &= \lambda_{l-1l-1l}(1-\xi^2)\chi_{(1\,l)}(g_{l-1})a_{(1\,l-2)}g_{l-1l-1l}.\end{aligned}$$

Hence, $[a_{(1\,l-2)}, [a_{l-1}, a_{(1\,l)}]_c]_c = -3\xi^2\chi_{(1\,l)}(g_{l-1})\lambda_{l-1l-1l}a^2_{(1\,l-2)}g_{l-1l-1l}$.

On the other hand, we have that, by item (1),

$$\begin{aligned}&\chi_{l-1}(g_{(1\,l)}g_{(1\,l-2)})a_{l-1}[a_{(1\,l-2)}, a_{(1\,l)}]_c \\ &\quad = -3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,l-2)})\chi_{l-1}(g_{(1\,l)}g_{(1\,l-2)})a_{l-1}a_{(3\,l)}a_{(2\,l-2)} \\ &\qquad + 3\lambda_{112}\chi_1(g_{(3\,l-2)})\chi_{l-1}(g_{(1\,l)}g_{(1\,l-2)})a_{l-1}a_{(3\,l-2)}a_{(2\,l)}.\end{aligned}$$

Now, by Lemma 5.28

$$\begin{aligned}[a_{(1\,l-2)}, a_{(1\,l)}]_c a_{l-1} &= -3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,l-2)})a_{(3\,l)}a_{(2\,l-2)}a_{l-1} \\ &\quad + 3\lambda_{112}\chi_1(g_{(3\,l-2)})a_{(3\,l-2)}a_{(2\,l)}a_{l-1} \\ &= -3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,l-2)})a_{(3\,l)}a_{(2\,l-1)} \\ &\quad - 3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,l-2)})\chi_{l-1}(g_{(2\,l-2)})a_{(3\,l)}a_{l-1}a_{(2\,l-2)} \\ &\quad - 3\lambda_{112}\lambda_{l-1l-1l}(1-\xi^2)\chi_1(g_{(3\,l-2)})a_{(3\,l-2)}a_{(2\,l-2)}g_{l-1l-1l} \\ &\quad + 3\lambda_{112}\chi_1(g_{(3\,l-2)})\chi_{l-1}(g_{(2\,l)})a_{(3\,l-2)}a_{l-1}a_{(2\,l)} \\ &= -3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,l-2)})a_{(3\,l)}a_{(2\,l-1)} \\ &\quad + 3\xi\lambda_{112}\lambda_{l-1l-1l}(1-\xi^2)\chi_1(g_{(2\,l-2)})a_{(3\,l-2)}a_{(2\,l-2)}g_{l-1l-1l} \\ &\quad - 3\xi^2\lambda_{112}\chi_{(1\,l)}(g_{(2\,l-2)})\chi_{l-1}(g_{(2\,l-2)})\chi_{l-1}(g_{(3\,l)})a_{l-1}a_{(3\,l)}a_{(2\,l-2)} \\ &\quad - 3\lambda_{112}\lambda_{l-1l-1l}(1-\xi^2)\chi_1(g_{(3\,l-2)})a_{(3\,l-2)}a_{(2\,l-2)}g_{l-1l-1l}\end{aligned}$$

$$+\ 3\lambda_{112}\chi_1(g_{(3\,l-2)})\chi_{l-1}(g_{(2\,l)})a_{(3\,l-1)}a_{(2\,l)}$$
$$+\ 3\lambda_{112}\chi_1(g_{(3\,l-2)})\chi_{l-1}(g_{(2\,l)})\chi_{l-1}(g_{(3\,l-2)})a_{l-1}a_{(3\,l-2)}a_{(2\,l)}.$$

Hence, using that $\chi_{l-1}(g_{112}) = 1$ and adding up the terms, we get (6). ■

Notice that for $0 \le p < q < l$, Lemmas 5.9 (1) and 5.10 give

$$[y_{(p+1\,l)}, y_{(q+1\,l)}]_c = \begin{cases} 0, & p+1 < q < l; \\ -3\lambda_{p+1p+2p+2}\chi_{p+2}(g_{(p+3\,l)})y^2_{(p+3\,l)}, & p+1 = q < l-1; \\ \lambda_{l-1ll}, & p+1 = q = l-1. \end{cases} \tag{5.35}$$

We fix $q = p+1$, $r = p+2$. Some of the identities computed in Lemma 5.31 and Proposition 5.32 become simpler when multiplied by a factor λ_{qrr}, using Lemma 5.1. This will be of great importance in the computations, as $\varsigma_{(i\,l)}$ is a linear combination of the elements $\varsigma^p(\boldsymbol{\lambda}, \boldsymbol{\mu})$ in 5.21 and each one of these terms is multiplied by λ_{qrr}.

We interpret these identities in the following corollary.

Corollary 5.33. The following identities hold in $\widetilde{\mathcal{L}}$.

(1) If $p < 3$, then

$$\lambda_{qrr}[a_{(1\,p)}, a_{(1\,p+1)}]_c = \lambda_{qrr}[a_{(1\,p)}, a_{(1\,p+2)}]_c = \lambda_{qrr}[a_{(1\,p+1)}, a_{(1\,p+2)}]_c = 0.$$

(2) If $4 \le s = p+1, p+2$, then

$$\lambda_{qrr}[a_{(1\,p)}, a_{(1\,s)}]_c = -\ 3\xi^2\lambda_{112}\lambda_{qrr}\chi_{(1\,s)}(g_{(2\,p)})a_{(3\,s)}a_{(2\,p)}$$
$$+\ 3\lambda_{112}\lambda_{qrr}\chi_1(g_{(3\,p)})a_{(3\,p)}a_{(2\,s)}.$$

(3) If $p \ge 3$, then

$$\lambda_{qrr}[a_{(1\,q)}, a_{(1\,r)}]_c = -3\lambda_{qrr}\xi^2\chi_{(1\,r)}(g_q)\lambda_{qqr}a^2_{(1\,p)}g_{qqr}$$
$$-\ 3\lambda_{qrr}\xi^2\lambda_{112}\chi_{(1\,r)}(g_{(2\,q)})a_{(3\,r)}a_{(2\,q)}$$
$$+\ 3\lambda_{qrr}\lambda_{112}\chi_1(g_{(3\,p)})\chi_q(g_{(2\,r)})\chi_{(1\,r)}(g_q)a_{(3\,q)}a_{(2\,r)}.$$ □

Proof. (1) follows using that $\lambda_{233}\lambda_{112} = \lambda_{344}\lambda_{112} = 0$.

(2) follows by Proposition 5.32 (1) using that $\lambda_{qrr}\lambda_{ppp+1} = 0$ by Lemma 5.1.

