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Jacobian algebras with periodic module category and exponential growth

Yadira Valdivieso-Díaz

Dean Funes 3350, Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Mar del Plata, Argentina

Abstract

Recently it was proven by Geiss, Labardini-Fragoso and Shöer in [1] that every Jacobian algebra associated to a triangulation of a closed surface S with a collection of marked points M is tame and Ladkani proved in [2] these algebras are (weakly) symmetric. In this work we show that for these algebras the Auslander-Reiten translation acts 2-periodically on objects. Moreover, we show that excluding only the case of a sphere with 4 (or less) punctures, these algebras are of exponential growth. These results imply that the existing characterization of symmetric tame algebras whose non-projective indecomposable modules are Ω-periodic, has at least a missing class (see [3, Theorem 6.2] or [4]).

As a consequence of the 2-periodical actions of the Auslander-Reiten translation on objects, we have that the Auslander-Reiten quiver of the generalized cluster category $\mathcal{C}_{(S,M)}$ consists only of stable tubes of rank 1 or 2.

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1. Introduction

Let k be an algebraically closed field. A potential W for a quiver Q is, roughly speaking, a linear combination of cyclic paths in the complete path algebra $k\langle\langle Q\rangle\rangle$. The Jacobian algebra $\mathcal{P}(Q, W)$ associated to a quiver with a potential (Q, W) is the quotient of the complete path algebra $k\langle\langle Q\rangle\rangle$ modulo the Jacobian ideal $J(W)$. Here, $J(W)$ is the topological closure closure of the ideal of $k\langle\langle Q\rangle\rangle$ which is generated by the cyclic derivatives of W with respect to the arrows of Q.

Quivers with potential were introduced in[5] in order to construct additive categorifications of cluster algebras with skew-symmetric exchange matrix. For the just mentioned categorification it is crucial that the potential for Q be non-degenerate, i.e. that it can be mutated along with the quiver arbitrarily, see [5] for more details on quivers with potentials.

Email address: valdivieso@mdp.edu.ar (Yadira Valdivieso-Díaz)

In [6] the authors introduced, under some mild hypothesis, for each oriented surface with marked points (S, M) a mutation finite cluster algebra with skew symmetric exchange matrices. More precisely, each triangulation $\mathbb{T}$ of (S, M) by tagged arcs corresponds to a cluster and the corresponding exchange matrix is conveniently coded into a quiver $Q(\mathbb{T})$. Labardini-Fragoso in [7] enhanced this construction by introducing potentials $W(\mathbb{T})$ and showed that these potentials are compatible with mutations. In particular, these potentials are non-degenerate. Ladkani showed that for surfaces with empty boundary and a triangulation $\mathbb{T}$ which has no self-folded triangles the Jacobian algebra $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$ is symmetric (and in particular finite-dimensional). It follows that for any triangulation $\mathbb{T}'$ of a closed surface (S, M) the Jacobian algebra $\mathcal{P}(Q(\mathbb{T}'), W(\mathbb{T}'))$ is weakly symmetric by [8]. In [1] it is shown by a degeneration argument that these algebras are tame.

Next, following Amiot [9, Sec. 3.4] and Labardini-Fragoso [10, Theorem 4.2] we have a 2-Calabi-Yau triangulated category $\mathcal{C}_{(S,M)}$ together with a family of cluster tilting objects $(T_{\mathbb{T}})_{\mathbb{T} \text{ triangulation of } (S,M)}$, related by Iyama-Yoshino mutations such that $\mathrm{End}_{\mathcal{C}_{(S,M)}}(T(\mathbb{T})) \cong \mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))^{\mathrm{op}}$ (see for example [11] for details of n-Calabi-Yau categories).

Theorem. *Let S be a closed oriented surface with a non-empty finite collection M of punctures, excluding only the case of a sphere with 4 (or less) punctures. For an arbitrary tagged triangulation $\mathbb{T}$, the Jacobian algebra $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$ is symmetric, tame, its stable Auslander-Reiten quiver consists only of stable tubes of rank 1 or 2 and it is an algebra of exponential growth.*

Recall that for symmetric algebras there is a relation between the Auslander-Reiten translation τ and the Heller translate or syzygy Ω, namely $\tau \cong \Omega^2$ (see [12, Section 2.5]), then if $\Lambda_{\mathbb{T}} = \mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$ is a Jacobian algebra as in the main Theorem, we have that the non-projective indecomposable $\Lambda_{\mathbb{T}}$-modules are Ω-periodic. Related to symmetric tame algebras, Erdmann and Skowroński have announced the following statement (see [3, Theorem 6.2] or [4]): *a non-simple indecomposable symmetric algebra Λ is tame with Ω-periodic modules if and only if Λ belongs to one of the following classes of algebras: a representation-finite symmetric algebra, a non domestic symmetric algebra of polynomial growth, or an algebra of quaternion type (in the sense of [13]).* We show at the end of this work that the main Theorem implies that the existing characterization has at least a missing class.

As another consequence of the Theorem we have the following result.

