

Gerstenhaber algebra structure on the Hochschild cohomology of quadratic string algebras

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Abstract We describe the Gerstenhaber algebra structure on the Hochschild cohomology $\mathrm{HH}^*(A)$ when A is a quadratic string algebra. First we compute the Hochschild cohomology groups using Barzdell's resolution and we describe generators of these groups. Then we construct comparison morphisms between the bar resolution and Bardzell's resolution in order to get formulae for the cup product and the Lie bracket. We find conditions on the bound quiver associated to string algebras in order to get non-trivial structures.

Keywords Hochschild cohomology, cup product, Lie bracket, string algebras

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1 Introduction

Let A be an associative, finite dimensional algebra over an algebraically closed field k . By a fundamental result in representation theory it is well known that there exists a finite quiver Q such that A is Morita equivalent to kQ/I , where kQ is the path algebra of Q and I is an admissible two-sided ideal of kQ , see for instance [2, Theorem 3.7]. The pair (Q, I) is called a bound quiver of A .

A finite dimensional algebra is called **biserial** if the radical of every projective indecomposable module is the sum of two uniserial modules whose intersection is simple or zero, see [14]. These algebras have been studied by several authors and from different points of view since there are a lot of natural examples of algebras which turn out to be of this kind: Nakayama algebras [23], Kawada algebras [19, 27], blocks of group algebras with cyclic or dihedral defect groups [13, 18, 26], iterated tilted algebras of type A and \tilde{A} [1, 3], Brauer graph algebras [31] and algebras

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appearing in the Gel'fand-Ponomarev classification of Harish-Chandra modules over the Lorentz group [15].

The representation theory of these algebras was first studied by Gel'fand and Ponomarev in [15]: they have provided the methods in order to classify all their indecomposable representations. This classification shows that biserial algebras are always tame, see also [12]. They are an important class of algebras whose representation theory has been very well described, see [3, 8].

The subclass of **special biserial algebras** was studied by Skowroński and Waschbüsch in [34] where they characterize the biserial algebras of finite representation type. The definition of these algebras can be given in terms of conditions on the bound quiver (Q, I) associated (see Section 2.2). A classification of the special biserial algebras which are minimal representation-infinite has been given by Ringel in [28]. This class of algebras has played an important rôle in the study of self-injective algebras, and their representation theory is well understood. There is a beautiful description of all finite-dimensional indecomposable modules over special biserial algebras: they are either string modules or band modules or non-uniserial projective-injective modules, see [8, 38].

Since A is an algebra over a field k , the **Hochschild cohomology groups** $\mathrm{HH}^n(A, M)$ with coefficients in an A -bimodule M can be identified with the groups $\mathrm{Ext}_{A-A}^n(A, M)$. In particular, if M is the A -bimodule A , we simply write $\mathrm{HH}^n(A)$. Even though the computation of the Hochschild cohomology groups $\mathrm{HH}^n(A)$ is rather complicated, some approaches have been successful when the algebra A is given by a quiver with relations. For instance, explicit formula for the dimensions of $\mathrm{HH}^n(A)$ in terms of those combinatorial data have been found in [6, 9–11, 17, 24]. In particular, Hochschild cohomology of special biserial algebras has been considered in [7, 20, 35].

In the particular case of monomial algebras, that is, algebras $A = kQ/I$ where I can be chosen as generated by paths, one has a detailed description of a minimal resolution of the A -bimodule A : Bardzell's resolution [5].

An algebra is called a **string algebra** if it is Morita equivalent to a monomial special biserial algebra kQ/I , and it is called **quadratic** if the ideal I is generated by paths of length two. In general, the computation of the Hochschild cohomology groups using Bardzell's resolution may lead to hard combinatorial computations. However, for quadratic string algebras the resolution, and the complex associated, are easier to handle.

The sum $\mathrm{HH}^*(A) = \bigoplus_{n \geq 0} \mathrm{HH}^n(A)$ is a Gerstenhaber algebra, that is, it is a graded commutative ring via the cup product, a graded Lie algebra via the bracket, and these two structures are related, see [16]. So far there are only a few classes of algebras in the literature where the Gerstenhaber algebra structure on Hochschild cohomology has been determined explicitly, see [7, 21, 22, 30, 32, 35–37, 39]. In [25] we compute the Hochschild cohomology groups of a triangular string algebra, and we prove that its ring structure is trivial.

The purpose of this paper is to study the Gerstenhaber algebra structure on the Hochschild cohomology of a quadratic string algebra A , that is, we describe explicitly the ring and the Lie algebra structure of the Hochschild cohomology of A , and we delete the triangular condition in order to get non-trivial structures. The main results show the dimension of the Hochschild cohomology groups $\mathrm{HH}^n(A)$, see Theorems 1 and 2. These computations are done using Bardzell's resolution, and allow us to give an explicit basis for each cohomology group. Furthermore,

we construct a comparison morphism between the bar resolution and Bardzell's resolution which lead us to the formulae for the cup product and the Lie bracket.

Finally we describe explicit non-zero elements in the Hochschild cohomology whose cup product and Lie bracket are non-zero, that is, we define \mathcal{G}_n to be the set of **gentle pairs** (see Definition 2(a)) and we prove the following theorems.

Theorem (Theorem 3) *Let $A = kQ/I$ be a quadratic string algebra and $\mathcal{G}_n \neq \emptyset$ for some $n > 0$. Then the cup product defined in $\mathrm{HH}^*(A)$ is non-trivial. More precisely,*

- (i) *if n is even and $\mathrm{char} k \neq 2$, $\mathrm{HH}^{s_1 n}(A) \cup \mathrm{HH}^{s_2 n}(A) \neq 0$;*
- (ii) *if n is odd and $\mathrm{char} k \neq 2$, $\mathrm{HH}^{2s_1 n}(A) \cup \mathrm{HH}^{2s_2 n}(A) \neq 0$;*
- (iii) *if $\mathrm{char} k = 2$, $\mathrm{HH}^{s_1 n}(A) \cup \mathrm{HH}^{s_2 n}(A) \neq 0$*

for any $s_1, s_2 \geq 1$.

Theorem (Theorem 4) *Let $\mathrm{char} k = 0$ and let $A = kQ/I$ be a quadratic string algebra such that $\mathcal{G}_n \neq \emptyset$ for some $n > 0$. Then the Lie bracket defined in $\mathrm{HH}^*(A)$ is non-trivial. More precisely,*

- (i) *if n is even, $[\mathrm{HH}^{s_1 n+1}(A), \mathrm{HH}^{s_2 n+1}(A)] \neq 0$;*
- (ii) *if n is odd, $[\mathrm{HH}^{2s_1 n+1}(A), \mathrm{HH}^{2s_2 n+1}(A)] \neq 0$*

for any $s_1, s_2 \geq 1$, $s_1 \neq s_2$.

These proofs are given in terms of representatives u_{sn} and $u_{s_{n+1}}$ of non-zero elements in $\mathrm{HH}^{sn}(A)$ and $\mathrm{HH}^{s_{n+1}}(A)$ respectively such that

$$u_{s_1 n} \cup u_{s_2 n} = u_{(s_1+s_2)n} \quad \text{and} \quad [u_{s_1 n+1}, u_{s_2 n+1}] = \lambda u_{(s_1+s_2)n+1}$$

for some scalar $\lambda \in k$.

The paper is organized as follows. In Section 2 we introduce all the necessary terminology. In Section 3 we recall the resolution given by Bardzell for monomial algebras in [5] and we present all the computations that lead us to Theorems 1 and 2 where we present the dimension of all the Hochschild cohomology groups of quadratic string algebras. In Section 4 we describe the ring and the Lie structure of the Hochschild cohomology of these algebras and we find conditions on the bound quiver associated in order to get non-trivial structures.

2 Preliminaries

2.1 Quivers and relations

Let Q be a connected finite quiver with a set of vertices Q_0 , a set of arrows Q_1 and $s, t : Q_1 \rightarrow Q_0$ be the maps associating to each arrow α its source $s(\alpha)$ and its target $t(\alpha)$. A path w of **length** l is a sequence of l arrows $\alpha_1 \dots \alpha_l$ such that $t(\alpha_i) = s(\alpha_{i+1})$. We denote by $|w|$ the length of the path w . We put $s(w) = s(\alpha_1)$ and $t(w) = t(\alpha_l)$. For any vertex x we consider e_x the trivial path of length zero and we put $s(e_x) = t(e_x) = x$. An oriented cycle is a non-trivial path w such that $s(w) = t(w)$.

The **path algebra** kQ is the k -vector space with basis the set of paths in Q ; the product on the basis elements is given by the concatenation of the sequences of

arrows of the paths w and w' if they form a path (namely, if $t(w) = s(w')$) and zero otherwise. Vertices form a complete set of orthogonal idempotents of kQ . Let F be the two-sided ideal of kQ generated by the arrows of Q . A two-sided ideal I of kQ is said to be **admissible** if there exists an integer $m \geq 2$ such that $F^m \subseteq I \subseteq F^2$. The elements in I are called **relations**, kQ/I is called a **monomial algebra** if the ideal I is generated by paths, and a relation is called **quadratic** if it is a path of length two.

By a fundamental result in representation theory it is well known that if A is a basic, indecomposable, finite dimensional algebra over an algebraically closed field k , then there exists a unique finite connected quiver Q and a surjective morphism of k -algebras $\nu : kQ \rightarrow A$, which is not unique in general, with $I_\nu = \text{Ker } \nu$ admissible, see for instance [2, Theorem 3.7]. The pair (Q, I_ν) is called a **presentation** of A .

2.2 String algebras

Recall from [34] that a bound quiver (Q, I) is **special biserial** if it satisfies the following conditions:

- S1) Each vertex in Q is the source of at most two arrows and the target of at most two arrows;
- S2) For an arrow α in Q there is at most one arrow β and at most one arrow γ such that $\alpha\beta \notin I$ and $\gamma\alpha \notin I$.

If the ideal I is generated by paths, the bound quiver (Q, I) is **string**.

An algebra is called **special biserial** (or **string**) if it is Morita equivalent to a path algebra kQ/I with (Q, I) a special biserial bound quiver (or a string bound quiver, respectively).