(3) is precisely Proposition 5.32 (2). ■

In particular, Corollary 5.33 gives

Corollary 5.34. Let $p \geq 3$. The following identities hold in $\widetilde{\mathcal{L}}$.

(1) $\lambda_{qrr}\lambda_{112}[a_{(3\,q)}, a_{(3\,r)}]_c = -3\lambda_{qrr}\lambda_{112}\lambda_{qqr}\xi^2\chi_{(3\,r)}(g_q)a^2_{(3\,p)}g_{qqr}$.

(2) $\lambda_{qrr}\lambda_{112}[a_{(3\,p)}, a_{(3\,r)}]_c = \lambda_{qrr}\lambda_{112}[a_{(3\,p)}, a_{(3\,q)}]_c = 0$. □

Corollary 5.35. Let $p \geq 3$, $s = q, r$. The following identities hold in $\widetilde{\mathcal{L}}$.

(1) $\lambda_{112}\lambda_{qrr}[a_{(1\,q)}, a_{(3\,p)}]_c = \lambda_{112}\lambda_{qrr}[a_{(1\,r)}, a_{(3\,p)}]_c = 0$.

(2) $\lambda_{112}\lambda_{qrr}[a_{(1\,r)}, a_{(3\,q)}]_c = 3\lambda_{112}\lambda_{qrr}\xi^2\lambda_{qqr}\chi_{(3\,p)}(g_{(1\,r)})a_{(3\,p)}a_{(1\,p)}g_{qqr}$.

(3) $\lambda_{112}\lambda_{qrr}[a_{(1\,q)}, a_{(3\,r)}]_c = \lambda_{112}\lambda_{qrr}\chi_{(3\,q)}(g_{(1\,q)})(1-\xi^2)a_{(3\,q)}a_{(1\,r)}$
$\qquad -3\lambda_{112}\lambda_{qrr}\lambda_{qqr}\chi_{(3\,p)}(g_{(1\,q)})a_{(3\,p)}a_{(1\,p)}g_{qqr}$.

(4) $\lambda_{112}\lambda_{qrr}[a_{(1\,p)}, a_{(3\,s)}]_c = -\lambda_{112}\lambda_{qrr}\chi_{(3\,p)}(g_{12})(1-\xi)a_{(3\,p)}a_{(1\,r)}$. □

Proof. (1) and (2) follow from Lemma 5.30. Also, q-Jacobi (2.2) and Lemma 5.30 together with Lemma 5.1 give

$$\begin{aligned}
\lambda_{112}\lambda_{qrr}[a_{(1\,q)}, a_{(3\,r)}]_c &= \lambda_{112}\lambda_{qrr}[a_{(1\,q)}, [a_{(3\,q)}, a_r]_c]_c \\
&= \lambda_{112}\lambda_{qrr}[[a_{(1\,q)}, a_{(3\,q)}]_c, a_r]_c \\
&\quad - \lambda_{112}\lambda_{qrr}\chi_r(g_{(3\,q)})a_{(1\,r)}a_{(3\,q)} + \lambda_{112}\lambda_{qrr}\chi_{(3\,q)}(g_{(1\,q)})a_{(3\,q)}a_{(1\,r)} \\
&= -\lambda_{112}\lambda_{qrr}\chi_r(g_{(3\,q)})[a_{(1\,r)}, a_{(3\,q)}]_c \\
&\quad + \lambda_{112}\lambda_{qrr}\chi_{(3\,q)}(g_{(1\,q)})(1-\xi^2)a_{(3\,q)}a_{(1\,r)}.
\end{aligned}$$

Hence now (3) follows using (2). (4) follows by Corollary 5.34 (2), using q-Jacobi. ■

Corollary 5.36. Let $p \geq 3$. The following identities hold in $\widetilde{\mathcal{L}}$.

(1) If $s = p, q, r$, then

$$\lambda_{112}\lambda_{qrr}[a_{(1\,s)}, a_{(2\,p)}]_c = -3\xi^2\lambda_{112}\lambda_{122}\lambda_{qrr}\chi_2(g_{(3\,s)})a_{(3\,s)}a_{(3\,p)}.$$

(2)

$$\lambda_{112}\lambda_{qrr}[a_{(1\,r)}, a_{(2\,q)}]_c = -3\xi^2\lambda_{112}\lambda_{122}\lambda_{qrr}\chi_2(g_{(3\,r)})a_{(3\,r)}a_{(3\,q)} + 3\lambda_{112}\lambda_{qrr}\xi^2\lambda_{qqr}\chi_{(2\,p)}(g_{(1\,r)})a_{(2\,p)}a_{(1\,p)}g_{qqr}.$$

(3)

$$\begin{aligned}\lambda_{112}\lambda_{qrr}[a_{(1\,q)}, a_{(2\,r)}]_c = &\ 9\xi\lambda_{112}\lambda_{122}\lambda_{qqr}\lambda_{qrr}\chi_2(g_{(3\,q)})\chi_{(3\,r)}(g_q)a^2_{(3\,p)}g_{qqr}\\ &+ 3(1+\xi)\lambda_{112}\lambda_{122}\lambda_{qrr}\chi_{(2\,r)}(g_{(3\,q)})a_{(3\,r)}a_{(3\,q)}\\ &+ \lambda_{112}\lambda_{qrr}\chi_{(2\,q)}(g_{(1\,q)})(1-\xi^2)a_{(2\,q)}a_{(1\,r)}\\ &- 3\lambda_{112}\lambda_{qrr}\lambda_{qqr}\chi_{(2\,p)}(g_{(1\,q)})a_{(2\,p)}a_{(1\,p)}g_{qqr}.\end{aligned}$$

(4) If $s = q, r$, then

$$\lambda_{112}\lambda_{qrr}[a_{(1\,p)}, a_{(2\,s)}]_c = -3\xi\lambda_{112}\lambda_{122}\lambda_{qrr}\chi_2(g_{(2\,p)})a_{(3\,s)}a_{(3\,p)} - \lambda_{112}\lambda_{qrr}\chi_{(2\,p)}(g_1)(1-\xi)a_{(2\,p)}a_{(1\,s)}.$$ □

Proof. (1) follows by Lemmas 5.1, 5.28 (1) and 5.30 and Corollary 5.33 (1) We use that $\lambda_{qrr}\lambda_{p-1p-1p} = \lambda_{qrr}\lambda_{p-1pp} = 0$.