Corollary. *Let S be a closed oriented surface with a non-empty finite collection M of punctures, excluding only the case of a sphere with 4 (or less) punctures.The Auslander-Reiten quiver of the generalized cluster category $\mathcal{C}_{(S,M)}$ consists only of stable tubes of rank 1 or 2.*

Notation 1.1. Let (S, M) be a marked surface with empty boundary and $\mathbb{T}$ be an tagged triangulation of (S, M). We construct the unreduced signed adjacency quiver $\widehat{Q}(\mathbb{T})$ of the triangulation $\mathbb{T}$ and following [7] we construct the unreduced potential $\widehat{W}(\mathbb{T})$. The quiver with potential $(Q(\mathbb{T}), W(\mathbb{T}))$ associated to the triangulation $\mathbb{T}$ of the marked surface (S, M) is the reduced part of $(\widehat{Q}(\mathbb{T}), \widehat{W}(\mathbb{T}))$.

Let $\mathcal{C}$ be the generalized cluster category of $(\mathcal{S}, M)$. We denote by Σ the suspension functor of $\mathcal{C}$ and by Σ_n the suspension functor in a n-angulated category $(\mathcal{F}, \Sigma_n, \pentagon)$ (cf. [14] for definition of n-angulated categories).

2. Proof of the results

We show first that the stable Auslander-Reiten quiver of a Jacobian algebra $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$ of an arbitrary tagged triangulation $\mathbb{T}$ of a closed oriented surface S consists only of stable tubes of rank 1 or 2 (see [15, Chapter VII] for details of Auslander-Reiten quiver). In order to prove it, we establish two preliminary results for 2-Calabi-Yau tilted symmetric algebras. Recall that any finite dimensional Jacobian algebra is a 2-Calabi-Yau tilted algebra (see [16, Corollary 3.6]).

Proposition 2.1. *Let $\mathcal{C}$ be a Hom-finite 2-Calabi-Yau triangulated category and $T \in \mathcal{C}$ be a cluster-tilting object. If $\mathrm{End}_{\mathcal{C}}(T)$ is symmetric, then its stable Auslander-Reiten quiver consists only of stable tubes of rank 1 or 2.*

Proof. By [17, Theorem I 2.4] there is a relation between the Serre functor $\mathbb{S}$, the Auslander-Reiten translation $\tau_{\mathcal{C}}$ and the suspension functor Σ of $\mathcal{C}$ on objects, that is, $\mathbb{S} = \Sigma\tau_{\mathcal{C}}$. Since $\mathcal{C}$ is 2-Calabi-Yau, we have $\Sigma\tau = \Sigma^2$, therefore $\tau_{\mathcal{C}} = \Sigma$. Denote by Λ the algebra $\mathrm{End}_{\mathcal{C}}(T)$ and by τ_Λ the Auslander-Reiten translation of $\mod \Lambda$. By hypothesis, Λ is a symmetric algebra, then by [18, Lema] the subcategory $\mathrm{add}(T)$ is closed under the Serre functor $\tau_{\mathcal{C}}^2 = \mathbb{S} = \Sigma^2$. Then by [14, Remark 6.3], the stable category $\underline{\mathrm{mod}}(\Lambda)$ is a 3-Calabi-Yau category. Hence by [17, Theorem I 2.4] we have $\Omega^{-1}\tau_\Lambda = \Omega^{-3}$, because Ω^{-1} is the suspension functor in $\underline{\mathrm{mod}}\Lambda$, therefore $\Omega^{-2} = \tau_\Lambda$.

On the other hand, Λ is symmetric, then $\Omega^2 = \tau_\Lambda$, therefore $\Omega^4 = \mathbb{1}_{\underline{\mathrm{mod}}\Lambda}$. Then $\Omega^4 = \tau_\Lambda^2 = \mathbb{1}_{\underline{\mathrm{mod}}\Lambda}$. □

Remark 2.2. Note that in proof of Proposition 2.1, we also show that symmetric 2-Calabi-Yau tilted algebras are algebras whose indecomposable non-projective modules are Ω-periodic and the period is a divisor of 4.

Proposition 2.3. *Let $\mathcal{C}$ be a Hom-finite 2-Calabi-Yau triangulated category. If there exists a cluster-tilting object $T \in \mathcal{C}$ such that $\mathrm{End}_{\mathcal{C}}(T)$ is symmetric, then the Auslander-Reiten quiver of the category $\mathcal{C}$ consists only of stable tubes of rank 1 or 2*

Proof. Let T be a cluster-tilting object in $\mathcal{C}$ such that the 2-Calabi-Yau tilted algebra $\Lambda = \mathrm{End}_{\mathcal{C}}(T)$ is symmetric, therefore by [14, Proposition 6.4] $\mathrm{proj}(\Lambda) = \mathrm{add}(T)$ is a 4-angulated category, with the Nakayama functor ν as suspension. It well known that the Nakayama functor of a symmetric algebra is the identity (see [13, Lemma I.3.5]).