Since Hochschild cohomology is invariant under Morita equivalence, whenever we deal with a string algebra A we will assume that it is given by a string presentation $A = kQ/I$ with (Q, I) satisfying the previous conditions.

In this paper we are interested in quadratic string algebras, that is, the ideal I is generated by paths of length two. We fix a minimal set \mathcal{R} of paths that generates the ideal I . Moreover, we denote by \mathcal{P} the set of paths in Q such that the set $\{\gamma + I, \gamma \in \mathcal{P}\}$ is a basis of $A = kQ/I$. It is clear that $Q_0 \cup Q_1 \subseteq \mathcal{P}$ since $I \subseteq F^2$.

3 Hochschild cohomology groups

3.1 Bardzell's resolution for quadratic string algebras

We recall that the Hochschild cohomology groups $\text{HH}^n(A)$ of an algebra A are the groups $\text{Ext}_{A-A}^n(A, A)$, $n \geq 0$. Since string algebras are monomial algebras, their Hochschild cohomology groups can be computed using a convenient minimal projective resolution of A as A -bimodule constructed by Bardzell in [5, Theorem 4.1]. In the particular case of quadratic string algebras, this minimal resolution is the following:

$$\dots \rightarrow A \otimes kAP_n \otimes A \xrightarrow{d_n} A \otimes kAP_{n-1} \otimes A \rightarrow \dots \rightarrow A \otimes kAP_0 \otimes A \xrightarrow{\mu} A \rightarrow 0$$

where $kAP_0 = kQ_0$, $kAP_1 = kQ_1$, for $n \geq 2$, kAP_n is the vector space generated by the set

$$AP_n = \{\alpha_1 \alpha_2 \cdots \alpha_n : \alpha_i \alpha_{i+1} \in I, 1 \leq i < n\},$$

all tensor products are taken over $E = kQ_0$, the subalgebra of A generated by the vertices, and the A -bimodule morphisms are

$$\begin{aligned} \mu(1 \otimes e_i \otimes 1) &= e_i, \\ d_1(1 \otimes \alpha \otimes 1) &= \alpha \otimes e_{t(\alpha)} \otimes 1 - 1 \otimes e_{s(\alpha)} \otimes \alpha, \\ d_n(1 \otimes \alpha_1 \cdots \alpha_n \otimes 1) &= \alpha_1 \otimes \alpha_2 \cdots \alpha_n \otimes 1 + (-1)^n 1 \otimes \alpha_1 \cdots \alpha_{n-1} \otimes \alpha_n. \end{aligned}$$

The E - A -bilinear map $c : A \otimes kAP_{n-1} \otimes A \rightarrow A \otimes kAP_n \otimes A$ defined by

$$c(a \otimes \alpha_1 \cdots \alpha_{n-1} \otimes 1) = \begin{cases} b \otimes \alpha_0 \alpha_1 \cdots \alpha_{n-1} \otimes 1 & \text{if } a = b\alpha_0 \text{ and } \alpha_0 \alpha_1 \in I, \\ 0 & \text{otherwise} \end{cases}$$

is a contracting homotopy, see [33, Theorem 1] for more details.

The Hochschild complex, obtained by applying $\text{Hom}_{A-A}(-, A)$ to the resolution we have just described and using the isomorphisms

$$\text{Hom}_{A-A}(A \otimes kAP_n \otimes A, A) \simeq \text{Hom}_{E-E}(kAP_n, A)$$

is

$$0 \longrightarrow \text{Hom}_{E-E}(kAP_0, A) \xrightarrow{F_1} \text{Hom}_{E-E}(kAP_1, A) \xrightarrow{F_2} \text{Hom}_{E-E}(kAP_2, A) \rightarrow \cdots$$

where

$$\begin{aligned} F_1(f)(\alpha) &= \alpha f(e_{t(\alpha)}) - f(e_{s(\alpha)})\alpha, \\ F_n(f)(\alpha_1 \cdots \alpha_n) &= \alpha_1 f(\alpha_2 \cdots \alpha_n) + (-1)^n f(\alpha_1 \cdots \alpha_{n-1})\alpha_n. \end{aligned}$$

3.2 Computations

In order to describe the ring and the Lie algebra structure of the Hochschild cohomology $\text{HH}^*(A)$ of a quadratic string algebra, we need a convenient description of the previous complex. In this section we will describe explicit basis of these k -vector spaces and study the behavior of the maps between them in order to get information about kernels and images.

For any pair X, Y of sets of paths in Q we denote by $(X//Y)$ the set of pairs $(\rho, \gamma) \in X \times Y$ such that ρ, γ are parallel paths in Q , that is

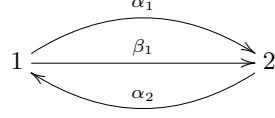
$$(X//Y) = \{(\rho, \gamma) \in X \times Y : s(\rho) = s(\gamma), t(\rho) = t(\gamma)\}.$$

Recall that we have fixed a set \mathcal{P} of paths in Q such that the set $\{\gamma + I : \gamma \in \mathcal{P}\}$ is a basis of $A = kQ/I$. For any $m \geq 0$ we denote by \mathcal{P}_m the set of paths in \mathcal{P} whose length is greater than or equal to m . Then

$$(X//\mathcal{P}_m) = \{(\rho, \gamma) \in X \times \mathcal{P} : s(\rho) = s(\gamma), t(\rho) = t(\gamma), |\gamma| \geq m\}$$

and hence $(X//\mathcal{P}) = (X//Q_0) \sqcup (X//\mathcal{P}_1) = (X//Q_0) \sqcup (X//Q_1) \sqcup (X//\mathcal{P}_2)$, where \sqcup depicts disjoint union.

Example 1 Let $A = kQ/I$ where Q is the quiver



and $I = \langle \alpha_1\alpha_2, \alpha_2\alpha_1, \beta_1\alpha_2 \rangle$. The sets $(AP_n//\mathcal{P})$ are

$$(AP_0//Q_0) = \{(e_1, e_1), (e_2, e_2)\}$$

$$(AP_0//\mathcal{P}_1) = \{(e_2, \alpha_2\beta_1)\}$$

and for $i \geq 0$,

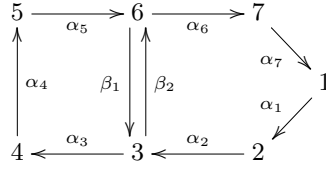
$$(AP_{2i}//Q_0) = \{((\alpha_1\alpha_2)^i, e_1), ((\alpha_2\alpha_1)^i, e_2), (\beta_1\alpha_2(\alpha_1\alpha_2)^{i-1}, e_1)\},$$

$$(AP_{2i}//\mathcal{P}_1) = \{((\alpha_2\alpha_1)^i, \alpha_2\beta_1)\},$$

$$(AP_{2i+1}//Q_0) = \emptyset,$$

$$(AP_{2i+1}//\mathcal{P}_1) = \{((\alpha_1\alpha_2)^i\alpha_1, \alpha_1), ((\alpha_1\alpha_2)^i\alpha_1, \beta_1), ((\alpha_2\alpha_1)^i\alpha_2, \alpha_2), (\beta_1(\alpha_2\alpha_1)^i, \alpha_1), (\beta_1(\alpha_2\alpha_1)^i, \beta_1)\}.$$

Example 2 Let $A = kQ/I$ where Q is the quiver



and $I = \langle \alpha_i\alpha_{i+1} \rangle_{\{i=1, \dots, 6\}} + \langle \beta_1\beta_2, \beta_2\beta_1 \rangle$. The sets $(AP_n//\mathcal{P})$ are

$$(AP_0//Q_0) = \{(e_i, e_i)\}_{\{i=1, \dots, 7\}}$$

$$(AP_0//\mathcal{P}_1) = \emptyset,$$

$$(AP_1//Q_0) = \emptyset,$$

$$(AP_1//\mathcal{P}_1) = \{(\alpha_i, \alpha_i)\}_{\{i=1, \dots, 7\}} \sqcup \{(\beta_i, \beta_i)\}_{\{i=1, 2\}},$$

$$(AP_{2i}//Q_0) = \{((\beta_1\beta_2)^i, e_6), ((\beta_2\beta_1)^i, e_3)\} \text{ for } i \geq 1,$$

$$(AP_4//\mathcal{P}_1) = \{(\alpha_2\alpha_3\alpha_4\alpha_5, \alpha_2\beta_2), (\alpha_3\alpha_4\alpha_5\alpha_6, \beta_2\alpha_6)\},$$

$$(AP_{2i}//\mathcal{P}_1) = \emptyset, \text{ for } i \geq 1, i \neq 2,$$

$$(AP_7//Q_0) = \{(\alpha_1\alpha_2 \dots \alpha_6\alpha_7, e_1)\},$$

$$(AP_{2i+1}//Q_0) = \emptyset \text{ for } i \geq 1, i \neq 3,$$

$$(AP_3//\mathcal{P}_1) = \{(\alpha_3\alpha_4\alpha_5, \beta_2), (\beta_1\beta_2\beta_1, \beta_1), (\beta_2\beta_1\beta_2, \beta_2)\},$$

$$(AP_5//\mathcal{P}_1) = \{(\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6, \alpha_2\beta_2\alpha_6), ((\beta_1\beta_2)^2\beta_1, \beta_1), ((\beta_2\beta_1)^2\beta_2, \beta_2)\},$$

$$(AP_{2i+1}//\mathcal{P}_1) = \{((\beta_1\beta_2)^i\beta_1, \beta_1), ((\beta_2\beta_1)^i\beta_2, \beta_2)\} \text{ for } i > 2.$$

Observe that the k -vector spaces $\text{Hom}_{E-E}(kAP_n, A)$ and $k(AP_n//\mathcal{P})$ are isomorphic: the basis element $(\rho, \gamma) \in (AP_n//\mathcal{P})$ corresponds to the morphism $f_{(\rho, \gamma)}$ in $\text{Hom}_{E-E}(kAP_n, A)$ defined by

$$f_{(\rho, \gamma)}(w) = \begin{cases} \gamma & \text{if } w = \rho, \\ 0 & \text{otherwise.} \end{cases}$$