(2) We have, using q-Jacobi (2.2),

$$\begin{aligned}\lambda_{112}\lambda_{qrr}[a_{(1\,r)}, a_{(2\,q)}]_c &= \lambda_{112}\lambda_{qrr}[a_{(1\,r)}, [a_2, a_{(3\,q)}]_c]_c\\ &= \lambda_{112}\lambda_{qrr}[[a_{(1\,r)}, a_2]_c, a_{(3\,q)}]_c + \lambda_{112}\lambda_{qrr}\chi_2(g_{(1\,r)})\big(a_2[a_{(1\,r)}, a_{(3\,q)}]_c\\ &\quad - \chi_{(3\,q)}(g_2)\chi_{(1\,r)}(g_2)[a_{(1\,r)}, a_{(3\,q)}]_c a_2\big).\end{aligned}$$

Now, by Lemma 5.28 (1) and Corollary 5.34,

$$\begin{aligned}&\lambda_{112}\lambda_{qrr}[[a_{(1\,r)}, a_2]_c, a_{(3\,q)}]_c\\ &= \lambda_{112}\lambda_{122}\lambda_{qrr}(1-\xi^2)\chi_2(g_{(3\,r)})\big(a_{(3\,r)}a_{(3\,q)} - \chi_{(3\,q)}(g_{(3\,r)}g_{122})a_{(3\,q)}a_{(3\,r)}\big)\\ &= -3\xi^2\lambda_{112}\lambda_{122}\lambda_{qrr}\chi_2(g_{(3\,r)})a_{(3\,r)}a_{(3\,q)}\\ &\quad - \lambda_{112}\lambda_{122}\lambda_{qrr}(1-\xi^2)\xi\chi_2(g_{(3\,r)})\chi_{(3\,q)}(g_{(3\,r)})[a_{(3\,q)}, a_{(3\,r)}]_c\\ &= -3\xi^2\lambda_{112}\lambda_{122}\lambda_{qrr}\chi_2(g_{(3\,r)})a_{(3\,r)}a_{(3\,q)}\\ &\quad + 3\lambda_{112}\lambda_{122}\lambda_{qrr}\lambda_{qqr}(1-\xi^2)\chi_{(2\,p)}(g_{(3\,r)})a^2_{(3\,p)}g_{qqr}.\end{aligned}$$

Set $s = 3\lambda_{112}\lambda_{qrr}\xi^2\lambda_{qqr}\chi_{(3\,p)}(g_{(1\,r)})$, by Corollary 5.35 (2),

$$\begin{aligned}
&\lambda_{112}\lambda_{qrr}\big(a_2[a_{(1\,r)}, a_{(3\,q)}]_c - \chi_{(3\,q)}(g_2)\chi_{(1\,r)}(g_2)[a_{(1\,r)}, a_{(3\,q)}]_c a_2\big)\\
&\quad = s\big(a_2 a_{(3\,p)} a_{(1\,p)} - \chi_{(3\,p)}(g_2)\chi_{(1\,p)}(g_2) a_{(3\,p)} a_{(1\,p)} a_2\big) g_{qqr}\\
&\quad = s(a_{(2\,p)} a_{(1\,p)} - \lambda_{122}(\xi^2 - \xi)\chi_{(1\,p)}(g_2) a_{(3\,p)}^2) g_{qqr}\\
&\qquad + s(\chi_{(3\,p)}(g_2) - \chi_{(3\,p)}(g_2)\chi_{(1\,p)}(g_2)\chi_2(g_{(1\,p)})) a_{(3\,p)} a_2 a_{(1\,p)} g_{qqr}\\
&\quad = s(a_{(2\,p)} a_{(1\,p)} + \lambda_{122}(1 - \xi)\chi_{(3\,p)}(g_2) a_{(3\,p)}^2) g_{qqr},
\end{aligned}$$

using once again Lemma 5.28 (1) and $1 = \chi_{(1\,p)}(g_2)\chi_2(g_{(1\,p)})$. Adding up,

$$\begin{aligned}
\lambda_{112}\lambda_{qrr}[a_{(1\,r)}, a_{(2\,q)}]_c &= -3\xi^2\lambda_{112}\lambda_{122}\lambda_{qrr}\chi_2(g_{(3\,r)}) a_{(3\,r)} a_{(3\,q)}\\
&\quad + \chi_2(g_{(1\,r)}) s a_{(2\,p)} a_{(1\,p)} g_{qqr}.
\end{aligned}$$

Here we have used that, as $\chi_{(3\,p)}(g_2)\chi_{(2\,p)}(g_{12}) = \chi_{(3\,p)}(g_{122})\xi^2 = 1$,

$$\begin{aligned}
&\lambda_{122}(1 - \xi)\chi_2(g_{(1\,r)})\chi_{(3\,p)}(g_2) s\\
&\quad = 3\lambda_{112}\lambda_{122}\lambda_{qqr}\lambda_{qrr}\xi^2(1 - \xi)\chi_2(g_{(1\,r)})\chi_{(3\,p)}(g_2)\chi_{(3\,p)}(g_{(1\,r)})\\
&\quad = -3\lambda_{112}\lambda_{122}\lambda_{qqr}\lambda_{qrr}(1 - \xi^2)\chi_{(3\,p)}(g_2)\chi_{(2\,p)}(g_{12})\chi_{(2\,p)}(g_{(3\,r)})\\
&\quad = -3\lambda_{112}\lambda_{122}\lambda_{qqr}\lambda_{qrr}(1 - \xi^2)\chi_{(2\,p)}(g_{(3\,r)}).
\end{aligned}$$

Hence the terms corresponding to $a_{(3\,p)}^2 g_{qqr}$ cancel.

(3) We have, using q-Jacobi,

$$\begin{aligned}
\lambda_{112}\lambda_{qrr}[a_{(1\,q)}, a_{(2\,r)}]_c &= \lambda_{112}\lambda_{qrr}[[a_{(1\,q)}, [a_2, a_{(3\,r)}]_c]_c\\
&= \lambda_{112}\lambda_{qrr}[[a_{(1\,q)}, a_2]_c, a_{(3\,r)}]_c - \lambda_{112}\lambda_{qrr}\chi_{(3\,r)}(g_2)[a_{(1\,q)}, a_{(3\,r)}]_c a_2\\
&\quad + \lambda_{112}\lambda_{qrr}\chi_2(g_{(1\,q)}) a_2[a_{(1\,q)}, a_{(3\,r)}]_c.
\end{aligned}$$

Now, by Lemma 5.28, and Corollary 5.34,

$$\begin{aligned}
\lambda_{112}\lambda_{qrr}[[a_{(1\,q)}, a_2]_c, a_{(3\,r)}]_c &= \lambda_{112}\lambda_{122}\lambda_{qrr}(1 - \xi^2)\chi_2(g_{(3\,q)})[a_{(3\,q)}, a_{(3\,r)}]_c\\
&\quad + \lambda_{112}\lambda_{122}\lambda_{qrr}(1 - \xi^2)\chi_2(g_{(3\,q)})\chi_{(3\,r)}(g_{(3\,q)})(1 - \xi) a_{(3\,r)} a_{(3\,q)}\\
&= -3\xi^2(1 - \xi^2)\lambda_{112}\lambda_{122}\lambda_{qqr}\lambda_{qrr}\chi_2(g_{(3\,q)})\chi_{(3\,r)}(g_q) a_{(3\,p)}^2 g_{qqr}\\
&\quad + 3\lambda_{112}\lambda_{122}\lambda_{qrr}\chi_{(2\,r)}(g_{(3\,q)}) a_{(3\,r)} a_{(3\,q)}.
\end{aligned}$$