By [14, Remark 6.3], then the suspension in the 4-angulated category $\mathrm{add}(T)$ satisfies $\Sigma_4 = \Sigma^2$, but also satisfies $\Sigma_4 = \nu = \mathbb{1}_{\mathrm{add}(T)}$, then $\Sigma^2 = \mathbb{1}_{\mathrm{add}(T)}$. Hence $\tau^2 = \Sigma^2 = \mathbb{1}_{\mathrm{add}(T)}$.

The result follows for Proposition 2.1 and the equivalence $\mathcal{C}/(\mathrm{add}(\Sigma T)) \cong \mathrm{mod}\Lambda$ proved in [19, Proposition, Section 2.1], which is induced by the functor $F : \mathcal{C} \longrightarrow \mathrm{mod}\Lambda$ which sends $X \in \mathcal{C}$ to the module $T \mapsto \mathrm{Hom}(T, X)$. □

As consequence of Proposition 2.3, we have a result about the Auslader-Reiten quiver of the generalized cluster category of a closed surface with punctures.

Corollary 2.4. *Let S be a closed oriented surface with a non-empty finite collection M of punctures, excluding only the case of a sphere with 4 (or less) punctures.The Auslander-Reiten quiver of the generalized cluster category $\mathcal{C}_{(S,M)}$ consists only of stable tubes of rank 1 or 2.*

Proof. It was proved in [2, Proposition 4.7] that there is a particular triangulation $\mathbb{T}$ of $(\mathcal{S}, M)$ such that the Jacobian algebra $\Lambda = \mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$ is symmetric.

By [6, Proposition 4.10], [20, Theorem 7.1] and [21, Theorem 3.2], the generalized cluster category $\mathcal{C}$ is independent of the choice of the triangulation $\mathbb{T}$, then the generalized cluster category $\mathcal{C}$ is equivalent to the generalized cluster category $\mathcal{C}_{(Q(\mathbb{T}),W(\mathbb{T}))}$. The result follows from Proposition 2.3. $\square$

Remark 2.5. A partial converse to Proposition 2.3 is given in [22, Lemma 2.2. (c)]. Moreover, this result and Corollary 2.4 imply that any Jacobian algebra of a tagged triangulation of a closed Riemann surface is not only weakly-symmetric, but is symmetric.

Now, we prove the first part of Theorem, that is, we prove that for an arbitrary tagged triangulation $\mathbb{T}$, the Auslander Reiten quiver of the Jacobian algebra $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$ consists only of stable tubes of rank 1 or 2.

Proof of periodic module category. Let $\mathbb{T}$ be a triangulation of the marked surface $(\mathcal{S}, M)$ and $\Lambda = \mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$ be the Jacobian Algebra of the triangulation $\mathbb{T}$. It was already known that Λ is a tame algebra (see [1]), and by Remark 2.5 Λ is symmetric. Therefore, it only remains to prove that its stable Auslander-Reiten quiver consists only of stable tubes of rank 1 or 2. Consider a cluster tilting object T of $\mathcal{C}$ such that the functor $F : \mathcal{C} \longrightarrow \mathrm{mod}\Lambda$ which sends $X \in \mathcal{C}$ to the module $T \mapsto \mathrm{Hom}(T, X)$ induces an equivalence $\mathcal{C}/(\mathrm{add}(\Sigma T)) \cong \mathrm{mod}\Lambda$ (see [19, Proposition, Section 2.1]). The functor F also induces an equivalence $\underline{\mathrm{mod}}\Lambda \cong \mathcal{C}/(T, \Sigma T)$ (see [19, Section 3.5]). Therefore the statement follows from Corollary 2.4. $\square$

Finally, before we prove the second part of the main result, we recall the definition of a tame algebra of exponential growth. Recall also that for the following definition the ground field k is assumed to be algebraically closed.

Following Drozd [23], an algebra A is *tame* if for every dimension $d \in \mathbb{N}$ there is a finite number of $A - k[X]-$bimodules $N_1, \ldots, N_{i(d)}$ such that each N_i is finitely generated free over the polynomial ring $k[X]$ and almost all d-dimensional indecomposable A-modules are isomorphic to $N_i \otimes_{k[X]} k[X]/(X - \lambda)$ for some $i \in \{1, \ldots, i(d)\}$ and some $\lambda \in k$. For a tame algebra A, we denote by $\mu_A(d)$ the smallest possible number of these N_i. Then A is said to be of *polynomial growth* [24] (respectively, *domestic* [25]) if there is a positive integer m such that $\mu_A(d) \leq d^m$ (respectively, $\mu_A(d) \leq m$) for all $d \geq 1$. We say that A is tame of *exponential growth* if $\mu_A(d) > r^d$ for infinitely many $d \in \mathbb{N}$ and some real number $r > 1$.

Also, we recall the definition of string and band in string algebras. Given an arrow $\alpha : i \to j$ in a quiver Q, we denote by $\alpha^{-1} : j \to i$ the formal inverse of α. Given such a formal inverse $l = \alpha^{-1}$, one writes $l^{-1} = \alpha$. Let $\bar{Q}_1$ be the set of all arrows and their formal inverses, the elements of $\bar{Q}_1$ are letters. A *string* w, of an algebra kQ/I, is a sequence $l_1 l_2 \cdots l_n$ of the elements of $\bar{Q}_1$ such that

(W1) We have $l_i^{-1} \neq l_{i+1}$, for all $1 \leq i < n$.