Taking into account these isomorphisms, for any $n \geq 0$ we can construct commutative diagrams

$$\begin{array}{ccc} \text{Hom}_{E-E}(kAP_n, A) & \xrightarrow{F_{n+1}} & \text{Hom}_{E-E}(kAP_{n+1}, A) \\ \downarrow \cong & & \downarrow \cong \\ k(AP_n//Q_0) \oplus k(AP_n//\mathcal{P}_1) & \xrightarrow{\begin{pmatrix} 0 & 0 \\ F_{n+1}^0 & 0 \\ 0 & F_{n+1}^1 \end{pmatrix}} & k(AP_{n+1}//Q_0) \oplus k(AP_{n+1}//Q_1) \oplus k(AP_{n+1}//\mathcal{P}_2) \end{array}$$

where

$$\begin{aligned} F_1^0(e_r, e_r) &= \sum_{\{\beta \in Q_1: t(\beta)=r\}} (\beta, \beta) - \sum_{\{\beta \in Q_1: s(\beta)=r\}} (\beta, \beta), \\ F_1^1(e_r, \gamma) &= \sum_{\{\beta \in Q_1: t(\beta)=r\}} (\beta, \beta\gamma) - \sum_{\{\beta \in Q_1: s(\beta)=r\}} (\beta, \gamma\beta), \end{aligned}$$

$$\begin{aligned} F_{n+1}^0(\alpha_1 \cdots \alpha_n, e_{s(\alpha_1)}) &= \sum_{\substack{\{\beta \in Q_1: \beta\alpha_1 \in I, \\ t(\beta)=s(\alpha_1)\}}} (\beta\alpha_1 \cdots \alpha_n, \beta) + (-1)^{n+1} \sum_{\substack{\{\beta \in Q_1: \alpha_n\beta \in I, \\ s(\beta)=s(\alpha_1)\}}} (\alpha_1 \cdots \alpha_n\beta, \beta) \\ F_{n+1}^1(\alpha_1 \cdots \alpha_n, \gamma) &= \sum_{\substack{\{\beta \in Q_1: \\ \beta\alpha_1 \in I, \beta\gamma \notin I\}}} (\beta\alpha_1 \cdots \alpha_n, \beta\gamma) + (-1)^{n+1} \sum_{\substack{\{\beta \in Q_1: \\ \alpha_n\beta \in I, \gamma\beta \notin I\}}} (\alpha_1 \cdots \alpha_n\beta, \gamma\beta). \end{aligned}$$

Now we will introduce several subsets of $(AP_n//\mathcal{P})$ in order to get a nice description of the kernel and the image of F_{n+1}^0 and F_{n+1}^1 .

For $n = 0$ we have that $(AP_0//\mathcal{P}) = (Q_0//\mathcal{P})$. Since A is finite dimensional and quadratic, if $(e_{s(\alpha_1)}, \alpha_1 \cdots \alpha_m) \in (Q_0//\mathcal{P}_1)$ then $\alpha_m\alpha_1 \in I$.

For $n = 1$, $(AP_1//\mathcal{P}) = (Q_1//\mathcal{P})$, and we consider the following partition

$$(Q_1//\mathcal{P}) = (1, 1)_1 \sqcup (0, 0)_1 \sqcup (1, 0)_1 \sqcup (0, 1)_1$$

defined as follows: if $(\alpha, \gamma) \in (Q_1//\mathcal{P})$, we wonder if the path γ starts or ends with the arrow α , that is,

$$\begin{aligned} (1, 1)_1 &= \{(\alpha, \alpha) : \alpha \in Q_1\} \\ (0, 0)_1 &= \{(\alpha, \gamma) \in (Q_1//\mathcal{P}) : \gamma \notin \alpha kQ \cup kQ\alpha\} \\ (1, 0)_1 &= \{(\alpha, \gamma) \in (Q_1//\mathcal{P}) : \gamma \in \alpha kQ, \gamma \notin kQ\alpha\} \\ (0, 1)_1 &= \{(\alpha, \gamma) \in (Q_1//\mathcal{P}) : \gamma \notin \alpha kQ, \gamma \in kQ\alpha\} \end{aligned}$$

For any $n \geq 2$, if $(\rho, \gamma) \in (AP_n//\mathcal{P})$ then ρ is a path of length greater than or equal to 2, that is, $\rho = \alpha_1 \hat{\rho} \alpha_2$ and wondering if the path γ starts with the arrow α_1 or ends with the arrow α_2 , we get the following partition

$$(AP_n//\mathcal{P}) = (0,0)_n \sqcup (1,0)_n \sqcup (0,1)_n \sqcup (1,1)_n$$

where

$$\begin{aligned} (0,0)_n &= \{(\rho, \gamma) \in (AP_n//\mathcal{P}) : \rho = \alpha_1 \hat{\rho} \alpha_2 \text{ and } \gamma \notin \alpha_1 kQ \cup kQ \alpha_2\} \\ (1,0)_n &= \{(\rho, \gamma) \in (AP_n//\mathcal{P}) : \rho = \alpha_1 \hat{\rho} \alpha_2 \text{ and } \gamma \in \alpha_1 kQ, \gamma \notin kQ \alpha_2\} \\ (0,1)_n &= \{(\rho, \gamma) \in (AP_n//\mathcal{P}) : \rho = \alpha_1 \hat{\rho} \alpha_2 \text{ and } \gamma \notin \alpha_1 kQ, \gamma \in kQ \alpha_2\} \\ (1,1)_n &= \{(\rho, \gamma) \in (AP_n//\mathcal{P}) : \rho = \alpha_1 \hat{\rho} \alpha_2 \text{ and } \gamma \in \alpha_1 kQ \cap kQ \alpha_2\}. \end{aligned}$$

We also have to distinguish elements inside each of the previous sets taking into account the following definitions:

$$\begin{aligned} {}^+(X//\mathcal{P}) &= \{(\rho, \gamma) \in (X//\mathcal{P}) : Q_1 \gamma \not\subset I\} \\ {}^-(X//\mathcal{P}) &= \{(\rho, \gamma) \in (X//\mathcal{P}) : Q_1 \gamma \subset I\} \end{aligned}$$

In an analogous way we define $(X//\mathcal{P})^+$, $(X//\mathcal{P})^-$, ${}^+(X//\mathcal{P})^+ = {}^+(X//\mathcal{P}) \cap (X//\mathcal{P})^+$ and so on.

Example 3 For the algebra presented in Example 1 we have that, for $n > 0$,

$$\begin{aligned} {}^+(0,0)_n^+ &= \begin{cases} (AP_n//Q_0) & \text{if } n \text{ is even,} \\ \emptyset & \text{otherwise,} \end{cases} \\ {}^+(0,0)_n^- &= \begin{cases} \{((\alpha_1 \alpha_2)^i \alpha_1, \beta_1)\} & \text{if } n = 2i + 1, \\ \emptyset & \text{otherwise,} \end{cases} \\ {}^-(0,0)_n^- &= \begin{cases} \{(\beta_1, \alpha_1)\} & \text{if } n = 1, \\ \emptyset & \text{otherwise,} \end{cases} \\ (1,0)_n^- &= \begin{cases} \emptyset, & \text{if } n = 1, \\ \{(\beta_1 (\alpha_2 \alpha_1)^i, \beta_1)\} & \text{if } n = 2i + 1, i \geq 1, \\ \{((\alpha_2 \alpha_1)^i, \alpha_2 \beta_1)\} & \text{if } n = 2i, \end{cases} \\ {}^-(0,1)_n &= \begin{cases} \{(\beta_1 (\alpha_2 \alpha_1)^i, \alpha_1)\} & \text{if } n = 2i + 1, i \geq 1, \\ \emptyset & \text{otherwise} \end{cases} \\ (1,1)_n &= \begin{cases} \{(\alpha_1, \alpha_1), (\alpha_2, \alpha_2), (\beta_1, \beta_1)\} & \text{if } n = 1, \\ \{((\alpha_1 \alpha_2)^i \alpha_1, \alpha_1), ((\alpha_2 \alpha_1)^i \alpha_2, \alpha_2)\} & \text{if } n = 2i + 1, i \geq 1, \\ \emptyset & \text{otherwise,} \end{cases} \\ {}^-(0,0)_n^+ = (1,0)_n^+ &= {}^+(0,1)_n = \emptyset, \end{aligned}$$

and in Example 2 we have that

$$\begin{aligned}
{}^+(0,0)_n^+ &= \begin{cases} (AP_n//Q_0) & \text{if } n \text{ is even,} \\ \{(\alpha_3\alpha_4\alpha_5, \beta_2)\} & \text{if } n = 3, \\ (AP_7//Q_0) & \text{if } n = 7, \\ \emptyset & \text{otherwise,} \end{cases} \\
(1,0)_n^+ &= \begin{cases} \{(\alpha_2 \dots \alpha_5, \alpha_2\beta_2)\} & \text{if } n = 4, \\ \emptyset, & \text{otherwise,} \end{cases} \\
{}^+(0,1)_n &= \begin{cases} \{(\alpha_3 \dots \alpha_6, \beta_2\alpha_6)\} & \text{if } n = 4, \\ \emptyset, & \text{otherwise,} \end{cases} \\
(1,1)_3 &= \{(\beta_1\beta_2)\beta_1, \beta_1), (\beta_2\beta_1\beta_2, \beta_2)\}, \\
(1,1)_n &= (AP_n//\mathcal{P}_1) \quad \text{if } n \text{ is odd, } n \neq 3, \\
(1,1)_n &= \emptyset \quad \text{if } n \text{ is even,} \\
{}^+(0,0)_n^- &= {}^-(0,0)_n^+ = {}^-(0,0)_n^- = (1,0)_n^- = {}^-(0,1)_n = \emptyset.
\end{aligned}$$

We start by describing the behavior of our maps F_{n+1}^1 restricted to the subsets we have just defined. With this aim in mind, for any $n \geq 1$ we define a map

$$\phi_n : (1,0)_n^+ \rightarrow {}^+(0,1)_n$$

as follows: given $(\rho, \gamma) \in (1,0)_n^+$ we have that $\rho = \alpha\rho'$, $\gamma = \alpha\gamma'$, ρ' and γ' do not share their ending arrows and there exists an arrow β such that $\gamma'\beta \notin I$. Since A is a string algebra we conclude that β is unique and that $(\rho'\beta, \gamma'\beta) \in {}^+(0,1)_n$. Hence we define $\phi_n(\alpha\rho', \alpha\gamma') = (\rho'\beta, \gamma'\beta)$.

