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Comment on: Exact solution of the inverse-square-root potential $V(r) = -\frac{\alpha}{\sqrt{r}}$

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HIGHLIGHTS

- The Schrödinger equation with inverse square-root potential can be solved exactly.
- Exact solutions are available for the symmetric one-dimensional model as well as for the central-field one.
- The odd solutions of the one-dimensional model are also the s states of the central-field one.
- The WKB approximate solutions for the one-dimensional problem are also approximate solutions for the central-field model.
- The Riccati-Padé method provides accurate eigenvalues for both models.

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ABSTRACT

We study the connection between the solutions of the Schrödinger equation with an inverse square-root potential in three and one dimension. In particular we show that an approximate analytical expression for the eigenvalues of the one-dimensional model also applies to the three-dimensional one for zero angular-momentum quantum number.

Contents

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1. Introduction

In a recent paper Li and Dai [1] showed that the Schrödinger equation with the potential $V(r) = -\alpha r^{-1/2}$, $\alpha > 0$, can be solved exactly in terms of Heun biconfluent functions. They obtained the eigenvalues $E_{\nu,l}$ for $\nu, l = 0, 1, \ldots, 5$, where ν and l are the well known radial and angular-momentum quantum numbers, respectively. However, the resulting quantization condition does not seem quite amenable for the accurate calculation of the eigenvalues as reflected by the low accuracy of the results.

Somewhat earlier Ishkhanyan [2] solved the Schrödinger equation with the potential $V(x) = V_0/\sqrt{x}$, $x > 0$, in terms of confluent hypergeometric and Hermite functions. From the resulting quantization condition he derived an approximate WKB-like expression for the eigenvalues. On commenting about these results Li and Dai simply state that "in one-dimensional problem there is no centrifugal potential" [1]. In this way they disregarded the fact that such results should agree with those of the three-dimensional problem with $l = 0$.

Some time ago Fernández et al. [3] proposed a modification of the Riccati–Padé method (RPM) [4,5] for the calculation of the eigenvalues of the Schrödinger equation with the potential $V(r) = gr^{\alpha}$, where $\alpha > -2$ is a rational exponent and $g\alpha > 0$. As an illustrative example they applied the most straightforward form of the RPM to the case $\alpha = -1/2$ and obtained the ground-state energy quite accurately. The first four digits of their estimate $E_{00} = -0.438041241942506$ [3] agree with the less accurate result of Li and Dai [1]: $E_{00} = -0.4380$.

The purpose of this comment is to connect and compare the results of the one-dimensional and three-dimensional problems wherever possible.

2. Comparison of the models

The eigenvalues $E_{\nu,l}$ of the central-field potential $V(r) = -r^{-1/2}$ depend on the radial $\nu = 0, 1, \ldots$ and angular-momentum $l = 0, 1, \ldots$ quantum numbers. On the other hand, the eigenvalues E_ν of the one-dimensional problem $V(x) = -x^{-1/2}, x > 0$ depend on only one quantum number $\nu = 0, 1, \ldots$ and it is well known that $E_{\nu,0} = E_\nu$. The boundary condition at origin for the eigenfunction is $\psi(0) = 0$. For this reason the WKB-like expression for the eigenvalues of the one-dimensional problem [2]

$$E_n = \frac{V_0}{2}\left(\frac{-mV_0}{\hbar^2}\right)^{1/3}\left(n - \frac{1}{2\pi}\right)^{-2/3}, \quad n = 1, 2, \ldots, \tag{1}$$

should provide an approximation to the eigenvalues $E_{\nu,0}$. In order to show the connection we simply substitute $V_0 = -1$, $m = 1/2$, $\hbar = 1$ and $n = \nu + 1$ into Eq. (1) and obtain

$$E_\nu = 2^{-4/3}\left(\nu + 1 - \frac{1}{2\pi}\right)^{-2/3}, \quad \nu = 0, 1, \ldots. \tag{2}$$

Table 1 shows some of the eigenvalues E_ν calculated by means of the RPM [4,5] (see also [6] for more details on the approach and [7] for additional calculations on this kind of models), the results of Li and Dai and those given by Eq. (2). Note that the agreement between the WKB-like and numerical results improves as ν increases.

We may go a step further and consider the parity-invariant potential $V(x) = -|x|^{-1/2}$. The solutions to the Schrödinger equation are either even or odd. The odd solutions are similar to those discussed above with eigenvalues $E_{2\nu+1} = E_{\nu,0}$. On the other hand, we can obtain the even solutions from the radial ones with $l = -1$. In this case the boundary condition at origin is $\psi'(0) = 0$. It is not clear to us if it is easy to derive such solutions from the expressions given by Li and Dai but it is not difficult to obtain the eigenvalues $E_{2\nu} = E_{\nu,-1}$ by simply setting $l = -1$ into the main RPM equations given earlier [3,7].

Table 2 shows the first few eigenvalues $E_{2\nu}$ calculated by means of the RPM for the central-field model with $l = -1$. We are not aware of earlier calculations of these eigenvalues that we may use for comparison. The results shown above clearly reveal the close connection between the central-field and one-dimensional models discussed by Li and Dai [1] and Ishkhanyan [2], respectively. The fact that the results of the latter apply to the former when $l = 0$ was overlooked by Li and Dai.

Table 1
Some odd-state eigenvalues $E_{2\nu+1}$ for the potential $V(x) = -|x|^{-1/2}$.

ν	RPM	Li and Dai	WKB
0	−0.438041241942506	−0.4380	−0.4455
1	−0.263203069697059	−0.2632	−0.2642
2	−0.197558399925621	−0.1976	−0.1978
3	−0.161704966236690	−0.1617	−0.1618
4	−0.138637391239232	−0.1386	−0.1387
5	−0.122345763853305	−0.1223	−0.1224
6	−0.110122627415274	–	−0.11013

Table 2
Some even-state eigenvalues for the potential $V(x) = -|x|^{-1/2}$.

ν	$E_{2\nu}$
0	−1.65348620403344841606862823385833854691523175506233724893250
1	−0.35908762566388513180568678315121190723640443405090
2	−0.23716811682524113123544244944196509896799 29
3	−0.18408254522333127880008411375790820222 5
4	−0.15328442645850039237586280219473 68
5	−0.13279806876069409227835136347
6	−0.11801901845130838082754 52

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