(W2) No proper subsequence of w or its inverse belongs to I.

(W3) $\text{end}(l_i) = \text{start}(l_{i+1})$ for all i.

If $w = l_1 l_2 \cdots l_n$ and $w' = l'_1 l'_2 \cdots l'_m$ are strings we say that the composition of w and w' is defined provide $l_1 l_2 \cdots l_n l'_1 l'_2 \cdots l'_m$ is a string, and write $ww' = l_1 l_2 \cdots l_n l'_1 l'_2 \cdots l'_m$.

A string w is said to be *cyclic* if all powers w^m, $m \in \mathbb{N}$ are strings. Given a cyclic string w, the powers w^m with $m \geq 2$ are said to be *proper powers*. A cyclic string w is said to be *primitive* provided it is not a proper power of some other string. A *band* is a cyclic primitive string.

Remark 2.6. From the proof of the [1, Theorem 3.6] follows that mutations of quivers with potential also preserve exponential growth, therefore it is enough to show a particular triangulation of each closed surface which induces a Jacobian algebra of exponential growth.

Lemma 2.7. *Let A be a finite dimensional algebra and A' be a quotient of A. If A' is of exponential growth, then so is A.*

The proof follows from the fact that any A'-module is also an A-module and if $L \cong N$ as $A - k[x]-$bimodules, then $L \cong N$ as $A' - k[x]-$bimodules. Therefore $\mu_{A'}(d) \leq \mu_A(d)$.

Our goal is to find a triangulation $\mathbb{T}$ for each closed surface with marked points (S, M) such that there is a quotient of the Jacobian algebra $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$ which is an algebra of exponential growth. We have to distinguish in two cases: the sphere with 5 punctures and the other closed surfaces.

Lemma 2.8. *For an arbitrary triangulation $\mathbb{T}$ of a sphere with 5 punctures, the Jacobian algebra $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$ is an algebra of exponential growth.*

Proof. Consider the skewed-gentle triangulation $\mathbb{T}$ of Figure 1 (see definition of skewed-gentle triangulation in [1, Section 6.7]) and the Jacobian algebra $\Lambda = \mathcal{P}(Q(\mathbb{T}), W(\mathbb{T})$. The skewed-gentle triangulation $\mathbb{T}$ was already studied in [1].

Let I be the ideal of $k\langle\langle Q(\mathbb{T})\rangle\rangle$ generated by the set $\partial(W')$ of cyclic derivatives of W', where $W' = b_5 c_5 a_2 b_1 c_1 + a_1 b_4 c_4 a_3$. Then the quotient algebra $\Lambda' = \Lambda/I$ is isomorphic $k\langle Q'\rangle/J$ where Q' is the quiver in Figure 2 and J is the ideal in $k\langle Q'\rangle$ generated by $\epsilon_i^2 - \epsilon_i$, $a_i b_i$, $b_i c_i$ and $c_i a_i$ for $i = 1, 2, 3$ and the set $\{b_2\epsilon_2 c_2 a_3,\ \epsilon_2 c_2 a_3 a_1,\ c_2 a_3 a_1 b_2,\ a_1 b_2 \epsilon_2 c_2,\ \epsilon_3 c_3 a_2 b_1 \epsilon_1 c_1,\ c_3 a_2 b_1 \epsilon_1 c_1 b_3,\ a_2 b_1 \epsilon_1 c_1 b_3 \epsilon_3,\ b_1 \epsilon_1 c_1 b_3 \epsilon_3 c_3,\ \epsilon_1 c_1 b_3 \epsilon_3 c_3 a_2,\ c_1 b_3 \epsilon_3 c_3 a_2 b_1,\ b_3 \epsilon_3 c_3 a_2 b_1 \epsilon_1\}$. Then the quotient Λ/I is a skewed-gentle algebra (see definition of skewed-gentle algebras in [26]).

The indecomposable representations of skewed-gentle (or more generally clannish) algebras are described by a combinatorial rule in terms of certain words; similar to the more