On the other hand, by Corollary 5.35, Lemma 5.28 and Corollary 5.34,

$$
\begin{aligned}
&\lambda_{112}\lambda_{qrr}[a_{(1\,q)}, a_{(3\,r)}]_c a_2 = \\
&\quad = \lambda_{112}\lambda_{qrr}\chi_{(3\,q)}(g_{(1\,q)})(1-\xi^2)a_{(3\,q)}a_{(1\,r)}a_2 \\
&\qquad - 3\lambda_{112}\lambda_{qrr}\lambda_{qqr}\chi_{(3\,p)}(g_{(1\,q)})a_{(3\,p)}a_{(1\,p)}a_2 g_{qqr} \\
&\quad = \lambda_{112}\lambda_{qrr}\chi_{(3\,q)}(g_{(1\,q)})\chi_2(g_{(1\,r)})(1-\xi^2)a_{(3\,q)}a_2 a_{(1\,r)} \\
&\qquad + \lambda_{112}\lambda_{122}\lambda_{qrr}(1-\xi^2)^2\chi_{(3\,q)}(g_{(1\,q)})\chi_2(g_{(3\,r)})a_{(3\,q)}a_{(3\,r)} \\
&\qquad - 3\lambda_{112}\lambda_{qrr}\lambda_{qqr}\chi_{(3\,p)}(g_{(1\,q)})\chi_2(g_{(1\,p)})a_{(3\,p)}a_2 a_{(1\,p)}g_{qqr} \\
&\qquad - 3\lambda_{112}\lambda_{122}\lambda_{qrr}\lambda_{qqr}(1-\xi^2)\chi_{(3\,p)}(g_{(1\,q)})\chi_2(g_{(3\,p)})a^2_{(3\,p)}g_{qqr} \\
&\quad = \lambda_{112}\lambda_{qrr}\chi_{(3\,q)}(g_{(1\,q)})\chi_2(g_{(1\,r)})(1-\xi^2)a_{(3\,q)}a_2 a_{(1\,r)} \\
&\qquad - 3\xi^2\lambda_{112}\lambda_{122}\lambda_{qrr}\chi_{(3\,q)}(g_{(1\,q)})\chi_2(g_{(3\,r)})\chi_{(3\,r)}(g_{(3\,q)})a_{(3\,r)}a_{(3\,q)} \\
&\qquad - 3\xi^2\lambda_{112}\lambda_{122}\lambda_{qqr}\lambda_{qrr}(1-\xi^2)^2\chi_{(3\,q)}(g_{(1\,q)})\chi_2(g_{(3\,r)})\chi_{(3\,r)}(g_q)a^2_{(3\,p)}g_{qqr} \\
&\qquad - 3\lambda_{112}\lambda_{qrr}\lambda_{qqr}\chi_{(3\,p)}(g_{(1\,q)})\chi_2(g_{(1\,p)})a_{(3\,p)}a_2 a_{(1\,p)}g_{qqr} \\
&\qquad - 3\lambda_{112}\lambda_{122}\lambda_{qrr}\lambda_{qqr}(1-\xi^2)\chi_{(3\,p)}(g_{(1\,q)})\chi_2(g_{(3\,p)})a^2_{(3\,p)}g_{qqr} \\
&\quad = -3\xi^2\lambda_{112}\lambda_{122}\lambda_{qrr}\chi_{(3\,q)}(g_{(1\,q)})\chi_2(g_{(3\,r)})\chi_{(3\,r)}(g_{(3\,q)})a_{(3\,r)}a_{(3\,q)} \\
&\qquad + (1-\xi^2)\lambda_{112}\lambda_{qrr}\chi_{(2\,q)}(g_{(1\,q)})\chi_2(g_r)a_{(3\,q)}a_2 a_{(1\,r)} \\
&\qquad + 3(1-\xi)\lambda_{112}\lambda_{122}\lambda_{qqr}\lambda_{qrr}\chi_{(3\,p)}(g_{(1\,q)})\chi_2(g_{(3\,p)})a^2_{(3\,p)}g_{qqr} \\
&\qquad - 3\lambda_{112}\lambda_{qrr}\lambda_{qqr}\chi_{(3\,p)}(g_{(1\,q)})\chi_2(g_{(1\,p)})a_{(3\,p)}a_2 a_{(1\,p)}g_{qqr}.
\end{aligned}
$$

Again, by Corollary 5.35,

$$
\begin{aligned}
\lambda_{112}\lambda_{qrr}a_2[a_{(1\,q)}, a_{(3\,r)}]_c = {} & \lambda_{112}\lambda_{qrr}\chi_{(3\,q)}(g_{(1\,q)})(1-\xi^2)a_{(2\,q)}a_{(1\,r)} \\
& + \lambda_{112}\lambda_{qrr}\chi_{(3\,q)}(g_{(1\,q)})\chi_{(3\,q)}(g_2)(1-\xi^2)a_{(3\,q)}a_2 a_{(1\,r)} \\
& - 3\lambda_{112}\lambda_{qrr}\lambda_{qqr}\chi_{(3\,p)}(g_{(1\,q)})a_{(2\,p)}a_{(1\,p)}g_{qqr} \\
& - 3\lambda_{112}\lambda_{qrr}\lambda_{qqr}\chi_{(3\,p)}(g_{(1\,q)})\chi_{(3\,p)}(g_2)a_{(3\,p)}a_2 a_{(1\,p)}g_{qqr}.
\end{aligned}
$$

Adding up,

$$
\begin{aligned}
\lambda_{112}\lambda_{qrr}[a_{(1\,q)}, a_{(2\,r)}]_c = {} & 9\xi\lambda_{112}\lambda_{122}\lambda_{qqr}\lambda_{qrr}\chi_2(g_{(3\,q)})\chi_{(3\,r)}(g_q)a^2_{(3\,p)}g_{qqr} \\
& + 3(1+\xi)\lambda_{112}\lambda_{122}\lambda_{qrr}\chi_{(2\,r)}(g_{(3\,q)})a_{(3\,r)}a_{(3\,q)}
\end{aligned}
$$

$$+ \lambda_{112}\lambda_{qrr}\chi_{(2\,q)}(g_{(1\,q)})(1-\xi^2)a_{(2\,q)}a_{(1\,r)}$$
$$- 3\lambda_{112}\lambda_{qrr}\lambda_{qqr}\chi_{(2\,p)}(g_{(1\,q)})a_{(2\,p)}a_{(1\,p)}g_{qqr}.$$

