

**Internal space structure generalization of the quintom cosmological scenario**Luis P. Chimento,<sup>1,\*</sup> Mónica Forte,<sup>1,†</sup> Ruth Lazkoz,<sup>2,‡</sup> and Martín G. Richarte<sup>1,§</sup><sup>1</sup>*Departamento de Física, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina*<sup>2</sup>*Fisika Teorikoa, Zientzia eta Teknologia Fakultatea, Euskal Herriko Unibertsitatea, 644 Posta Kutxatila, 48080 Bilbao, Spain*

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We introduce the Lagrangian for a multiscalar-field configuration in an  $N$ -dimensional internal space endowed with a constant metric  $Q_{ik}$ , and generalize the quintom cosmological scenario. We find the energy momentum tensor of the model and show that the set of dual transformations, which preserve the form of the Einstein equations in the Friedmann-Robertson-Walker cosmology, is enlarged. We show that the stability of the power-law solutions leads to an exponential potential which is invariant under linear transformations in the internal space. Moreover, we obtain the general exact solution of the Einstein-Klein-Gordon equations for that potential. There exist solutions that cross the phantom divide and solutions that blow up at a finite time, exhibiting a superaccelerated behavior and ending in a big rip. We show that the quintom model with a separable potential can be identified with a mixture of several fluids. This framework includes the  $\Lambda$ CDM model, a bouncing model, and a setting sourced by a cosmic string network plus a cosmological constant. Then we concentrate on the case where the dimension of the internal quintessence sector  $N_q$  exceeds the dimension of the internal phantom sector  $N_{ph}$ . For  $(N_q, N_{ph}) = (2, 1)$  the dark energy density derived from the 3-scalar field crosses the phantom divide, and its negative component plays the role of the negative part of a classical Dirac field.

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**I. INTRODUCTION**

A large number of recent cosmological data, including type Ia supernovae [1], large scale structure [2,3], and cosmic microwave background anisotropies [4], have provided strong evidences for a spatially flat and accelerated expanding universe at the present time. In the case of Friedmann-Robertson-Walker (FRW) cosmologies, such an accelerated expansion implies the existence of mysterious exotic matter with large enough negative pressure, i.e. dark energy, whose energy density has been dominating the recent stage of the universe. The astrophysical feature of dark energy is that it remains unclustered at all scales where gravitational clustering of baryons and nonbaryonic cold dark matter can be seen. The combined analysis of the different cosmological observations suggests that the universe consists of about 70% dark energy, 30% dust matter (composed by cold dark matter and baryons), and negligible radiation. Although the nature and the origin of dark energy are unknown, several candidates have been proposed to describe it. The simplest theoretical candidate is the cosmological constant  $\Lambda$  [5,6] whose equation of state is  $p = -\rho$ . However, this proposal suffers from the well-known cosmological problem according to which there is a unexplained extraordinary discrepancy (of about 120 orders of magnitude) between the observed value of  $\Lambda$  and the value predicted by quantum field theory [7,8]. For this reason, alternative routes have been suggested to study the

dark energy and its dynamical evolution [8]. For example, a wide variety of scalar-field dark energy models have been studied, including quintessence [9],  $k$ -essence [10], tachyon [11], phantom [12], ghost condensate [13] and more recently, quintom dark energy [14–16]. In addition, other possibilities include interacting dark energy [17], braneworld models [18], Chaplygin gas models [19], holographic dark energy [20], and many others.

The quintom model is a hybrid with a quintessence component, usually constructed by a scalar field and a phantom scalar field minimally coupled to gravity [21]. This model has aroused interest due to a debate about the possibility that the dark energy equation of state crossed the phantom divide in the near past [22]. Notice that in the quintessence models the kinetic energy is always positive definite for a positive definite potential, whereas one has exactly the opposite in phantom models. So, there is no way to use a single phantom field or quintessence field for crossing the divide line [23].

Many authors have studied the quintom paradigm with several potentials from the dynamical autonomous system perspective [15,24]. In this context, some of them have performed a general analysis by including the contribution of both a quintom component (dark energy) and a matter component into the Friedmann equation. For example, in Ref. [15] it was found that the phantom attractor is not generic, which means that in a quintom model there may exist either de Sitter attractors associated with the saddle points of the potential, or tracking attractors in the asymptotic regime where the scalar fields diverge.

Finally, we want to stress that, motivated by string theory, a quintom scenario has been found by taking into

\*chimento@df.uba.ar

†monicaforte@fibertel.com.ar

‡ruth.lazkoz@ehu.es

§martin@df.uba.ar

account the nonperturbative effects of a generalized Dirac-Born-Infeld action [16]. Moreover, a “no-go theorem” has been proven to constrain the building of a quintom scenario [25]. According to this theorem, in order to achieve the crossing of the phantom divide, it is necessary to have more than one degree of freedom. Clearly, this theorem offers a concrete theoretical justification for the quintom paradigm.

In this paper we present a multiscalar-field model embedded in an  $N$ -dimensional internal space structure endowed with a constant internal metric  $Q_{ik}$ . In Sec. II, we introduce the Lagrangian for the multiscalar-field model, calculate the energy momentum tensor, and obtain the energy density and pressure of the multiscalar field. We also study how the dual transformation that links contracting and expanding cosmologies is realized when the internal structure is taken into account, and we then perform the stability analysis for power-law solutions. In Sec. III, we report the general exact solution of the Einstein-Klein-Gordon equations for the exponential potential and illustrate the existence of the crossing of the phantom divide, showing that the model has a smooth transition from the quintessence to the phantom era. In Sec. IV we study quintom and three-dimensional scalar-field models driven by separable potentials. Finally, in Sec. V the conclusions are stated. Throughout this paper we shall use natural units ( $G = c = 1$ ).