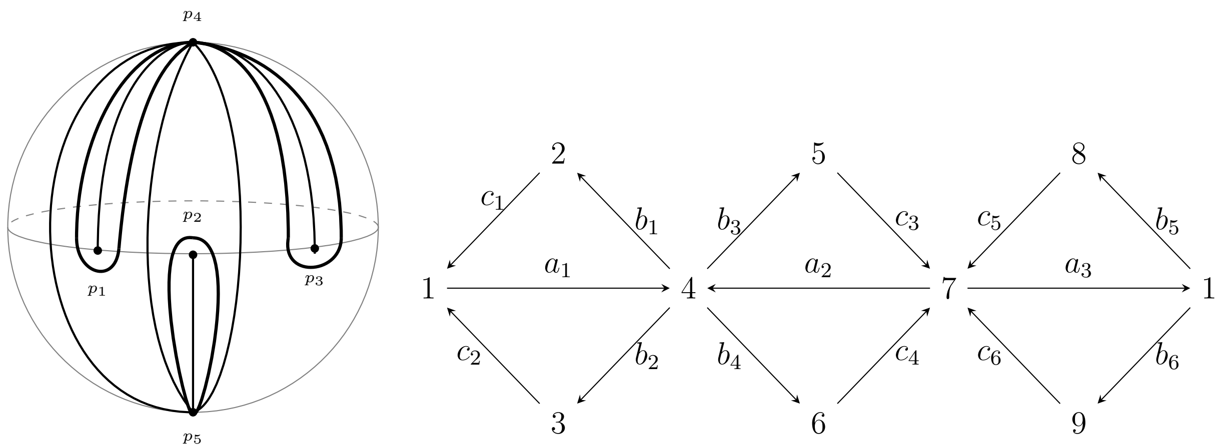

Figure 1: Triangulation $\mathbb{T}$ of a sphere with 5 punctures

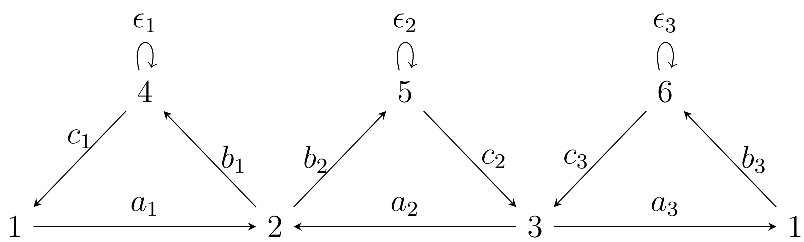

Figure 2: Skewed-gentle quiver Q'

widely known special biserial algebras, see [27] for more details. Thus in our situation it is enough to show that the corresponding clan $C = (k, Q', S_p, (q_b), \leq)$ (see [27, Definition 1.1]) admits two bands such that any arbitrary sequence of composition of them is again a band.

Let $C = (k, Q', S_p, (q_b \mid b \in S_p), \leq)$ be the clan of the algebra Λ', where the special loops S_p are ϵ_1, ϵ_2 and ϵ_3 and with the following relations: $a_i < b_i^{-1}$, $b_i < c_i^{-1}$ and $c_i < a_i^{-1}$ for $i = 1, 2, 3$.

Recall, a word in a clan C is a formal sequence $w_1 w_2 \dots w_n$ of letters with $n > 0$ such that for each $1 \leq i < n$ we have $\mathrm{end}(w_i) = \mathrm{start}(w_{i+1})$ and the letters w_i^{-1} and w_{i+1} are incomparable.

Now, consider the bands $\alpha = a_1 a_2^{-1} a_3$ and $\beta = a_1 b_2 \epsilon_2^* c_2 c_3^{-1} \epsilon_3^* b_3^{-1}$. Observe that a_3^{-1} and a_1 are incomparable and also b_3 and a_1 are incomparable, then the products $\alpha\beta$ and $\beta\alpha$ are well defined and they are also bands, therefore Λ' is an algebra of exponential growth and by Lemma 2.7 then Λ is so. One needs some number theory argument to prove that the string algebras or skewed-gentle algebras having two bands as before are of exponential growth, these number theory argument was given by Skowroński for a particular case in [24, Lemma 1]. Finally, as we mention in Remark 2.6 mutations of quiver with potential preserve exponential growth property and flips of tagged triangulations are compatible with mutations of quiver with potential (cf. [7, Theorem 30]), then any Jacobian algebra associated to a triangulation of a sphere with 5 punctures is also of exponential growth. $\square$

For an ideal triangulation $\mathbb{T}$, the *valency* $\mathrm{val}_{\mathbb{T}}(p)$ of a puncture $p \in \mathbb{P}$ is the number of arcs in $\mathbb{T}$ incident to p, where each loop at p is counted twice.

Remark 2.9. Let $\mathbb{T}$ be a triangulation with no self-folded triangles of a marked surface with empty boundary $(S, \mathbb{M})$ and $(Q(\mathbb{T}), W(\mathbb{T}))$ be the quiver with potential associated to the triangulation $\mathbb{T}$.

1. The quiver $Q(\mathbb{T})$ is a block-decomposable graph of blocks of type II, where a block of type II is a 3-cycle. See [6] for details.

2. If every puncture $p \in M$ has valency at least 3, then any arrow α of $Q(\mathbb{T})$ has exactly two arrows β, γ starting at the terminal vertex of α and exactly one arrow δ ending at the terminal vertex of α. Following Ladkani in [2] there are two functions $f, g : Q_1(\mathbb{T}) \to Q_1(\mathbb{T})$ such that $\alpha f(\alpha) f^2(\alpha)$ is a 3-cycle arising from a triangle in $\mathbb{T}$ and $(g^{n_\alpha - 1}(\alpha))(g^{n_\alpha - 2}(\alpha)) \ldots (g(\alpha))(\alpha)$ is a cycle surrounding a puncture q, where $n_\alpha = \min\{r > 0 \mid g^r(\alpha) = \alpha\}$.

3. If every puncture has valency at least four, then there are no commutativity relations involving only paths of length two.