The proof of the following lemma is analogous to the proof of [25, Lemma 4.1] concerning triangular string algebras.

Lemma 1 *For any $n \geq 1$ we have*

- (a) ${}^-(0,0)_n^- \sqcup (1,0)_n^- \sqcup {}^-(0,1)_n \sqcup (1,1)_n \subset \text{Ker } F_{n+1}^1$;
- (b) the function F_{n+1}^1 induces a bijection from ${}^-(0,0)_n^+$ to ${}^-(0,1)_{n+1} \cap (AP_{n+1}//\mathcal{P}_2)$;
- (c) the function F_{n+1}^1 induces a bijection from ${}^+(0,0)_n^-$ to $(1,0)_{n+1}^- \cap (AP_{n+1}//\mathcal{P}_2)$;
- (d) the maps $\phi_n : (1,0)_n^+ \rightarrow {}^+(0,1)_n$ are bijections and

$$\begin{aligned}
&(id + (-1)^n \phi_n)((1,0)_n^+) \subset \text{Ker } F_{n+1}^1, \\
&(-1)^{n+1} F_{n+1}^1((1,0)_n^+) = (1,1)_{n+1} \cap (AP_{n+1}//\mathcal{P}_2), \\
&F_{n+1}^1({}^+(0,0)_n^+ \cap (AP_n//\mathcal{P}_1)) = (id + (-1)^{n+1} \phi_{n+1})((1,0)_{n+1}^+) \cap k(AP_{n+1}//\mathcal{P}_2).
\end{aligned}$$

Proof (a) In order to check that $(\rho, \gamma) \in (AP_n//\mathcal{P}_1)$ belongs to $\text{Ker } F_{n+1}^1$ we have to prove that if $\beta_1\rho$ or $\rho\beta_2$ belong to AP_{n+1} then $\beta_1\gamma \in I$ and $\gamma\beta_2 \in I$.

If $(\rho, \gamma) \in {}^-(0,0)_n^-$ then the statement is clear because $Q_1\gamma \subset I$ and $\gamma Q_1 \subset I$.

If $(\rho, \gamma) \in (1,0)_n^-$, namely $(\rho, \gamma) = (\alpha\hat{\rho}, \alpha\hat{\gamma})$ with $\gamma Q_1 \subset I$, then $\gamma\beta_2 \in I$. On the other hand, $\beta_1\rho \in AP_{n+1}$ implies that $\beta_1\alpha \in I$ and then $\beta_1\gamma \in I$.

The proof for ${}^-(0,1)_n$ is analogous.

Finally, if $(\rho, \gamma) \in (1,1)_n$, this is, $(\rho, \gamma) = (\alpha, \alpha)$ if $n = 1$ and if $n \geq 2$, $(\rho, \gamma) = (\alpha_1\hat{\rho}\alpha_2, \alpha_1\hat{\gamma}\alpha_2)$, again $\beta_1\rho, \rho\beta_2 \in AP_{n+1}$ imply directly $\beta_1\alpha, \alpha\beta_2 \in I$ in the first case, and for $n \geq 2$, $\beta_1\alpha_1, \alpha_2\beta_2 \in I$, so $\beta_1\gamma \in I$ and $\gamma\beta_2 \in I$.

- (b) Since A is a string algebra if $(\rho, \gamma) \in {}^-(0, 0)_n^+$ there exists a unique arrow β such that $\gamma\beta \in \mathcal{P}$ and then $\rho\beta \in AP_{n+1}$. It is clear that $(\rho\beta, \gamma\beta) \in {}^-(0, 1)_{n+1} \cap (AP_{n+1}/\mathcal{P}_2)$ and $F_{n+1}^1((\rho, \gamma)) = (-1)^{n+1}(\rho\beta, \gamma\beta)$.
- (c) Analogous to the previous one.
- (d) By construction it is clear that the map ϕ_n is a bijection. The first statements follow from the fact that if $(\alpha\rho', \alpha\gamma') \in (1, 0)_n^+$, then $\phi_n(\alpha\rho', \alpha\gamma') = (\rho'\beta, \gamma'\beta)$, $(\alpha\rho'\beta, \alpha\gamma'\beta) \in (1, 1)_{n+1} \cap (AP_{n+1}/\mathcal{P}_2)$ and

$$F_{n+1}^1(\rho'\beta, \gamma'\beta) = (\alpha\rho'\beta, \alpha\gamma'\beta) = (-1)^{n+1}F_{n+1}^1(\alpha\rho', \alpha\gamma').$$

Finally if $(\rho, \gamma) \in {}^+(0, 0)_n^+ \cap (AP_n/\mathcal{P}_1)$ there exist unique arrows α, β such that $\alpha\gamma\beta \in \mathcal{P}$. So $(\alpha\rho, \alpha\gamma) \in (1, 0)_{n+1}^+ \cap (AP_{n+1}/\mathcal{P}_2)$, $\phi_{n+1}(\alpha\rho, \alpha\gamma) = (\rho\beta, \gamma\beta) \in {}^+(0, 1)_{n+1} \cap (AP_{n+1}/\mathcal{P}_2)$ and

$$F_{n+1}((\rho, \gamma)) = (\alpha\rho, \alpha\gamma) + (-1)^{n+1}(\rho\beta, \gamma\beta).$$

□

Proposition 1 For any $n \geq 1$ we have that

$$\begin{aligned} \dim_k \text{Ker } F_{n+1}^1 &= |{}^-(0, 0)_n^-| + |{}^-(0, 1)_n| + |(1, 1)_n| + |(1, 0)_n|, \\ \dim_k \text{Im } F_{n+1}^1 &= |({}^-(0, 1)_{n+1} \sqcup (1, 0)_{n+1} \sqcup (1, 1)_{n+1}) \cap (AP_{n+1}/\mathcal{P}_2)|. \end{aligned}$$

Proof It follows directly since the previous lemma implies that the set

$${}^-(0, 0)_n^- \sqcup (1, 0)_n^- \sqcup {}^-(0, 1)_n \sqcup (1, 1)_n \sqcup (id + (-1)^n \phi_n)((1, 0)_n^+)$$

is a basis of $\text{Ker } F_{n+1}^1$, the set

$$({}^-(0, 1)_{n+1} \sqcup (1, 0)_{n+1}^- \sqcup (1, 1)_{n+1} \sqcup (id + (-1)^{n+1} \phi_{n+1})((1, 0)_{n+1}^+)) \cap k(AP_{n+1}/\mathcal{P}_2)$$

is a basis of $\text{Im } F_{n+1}^1$ and

$$(1, 0)_n = (1, 0)_n^+ \sqcup (1, 0)_n^-.$$

□

The following result will be used in the description of the cup product defined in $\text{HH}^*(A)$. For any cocycle f in $\text{Ker } F_{n+1}$ we denote by \bar{f} its equivalence class in $\text{HH}^n(A)$.

Proposition 2 Let $n > 1$ and let

$$f = \sum_{i=1}^m \lambda_i(\rho_i, \gamma_i) \in \text{Ker } F_{n+1}^1 \cap k(AP_n/\mathcal{P}_2)$$

such that $(\rho_i, \gamma_i) \notin {}^-(0, 0)_n^-$ for all i with $1 \leq i \leq m$. Then $\bar{f} = 0$ in $\text{HH}^n(A)$.

Proof From Lemma 1 we know that f is a linear combination of elements in

$$\left((1, 0)_n^- \sqcup (0, 1)_n \sqcup (1, 1)_n \sqcup (id + (-1)^n \phi_n)((1, 0)_n^+) \right) \cap k(AP_n // \mathcal{P}_2).$$

Now,

$$\begin{aligned} (1, 0)_n^- \cap (AP_n // \mathcal{P}_2) &= F_n^1(+(0, 0)_{n-1}^-) \\ (0, 1)_n \cap (AP_n // \mathcal{P}_2) &= F_n^1(-(0, 0)_{n-1}^+) \\ (1, 1)_n \cap (AP_n // \mathcal{P}_2) &= (-1)^n F_n^1((1, 0)_{n-1}^+) \\ (id + (-1)^n \phi_n)((1, 0)_n^+) \cap k(AP_n // \mathcal{P}_2) &= F_n^1(+(0, 0)_{n-1}^+) \cap (AP_{n-1} // \mathcal{P}_1) \end{aligned}$$

and these equalities imply that f belongs to the image of F_n . \square

In order to describe the behavior of F_{n+1}^0 in $k(AP_n // Q_0)$ we need a description of the basis elements of this vector space. Observe that $(AP_n // Q_0) \subset +(0, 0)_n^+$.

Definition 1 A pair $(\alpha_1 \cdots \alpha_n, e_{s(\alpha_1)}) \in (AP_n // Q_0)$ is called **incomplete** if $\alpha_n \alpha_1 \notin I$, and it is called **complete** if $\alpha_n \alpha_1 \in I$. We denote by \mathcal{I}_n and by \mathcal{C}_n the set of incomplete and complete pairs in $(AP_n // Q_0)$, respectively.

The cyclic group $\mathbb{Z}_n = \langle \tau \rangle$ of order n acts on \mathcal{C}_n , with the action given by

$$\tau(\alpha_1 \cdots \alpha_n, e_{s(\alpha_1)}) = (\alpha_n \alpha_1 \cdots \alpha_{n-1}, e_{s(\alpha_n)}).$$

For any $(\rho, e) \in \mathcal{C}_n$ we define its **order** as the first natural number r such that $\tau^r(\rho, e) = (\rho, e)$, and consider the norm of this element defined as follows

$$N(\rho, e) = \sum_{i=0}^{r-1} \tau^i(\rho, e).$$

Clearly τ and N induce linear maps $\tau, N : k\mathcal{C}_n \rightarrow k\mathcal{C}_n$. We consider the following subset of complete pairs

$$\begin{aligned} \mathcal{C}_n(0) &= \{(\alpha_1 \cdots \alpha_n, e_{s(\alpha_1)}) \in \mathcal{C}_n : \\ &\quad \nexists \gamma \in Q_1 \setminus \{\alpha_n\} \text{ and } \nexists \beta \in Q_1 \setminus \{\alpha_1\} \text{ with } \alpha_n \beta, \gamma \alpha_1 \in I\}. \end{aligned}$$

Definition 2

- (a) A complete pair $(\alpha_1 \cdots \alpha_n, e_{s(\alpha_1)})$ is called **gentle** if $\tau^m(\alpha_1 \cdots \alpha_n, e_{s(\alpha_1)}) \in \mathcal{C}_n(0)$ for any $m \in \mathbb{Z}$;
- (b) An incomplete pair $(\alpha_1 \cdots \alpha_n, e_{s(\alpha_1)})$ is called **empty** if there is no relation $\beta \gamma \in I$ with $t(\beta) = s(\alpha_1) = s(\gamma)$.