## II. MULTISCALAR-FIELD COSMOLOGIES

We wish to construct a general class of dark energy models with late-time phantom behavior motivated by the fact that this peculiar sort of expansion fits comfortably within current observational results. Accelerated expansion cannot extend indefinitely back in time because it would spoil the formation of structures, and well-founded theoretical arguments support the view that our universe was dark matter dominated over a long time span before it entered the current dark energy dominated accelerated phase. In addition, the cosmological evidences indicate a superaccelerated regime provided by a dark energy source. This latter fact means that dark energy is phantomlike. And, therefore, the dark energy may have crossed the phantom divide just recently.

The mainstream approach (for simplicity and because there is no evidence in favor of the contrary) is to consider that dark energy and dark matter interact only gravitationally, so they “see” each other by their effects in the expansion. Intuition suggests, then, that the ability for a dark energy fluid to cross the divide is not ruined by the presence of dark matter. For that reason, even if one’s goal is to try and construct cosmological scenarios containing dark matter and dark energy featuring a crossing of the phantom divide, it makes sense to start off by considering the dark energy fluid alone. Put another way, it is very likely that the construction of a dark energy source with the desired property leads, when combined with dark matter, to

a cosmological scenario in which the dark energy bit retains the same feature.

Quite a few scalar-field model configurations with the ability of crossing the phantom divide can be found in the literature, and perhaps the most populated class is that of the generically dubbed quintom models [14–16]. Their main feature is that the kinetic energy is not a definite positive quantity. Quintom scenarios have attracted considerable attention because the mentioned property of the kinetic energy allows one to give a unified treatment of phantom and conventional scalar fields.

Taking quintom models as an inspiration, we assume an  $N$ -dimensional internal space structure endowed with a constant metric  $Q_{ik}$  and propose a new theoretical setup with multiple fields  $\phi_i(x)$  where  $i = 1 \dots N$ . This composite field can be thought of as a vector within the internal space. For any two vectors  $u_i$  and  $w_k$  belonging to this space, their scalar product is  $\mathbf{u} \cdot \mathbf{w} \equiv Q_{ik}u_iw_k$  and we denote the norm as  $\mathbf{w} \cdot \mathbf{w} \equiv w^2 = Q_{ik}w_iw_k$ . In what follows we use Latin indices for the internal space, and summation over repeated indices will be assumed.

The Lagrangian for this model is given by

$$\mathcal{L} = \frac{1}{2}Q_{ik}\partial_\mu\phi_i(x)\partial^\mu\phi_k(x) - V(\phi_1, \phi_2, \dots, \phi_N), \quad (1)$$

where  $V(\phi_1, \phi_2, \dots, \phi_N)$  is a self-interacting scalar potential depending on the fields  $\phi_1, \phi_2, \dots, \phi_N$  only. The energy momentum  $T_{\mu\nu}$  associated with the multiscalar-field Lagrangian (1) is obtained through the standard equation  $T_{\mu\nu} = 2\delta\mathcal{L}/\delta g^{\mu\nu} - g_{\mu\nu}\mathcal{L}$ , so we get

$$T_{\mu\nu} = Q_{ik}\partial_\mu\phi_i\partial_\nu\phi_k - g_{\mu\nu}[\frac{1}{2}Q_{ik}\partial_\xi\phi_i\partial^\xi\phi_k - V]. \quad (2)$$

Our framework is that of spatially flat FRW cosmologies, and the energy density and pressure of the dark energy fluid are

$$\rho = \frac{1}{2}Q_{ik}\dot{\phi}_i\dot{\phi}_k + V, \quad p = \frac{1}{2}Q_{ik}\dot{\phi}_i\dot{\phi}_k - V, \quad (3)$$

where we have assumed homogeneous scalar fields  $\phi_i = \phi_i(t)$ . The evolution of a universe sourced by that configuration of multiple interacting fields in the fashion above is governed by the equations

$$3H^2 = \frac{1}{2}Q_{ik}\dot{\phi}_i\dot{\phi}_k + V, \quad (4)$$

$$\ddot{\phi}_i + 3H\dot{\phi}_i + Q_{ik}^{-1}\frac{\partial V}{\partial\phi_k} = 0, \quad (5)$$

and they get combined to give

$$\dot{H} = -\frac{1}{2}Q_{ik}\dot{\phi}_i\dot{\phi}_k. \quad (6)$$

Our proposal is to use this extension which, due to the presence of an internal space, allows us to consider a richer structure than in the precursor quintom cosmologies. The usual quintom model fits in our description for the choice  $Q_{ik} = \text{diag}(1, -1)$ . In the following cases, the internal

metrics  $Q_{ik} = \text{diag}(1, 0)$  and  $Q_{ik} = \text{diag}(0, -1)$  reproduce the quintessence and phantom cosmologies, respectively.

### A. Duality

Interestingly, the internal space structure also allows us to interpret duality transformations linking contracting and expanding conventional cosmologies in a very intuitive and elegant fashion. In fact, the  $a \rightarrow 1/a$  duality transformation implies  $H \rightarrow -H$ , and from the Einstein equations (4) and (6) it follows that there are two alternative ways to achieve the transformation that leaves those equations unaltered:

(i) Case 1,

$$\dot{\phi}_k \rightarrow i\dot{\phi}_k, \quad V \rightarrow Q_{ik}\dot{\phi}_i\dot{\phi}_k + V(\phi_i). \quad (7)$$

(ii) Case 2,

$$Q_{ik} \rightarrow -Q_{ik}, \quad V \rightarrow Q_{ik}\dot{\phi}_i\dot{\phi}_k + V(\phi_i). \quad (8)$$

A realization of the duality as given by case (1) requires potentials which remain real under the transformation of the fields  $\dot{\phi}_k \rightarrow i\dot{\phi}_k$ . However, the realization induced by the internal structure and given by case (2) is of wider applicability, as the former requirement on the potential is not necessary. In addition, both the transformed and non-transformed fields are real. For example, phantom cosmologies characterized by a future singularity occurring in a finite time can be obtained by performing the duality transformation  $a \rightarrow 1/a$  onto cosmologies with a final big crunch. For case (1), one of them (phantom or big crunch cosmology) will be associated with imaginary fields. However, for case (2) both cosmologies are described by real fields.