(4) follows using (1) together with q-Jacobi and Lemma 5.1. ■

Next, we order the elements 5.29 in terms of the PBW basis. We have

Proposition 5.37. Assume $p \geq 3$. Then

$$\begin{aligned}
\lambda_{qrr}a_{p,q,r} = {} & \lambda_{qrr}\xi\,\chi_q(g_{(1\,p)})a_{(1\,r)}a_{(1\,q)}a_{(1\,p)} \\
& - 3\xi^2\lambda_{qrr}\lambda_{qqr}\chi_{(1\,r)}(g_q)a^3_{(1\,p)}g_{qqr} \\
& - 3\xi^2\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_{(1\,r)}(g_{(2\,q)})a_{(3\,r)}a_{(2\,q)}a_{(1\,p)} \\
& + 3\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(3\,q)})a_{(3\,q)}a_{(2\,r)}a_{(1\,p)} \\
& - 3\xi^2\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_{(3\,r)}(g_{(2\,q)})\chi_{(2\,p)}(g_{(1\,q)})a_{(3\,r)}a_{(2\,p)}a_{(1\,q)} \\
& + 3\lambda_{qrr}\lambda_{112}\chi_{(3\,p)}(g_{12})\chi_r(g_{(1\,q)})a_{(3\,p)}a_{(2\,r)}a_{(1\,q)} \\
& + 3\lambda_{qrr}\lambda_{112}(1-\xi^2)\chi_{(3\,p)}(g_{12})a_{(3\,p)}a_{(2\,q)}a_{(1\,r)} \\
& + 6\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(2\,p)})a_{(3\,q)}a_{(2\,p)}a_{(1\,r)}.
\end{aligned} \tag{5.36}$$

$$\begin{aligned}
\lambda_{qrr}a_{p,r,q} = {} & \lambda_{qrr}\xi^2\chi_{qqr}(g_{(1\,p)})a_{(1\,r)}a_{(1\,q)}a_{(1\,p)} \\
& - 3\xi^2\lambda_{112}\lambda_{qrr}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,q)}(g_{(1\,r)})\chi_r(g_1)a_{(3\,q)}a_{(2\,p)}a_{(1\,r)} \\
& + 3\lambda_{112}\lambda_{qrr}\chi_1(g_{(3\,p)})\chi_{(1\,q)}(g_r)a_{(3\,p)}a_{(2\,q)}a_{(1\,r)} \\
& - 3\xi^2\lambda_{112}\lambda_{qrr}\chi_{(1\,r)}(g_{(2\,p)})a_{(3\,r)}a_{(2\,p)}a_{(1\,q)} \\
& + 3\lambda_{112}\lambda_{qrr}\chi_1(g_{(3\,p)})a_{(3\,p)}a_{(2\,r)}a_{(1\,q)}.
\end{aligned} \tag{5.37}$$

$$\begin{aligned}
\lambda_{qrr}a_{r,p,q} = {} & \xi\,\chi_q(g_{(1\,p)})\lambda_{qrr}a_{(1\,r)}a_{(1\,q)}a_{(1\,p)} \\
& - 3\xi^2\lambda_{112}\lambda_{qrr}\chi_{(3\,q)}(g_{(1\,r)})\chi_{(2\,p)}(g_r)a_{(3\,q)}a_{(2\,p)}a_{(1\,r)} \\
& + 3\lambda_{112}\lambda_{qrr}\chi_{(3\,p)}(g_{(2\,r)})\chi_{(2\,q)}(g_{(1\,r)})a_{(3\,p)}a_{(2\,q)}a_{(1\,r)}.
\end{aligned} \tag{5.38}$$

□

Proof. We will go through the description of $\lambda_{qrr}a_{p,q,r}$ step by step, following the identities in the lemmas. The other two summands are simpler and will be presented in their final form. Every time there is a monomial that needs to be ordered, we shall highlight it on bold letters, for the reader to identify which is the bracket that needs to be computed for the next step. To do this, we shall use Corollaries 5.33, 5.35, 5.36, and 5.34.

We shall also reduce some of the scalars, for instance, we consider

$$\lambda_{qrr}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(1\,r)}(g_{(1\,p)}g_{(1\,q)}) = \lambda_{qrr}\chi_r(g_{(1\,q)}).$$

However, we leave a full reduction to the end.

We have, using Corollary 5.33,

$$\begin{aligned}
\lambda_{qrr}a_{p,q,r} &= \lambda_{qrr}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(1\,r)}(g_{(1\,p)}g_{(1\,q)})a_{(1\,r)}a_{(1\,q)}a_{(1\,p)}\\
&\quad + \chi_{(1\,q)}(g_{(1\,p)})\chi_{(1\,r)}(g_{(1\,p)})\lambda_{qrr}[a_{(1\,q)}, a_{(1\,r)}]_c a_{(1\,p)}\\
&\quad + \chi_{(1\,q)}(g_{(1\,p)})\lambda_{qrr}a_{(1\,q)}[a_{(1\,p)}, a_{(1\,r)}]_c + \lambda_{qrr}[a_{(1\,p)}, a_{(1\,q)}]_c a_{(1\,r)}\\
&= \lambda_{qrr}\xi\chi_q(g_{(1\,p)})a_{(1\,r)}a_{(1\,q)}a_{(1\,p)}\\
&\quad - \chi_{(1\,q)}(g_{(1\,p)})\chi_{(1\,r)}(g_{(1\,p)})3\lambda_{qrr}\xi^2\chi_{(1\,r)}(g_q)\lambda_{qqr}\chi_{(1\,p)}(g_{qqr})a^3_{(1\,p)}g_{qqr}\\
&\quad - \chi_{(1\,q)}(g_{(1\,p)})\chi_{(1\,r)}(g_{(1\,p)})3\lambda_{qrr}\xi^2\lambda_{112}\chi_{(1\,r)}(g_{(2\,q)})a_{(3\,r)}a_{(2\,q)}a_{(1\,p)}\\
&\quad + \chi_{(1\,q)}(g_{(1\,p)})\chi_{(1\,r)}(g_{(1\,p)})3\lambda_{qrr}\lambda_{112}\chi_1(g_{(3\,p)})\chi_q(g_{(2\,r)})\chi_{(1\,r)}(g_q)\\
&\quad a_{(3\,q)}a_{(2\,r)}a_{(1\,p)}\\
&\quad - \chi_{(1\,q)}(g_{(1\,p)})\lambda_{qrr}3\xi^2\lambda_{112}\chi_{(1\,r)}(g_{(2\,p)})\mathbf{a}_{(\mathbf{1\,q})}\mathbf{a}_{(\mathbf{3\,r})}a_{(2\,p)}\\
&\quad + \chi_{(1\,q)}(g_{(1\,p)})\lambda_{qrr}3\lambda_{112}\chi_1(g_{(3\,p)})\mathbf{a}_{(\mathbf{1\,q})}\mathbf{a}_{(\mathbf{3\,p})}a_{(2\,r)}\\
&\quad - 3\xi^2\lambda_{112}\lambda_{qrr}\chi_{(1\,q)}(g_{(2\,p)})a_{(3\,q)}a_{(2\,p)}a_{(1\,r)}.
\end{aligned}$$