We denote \mathcal{G}_n and \mathcal{E}_n the set of gentle and empty pairs in $(AP_n // Q_0)$ respectively, and $N\mathcal{G}_n, N\mathcal{E}_n$ their corresponding complements in \mathcal{C}_n and \mathcal{I}_n respectively, that is

$$(AP_n // Q_0) = \mathcal{C}_n \sqcup \mathcal{I}_n, \quad \mathcal{C}_n = \mathcal{G}_n \sqcup N\mathcal{G}_n, \quad \mathcal{I}_n = \mathcal{E}_n \sqcup N\mathcal{E}_n.$$

Example 4 For the algebra presented in Example 1 we have that, for any $n > 0$,

$$\begin{aligned} N\mathcal{G}_n &= \begin{cases} \{(\alpha_1\alpha_2)^i, e_1), (\alpha_2\alpha_1)^i, e_2)\} & \text{if } n = 2i, \\ \emptyset & \text{otherwise,} \end{cases} \\ \mathcal{G}_n &= \emptyset, \\ N\mathcal{E}_n &= \begin{cases} \{\beta_1(\alpha_2\alpha_1)^{i-1}\alpha_2, e_1)\} & \text{if } n = 2i, \\ \emptyset & \text{otherwise,} \end{cases} \\ \mathcal{E}_n &= \emptyset, \end{aligned}$$

and in Example 2 we have that

$$\begin{aligned} \mathcal{G}_n &= \begin{cases} (AP_n//Q_0) & \text{if } n = 2i, \\ \emptyset, & \text{otherwise,} \end{cases} \\ N\mathcal{G}_n &= \emptyset, \\ \mathcal{E}_n &= \begin{cases} \{(\alpha_1 \dots \alpha_7, e_1)\} & \text{if } n = 7, \\ \emptyset & \text{otherwise,} \end{cases} \\ N\mathcal{E}_n &= \emptyset. \end{aligned}$$

In order to describe the map $F_{n+1}^0 : k(AP_n//Q_0) \rightarrow k(AP_{n+1}//Q_1)$ it is enough to study its behavior in each direct summand $k\mathcal{G}_n, kN\mathcal{G}_n, k\mathcal{E}_n$ and $kN\mathcal{E}_n$.

Observe that any element in the set $(1, 1)_{n+1} \cap (AP_{n+1}//Q_1)$ is of the form

$$(\alpha_1\alpha_2 \dots \alpha_n\alpha_1, \alpha_1)$$

where $(\alpha_1\alpha_2 \dots \alpha_n, e_{s(\alpha_1)})$ is a complete pair in \mathcal{C}_n , belonging to either \mathcal{G}_n or $N\mathcal{G}_n$. Hence we can decompose $(1, 1)_{n+1} \cap (AP_{n+1}//Q_1)$ in the disjoint union

$$(1, 1)_{n+1}^{\mathcal{G}} \sqcup (1, 1)_{n+1}^{N\mathcal{G}}.$$

Lemma 2 *The sequence*

$$k\mathcal{G}_n \xrightarrow{N} k\mathcal{G}_n \xrightarrow{(1-\tau)} k\mathcal{G}_n \xrightarrow{N} k\mathcal{G}_n$$

is exact.

Proof It is clear that $N(1-\tau) = (1-\tau)N = 0$. Any $x \in k\mathcal{G}_n$ can be written as follows

$$x = \sum_i \sum_{j=0}^{m_i-1} \lambda_{ij} \tau^j(\rho_i, e_{s_i}),$$

with m_i the order of (ρ_i, e_{s_i}) , $\lambda_{ij} \in k$ and $(\rho_k, e_{s_k}) \neq \tau^j(\rho_i, e_{s_i})$ if $k \neq i$. Now, $N(x) = 0$ implies that

$$0 = \sum_{i,j} \lambda_{ij} N\tau^j(\rho_i, e_{s_i}) = \sum_{i,j} \lambda_{ij} N(\rho_i, e_{s_i})$$

because $N\tau = N$. This implies that $\sum_{j=0}^{m_i-1} \lambda_{ij} = 0$, and hence

$$x = \sum_{i,j} \lambda_{ij} \tau^j(\rho_i, e_{s_i}) - \sum_i \left(\sum_{j=0}^{m_i-1} \lambda_{ij} \right) (\rho_i, e_{s_i}) = \sum_i \sum_{j=1}^{m_i-1} \lambda_{ij} (\tau^j - 1)(\rho_i, e_{s_i}) \in \text{Im}(1 - \tau).$$

On the other hand, if $(1 - \tau)x = 0$ then

$$\sum_i \sum_{j=0}^{m_i-1} \lambda_{ij} \tau^j(\rho_i, e_{s_i}) = \sum_i \sum_{j=1}^{m_i} \lambda_{i(j-1)} \tau^j(\rho_i, e_{s_i})$$

so $\lambda_{i0} = \lambda_{ij}$ for any j , and hence

$$x = \sum_i \sum_{j=0}^{m_i-1} \lambda_{i0} \tau^j(\rho_i, e_{s_i}) = \sum_i \lambda_{i0} N(\rho_i, e_{s_i}) \in \text{Im } N.$$

□

Lemma 3 *If $n \geq 1$ then $F_{n+1}^0 = G_{n+1}^1 \oplus G_{n+1}^2 \oplus G_{n+1}^3$ with*

$$G_{n+1}^1 : k\mathcal{E}_n \rightarrow 0$$

$$G_{n+1}^2 : k\mathcal{G}_n \rightarrow k(1, 1)_{n+1}^{\mathcal{G}}$$

$$G_{n+1}^3 : k(N\mathcal{E}_n \sqcup N\mathcal{G}_n) \rightarrow k(1, 1)_{n+1}^{N\mathcal{G}} \oplus k(((1, 0)_{n+1} \sqcup (0, 1)_{n+1}) \cap (AP_{n+1} // Q_1)).$$

Moreover, G_{n+1}^3 is injective, $\text{Ker } G_{2m+1}^2 = k\mathcal{G}_{2m} / \text{Im}(1 - \tau)$ and

$$\text{Ker } G_{2m}^2 = \begin{cases} 0 & \text{if char } k \neq 2 \\ k\mathcal{G}_{2m-1} / \text{Im}(1 - \tau) & \text{if char } k = 2. \end{cases}$$

Proof Recall that

$$\begin{aligned} F_{n+1}^0(\alpha_1 \cdots \alpha_n, e_{s(\alpha_1)}) \\ = \sum_{\substack{\{\beta \in Q_1 : \beta \alpha_1 \in I, \\ t(\beta) = s(\alpha_1)\}}} (\beta \alpha_1 \cdots \alpha_n, \beta) + (-1)^{n+1} \sum_{\substack{\{\beta \in Q_1 : \alpha_n \beta \in I, \\ s(\beta) = s(\alpha_1)\}}} (\alpha_1 \cdots \alpha_n \beta, \beta) \end{aligned}$$

so it is clear that $F_{n+1}^0(\mathcal{E}_n) = 0$, and if $(\alpha_1 \cdots \alpha_n, e_{s(\alpha_1)}) \in \mathcal{G}_n$ then

$$G_{n+1}^2(\alpha_1 \cdots \alpha_n, e_{s(\alpha_1)}) = (\alpha_n \alpha_1 \cdots \alpha_n, \alpha_n) + (-1)^{n+1} (\alpha_1 \cdots \alpha_n \alpha_1, \alpha_1) \in k(1, 1)_{n+1}^{\mathcal{G}}.$$

Now we shall prove that $\text{Ker } G_{2m+1}^2 = k\mathcal{G}_{2m} / \text{Im}(1 - \tau)$. From the commutativity of the diagram

$$\begin{array}{ccc} k\mathcal{G}_{2m} & \xrightarrow{G_{2m+1}^2} & k(1, 1)_{2m+1}^{\mathcal{G}} \\ \parallel & & \uparrow -\psi \\ k\mathcal{G}_{2m} & \xrightarrow{(1-\tau)} & k\mathcal{G}_{2m} \end{array}$$

where $\psi(\alpha_1 \cdots \alpha_{2m}, e_s(\alpha_1)) = (\alpha_1 \cdots \alpha_{2m} \alpha_1, \alpha_1)$ is an isomorphism, we get that

$$\text{Ker } G_{2m+1}^2 = \text{Ker}(1 - \tau).$$

The equalities

$$\text{Ker}(1 - \tau) = \text{Im } N = k\mathcal{G}_{2m} / \text{Ker } N = k\mathcal{G}_{2m} / \text{Im}(1 - \tau)$$

follow from the exactness of the sequence

$$k\mathcal{G}_{2m} \xrightarrow{N} k\mathcal{G}_{2m} \xrightarrow{(1-\tau)} k\mathcal{G}_{2m} \xrightarrow{N} k\mathcal{G}_{2m},$$

proved in Lemma 2. If $\text{char } k = 2$ the same proof works for $\text{Ker } G_{2m}^2$. If $\text{char } k \neq 2$, we define $T : k(1, 1)_{2m}^{\mathcal{G}} \rightarrow k\mathcal{G}_{2m-1}$ as follows:

$$T(\alpha_1 \cdots \alpha_{2m-1} \alpha_1, \alpha_1) = \frac{1}{2} \sum_{i=0}^{2m-2} (-1)^i \tau^i(\alpha_1 \cdots \alpha_{2m-1}, e_s(\alpha_1))$$

and we get that $T \circ G_{2m}^2 = \text{id}$ and $G_{2m}^2 \circ T = \text{id}$, hence G_{2m}^2 is bijective.