### B. Stability of power-law solutions

We are now going to investigate the existence of asymptotic power-law solutions by means of a structural stability analysis; in other words, we are going to unveil the asymptotic nature of potentials allowing such stable solutions. To this end, we introduce the barotropic index  $\gamma = (\rho + p)/\rho$ , and from the definitions above and the Einstein equations (4) and (6), one gets the dynamical equations for  $\gamma = -2\dot{H}/3H^2$ , namely,

$$\dot{\gamma} = (\gamma - 2)\left(3H\gamma + \frac{\dot{V}}{V}\right). \quad (9)$$

Clearly, the solution  $\gamma = 2$  represents an equilibrium point, but it corresponds to nonaccelerated expansion, specifically,  $a \propto t^{1/3}$ . Given that practically any other interesting cosmological model one can think of will have  $\gamma < 2$ , let us demand then that Eq. (9) admits another equilibrium point with  $\gamma \equiv \gamma_0$  representing, in consequence, a solution with  $a \propto t^{2/3\gamma_0}$ . In particular, a solution

representing a phantom late-time attractor would be characterized by a constant and negative value of  $\gamma$ . Below, we will show that the requirement of the existence of such an equilibrium point restricts the functional form of the  $V(\phi_i)$  potential. Then we formulate a structural stability analysis of the equation governing the evolution of the parameter  $\gamma$ . This is equivalent to imposing the asymptotic condition on the potential  $\dot{V} + 3\gamma_0 HV = 0$ , which can be integrated to give  $V = V_0 a^{-3\gamma_0}$  with  $V_0$  a positive integration constant. By choosing this potential, the asymptotic regime of  $\gamma$  is governed by the equation

$$\dot{\gamma} = 3H(\gamma - 2)(\gamma - \gamma_0). \quad (10)$$

Let us now define  $\gamma = 2 - \epsilon$  and recast Eq. (10) in terms of this new definition. By expanding the right-hand side of this equation up to order  $\epsilon$ , for expanding universes ( $H > 0$ ), the solution  $\gamma = 2$  is a repeller. A similar analysis can be done for the case  $\gamma = \gamma_0 \neq 0$  (the particular case  $\gamma_0 = 0$  will be investigated at the end of this section). If one defines  $\gamma = \gamma_0 - \epsilon$ , it turns out that the  $\gamma = \gamma_0$  solution is an attractor provided  $\gamma_0 < 2$ , which is, on the other hand, our working hypothesis. If one then calculates  $V$  and  $H$  as functions of  $t$  for  $a = t^{2/3\gamma_0}$ , it can be inferred from Eqs. (4) and (6) that

$$V = \frac{V_0}{t^2}, \quad V_0 = \frac{2(2 - \gamma_0)}{3\gamma_0^2}. \quad (11)$$

As it has been shown, in spite of having started from a general formulation, we have been able to calculate the value of  $V_0$  and the asymptotic dependence of the potential on cosmic time. However, as far as the reconstruction of the potential is concerned, we need to make contact with standard formulations in physical cosmology and to specify the kinetic energy as a function of the fields and their derivatives.

For the attractor solution (which satisfies a power law) it can be seen that a satisfactory choice for the asymptotic behavior of the scalar fields is  $\phi = \phi_0 \log t$  with  $\phi_0$  a constant vector in the internal space. Besides, we are considering that physical quantities such as the potential energy depend only on scalar quantities in this internal space, so for simplicity we will consider that the potential depends on the scalar fields through a linear form of the fields. To that end, we introduce a constant vector  $\alpha$  in the internal space. Under these assumptions we build the scalar quantity  $\phi = \alpha \cdot \phi_0 \log t$ , which leads to

$$V = V_0 e^{-\alpha \cdot \phi}, \quad (12)$$

where we have fixed  $\alpha \cdot \phi_0 = 2$ . One can also find the following relations:

$$\phi_0 = \frac{2\alpha}{\alpha^2}, \quad \alpha^2 = 3\gamma_0. \quad (13)$$

Finally, in terms of the parameters of our framework the attractor solution gets recast as

$$a = t^{2/\alpha^2}. \quad (14)$$

Because of the internal space structure of our model, the exponent of the latter can take either sign, so both phantom and nonphantom power-law solutions are available.

Finally, we conclude this section with the investigation of the stability of the solution  $\gamma_0 = 0$ . If we impose the fulfillment of Eq. (9) asymptotically, one obtains  $\dot{V}/V = 0$  with  $V = V_0$  a constant. Now defining  $\gamma = -\epsilon$  and using again (10), it follows that the de Sitter solution ( $H = H_0$ ) is asymptotically stable.