Using $\chi_{(1\,q)}(g_{(1\,p)}) = \xi\chi_q(g_{(1\,p)})$, and

$$\begin{aligned}
\lambda_{qqr}\chi_{(1\,p)}(g_{qqr}) &= \lambda_{qqr}\xi,\\
\lambda_{qqr}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(1\,r)}(g_{(1\,p)}) &= \lambda_{qqr}\xi^2\\
\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(1\,r)}(g_{(1\,p)}) &= \lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)}),\\
\chi_1(g_{(3\,p)})\chi_q(g_{(2\,r)})\chi_{(1\,r)}(g_q) &= \chi_1(g_{(3\,q)}),\\
\chi_{(1\,q)}(g_{(1\,p)})\chi_{(1\,r)}(g_{(2\,p)}) &= \xi\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)}),
\end{aligned}$$

this becomes, applying Corollary 5.35,

$$\begin{aligned}
\lambda_{qrr}a_{p,q,r} &= \lambda_{qrr}\chi_r(g_{(1\,q)})a_{(1\,r)}a_{(1\,q)}a_{(1\,p)}\\
&\quad - 3\xi^2\lambda_{qrr}\lambda_{qqr}\chi_{(1\,r)}(g_q)a^3_{(1\,p)}g_{qqr}
\end{aligned}$$

$$
\begin{aligned}
&- 3\xi^2\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_{(1\,r)}(g_{(2\,q)})a_{(3\,r)}a_{(2\,q)}a_{(1\,p)}\\
&+ 3\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(3\,q)})a_{(3\,q)}a_{(2\,r)}a_{(1\,p)}\\
&- 3\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,r)}(g_{(1\,q)})a_{(3\,r)}\mathbf{a}_{(\mathbf{1}\,\mathbf{q})}\mathbf{a}_{(\mathbf{2}\,\mathbf{p})}\\
&- 3\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})[a_{(1\,q)}, a_{(3\,r)}]_c a_{(2\,p)}\\
&+ 3\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(1\,p)})\chi_1(g_{(3\,p)})\chi_{(3\,p)}(g_{(1\,q)})a_{(3\,p)}\mathbf{a}_{(\mathbf{1}\,\mathbf{q})}\mathbf{a}_{(\mathbf{2}\,\mathbf{r})}\\
&+ 3\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(1\,p)})\chi_1(g_{(3\,p)})[a_{(1\,q)}, a_{(3\,p)}]_c a_{(2\,r)}\\
&- 3\xi^2\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(2\,p)})a_{(3\,q)}a_{(2\,p)}a_{(1\,r)}\\
= {}& \lambda_{qrr}\chi_r(g_{(1\,q)})a_{(1\,r)}a_{(1\,q)}a_{(1\,p)}\\
&- 3\xi^2\lambda_{qrr}\lambda_{qqr}\chi_{(1\,r)}(g_q)a_{(1\,p)}^3 g_{qqr}\\
- {}& 3\xi^2\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_{(1\,r)}(g_{(2\,q)})a_{(3\,r)}a_{(2\,q)}a_{(1\,p)}\\
&+ 3\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(3\,q)})a_{(3\,q)}a_{(2\,r)}a_{(1\,p)}\\
&- 3\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,r)}(g_{(1\,q)})a_{(3\,r)}\mathbf{a}_{(\mathbf{1}\,\mathbf{q})}\mathbf{a}_{(\mathbf{2}\,\mathbf{p})}\\
&- 3(1-\xi^2)\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,q)}(g_{(1\,q)})a_{(3\,q)}\mathbf{a}_{(\mathbf{1}\,\mathbf{r})}\mathbf{a}_{(\mathbf{2}\,\mathbf{p})}\\
&+ 9\xi\lambda_{qrr}\lambda_{112}\lambda_{qqr}\chi_1(g_{(1\,r)})\chi_{(3\,p)}(g_{(1\,q)})a_{(3\,p)}\mathbf{a}_{(\mathbf{1}\,\mathbf{p})}\mathbf{a}_{(\mathbf{2}\,\mathbf{p})}g_{qqr}\\
&+ 3\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})a_{(3\,p)}\mathbf{a}_{(\mathbf{1}\,\mathbf{q})}\mathbf{a}_{(\mathbf{2}\,\mathbf{r})}\\
&- 3\xi^2\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(2\,p)})a_{(3\,q)}a_{(2\,p)}a_{(1\,r)}.
\end{aligned}
$$

We have also used

$$
\lambda_{qqr}\chi_{qqr}(g_{(1\,p)})\chi_{(2\,p)}(g_{qqr}) = \lambda_{qqr}\xi, \; \chi_1(g_{(3\,p)})\chi_{(3\,p)}(g_{(1\,q)}) = \chi_{(3\,p)}(g_{(2\,q)}).
$$

We obtain

$$
\begin{aligned}
\lambda_{qrr}a_{p,q,r} = {}& \lambda_{qrr}\chi_r(g_{(1\,q)})a_{(1\,r)}a_{(1\,q)}a_{(1\,p)}\\
&- 3\xi^2\lambda_{qrr}\lambda_{qqr}\chi_{(1\,r)}(g_q)a_{(1\,p)}^3 g_{qqr}\\
&- 3\xi^2\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_{(1\,r)}(g_{(2\,q)})a_{(3\,r)}a_{(2\,q)}a_{(1\,p)}\\
&+ 3\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(3\,q)})a_{(3\,q)}a_{(2\,r)}a_{(1\,p)}\\
&- 3\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,r)}(g_{(1\,q)})\chi_{(2\,p)}(g_{(1\,q)})a_{(3\,r)}a_{(2\,p)}a_{(1\,q)}\\
&- 3\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,r)}(g_{(1\,q)})a_{(3\,r)}[a_{(1\,q)}, a_{(2\,p)}]_c\\
&- 3(1-\xi^2)\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,q)}(g_{(1\,q)})\chi_{(2\,p)}(g_{(1\,r)})a_{(3\,q)}a_{(2\,p)}a_{(1\,r)}
\end{aligned}
$$

$$
\begin{aligned}
&- 3(1-\xi^2)\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,q)}(g_{(1\,q)})a_{(3\,q)}[a_{(1\,r)}, a_{(2\,p)}]_c \\
&+ 9\xi\lambda_{qrr}\lambda_{112}\lambda_{qqr}\chi_1(g_{(1\,r)})\chi_{(2\,p)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(1\,q)})a_{(3\,p)}a_{(2\,p)}a_{(1\,p)}g_{qqr} \\
&+ 9\xi\lambda_{qrr}\lambda_{112}\lambda_{qqr}\chi_1(g_{(1\,r)})\chi_{(3\,p)}(g_{(1\,q)})a_{(3\,p)}[a_{(1\,p)}, a_{(2\,p)}]_c g_{qqr} \\
&+ 3\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})\chi_{(2\,r)}(g_{(1\,q)})a_{(3\,p)}a_{(2\,r)}a_{(1\,q)} \\
&+ 3\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})a_{(3\,p)}[a_{(1\,q)}, a_{(2\,r)}]_c \\
&- 3\xi^2\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(2\,p)})a_{(3\,q)}a_{(2\,p)}a_{(1\,r)}.
\end{aligned}
$$