If $(\alpha_1 \cdots \alpha_n, e_s(\alpha_1)) \in N\mathcal{E}_n$ then

$$G_{n+1}^3(\alpha_1 \cdots \alpha_n, e_s(\alpha_1)) = \begin{cases} (\beta \alpha_1 \cdots \alpha_n, \beta) & \in k(1, 0)_{n+1} \\ (-1)^{n+1}(\alpha_1 \cdots \alpha_n \gamma, \gamma) & \in k(0, 1)_{n+1} \\ (\beta \alpha_1 \cdots \alpha_n, \beta) + (-1)^{n+1}(\alpha_1 \cdots \alpha_n \gamma, \gamma) & \in k((1, 0)_{n+1} \sqcup (0, 1)_{n+1}) \end{cases}$$

depending on the existence of β and γ satisfying $\beta \neq \alpha_n$ and $\beta \alpha_1 \in I$, $\gamma \neq \alpha_1$ and $\alpha_n \gamma \in I$. Finally, if $(\alpha_1 \cdots \alpha_n, e_s(\alpha_1)) \in N\mathcal{G}_n$ then

$$G_{n+1}^3(\alpha_1 \cdots \alpha_n, e_s(\alpha_1)) = \begin{cases} (\alpha_n \alpha_1 \cdots \alpha_n, \alpha_n) + (-1)^{n+1}(\alpha_1 \cdots \alpha_n \alpha_1, \alpha_1) \\ (\alpha_n \alpha_1 \cdots \alpha_n, \alpha_n) + (-1)^{n+1}(\alpha_1 \cdots \alpha_n \alpha_1, \alpha_1) + (\beta \alpha_1 \cdots \alpha_n, \beta) \\ (\alpha_n \alpha_1 \cdots \alpha_n, \alpha_n) + (-1)^{n+1}(\alpha_1 \cdots \alpha_n \alpha_1, \alpha_1) + (-1)^{n+1}(\alpha_1 \cdots \alpha_n \gamma, \gamma) \\ (\alpha_n \alpha_1 \cdots \alpha_n, \alpha_n) + (-1)^{n+1}(\alpha_1 \cdots \alpha_n \alpha_1, \alpha_1) + (\beta \alpha_1 \cdots \alpha_n, \beta) + (-1)^{n+1}(\alpha_1 \cdots \alpha_n \gamma, \gamma) \end{cases}$$

depending on the existence of β and γ in Q_1 satisfying $\beta \neq \alpha_n$ and $\beta \alpha_1 \in I$, $\gamma \neq \alpha_1$ and $\alpha_n \gamma \in I$. Hence

$$G_{n+1}^3(kN\mathcal{E}_n) \subset k(((1, 0)_{n+1} \sqcup (0, 1)_{n+1}) \cap (AP_{n+1}/Q_1)),$$

$$G_{n+1}^3(kN\mathcal{G}_n) \subset k(((1, 1)_{n+1}^{\mathcal{G}} \sqcup (1, 0)_{n+1} \sqcup (0, 1)_{n+1}) \cap (AP_{n+1}/Q_1)).$$

Now we define the linear map

$$T : k(((1, 1)_{n+1}^{\mathcal{G}} \sqcup (1, 0)_{n+1} \sqcup (0, 1)_{n+1}) \cap (AP_{n+1}/Q_1)) \rightarrow k(N\mathcal{E}_n \sqcup N\mathcal{G}_n)$$

as follows:

$$T(\alpha_1 \cdots \alpha_n \alpha_1, \alpha_1) = (-1)^{n+1} \sum_{i=0}^{\mu(w)-1} (-1)^{in} \tau^i(w)$$

$$T(\beta \alpha_1 \cdots \alpha_n, \beta) = \begin{cases} w & \text{if } \alpha_n \alpha_1 \notin I \\ 0 & \text{if } \alpha_n \alpha_1 \in I \text{ and } \exists \gamma \neq \alpha_1 \\ & \text{such that } \alpha_n \gamma \in I \\ (-1)^n \sum_{i=0}^{\mu(\tau(w))-1} (-1)^{in} \tau^{i+1}(w) & \text{otherwise} \end{cases}$$

$$T(\alpha_1 \cdots \alpha_n \gamma, \gamma) = \begin{cases} - \sum_{i=0}^{\mu(\tau(w))-1} (-1)^{in} \tau^{i+1}(w) & \text{if } \alpha_n \alpha_1 \in I \\ 0 & \text{if } \alpha_n \alpha_1 \notin I \text{ and } \exists \beta \neq \alpha_n \\ & \text{such that } \beta \alpha_1 \in I \\ (-1)^{n+1} w & \text{otherwise} \end{cases}$$

where $w = (\alpha_1 \cdots \alpha_n, e_{s(\alpha_1)})$ and $\mu(w)$ is the first natural number such that $\tau^{\mu(w)-1}(w) \notin \mathcal{C}_n(0)$. A direct computation shows that $T \circ G_{n+1}^3 = \text{id}$, and hence G_{n+1}^3 is injective. \square

Proposition 3 *If $n \geq 1$ then*

$$\dim_k \text{Ker } F_{n+1}^0 = \begin{cases} |\mathcal{E}_n| + \dim_k k\mathcal{G}_n / \text{Im}(1 - \tau) & \text{if } n \text{ is even and } \text{char } k \neq 2, \\ |\mathcal{E}_n| & \text{if } n \text{ is odd and } \text{char } k \neq 2, \\ |\mathcal{E}_n| + \dim_k k\mathcal{G}_n / \text{Im}(1 - \tau) & \text{if } \text{char } k = 2, \end{cases}$$

$$\dim_k \text{Im } F_{n+1}^0 = \begin{cases} |\mathcal{C}_n| + |N\mathcal{E}_n| - \dim_k k\mathcal{G}_n / \text{Im}(1 - \tau) & \text{if } n \text{ is even and } \text{char } k \neq 2, \\ |\mathcal{C}_n| + |N\mathcal{E}_n| & \text{if } n \text{ is odd and } \text{char } k \neq 2, \\ |\mathcal{C}_n| + |N\mathcal{E}_n| - \dim_k k\mathcal{G}_n / \text{Im}(1 - \tau) & \text{if } \text{char } k = 2. \end{cases}$$

Proof The formula for the dimension of $\text{Ker } F_{n+1}^0$ follows by a direct computation using Lemma 3. The equalities

$$\dim_k \text{Im } F_{n+1}^0 = |(AP_n // Q_0)| - \dim_k \text{Ker } F_{n+1}^0$$

and

$$|(AP_n // Q_0)| = |\mathcal{G}_n| + |N\mathcal{G}_n| + |\mathcal{E}_n| + |N\mathcal{E}_n| = |\mathcal{C}_n| + |\mathcal{E}_n| + |N\mathcal{E}_n|$$

imply the formula for $\dim_k \text{Im } F_{n+1}^0$. \square

3.3 Dimensions of the Hochschild cohomology groups

We start this section with the computation of the first Hochschild cohomology groups.

Theorem 1 *Let $A = kQ/I$ be a quadratic string algebra. Then*

$$\dim_k \text{HH}^0(A) = |{}^-(Q_0 // \mathcal{P}_1)^-| + 1,$$

$$\dim_k \text{HH}^1(A) = \begin{cases} |{}^-(0, 0)_1^-| + |Q_1| - |Q_0| + 1 & \text{if } \text{char } k \neq 2, \\ |{}^-(0, 0)_1^-| + |Q_1| - |Q_0| + 1 + |\mathcal{G}_1| & \text{if } \text{char } k = 2. \end{cases}$$

Proof Recall that the maps F_1^0 and F_1^1 appearing in the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{E-E}(kQ_0, A) & \xrightarrow{F_1} & \mathrm{Hom}_{E-E}(kQ_1, A) \\ \downarrow \cong & & \downarrow \cong \\ k(Q_0//Q_0) \oplus k(Q_0//\mathcal{P}_1) & \xrightarrow{\begin{pmatrix} 0 & 0 \\ F_1^0 & 0 \\ 0 & F_1^1 \end{pmatrix}} & k(Q_1//Q_0) \oplus k(Q_1//Q_1) \oplus k(Q_1//\mathcal{P}_2) \end{array}$$

are given by

$$\begin{aligned} F_1^0(e_r, e_r) &= \sum_{\{\beta \in Q_1: t(\beta)=r\}} (\beta, \beta) - \sum_{\{\beta \in Q_1: s(\beta)=r\}} (\beta, \beta), \\ F_1^1(e_r, \gamma) &= \sum_{\{\beta \in Q_1: t(\beta)=r\}} (\beta, \beta\gamma) - \sum_{\{\beta \in Q_1: s(\beta)=r\}} (\beta, \gamma\beta). \end{aligned}$$

Then

$$F_1^0\left(\sum_{r \in Q_0} \lambda_r(e_r, e_r)\right) = \sum_{\beta \in Q_1} (\lambda_{t(\beta)} - \lambda_{s(\beta)})(\beta, \beta) = 0$$

implies that $\lambda_i = \lambda_j$ whenever there exists an arrow $\beta : i \rightarrow j$. Since Q is connected, we have that $\lambda_i = \lambda_j$ for any i, j . Hence $\dim_k \mathrm{Ker} F_1^0 = 1$.

On the other hand, $(Q_0//\mathcal{P}_1)^- = (Q_0//\mathcal{P}_1)^- \sqcup^- (Q_0//\mathcal{P}_1)^+ \sqcup^+ (Q_0//\mathcal{P}_1)^- \sqcup^+ (Q_0//\mathcal{P}_1)^+$ and we have that:

- (i) $F_1^1((Q_0//\mathcal{P}_1)^-) = 0$;
- (ii) F_1^1 induces a bijection from $^-(Q_0//\mathcal{P}_1)^+$ to $^-(0, 1)_1$;
- (iii) F_1^1 induces a bijection from $^+(Q_0//\mathcal{P}_1)^-$ to $(1, 0)_1^-$;
- (iv) there exists a bijection $\phi_1 : (1, 0)_1^+ \rightarrow ^+(0, 1)_1$ given by $\phi_1(\alpha, \alpha\gamma) = (\beta, \gamma\beta)$ such that $F_1^1(^+(Q_0//\mathcal{P}_1)^+) = (id - \phi_1)((1, 0)_1^+)$.