### III. THE EXPONENTIAL POTENTIAL

We have investigated the stability of the power-law solution with a exponential potential and the possibility of phantom and no-phantom behavior for the scale factor. In this section we shall find the general exact solution for our  $N$ -scalar-field model driven by the exponential potential (12) in the spatially flat FRW spacetime. In this case Eqs. (4) and (5) represent a nonlinear coupled system of  $(N + 1)$  differential equations for the scale factor  $a(t)$  and the  $N$  scalar fields  $\phi_i(t)$ . For a nonphantom scalar field driven by an exponential potential and a free scalar field, the general solution of the system of equations (4) and (5) was found in Ref. [26]. Using the Einstein equation (6) it is easy to prove that the geometrical object

$$\dot{\phi}_k = \alpha_k H + \frac{c_k}{a^3}, \quad (15)$$

with  $c_k$  a constant vector in the internal space, is a first integral of Eq. (5). Inserting the latter into Eq. (6) and expanding, we obtain

$$-2\dot{H} = \alpha^2 H^2 + 2\boldsymbol{\alpha} \cdot \mathbf{c} \frac{H}{a^3} + \frac{c^2}{a^6}. \quad (16)$$

Making the change of variables

$$s = a^{-3/n}, \quad \tau = \boldsymbol{\alpha} \cdot \mathbf{c} t, \quad n = -\frac{6}{\alpha^2} \quad (17)$$

in Eq. (16), it becomes a nonlinear second order differential equation for  $s(\tau)$ :

$$\ddot{s} + s^n \dot{s} + \frac{1}{4\cos^2\theta} s^{2n+1} = 0, \quad (18)$$

where the dot stands for differentiation with respect to the argument of the function and

$$\cos\theta = \frac{\boldsymbol{\alpha} \cdot \mathbf{c}}{\alpha c}. \quad (19)$$

The sign of the parameter  $n$  depends exclusively on the internal metric  $Q_{ik}$ . In particular, for a Euclidean metric this parameter becomes negative. After the new variable  $s$  is calculated, we obtain the scale factor  $a$  in terms of the cosmic time  $t$  by using Eqs. (17).

Inserting the first integral (15) into the Friedmann equation (4), we get a quadratic equation in the expansion rate

$H$ , which has real solutions only when its discriminant is positive definite; this condition can be recast as  $(\boldsymbol{\alpha} \cdot \mathbf{c})^2 + (6 - \alpha^2)(c^2 + 2a^6 V) > 0$ . It gives a relation among the integration constants and the exponent of the exponential potential.

Equation (18) can be linearized and solved using the form invariance group of this nonlinear differential equation under a nonlocal change of variable. This technique relates a class of nonlinear differential equations with the damped harmonic oscillator one. In fact, Eq. (18) belongs to the following general class of second order nonlinear ordinary differential equations:

$$\ddot{s} + \mu F(s)\dot{s} + \nu F(s) \int F(s)ds + \delta F(s) = 0, \quad (20)$$

where  $s = s(t)$ ,  $F(s)$  is a real arbitrary function, and  $\mu, \nu, \delta$  are constant parameters.

By using the new pair of variables  $(h, \eta)$ , defined by the nonlocal transformation

$$F(s)ds = G(h)dh, \quad F(s)dt = G(h)d\eta, \quad (21)$$

Eq. (20) becomes the nonlinear ordinary differential equation

$$h'' + \mu G(h)h' + \nu G(h) \int G(h)dh + \delta G(h) = 0, \quad (22)$$

where  $'$  denotes differentiation with respect to  $\eta$ . It is easy to see that Eqs. (20) and (22) are related by means of the formal changes  $F \leftrightarrow G$  and  $s \leftrightarrow h$ . Then, the nonlocal transformation (21) preserves the form of these equations, mapping solutions of Eq. (20) into solutions of Eq. (22) for any function  $F(s)$  and  $G(\eta)$ , thus linking solutions of two different physical configurations. Taking into account the nature of this nonlocal change of variables, it is not always possible to find explicit solutions. However, we can use the form invariance to relate solutions of a nonlinear equation (20) with a linear one. In our case, choosing the specific parametrization  $F(s) = s^n$  and  $G(h) = 1$  in the change of variables (21)

$$h = \int s^n ds, \quad \eta = \int s^n dt \quad (23)$$

and introducing these new variables in Eqs. (20) and (22), they become

$$\ddot{s} + \mu s^n \dot{s} + \frac{\nu}{n+1} s^{2n+1} + \delta s^n = 0, \quad (24)$$

$$h'' + \mu h' + \nu h + \delta = 0. \quad (25)$$

This shows that the nonlocal change of variable (23) transforms the nonlinear differential equation (24) into the linear damped harmonic oscillator equation (25).

Comparing our Eq. (18) with Eqs. (24) and (25), for  $\mu = 1$ , we have two different cases to analyze according to the  $n$  value,

$$(i) \quad n \neq -1 \text{ and } \delta = 0 \Rightarrow F(s) = s^n, \quad \nu = \frac{(n+1)}{4\cos^2\theta},$$



$$(ii) \quad n = -1 \text{ and } \nu = 0 \Rightarrow F(s) = \frac{1}{s}, \quad \delta = \frac{1}{4\cos^2\theta}.$$

Below, we find the general solution of the Einstein-Klein-Gordon equations for the exponential potential.

### A. Explicit solution for $n \neq -1, -2$ and $\nu = \frac{(n+1)}{(n+2)^2}$

In this case explicit  $\tau$ -dependent solutions can be found, introducing the function  $v(\tau)$  (for more details, see Ref. [26]) and making the substitution

$$s^n = \left(\frac{n+2}{n}\right) \frac{v^n}{k_1 + \int v^n d\tau} \quad (26)$$

into Eq. (18). This reduces it to  $\ddot{v} = 0$ , whose solution is  $v(\tau) = k_2 + k_3\tau$ . After inserting the solution  $v$  in Eq. (26), integrating, and using  $s(a)$  as given by Eq. (17), we obtain the scale factor

$$a(v) = \left[ \frac{n}{(n+1)(n+2)} \right]^{1/3} [v + k|v|^{-n}]^{1/3}, \quad (27)$$

where the constant  $k$  has been expressed in terms of the old one [27]. Finally, we can get the components  $\phi^i$  of the multiscalar field  $\boldsymbol{\phi}$  by integrating Eq. (15),

$$\frac{d\phi_k}{dv} = \alpha_k \frac{d \ln a}{dv} + \frac{c_k a^{-3}}{\boldsymbol{\alpha} \cdot \mathbf{c}}. \quad (28)$$