Observe that $\chi_1(g_{(1\,r)})\chi_{(2\,p)}(g_{(1\,r)}) = \chi_{(1\,p)}(g_{(1\,r)})$ and that we can add the two terms $a_{(3\,q)}a_{(2\,p)}a_{(1\,r)}$ and the corresponding scalar becomes

$$
\begin{aligned}
&- 3(1-\xi^2)\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_{(3\,q)}(g_{(1\,q)})\chi_{(1\,p)}(g_{(1\,r)}) \\
&- 3\xi^2\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(2\,p)}) = -3\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(2\,p)})\xi^2 \times \\
&\quad \times \Big((1-\xi^2)\xi\,\chi_{qqr}(g_{(1\,p)})\chi_{(3\,q)}(g_{(1\,q)})\chi_{(1\,p)}(g_{(1\,r)})\chi_{(2\,p)}(g_{(1\,q)}) + 1\Big) \\
&= -3\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(2\,p)})\xi^2\Big((1-\xi^2)\xi^2 + 1\Big) = 6\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(2\,p)}).
\end{aligned}
$$

Now, we apply Corollary 5.36:

$$
\begin{aligned}
\lambda_{qrr}a_{p,q,r} = {}& \lambda_{qrr}\chi_r(g_{(1\,q)})a_{(1\,r)}a_{(1\,q)}a_{(1\,p)} \\
&- 3\xi^2\lambda_{qrr}\lambda_{qqr}\chi_{(1\,r)}(g_q)a_{(1\,p)}^3 g_{qqr} \\
&- 3\xi^2\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_{(1\,r)}(g_{(2\,q)})a_{(3\,r)}a_{(2\,q)}a_{(1\,p)} \\
&+ 3\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(3\,q)})a_{(3\,q)}a_{(2\,r)}a_{(1\,p)} \\
&- 3\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,r)}(g_{(1\,q)})\chi_{(2\,p)}(g_{(1\,q)})a_{(3\,r)}a_{(2\,p)}a_{(1\,q)} \\
&+ 9\xi^2\lambda_{qrr}\lambda_{112}\lambda_{122}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,r)}(g_{(1\,q)})\chi_2(g_{(3\,q)})a_{(3\,r)}a_{(3\,q)}a_{(3\,p)} \\
&+ 9\xi^2(1-\xi^2)\lambda_{qrr}\lambda_{112}\lambda_{122}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,q)}(g_{(1\,q)})\chi_2(g_{(3\,r)})\mathbf{a}_{(\mathbf{3\,q})}\mathbf{a}_{(\mathbf{3\,r})}a_{(3\,p)} \\
&+ 9\xi\lambda_{qrr}\lambda_{112}\lambda_{qqr}\chi_1(g_{(1\,r)})\chi_{(2\,p)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(1\,q)})a_{(3\,p)}a_{(2\,p)}a_{(1\,p)}g_{qqr} \\
&- 27\lambda_{qrr}\lambda_{112}\lambda_{122}\lambda_{qqr}\chi_1(g_{(1\,r)})\chi_{(3\,p)}(g_{(1\,q)})\chi_2(g_{(3\,p)})a_{(3\,p)}^3 g_{qqr} \\
&+ 3\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})\chi_{(2\,r)}(g_{(1\,q)})a_{(3\,p)}a_{(2\,r)}a_{(1\,q)} \\
&+ 27\xi\lambda_{qrr}\lambda_{112}\lambda_{122}\lambda_{qqr}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})\chi_2(g_{(3\,q)})\chi_{(3\,r)}(g_q)a_{(3\,p)}^3 g_{qqr} \\
&+ 9(1+\xi)\lambda_{qrr}\lambda_{112}\lambda_{122}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})\chi_{(2\,r)}(g_{(3\,q)})\mathbf{a}_{(\mathbf{3\,p})}\mathbf{a}_{(\mathbf{3\,r})}a_{(3\,q)} \\
&+ 3\lambda_{qrr}\lambda_{112}(1-\xi^2)\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})\chi_{(2\,q)}(g_{(1\,q)})a_{(3\,p)}a_{(2\,q)}a_{(1\,r)}
\end{aligned}
$$

$$- 9\lambda_{qrr}\lambda_{112}\lambda_{qqr}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})\chi_{(2\,p)}(g_{(1\,q)})a_{(3\,p)}a_{(2\,p)}a_{(1\,p)}g_{qqr}$$
$$+ 6\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(2\,p)})a_{(3\,q)}a_{(2\,p)}a_{(1\,r)}.$$

Observe that the terms $a^3_{(3\,p)}g_{qqr}$ have the scalar

$$27(1-\xi^2)\lambda_{qrr}\lambda_{112}\lambda_{122}\lambda_{qqr}\chi_{(3\,p)}(g_q)\chi_q(g_1).$$

On the other hand, the terms $a_{(3\,p)}a_{(2\,p)}a_{(1\,p)}g_{qqr}$ cancel with each other.