Hence $^-(Q_0//\mathcal{P}_1)^-$ is a basis for $\mathrm{Ker} F_1^1$ and then

$$\dim_k \mathrm{HH}^0(A) = \dim_k \mathrm{Ker} F_1 = \dim_k \mathrm{Ker} F_1^0 + \dim_k \mathrm{Ker} F_1^1 = 1 + |^-(Q_0//\mathcal{P}_1)^-|$$

From these computations we get that

$$\begin{aligned} \dim_k \mathrm{Im} F_1^0 &= |(Q_0//Q_0)| - \dim_k \mathrm{Ker} F_1^0 = |Q_0| - 1 \\ \dim_k \mathrm{Im} F_1^1 &= |(Q_0//\mathcal{P}_1)| - \dim_k \mathrm{Ker} F_1^1 \\ &= |^-(0, 1)_1| + |(1, 0)_1^-| + |(1, 0)_1^+|. \end{aligned}$$

From Proposition 1 and Proposition 3 we have that

$$\begin{aligned} \dim_k \mathrm{Ker} F_2 &= \dim_k \mathrm{Ker} F_2^0 + \dim_k \mathrm{Ker} F_2^1 \\ &= |\mathcal{E}_1| + |^-(0, 0)_1^-| + |^-(0, 1)_1| + |(1, 1)_1| + |(1, 0)_1| \end{aligned}$$

if $\mathrm{char} k \neq 2$ and

$$\dim_k \mathrm{Ker} F_2 = |\mathcal{E}_1| + |^-(0, 0)_1^-| + |^-(0, 1)_1| + |(1, 1)_1| + |(1, 0)_1| + \dim_k k\mathcal{G}_1 / \mathrm{Im}(1 - \tau)$$

if $\text{char } k = 2$. Now $|(1, 1)_1| = |Q_1|$, $\mathcal{E}_1 = \emptyset$ since A is finite dimensional and

$$k\mathcal{G}_1/\text{Im}(1 - \tau) = k\mathcal{G}_1,$$

so

$$\dim_k \text{HH}^1(A) = \begin{cases} |^-(0, 0)_1^-| + |Q_1| - |Q_0| + 1 & \text{if } \text{char } k \neq 2, \\ |^-(0, 0)_1^-| + |Q_1| - |Q_0| + 1 + |\mathcal{G}_1| & \text{if } \text{char } k = 2. \end{cases}$$

□

The dimensions that have been computed in Propositions 1 and 3 lead us to the following theorem.

Theorem 2 *If $A = kQ/I$ is a quadratic string algebra and $n \geq 2$ then*

$$\begin{aligned} \dim_k \text{HH}^n(A) &= |^-(0, 0)_n^-| + |\mathcal{E}_n| - |N\mathcal{E}_{n-1}| + |((1, 0)_n \sqcup ^-(0, 1)_n) \cap (AP_n//Q_1)| \\ &+ \begin{cases} \dim_k k\mathcal{G}_n/\text{Im}(1 - \tau) & \text{if } n \text{ is even and } \text{char } k \neq 2 \\ \dim_k k\mathcal{G}_{n-1}/\text{Im}(1 - \tau) & \text{if } n \text{ is odd and } \text{char } k \neq 2, \\ \dim_k k\mathcal{G}_n/\text{Im}(1 - \tau) + \dim_k k\mathcal{G}_{n-1}/\text{Im}(1 - \tau) & \text{if } \text{char } k = 2. \end{cases} \end{aligned}$$

Proof From Propositions 3 and 1 we get that if n is even and $\text{char } k \neq 2$ then

$$\begin{aligned} \dim_k \text{HH}^n(A) &= \dim_k \text{Ker } F_{n+1} - \dim_k \text{Im } F_n \\ &= |^-(0, 0)_n^-| + |^-(0, 1)_n^-| + |(1, 1)_n| + |(1, 0)_n| \\ &- |(^-(0, 1)_n \sqcup (1, 0)_n \sqcup (1, 1)_n) \cap (AP_n//\mathcal{P}_2)| \\ &+ |\mathcal{E}_n| + \dim_k k\mathcal{G}_n/\text{Im}(1 - \tau) - |\mathcal{C}_{n-1}| - |N\mathcal{E}_{n-1}| \end{aligned}$$

and the desired formula follows since the identification

$$(\alpha_1 \cdots \alpha_{n-1} \alpha_1, \alpha_1) \leftrightarrow (\alpha_1 \cdots \alpha_{n-1}, e_{s(\alpha_1)})$$

implies that $|(1, 1)_n \cap (AP_n//Q_1)| = |\mathcal{C}_{n-1}|$. Similarly one can deduce the other formulae. □

Definition 3 A string algebra $A = kQ/I$ is called a **gentle algebra** if in addition (Q, I) satisfies :

- G1) For an arrow α in Q there exists at most one arrow β and at most one arrow γ such that $\alpha\beta \in I$ and $\gamma\alpha \in I$.
- G2) I is quadratic.

The Hochschild cohomology groups of gentle algebras have already been computed in [20], and these results have been expressed in terms of the derived invariant introduced by Avella-Alaminos and Geiss in [4]. As a consequence of our previous theorem, we recover the results in [20].

Corollary 1 *If $A = kQ/I$ is a gentle algebra, then*

$$\dim_k \mathrm{HH}^0(A) = |\bar{-(Q_0//\mathcal{P}_1)}^-| + 1$$

$$\dim_k \mathrm{HH}^1(A) = \begin{cases} |\bar{-(0,0)}_1^-| + |Q_1| - |Q_0| + 1 & \text{if } \mathrm{char} k \neq 2, \\ |\bar{-(0,0)}_1^-| + |Q_1| - |Q_0| + 1 + |(Q_1//Q_0)| & \text{if } \mathrm{char} k = 2, \end{cases}$$

$$\dim_k \mathrm{HH}^n(A) = |\bar{-(0,0)}_n^-| + |\mathcal{E}_n| + a \dim_k k\mathcal{G}_n / \mathrm{Im}(1 - \tau) + b \dim_k \mathcal{G}_{n-1} / \mathrm{Im}(1 - \tau)$$

where

$$(a, b) = \begin{cases} (1, 0) & \text{if } n \geq 2, n \text{ even, } \mathrm{char} k \neq 2 \\ (0, 1) & \text{if } n \geq 2, n \text{ odd, } \mathrm{char} k \neq 2, \\ (1, 1) & \text{if } n \geq 2, \mathrm{char} k = 2. \end{cases}$$

Proof From Theorems 1 and 2 it is clear that we only have to prove that

$$|N\mathcal{E}_{n-1}| = |((1,0)_n \sqcup \bar{-(0,1)}_n) \cap (AP_n//Q_1)|.$$

Since A is gentle, $N\mathcal{G}_{n-1} = \emptyset$, and in this case the injective map

$$G_n^3 : k(N\mathcal{E}_{n-1}) \rightarrow k(((1,0)_n \sqcup \bar{-(0,1)}_n) \cap (AP_n//Q_1))$$

studied in Lemma 3 satisfies

$$G_n^3(\alpha_1 \cdots \alpha_{n-1}, e_r) = \begin{cases} (\beta\alpha_1 \cdots \alpha_{n-1}, \beta) & \in k(1,0)_n^- \\ (-1)^n(\alpha_1 \cdots \alpha_{n-1}\gamma, \gamma) & \in k(\bar{-(0,1)}_n) \\ (\beta\alpha_1 \cdots \alpha_{n-1}, \beta) + (-1)^n(\alpha_1 \cdots \alpha_{n-1}\gamma, \gamma) & \in k((id + (-1)^n\phi_n)(1,0)_n^+) \end{cases}$$

depending on the existence of β and γ . So

$$|N\mathcal{E}_{n-1}| = \dim_k \mathrm{Im} G_n^3 = |((1,0)_n \sqcup \bar{-(0,1)}_n) \cap (AP_n//Q_1)|.$$

□

Example 5 The algebra presented in Example 1 is a quadratic string algebra, then by Theorems 1 and 2 we have that

$$\dim_k \mathrm{HH}^n(A) = \begin{cases} 2 & \text{if } n = 0, \\ 3 & \text{if } n = 1, \\ 1 & \text{if } n = 2i + 1, i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The algebra in Example 2 is a gentle algebra, then we can compute its cohomology using Corollary 1,

$$\dim_k \mathrm{HH}^n(A) = \begin{cases} 1 & \text{if } n = 0, \\ 3 & \text{if } n = 1, \\ 2 & \text{if } n = 7, \\ 1 & \text{otherwise.} \end{cases}$$

The following results will be used in the description of the Lie bracket defined in $\mathrm{HH}^*(A)$.

Proposition 4 *Let A be a gentle algebra, $n > 1$ and let*

$$f = \sum_{i=1}^m \lambda_i(\rho_i, \gamma_i) \in \mathrm{Ker} F_{n+1}^1$$

such that $(\rho_i, \gamma_i) \notin {}^-(0, 0)_n^-$, $(\rho_i, \gamma_i) \notin (1, 1)_n \cap (AP_n//Q_1)$ for all i with $1 \leq i \leq m$. Then $\bar{f} = 0$ in $\mathrm{HH}^n(A)$.

Proof From Lemma 1 we know that f can be written as $f = f_1 + f_2$ where

$$f_1 \in \mathrm{Ker} F_{n+1}^1 \cap k(AP_n//Q_1) \quad \text{and} \quad f_2 \in \mathrm{Ker} F_{n+1}^1 \cap k(AP_n//\mathcal{P}_2).$$

By Proposition 2 we have that $\bar{f}_2 = 0$ in $\mathrm{HH}^n(A)$. On the other hand, f_1 is a linear combination of elements in

$$\left((1, 0)_n^- \sqcup {}^-(0, 1)_n \sqcup (id + (-1)^n \phi_n)((1, 0)_n^+) \right) \cap k(AP_n//Q_1).$$

The proof of Corollary 1 shows that

$$\left((1, 0)_n^- \sqcup {}^-(0, 1)_n \sqcup (id + (-1)^n \phi_n)(1, 0)_n^+ \right) \cap k(AP_n//Q_1) = \mathrm{Im} G_n^3 \subset \mathrm{Im} F_n^0.$$

Hence $\bar{f}_1 = 0$ in $\mathrm{HH}^n(A)$. \square

Corollary 2 *Let A be a gentle algebra, $n > 1$ and let*

$$f = \sum_{i=1}^m \lambda_i(\rho_i, \gamma_i) \in \mathrm{Ker} F_{n+1}^1$$

such that $(\rho_i, \gamma_i) \notin {}^-(0, 0)_n^-$ for all i with $1 \leq i \leq m$. If $\mathcal{G}_{n-1} = \emptyset$ then $\bar{f} = 0$ in $\mathrm{HH}^n(A)$.