To this end, we rewrite the second term of Eq. (28) using the definition given in Eq. (17) as

$$a^{-3} = \frac{n+2}{n} \frac{d}{dv} \left[ \ln \left( k_1 + \int v^n d\tau \right) \right]. \quad (29)$$

Inserting the latter into Eq. (28), we obtain  $\phi^i(v)$ ,

$$\Delta\phi_i = \alpha_i \ln|a| + \frac{(n+2)c_i}{n(\boldsymbol{\alpha} \cdot \mathbf{c})} \ln \left[ \frac{n+2}{n} v^n a^3 \right], \quad (30)$$

where  $\Delta\phi_i(v) = \phi_i(v) - \phi_{0i}$  and  $\phi_{0i}$  are  $N$  integration constants [28]. For positive  $n$ , when  $v \rightarrow 0$ , the leading terms in the scale factor are given by

$$a \propto k|v|^{-n/3}. \quad (31)$$

So, the scale factor blows up near the origin  $v = 0$  and the components of the multiscalar field have a logarithmic divergence  $\Delta\phi_i \propto \ln|v|$  for all  $n$ . We find a resemblance to the big rip behavior in the phantom universe reported recently by Wei [29] with an inverse power potential  $V = V_0\phi^{-1}$ .

As we have integrated the second order differential for the scale factor (6), instead of the original Friedmann equation (4), we have to constrain the integration constants. By using Eqs. (4) and (6), we obtain  $3H^2 + \dot{H} = V$ , which can be rewritten in terms of the variable  $u = a^3$  as

$$\frac{d^2u}{dv^2} = \frac{3uV}{(\boldsymbol{\alpha} \cdot \mathbf{c})^2}. \quad (32)$$

After replacing the solutions  $u(v)$  and  $\phi(v)$  in Eq. (32), we obtain the following relation between integration constants:

$$k = -\frac{V_0}{2c^2|\phi_0|\cos^2\theta} \text{sign}\left(\frac{n+2}{n}\right). \quad (33)$$

Hence, the scale factor (27) and the multiscalar field (30), whose integration constants verify the relation (33), are exact solutions of the Einstein-Klein-Gordon equations (4) and (5).

### B. Explicit solution for $n = -1$ and $\nu = 0$

In the  $n = -1$  case, following the same steps we get

$$a = [|v|b_1 + \ln|v|]^{1/3}, \quad (34)$$

$$\Delta\phi_i = \frac{\alpha_i}{3} \ln|v| [b_1 + \ln|v|] - \frac{c_i}{\boldsymbol{\alpha} \cdot \mathbf{c}} \ln|b_1 + \ln|v||, \quad (35)$$

where  $b_1$  is an integration constant. Inserting Eqs. (34) and (35) into Eq. (32), we find the relation  $V_0 = 2|\phi_0|c^2\cos^2\theta$ , where  $\alpha^2 = 6$  and the constant  $\phi_0$  has been renamed properly.

### C. Implicit solution

The general implicit solution of Eq. (18) can be found for arbitrary choices of the parameters by solving Eq. (25) and using Eqs. (17) and (23). In this case we have two different sets of solutions according to the value of  $n$ . For  $n \neq -1$ ,  $\delta = 0$ , and  $\mu = 1$ , the scale factor is given by

$$a(\eta) = [\sqrt{(n+1)}(b_1 \exp\lambda_- \eta + b_2 \exp\lambda_+ \eta)]^{-n/3(n+1)}, \quad (36)$$

where  $\lambda_{\pm}$  are the roots of the characteristic polynomial of Eq. (25),  $\lambda_{\pm} = (-1 \pm \sqrt{1-4\nu})/2$ . Also, from Eq. (15) we calculate the multiscalar field

$$\phi_i(\eta) = \phi_{i0} + \alpha_i \ln a(\eta) + \frac{c_i}{\boldsymbol{\alpha} \cdot \mathbf{c}} \eta, \quad (37)$$

where  $\phi_{i0}$  are integration constants. Now, inserting Eqs. (36) and (37) into the Friedmann equation (32), the integration constants are constrained to

$$b_1 b_2 = \frac{V_0}{2(1-4\nu)c^2|\phi_0|\cos^2\theta}. \quad (38)$$

Finally, in the  $n = -1$ ,  $\nu = 0$  and  $\delta = (4\cos^2\theta)^{-1}$  case, we have

$$a(\eta) = \exp\left[\frac{1}{3}(b_2 - \delta\eta + b_1 e^{-\eta})\right], \quad (39)$$

$$\phi_i(\eta) = \phi_{i0} + \alpha_i \ln a(\eta) + \frac{c_i}{\boldsymbol{\alpha} \cdot \mathbf{c}} \eta, \quad (40)$$

$$b_1 = \frac{V_0}{2|\phi_0|c^2\cos^2\theta}. \quad (41)$$

By taking  $\theta = 0$  we obtain the standard formula in the case of two scalar fields driven by an exponential potential (see Ref. [26]).

#### IV. TWO- AND THREE-DIMENSIONAL INTERNAL SPACE

Below we investigate several interesting examples with different internal metrics.