Proof of exponential growth property. It was proven in [2, Proposition 5.1], that if the marked surface (S,M) is not a sphere with 4 or 5 punctures, it has a triangulation $\mathbb{T}$ with no self-folded triangles and in which each puncture $p \in M$ has valency at least four.

Let $f, g : Q_1(\mathbb{T}) \to Q_1(\mathbb{T})$ be functions as in Remark 2.9 part 2). Let I be the ideal in A generated by the relations $\alpha f(\alpha)$ for every arrow α in $Q_1(\mathbb{T})$ and consider the quotient $A' = A/I$. It is clear that this quotient is a string algebra.

To prove that A' is an algebra of exponential growth we use the the argument as in the case of a sphere with 5 puncture, then it is enough to prove that A' admits two bands ξ and η such that any arbitrary combination of them is again a band.

Consider an arrow $\alpha : i \to j$ of the quiver $Q(\mathbb{T})$, we denote by $\mathbb{T}_\alpha$ the following piece of the triangulation $\mathbb{T}$. The vertices of α are two arcs of a triangle $\triangle_\alpha$ of $\mathbb{T}$, and if q and p are the endpoints of the remaining arc of this triangle, we let $\mathbb{T}_\alpha$ the set of arcs of $\mathbb{T}$ that have q or p as one of its endpoints. We observe that the arc with endpoints p and q is a side of exactly two triangles, one of them is the triangle $\triangle_\alpha$ and we denote by $\triangle_\delta$ the other one. In Figure 3, we show the piece of triangulation $\mathbb{T}_\alpha$.

Denote by $\alpha\gamma\beta$ the 3-cycle arising from the triangle $\triangle_\alpha$. Surrounding the puncture q, there is a cycle $w(q)$ starting at j which is denoted by

$$(\gamma)(g\gamma)(g^2\gamma)(g^3\gamma)\cdots(g^{n_\gamma - 1}\gamma),$$

and surrounding p there is a cycle $w(p)$ starting at i which is denoted by

$$(g\beta)(g^2\beta)\cdots(g^{n_\beta - 2}\beta)(g^{n_\beta - 1}\beta)(\beta).$$

We denote by $\rho_1(\alpha)$ the word

$$(g^2\gamma)(g^3\gamma)\cdots(g^{n_\gamma - 1}\gamma),$$

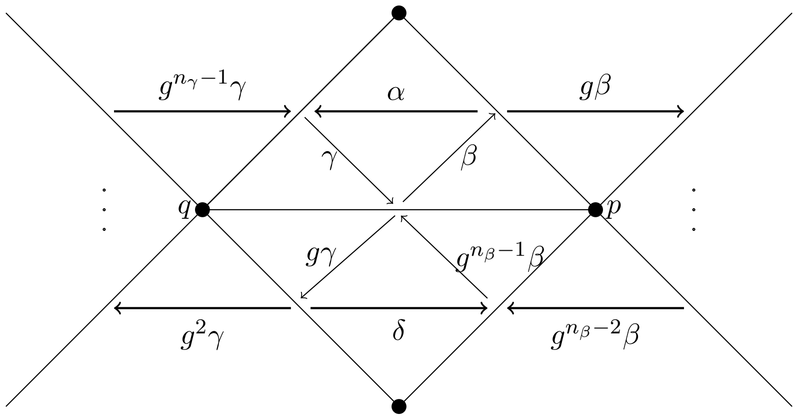

Figure 3: The piece of triangulation $\mathbb{T}_\alpha$

by $\rho_2(\alpha)$ the word

$$(g\beta)(g^2\beta)\cdots(g^{n_\beta-2}\beta)$$

and by δ the arrow $fg\gamma$, which is part of the 3-cycle arising from the triangle $\triangle_\delta$.

Then denote by $\xi(\alpha)$ the word

$$(\alpha)(\rho_1(\alpha))^{-1}(\delta)(\rho_2(\alpha))^{-1}.$$

Observe that $\rho_1(\alpha)$ and $\rho_2(\alpha)$ are defined for any arrow $\alpha \in Q_1$, therefore we can define $\xi(\alpha)$ for any arrow α. In particular, we also have the string $\xi(g\beta)$. We define $\eta = \xi(g\beta)^{-1}$

Observe that both strings $\xi(\alpha)$ and η are actually bands, moreover, we have that the compositions $\xi(\alpha)\eta$ and $\eta\xi(\alpha)$ are well defined bands and so is any word formed by (arbitrary) compositions of the bands $\xi(\alpha)$ and η. Therefore A' is an algebra of exponential growth, and by Lemma 2.7 it follows that A is an algebra of exponential growth. The result follows from the fact that exponential growth is preserved by mutations (see Remark 2.6). $\square$

Finally, in order to show the main Theorem shows that the existing characterization of symmetric tame algebras whose non-projective indecomposable modules are Ω-periodic was not complete, we first recall some definitions.

Definition 2.10. Let Λ and Λ' be self injective algebras.