We order the terms $a_{(3\,*)}a_{(3\,*)}a_{(3\,*)}$ using Corollary 5.34 and we get

$$\begin{aligned}
\lambda_{qrr}a_{p,q,r} = {} & \lambda_{qrr}\chi_r(g_{(1\,q)})a_{(1\,r)}a_{(1\,q)}a_{(1\,p)} \\
& - 3\xi^2\lambda_{qrr}\lambda_{qqr}\chi_{(1\,r)}(g_q)a^3_{(1\,p)}g_{qqr} \\
& - 3\xi^2\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_{(1\,r)}(g_{(2\,q)})a_{(3\,r)}a_{(2\,q)}a_{(1\,p)} \\
& + 3\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(3\,q)})a_{(3\,q)}a_{(2\,r)}a_{(1\,p)} \\
& - 3\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,r)}(g_{(1\,q)})\chi_{(2\,p)}(g_{(1\,q)})a_{(3\,r)}a_{(2\,p)}a_{(1\,q)} \\
& + 9\xi^2\lambda_{qrr}\lambda_{112}\lambda_{122}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,r)}(g_{(1\,q)})\chi_2(g_{(3\,q)})a_{(3\,r)}a_{(3\,q)}a_{(3\,p)} \\
& + 9\xi^2(1-\xi^2)\lambda_{qrr}\lambda_{112}\lambda_{122}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,q)}(g_{(1\,q)})\chi_2(g_{(3\,r)}) \\
& \chi_{(3\,r)}(g_{(3\,q)})a_{(3\,r)}a_{(3\,q)}a_{(3\,p)} \\
& + 9\xi^2(1-\xi^2)\lambda_{qrr}\lambda_{112}\lambda_{122}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,q)}(g_{(1\,q)})\chi_2(g_{(3\,r)}) \\
& [a_{(3\,q)}, a_{(3\,r)}]_c a_{(3\,p)} \\
& + 27(1-\xi^2)\lambda_{qrr}\lambda_{112}\lambda_{122}\lambda_{qqr}\chi_{(3\,p)}(g_q)\chi_q(g_1)a^3_{(3\,p)}g_{qqr} \\
& + 3\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})\chi_{(2\,r)}(g_{(1\,q)})a_{(3\,p)}a_{(2\,r)}a_{(1\,q)} \\
& + 9(1+\xi)\lambda_{qrr}\lambda_{112}\lambda_{122}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})\chi_{(2\,r)}(g_{(3\,q)})\chi_{(3\,r)}(g_{(3\,p)}) \\
& \chi_{(3\,q)}(g_{(3\,p)})a_{(3\,r)}a_{(3\,q)}a_{(3\,p)} \\
& + 9(1+\xi)\lambda_{qrr}\lambda_{112}\lambda_{122}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})\chi_{(2\,r)}(g_{(3\,q)})\chi_{(3\,r)}(g_{(3\,p)}) \\
& a_{(3\,r)}[a_{(3\,p)}, a_{(3\,q)}]_c \\
& + 9(1+\xi)\lambda_{qrr}\lambda_{112}\lambda_{122}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})\chi_{(2\,r)}(g_{(3\,q)}) \\
& [a_{(3\,p)}, a_{(3\,r)}]_c a_{(3\,q)} \\
& + 3\lambda_{qrr}\lambda_{112}(1-\xi^2)\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})\chi_{(2\,q)}(g_{(1\,q)})a_{(3\,p)}a_{(2\,q)}a_{(1\,r)} \\
& + 6\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(2\,p)})a_{(3\,q)}a_{(2\,p)}a_{(1\,r)}.
\end{aligned}$$

That is,

$$
\begin{aligned}
\lambda_{qrr} a_{p,q,r} = {} & \lambda_{qrr}\chi_r(g_{(1\,q)})a_{(1\,r)}a_{(1\,q)}a_{(1\,p)} \\
& - 3\xi^2\lambda_{qrr}\lambda_{qqr}\chi_{(1\,r)}(g_q)a^3_{(1\,p)}g_{qqr} \\
& - 3\xi^2\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_{(1\,r)}(g_{(2\,q)})a_{(3\,r)}a_{(2\,q)}a_{(1\,p)} \\
& + 3\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(3\,q)})a_{(3\,q)}a_{(2\,r)}a_{(1\,p)} \\
& - 3\lambda_{qrr}\lambda_{112}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,r)}(g_{(1\,q)})\chi_{(2\,p)}(g_{(1\,q)})a_{(3\,r)}a_{(2\,p)}a_{(1\,q)} \\
& + 9\xi^2\lambda_{qrr}\lambda_{112}\lambda_{122}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,r)}(g_{(1\,q)})\chi_2(g_{(3\,q)})a_{(3\,r)}a_{(3\,q)}a_{(3\,p)} \\
& + 9\xi^2(1-\xi^2)\lambda_{qrr}\lambda_{112}\lambda_{122}\chi_{qqr}(g_{(1\,p)})\chi_1(g_{(1\,r)})\chi_{(3\,q)}(g_{(1\,q)})\chi_2(g_{(3\,r)}) \\
& \chi_{(3\,r)}(g_{(3\,q)})a_{(3\,r)}a_{(3\,q)}a_{(3\,p)} \\
& - 27\xi^2(1-\xi^2)\lambda_{qrr}\lambda_{112}\lambda_{122}\lambda_{qqr}\chi_1(g_{(1\,r)})\chi_{(3\,q)}(g_{(1\,q)})\chi_2(g_{(3\,r)}) \\
& \chi_{(3\,r)}(g_q)a^3_{(3\,p)}g_{qqr} \\
& + 27(1-\xi^2)\lambda_{qrr}\lambda_{112}\lambda_{122}\lambda_{qqr}\chi_{(3\,p)}(g_q)\chi_q(g_1)a^3_{(3\,p)}g_{qqr} \\
& + 3\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})\chi_{(2\,r)}(g_{(1\,q)})a_{(3\,p)}a_{(2\,r)}a_{(1\,q)} \\
& + 9(1+\xi)\lambda_{qrr}\lambda_{112}\lambda_{122}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})\chi_{(2\,r)}(g_{(3\,q)})\chi_{(3\,r)}(g_{(3\,p)}) \\
& \chi_{(3\,q)}(g_{(3\,p)})a_{(3\,r)}a_{(3\,q)}a_{(3\,p)} \\
& + 3\lambda_{qrr}\lambda_{112}(1-\xi^2)\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})\chi_{(2\,q)}(g_{(1\,q)})a_{(3\,p)}a_{(2\,q)}a_{(1\,r)} \\
& + 6\lambda_{qrr}\lambda_{112}\chi_{(1\,q)}(g_{(2\,p)})a_{(3\,q)}a_{(2\,p)}a_{(1\,r)}.
\end{aligned}
$$

On the one hand, the terms involving $a^3_{(3\,p)}g_{qqr}$ cancel with each other, and so do the ones involving $a_{(3\,r)}a_{(3\,q)}a_{(3\,p)}$. We use

$$
\begin{aligned}
\lambda_{112}\chi_1(g_{(1\,r)})\chi_{(3\,r)}(g_{(1\,q)}) &= \lambda_{112}\xi^2\chi_{(3\,r)}(g_{(2\,q)}) \\
\lambda_{112}\chi_{(1\,q)}(g_{(1\,p)})\chi_{(3\,p)}(g_{(2\,q)})\chi_{(2\,q)}(g_{(1\,q)}) &= \lambda_{112}\chi_{(3\,p)}(g_{12})
\end{aligned}
$$

to simplify the scalars and we end up with 5.36.

Similar computations lead to 5.37 and 5.38. ■

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