##### A. Big rip solution

Let us explore a new class of quintom models associated with the internal metric

$$Q_{ik} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (42)$$

As the sign of the kinetic term is nondefinite, a new feature arises—the possibility of having a smooth transition between quintessence and phantom scenarios (see Fig. 1). For  $\alpha^2 < 0$  the scale factor has two branches. In the branch  $\Delta\tau < 0$  the universe expands and has a finite time span; it begins to evolve from an initial singularity and has a transient phase to a superaccelerated expanding scenario ending in a big rip. Initially, the universe is dominated by a quintessence scenario with kinetic energy  $T(\phi_1, \phi_2) > 0$  and barotropic index  $\gamma = (p + \rho)/\rho > 0$ ; after that, the universe evolves toward a dominated phantom stage with  $T(\phi_1, \phi_2) < 0$  and  $\gamma < 0$  until the big rip occurs when  $\Delta\tau \rightarrow 0^-$  and  $a \rightarrow \infty^+$ . The point where the kinetic term changes its sign corresponds to the inflexion point of the scale factor (see Fig. 1). The branch  $\Delta\tau > 0$  represents a bouncing universe with a nonvanishing absolute minimum.

##### B. Quintom

We consider the standard quintom model with internal metric  $Q_{ik} = \text{diag}(1, -1)$  and assume that the scalar fields  $\phi_1$  and  $\phi_2$  are coupled through the potential  $V(\phi_1, \phi_2)$ . So, the Friedmann equation reads

$$3H^2 = \frac{1}{2}(\dot{\phi}_1^2 - \dot{\phi}_2^2) + V(\phi_1, \phi_2), \quad (43)$$

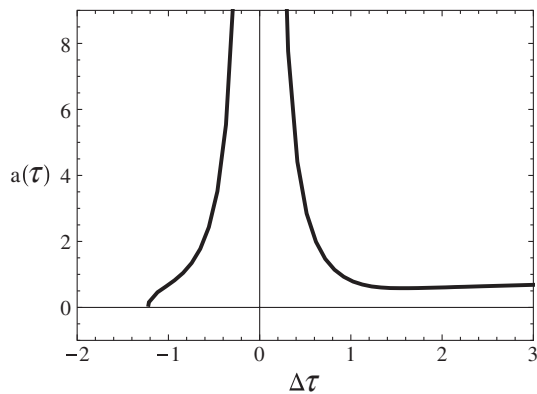


FIG. 1. The two branches of the scale factor for  $\alpha^2 = -1$ .

where  $\phi_1$  represents the quintessence and  $\phi_2$  plays the role of a phantom field. The system is closed with the Klein-Gordon equations, one for each field.

$$\ddot{\phi}_1 + 3H\dot{\phi}_1 + \partial_{\phi_1}V(\phi_1, \phi_2) = 0, \quad (44)$$

$$\ddot{\phi}_2 + 3H\dot{\phi}_2 - \partial_{\phi_2}V(\phi_1, \phi_2) = 0. \quad (45)$$

We propose a potential which can be written as follows,

$$V(\phi_1, \phi_2) = V_0 + \frac{\sigma}{2}\dot{\phi}_1^2 + \frac{\beta}{2}\dot{\phi}_2^2, \quad (46)$$

where  $\sigma$  and  $\beta$  are constant and  $V_0 > 0$ . It can mimic a simple model containing a mixture of dark matter and dark energy, where the former is the clustering component and the latter gives rise to the current acceleration. With the help of Eq. (46) we find that

$$\partial_{\phi_1}V = \sigma\ddot{\phi}_1, \quad \partial_{\phi_2}V = \beta\ddot{\phi}_2, \quad (47)$$

where we have assumed separability of the potential  $V(\phi_1, \phi_2) = V(\phi_1) + V(\phi_2)$ . Thus, Eqs. (44) and (45) are easily integrated,

$$\dot{\phi}_1 = \dot{\phi}_{01}a^{-3/(1+\sigma)}, \quad \dot{\phi}_2 = \dot{\phi}_{02}a^{-3/(1-\beta)}, \quad (48)$$

with  $\dot{\phi}_{01}$  and  $\dot{\phi}_{02}$  integration constants. Combining Eqs. (43), (46), and (48), the Friedmann equation becomes

$$3H^2 = V_0 + \frac{1}{2}\frac{(\sigma+1)\dot{\phi}_{01}^2}{a^{6/(1+\sigma)}} + \frac{1}{2}\frac{(\beta-1)\dot{\phi}_{02}^2}{a^{6/(1-\beta)}}. \quad (49)$$

On the other hand, the effective barotropic index  $\gamma = (\rho + p)/\rho$  of this mixture,

$$\gamma = \frac{\dot{\phi}_{01}^2 a^{-6/(1+\sigma)} - \dot{\phi}_{02}^2 a^{-6/(1-\beta)}}{\rho}, \quad (50)$$

shows the occurrence of a smooth transition from the no-phantom regime to the phantom one. This transition is achieved when  $\gamma(a_{\text{ph}}) = 0$ , that is, at  $a_{\text{ph}} = (\dot{\phi}_{02}/\dot{\phi}_{01})^{(1-\beta)(1+\sigma)/3(\sigma+\beta)}$ .