- The algebras Λ and Λ' are said to be *socle equivalent* if the factor algebras $\Lambda/\mathrm{Soc}(\Lambda)$ and $\Lambda'/\mathrm{Soc}(\Lambda')$ are isomorphic.

- The algebra Λ is said to be a *self injective algebra of Dynkin type* Δ if Λ is isomorphic to an orbit algebra $\hat{B}/G$, where $\hat{B}$ is the repetitive algebra of a tilted algebra B of Dynkin type Δ and G is an admissible group of automorphisms of $\hat{B}$.

- The algebra Λ is said to be a *self injective algebra of tubular type* if Λ is isomorphic to an orbit algebra $\hat{B}/G$, where $\hat{B}$ is the repetitive algebra of a tubular algebra B and G is an admissible group of automorphisms of $\hat{B}$.

- The algebra Λ is said to be of *quaternion type* if the following conditions are satisfied:
  - Λ is symmetric, indecomposable, tame of infinite representation type.
  - The indecomposable nonprojective finite dimensional Λ-modules are Ω-periodic of period dividing 4.
  - The Cartan matrix of Λ is non-singular.

We say that an algebra A is of *pure quaternion type* if it is of quaternion type and not of polynomial growth

According to the characterization of Erdmann and Skowroński (see [4] or [3, Theorem 6.2]) any symmetric tame algebras with Ω-periodic modules is one of the following type of algebra:

i) socle equivalent to a symmetric algebra of Dynkin type (or equivalently a representation finite symmetric algebra);

ii) socle equivalent to a symmetric algebra of tubular type (or equivalently a non-domestic symmetric algebra of exponential growth);

iii) an algebra of pure quaternion type.

In the following Remark we show that any Jacobian algebra arising from a closed surface with marked points, excluding only a sphere with 4 punctures, does not belong to none of one of the previous classes of algebras.

Remark 2.11. (1) A combination of [1, Theorem 7.1], [2, Theorem], [6, Proposition 2.10] and the Theorem of this work, yields that for each closed surface S of genus g with p punctures, excluding only the case of a sphere with 4 punctures, and each tagged triangulation $\mathbb{T}$, the Jacobian algebra $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$ is symmetric and tame with Ω-periodic module category and with $6(g-1)+3p$ simple modules. However, $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}))$ is neither of the following algebras:

a) an algebra of blocks of quaternion type, because in the case of a torus with one puncture, it was shown by Ladkani in [2], that the Cartan matrix of the associated Jacobian algebra is singular, therefore it is not blocks of quaternion type by definition. In the other cases, the module category of the Jacobian algebra has at least 6 simple modules, in contrast to algebras of quaternion type which have at most 3 simple modules (see [28, Theorem]).

b) an algebra socle equivalent to an algebra of tubular type, because this algebra is an algebra of polynomial growth.

c) socle equivalent to an algebra of Dynkin type, algebras which are of finite representation type.

Therefore, the existing characterization of algebras which are symmetric, tame and with the non-projective indecomposable modules Ω-periodic, was not complete (see [3, Theorem 6.2] or [4]). These family of Jacobian algebras forms a new family with these properties.

(2) A similar statement to the theorem is known to hold for the sphere with 4 punctures, except that in this case the potentials depend also on the choice of a parameter $\lambda \in k \setminus \{0, 1\}$ and in this case the Jacobian algebras $\mathcal{P}(Q(\mathbb{T}), W(\mathbb{T}, \lambda))$ are (weakly) symmetric of tubular type $(2, 2, 2, 2)$ and this is of polynomial (lineal) growth, and the category $\mathcal{C}_{(Q,M,\lambda)}$ is a tubular cluster category of type $(2, 2, 2, 2)$, see [29] and [30].

(3) Similar result to Theorem and Corollary was announced by Ladkani at the abstract of the Second ARTA conference, see [31].

(4) The results of this work were presented at the Second and Third ARTA conference.

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References

[1] C. Geiss, D. Labardini-Fragoso, J. Schröer, The representation type of Jacobian algebras arXiv:1308.0478.

[2] S. Ladkani, On Jacobian algebras from closed surfaces arXiv:1207.3778.

[3] K. Erdmann, A. Skowroński, Periodic algebras, in: Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 201–251. `doi:10.4171/062-1/5`.
URL `http://dx.doi.org/10.4171/062-1/5`

[4] K. Erdmann, A. Skowroński, The stable Calabi-Yau dimension of tame symmetric algebras, J. Math. Soc. Japan 58 (1) (2006) 97–128.
URL `http://projecteuclid.org/euclid.jmsj/1145287095`

[5] H. Derksen, J. Weyman, A. Zelevinsky, Quivers with potentials and their representations. I. Mutations, Selecta Math. (N.S.) 14 (1) (2008) 59–119. `doi:10.1007/s00029-008-0057-9`.
URL `http://dx.doi.org/10.1007/s00029-008-0057-9`

[6] S. Fomin, M. Shapiro, D. Thurston, Cluster algebras and triangulated surfaces. I. Cluster complexes, Acta Math. 201 (1) (2008) 83–146. `doi:10.1007/s11511-008-0030-7`.
URL `http://dx.doi.org/10.1007/s11511-008-0030-7`

[7] D. Labardini-Fragoso, Quivers with potentials associated to triangulated surfaces, Proc. Lond. Math. Soc. (3) 98 (3) (2009) 797–839. doi:10.1112/plms/pdn051.
URL http://dx.doi.org/10.1112/plms/pdn051

[8] M. Herschend, O. Iyama, Selfinjective quivers with potential and 2-representation-finite algebras, Compos. Math. 147 (6) (2011) 1885–1920. doi:10.1112/S0010437X11005367.
URL http://dx.doi.org/10.1112/S0010437X11005367

[9] C. Amiot, On generalized cluster categories arXiv:1101.3675.