In the case  $\sigma = 1$  and  $\beta = 3$ , we associate the second term of the Friedmann equation (49) with an attractive component  $\rho_m = \dot{\phi}_{01}^2/a^3$ , which includes both baryonic and nonbaryonic matter, having the equation of state  $p_m \approx 0$ , while the remaining two terms constitute the dark energy component  $\rho_{\text{de}} = V_0 + \dot{\phi}_{02}^2 a^3$ . Equation (49) with  $\dot{\phi}_{02}^2 = V_0$  and  $\dot{\phi}_{01}^2 = V_0/4$  is solved by changing the variable to  $a = v^{2/3}$ . It turns into the equation  $4\dot{v} = \pm\sqrt{3V_0}(1 + 2v^2)$ , whose solution gives the scale factor

$$a = \left[ \frac{1}{\sqrt{2}} \tan \frac{\sqrt{6V_0}\Delta t}{4} \right]^{2/3}, \quad (51)$$

where  $\Delta t = t - t_0$ . Besides, the scalar fields  $\phi_1$  and  $\phi_2$  are obtained after integrating Eq. (48),

$$\phi_1 = \frac{2}{\sqrt{3}} \ln \left[ \sin \frac{\sqrt{6V_0} \Delta t}{4} \right], \quad (52)$$

$$\phi_2 = -\frac{2}{\sqrt{3}} \ln \left[ \cos \frac{\sqrt{6V_0} \Delta t}{4} \right]. \quad (53)$$

The scale factor given by (51) represents a universe starting as dust dominated by the scalar field  $\phi_1$  with  $a \approx \Delta t^{2/3}$ . After that, it expands and ends in a big rip singularity at a finite time. The big rip is reached where  $\phi_2$  blows up.

Finally, using Eq. (48) we are able to obtain the potential (46),

$$V(\phi_1, \phi_2) = \frac{V_0}{4} [e^{-\sqrt{3}\phi_1} + 3e^{\sqrt{3}\phi_2}], \quad (54)$$

as a sum of two separate potentials, each one depending on separate fields  $V(\phi_1, \phi_2) = V_1(\phi_1) + V_2(\phi_2)$ . This is in agreement with the precursor quintom model proposed in the literature [21].

Now, we investigate the degenerate case  $\beta = -\sigma$  with  $\sigma > 0$ . It gives rise to singular and bouncing solutions with a final de Sitter behavior. The Friedmann equation (49) reads

$$3H^2 = V_0 + \frac{\sigma + 1}{2} (\dot{\phi}_{01}^2 - \dot{\phi}_{02}^2) a^{-(6/(1+\sigma))}. \quad (55)$$

Here there is a degeneration because the contributions of both fields are proportional. When  $\dot{\phi}_{01}^2 > \dot{\phi}_{02}^2$ , we recover the  $\Lambda$ CDM cosmological model for  $\sigma = 1$  and the cosmic string for  $\sigma = 2$ . When  $\dot{\phi}_{01}^2 < \dot{\phi}_{02}^2$ , the solution bounces where the total energy density vanishes. Solving the Friedmann equation (55), we obtain the solutions

$$a^+ = [\sqrt{b} \sinh \omega \Delta t]^{(\sigma+1)/3}, \quad b > 0, \quad (56)$$

$$a^- = [\sqrt{-b} \cosh \omega \Delta t]^{(\sigma+1)/3}, \quad b < 0, \quad (57)$$

where  $\omega^2 = 3V_0/(1+\sigma)^2$  and  $b = (\sigma+1) \times (\dot{\phi}_{01}^2 - \dot{\phi}_{02}^2)/2V_0$ . The scale factor (56) evolves like  $a \propto t^{(\sigma+1)/3}$  near the singularity, having an inflationary phase for  $\sigma > 2$  and ending in a de Sitter stage. The bouncing solution (57) begins and ends with de Sitter phases. From Eqs. (48), (56), and (57), after integrating, we find the fields in terms of the cosmic time,

$$\phi_1^+ = \dot{\phi}_{01} \phi^+, \quad \phi^2 = \dot{\phi}_{02} \phi^+, \quad (58)$$

$$\phi_1^- = \dot{\phi}_{01} \phi^-, \quad \phi_2 = \dot{\phi}_{02} \phi^-, \quad (59)$$

with

$$\phi^+ = \frac{1}{\omega\sqrt{b}} \operatorname{Intanh} \frac{\omega \Delta t}{2}, \quad \phi^- = \frac{1}{\omega\sqrt{-b}} \tan^{-1} e^{\omega \Delta t}. \quad (60)$$

Hence, coming back to Eq. (46) we get the potential corresponding to either case, quintessence or phantom

dominated models,

$$V = V_0 \left[ 1 + \frac{\sigma}{1+\sigma} \sinh^2 \sqrt{b} \omega \phi^+ \right], \quad (61)$$

$$V = V_0 \left[ 1 - \frac{\sigma}{1+\sigma} \sin^2 \sqrt{-b} \omega \phi^- \right]. \quad (62)$$

In these models there is no transition from nonphantom to phantom behavior because the barotropic index has a definite sign.

### C. Three-dimensional internal space

Now, we study the case of a multiscalar-field model containing three fields with the following internal metric:

$$Q_{ik} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (63)$$

It represents a scalar-field configuration where the dimension of the internal space associated with the quintessence sector  $N_q$  exceeds the dimension of the respective phantom sector  $N_{ph}$ , so  $N_q > N_{ph}$ . The case where the phantom dominates over quintessence components  $N_{ph} > N_q$  can be obtained from the latter by changing the sign of the metric  $Q_{ik} \rightarrow -Q_{ik}$ .

We focus on the case where the multiscalar field is driven by the potential

$$V(\phi_1, \phi_2, \phi_3) = V_0 + \frac{1}{2} \dot{\phi}_2^2 - \frac{1}{2} \dot{\phi}_3^2, \quad (64)$$

so the Friedmann equation reads

$$3H^2 = V_0 + \frac{\dot{\phi}_{10}^2}{a^6} + \frac{\dot{\phi}_{20}^2}{a^3} - \frac{\dot{\phi}_{30}^2}{a^3}. \quad (65)$$

The total energy density [Eq. (65)] can be thought of as a mixture of two fluids. The corresponding energy densities (of the two fluids) are defined as positive and are conserved separately,

$$\rho_b = \frac{\dot{\phi}_{20}^2}{a^3}, \quad \rho_{de} = V_0 + \frac{\dot{\phi}_{10}^2}{a^6} - \frac{\dot{\phi}_{30}^2}{a^3}, \quad (66)$$

$$\dot{\rho}_b + 3H\rho_b = 0, \quad \dot{\rho}_{de} + 3H\gamma_{de}\rho_{de} = 0, \quad (67)$$

$$\gamma_{de} = \frac{2\dot{\phi}_{10}^2 - \dot{\phi}_{30}^2 a^3}{\dot{\phi}_{10}^2 - \dot{\phi}_{30}^2 a^3 + V_0 a^6}, \quad (68)$$

and  $\dot{\phi}_{30}^2 \leq 4V_0 \dot{\phi}_{10}^2$ . We have identified  $\rho_b$  and  $\rho_{de}$  as the baryonic and dark energy densities. The latter component comprises vacuum energy density  $V_0$ , a stiff fluid term  $\rho_s = \dot{\phi}_{10}^2/a^6$ , and a kind of pressureless perfect fluid with negative energy density,  $\rho_D = -\dot{\phi}_{30}^2/a^3$ . The last term mimics the negative part of the classical Dirac field [30]. Hence, the internal space permits us to incorporate

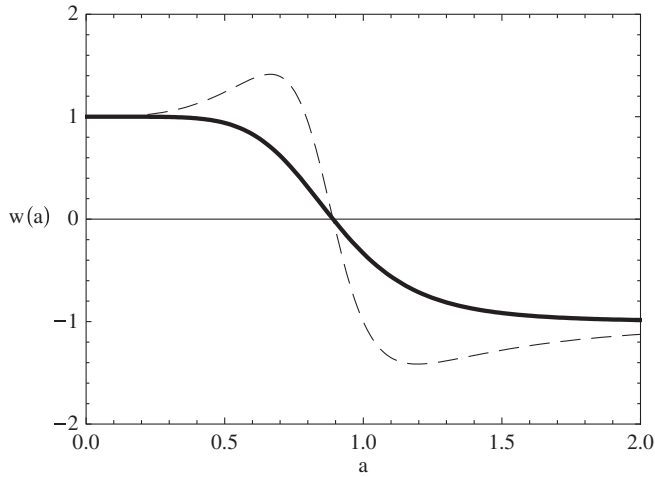


FIG. 2. The state parameter  $w_{\text{de}}$  as a function of the scale factor. The dashed line includes the Dirac term, and the solid line does not include it.

the negative part of the classical Dirac field as a source of the Einstein equations in a natural way.

The general solution of the Friedmann equation (65) takes the following form:

$$a^3 = \sqrt{\frac{\dot{\phi}_{10}^2}{V_0}} \sinh \sqrt{3V_0} t + \frac{\dot{\phi}_{20}^2 - \dot{\phi}_{30}^2}{2V_0} [\cosh \sqrt{3V_0} t - 1], \quad (69)$$

where the initial singularity was fixed at  $t = 0$ . The dark energy crosses the phantom divide at  $a_c = (2\dot{\phi}_{10}^2/\dot{\phi}_{30}^2)^{1/3}$  where it reaches its minimum value  $\rho_{\text{dec}} = V_0 - \dot{\phi}_{30}^4/4\dot{\phi}_{10}^2$ .

Basically, the dark energy crosses the phantom divide due to the term  $\dot{\phi}_{30}^2$  [30]. Concerning the nature of this parameter, note that the dominant energy condition is violated in this cosmological scenario thanks to the term  $\dot{\phi}_{30}^2$ . When it is absent the dark energy state parameter  $w_{\text{de}}$  always remains in the range  $[-1, 1]$  (see Fig. 2), implying that the dark energy component satisfies the dominant energy condition.

## V. SUMMARY AND CONCLUSION

We have extended the quintom model by introducing a multiscalar-field configuration along with an  $N$ -dimensional internal space endowed with a constant internal metric and given the Lagrangian formulation of this new model. This formulation enriches the duality between contracting and expanding cosmologies. We

have shown that power-law solutions are asymptotically stable when the multiscalar field is driven by the exponential potential with the same internal symmetry as the kinetic energy term, generalizing similar results obtained for the quintessence cosmology.

We have found the general solution of the nonlinear Einstein-Klein-Gordon equations for the exponential potential  $V = V_0 e^{-\alpha\phi}$ , extending previous works with standard scalar fields [26] and  $k$ -essence [31]. The richness introduced by the  $N$ -dimensional internal space and the internal metric  $Q_{ik}$  leads, in a natural way, to new singular solutions with a future big rip. For instance, in the explicit solution case, the scale factor blows up at a finite time for  $\alpha^2 < 0$ , exhibiting a superaccelerated behavior and ending in a final big rip. It has a resemblance with the big rip behavior in the phantom universe reported recently by Wei [29] for the potential  $V = V_0 \Phi^{-1}$ .

We have shown that the quintom model with a separable potential can be interpreted as a system of coupled fluids. This framework contains the  $\Lambda$ CDM model, conduces to bouncing universes, and mimics the cosmic string cosmological model with a cosmological constant.

To gain insight into the internal space structure, we have studied a model with a metric  $Q_{ik}$ , representing a configuration where the dimension of the internal space associated with the quintessence sector  $N_q$  exceeds the dimension of the respective phantom sector  $N_{\text{ph}}$ . Choosing  $(N_q, N_{\text{ph}}) = (2, 1)$  we have accommodated a 3-scalar field with the following three components: stiff fluid, dust, and dust with negative energy density. Furthermore, the dark energy density, composed of dust with negative energy density, stiff fluid, and vacuum energy, crosses the phantom divide. Comparing with Ref. [30] we see that the dust component with negative energy density plays the same role as the negative part of a classical Dirac field.

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