[10] D. Labardini-Fragoso, On triangulations, quivers with potentials and mutations arXiv:1302.1936.

[11] B. Keller, Calabi-Yau triangulated categories, in: Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 467–489. doi:10.4171/062-1/11.
URL http://dx.doi.org/10.4171/062-1/11

[12] P. Gabriel, Auslander-Reiten sequences and representation-finite algebras, in: Representation theory, I (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979), Vol. 831 of Lecture Notes in Math., Springer, Berlin, 1980, pp. 1–71.

[13] K. Erdmann, Blocks of tame representation type and related algebras, Vol. 1428 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1990.

[14] C. Geiss, B. Keller, S. Oppermann, n-angulated categories, J. Reine Angew. Math. 675 (2013) 101–120.

[15] M. Auslander, I. Reiten, S. S. O., Representation theory of Artin algebras, Vol. 36 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1997, corrected reprint of the 1995 original.

[16] C. Amiot, Cluster categories for algebras of global dimension 2 and quivers with potential, Ann. Inst. Fourier (Grenoble) 59 (6) (2009) 2525–2590.
URL http://aif.cedram.org/item?id=AIF_2009__59_6_2525_0

[17] I. Reiten, M. Van den Bergh, Noetherian hereditary abelian categories satisfying Serre duality, J. Amer. Math. Soc. 15 (2) (2002) 295–366. doi:10.1090/S0894-0347-02-00387-9.
URL http://dx.doi.org/10.1090/S0894-0347-02-00387-9

[18] C. M. Ringel, The self-injective cluster-tilted algebras, Archiv der Mathematik 91 (3) (2008) 218–225.

[19] B. Keller, I. Reiten, Cluster-tilted algebras are Gorenstein and stably Calabi-Yau, Adv. Math. 211 (1) (2007) 123–151. doi:10.1016/j.aim.2006.07.013.
URL http://dx.doi.org/10.1016/j.aim.2006.07.013

[20] D. Labardini-Fragoso, Quivers with potentials associated to triangulated surfaces, part IV: Removing boundary assumptions arXiv:1206.1798v1.

[21] B. Keller, D. Yang, Derived equivalences from mutations of quivers with potential, Adv. Math. 226 (3) (2011) 2118–2168. doi:10.1016/j.aim.2010.09.019.
URL http://dx.doi.org/10.1016/j.aim.2010.09.019

[22] I. Burban, O. Iyama, B. Keller, I. Reiten, Cluster tilting for one-dimensional hypersurface singularities, Adv. Math. 217 (6) (2008) 2443–2484. doi:10.1016/j.aim.2007.10.007.
URL http://dx.doi.org/10.1016/j.aim.2007.10.007

[23] J. A. Drozd, Tame and wild matrix problems, in: Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Vol. 832 of Lecture Notes in Math., Springer, Berlin, 1980, pp. 242–258.

[24] A. Skowroński, Group algebras of polynomial growth, manuscripta mathematica 59 (4) (1987) 499–516.

[25] W. Crawley-Boevey, Tame algebras and generic modules, Proc. London Math. Soc. (3) 63 (2) (1991) 241–265. doi:10.1112/plms/s3-63.2.241.
URL http://dx.doi.org/10.1112/plms/s3-63.2.241

[26] C. Geiss, J. A. de la Peña, Auslander-Reiten components for clans, Bol. Soc. Mat. Mexicana (3) 5 (2) (1999) 307–326.

[27] W. Crawley-Boevey, Functorial filtrations II: Clans and the gelfand problem, Journal of the London Mathematical Society 2 (1) (1989) 9–30.

[28] K. Erdmann, On the number of simple modules of certain tame blocks and algebras, Arch. Math. (Basel) 51 (1) (1988) 34–38. doi:10.1007/BF01194151.
URL http://dx.doi.org/10.1007/BF01194151

[29] C. Geiss, R. González-Silva, Tubular Jacobian algebras, Algebr. Represent. Theory 18 (1) (2015) 161–181. doi:10.1007/s10468-014-9486-7.
URL http://dx.doi.org/10.1007/s10468-014-9486-7

[30] M. Barot, C. Geiss, Tubular cluster algebras I: categorification, Mathematische Zeitschrift 271 (3-4) (2012) 1091–1115.

[31] S. Ladkani, On symmetric Jacobian algebras Second ARTA conference, Torun 2013.
URL arta2013.mat.umk.pl/abstracts/abstr_23.pdf