Proof By assumption $\mathcal{G}_{n-1} = \emptyset$ and, since A is gentle, $N\mathcal{G}_{n-1} = \emptyset$. Then

$$(1, 1)_n \cap (AP_n//Q_1) = (1, 1)_n^{\mathcal{G}} \sqcup (1, 1)_n^{N\mathcal{G}} = \emptyset$$

and hence the desired result follows from the previous proposition. \square

4 Gerstenhaber algebra

4.1 Comparison morphisms

The cup product and the Lie bracket in the cohomology $\mathrm{HH}^*(A)$ are induced by operations defined using the bar resolution, and we have made the computations of $\mathrm{HH}^n(A)$ using Bardzell's resolution. In this section we will construct comparison morphisms between the bar resolution and Bardzell's resolution in order to get formulae for the cup product and the Lie bracket.

We start by recalling the definition of these structures at the level of cochains using the bar resolution $(A^{\otimes n}, b_n)_{n \geq 0}$. Given $f \in \text{Hom}_{E-E}(A^{\otimes n}, A)$ and $g \in \text{Hom}_{E-E}(A^{\otimes m}, A)$ we have

$$f \cup g \in \text{Hom}_{E-E}(A^{\otimes m+n}, A) \quad \text{and} \quad [f, g] \in \text{Hom}_{E-E}(A^{\otimes m+n-1}, A)$$

defined by

$$f \cup g(v_1 \otimes \cdots \otimes v_{n+m}) = f(v_1 \otimes \cdots \otimes v_n)g(v_{n+1} \otimes \cdots \otimes v_{n+m})$$

and

$$[f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f$$

where

$$f \circ g = \sum_{i=1}^n (-1)^{(i-1)(m-1)} f \circ_i g \quad (1)$$

and

$$f \circ_i g(v_1 \otimes \cdots \otimes v_{n+m-1}) = f(v_1 \otimes \cdots \otimes v_{i-1} \otimes g(v_i \otimes \cdots \otimes v_{i+m-1}) \otimes v_{i+m} \otimes \cdots \otimes v_{n+m-1}).$$

These structures are easily carried to Bardzell's resolution using the comparison morphisms

$$A \otimes kAP_n \otimes A \begin{array}{c} \xrightarrow{U_n} \\ \xleftarrow{V_n} \end{array} A \otimes A^{\otimes n} \otimes A$$

given by the A -bimodule morphisms

$$U_0(1 \otimes e \otimes 1) = e \otimes 1$$

$$U_n(1 \otimes \alpha_1 \alpha_2 \cdots \alpha_n \otimes 1) = 1 \otimes \alpha_1 \otimes \cdots \otimes \alpha_n \otimes 1, \quad n \geq 1$$

$$V_0(1 \otimes 1) = 1 \otimes 1 \otimes 1 = \sum_{i \in Q_0} 1 \otimes e_i \otimes 1$$

$$V_1(1 \otimes \gamma \otimes 1) = \begin{cases} 0 & \text{if } |\gamma| = 0 \\ \sum_{i=1}^s \alpha_1 \cdots \alpha_{i-1} \otimes \alpha_i \otimes \alpha_{i+1} \cdots \alpha_s & \text{if } \gamma = \alpha_1 \cdots \alpha_s \end{cases}$$

$$V_n(1 \otimes v_1 \otimes \cdots \otimes v_n \otimes 1)$$

$$= \begin{cases} v'_1 \otimes \alpha_1 v_2 \cdots v_{n-1} \alpha_n \otimes v'_n & \text{if } v_1 = v'_1 \alpha_1, v_n = \alpha_n v'_n \\ & \text{and } \alpha_1 v_2 \cdots v_{n-1} \alpha_n \in AP_n, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4 *The maps $U = (U_n)_{n \geq 0}$ and $V = (V_n)_{n \geq 0}$ are morphism of complexes and $V \circ U = \text{id}$.*

Using the isomorphisms

$$\text{Hom}_{A-A}(A \otimes A^{\otimes n} \otimes A, A) \simeq \text{Hom}_{E-E}(A^{\otimes n}, A),$$

$$\text{Hom}_{A-A}(A \otimes kAP_n \otimes A, A) \simeq \text{Hom}_{E-E}(kAP_n, A)$$

we get that the morphisms U and V induce quasi-isomorphisms

$$U^\bullet = \left(U^n : \text{Hom}_{E-E}(A^{\otimes n}, A) \longrightarrow \text{Hom}_{E-E}(kAP_n, A) \right)_{n \geq 0} \quad \text{and}$$

$$V^\bullet = \left(V^n : \text{Hom}_{E-E}(kAP_n, A) \longrightarrow \text{Hom}_{E-E}(A^{\otimes n}, A) \right)_{n \geq 0}.$$

With these quasi-isomorphisms, the cup product and the Lie bracket, which we still denote \cup and $[\ , \]$, can be defined as follows: given $f \in \text{Hom}_{E-E}(kAP_n, A)$ and $g \in \text{Hom}_{E-E}(kAP_m, A)$,

$$f \cup g \in \text{Hom}_{E-E}(kAP_{n+m}, A) \quad \text{and} \quad f \circ_i g \in \text{Hom}_{E-E}(kAP_{n+m-1}, A), 1 \leq i \leq n$$

are defined by

$$f \cup g = U^{n+m}(V^n(f) \cup V^m(g)) \quad \text{and} \quad f \circ_i g = U^{m+n-1}(V^n(f) \circ_i V^m(g)).$$

As usual,

$$F_{n+m+1}(f \cup g) = F_{n+1}(f) \cup g + (-1)^n f \cup F_{m+1}(g)$$

$$F_{n+m}([f, g]) = [f, F_{m+1}(g)] + (-1)^{m-1} [F_{n+1}(f), g]$$

so the products \cup and $[-, -]$ defined in the complex $(\text{Hom}_{E-E}(kAP_n, A), F_n)_{n \geq 0}$ induce products at the cohomology level.

4.2 Formulae for the cup product and the Lie bracket

The following detailed computations will be useful in order to describe the desired structures:

$$f \cup g(\alpha_1 \cdots \alpha_{n+m}) = f(\alpha_1 \cdots \alpha_n)g(\alpha_{n+1} \cdots \alpha_{n+m})$$

and, if $g \in \text{Hom}_{E-E}(kAP_m, A)$ is a basis element, that is, g sends a fix basis element in AP_m to a basis element in \mathcal{P} , and it is zero otherwise, then

$$f \circ_1 g(\alpha_1 \cdots \alpha_{n+m-1}) = \mu f(\beta \alpha_{m+1} \cdots \alpha_{n+m-1})$$

if $g(\alpha_1 \cdots \alpha_m) = \mu \beta \in \mathcal{P}_1$ and $\beta \alpha_{m+1} \in I$,

$$f \circ_n g(\alpha_1 \cdots \alpha_{n+m-1}) = f(\alpha_1 \cdots \alpha_{n-1} \beta) \mu$$

if $g(\alpha_n \cdots \alpha_{n+m-1}) = \beta \mu \in \mathcal{P}_1$ and $\alpha_{n-1} \beta \in I$,

$$f \circ_i g(\alpha_1 \cdots \alpha_{n+m-1}) = f(\alpha_1 \cdots \alpha_{i-1} \beta \alpha_{i+m} \cdots \alpha_{n+m-1})$$

if $g(\alpha_i \cdots \alpha_{i+m-1}) = \beta \in Q_1$ and $\beta \alpha_{i+m}, \alpha_{i-1} \beta \in I$, and it is zero otherwise.

Now, using the identification $\text{Hom}_{E-E}(kAP_{m+n}, A) \simeq k(AP_{m+n}/\mathcal{P})$ given by

$$f_{(\rho, \gamma)} \longleftrightarrow (\rho, \gamma),$$

we get that, given

$$(\rho, \gamma) = (\alpha_1 \cdots \alpha_n, \gamma) \in (AP_n/\mathcal{P}) \quad \text{and} \quad (\rho', \gamma') = (\beta_1 \cdots \beta_m, \gamma') \in (AP_m/\mathcal{P})$$

we have that

- if $\alpha_n \beta_1 \in I$ and $\gamma \gamma' \notin I$ then $(\rho, \gamma) \cup (\rho', \gamma') = (\rho \rho', \gamma \gamma')$;
- if $\gamma' \in Q_1$, $\gamma' = \alpha_i$, $\alpha_{i-1} \beta_1 \in I$ and $\beta_m \alpha_{i+1} \in I$ then

$$(\rho, \gamma) \circ_i (\rho', \gamma') = (\alpha_1 \dots \alpha_{i-1} \rho' \alpha_{i+1} \dots \alpha_n, \gamma);$$
- if $\gamma' \in \mathcal{P}_2$, $\gamma' = \mu \alpha_1$ and $\beta_m \alpha_2 \in I$ then $(\rho, \gamma) \circ_1 (\rho', \gamma') = (\rho' \alpha_2 \dots \alpha_n, \mu \gamma)$;
- if $\gamma' \in \mathcal{P}_2$, $\gamma' = \alpha_n \mu$ and $\alpha_{n-1} \beta_1 \in I$ then $(\rho, \gamma) \circ_n (\rho', \gamma') = (\alpha_1 \dots \alpha_{n-1} \rho', \gamma \mu)$

and all the other cases are zero.

4.3 Vanishing of the cup product and the Lie bracket

In this section we find conditions on the presentation (Q, I) of the quadratic string algebra that ensures the vanishing of the cup product and the Lie bracket. We start with a remark that will be used throughout this section.

Remark 1 Any $\bar{f} = \overline{\sum_i \lambda_i(\rho_i, \gamma_i)}$ in $\text{HH}^n(A)$ can be written as $\bar{f} = \bar{f}_1 + \bar{f}_2$ with

$$f_1 = \sum_{|\gamma_i|=0} \lambda_i(\rho_i, \gamma_i) \in \text{Ker } F_{n+1}^0, \quad f_2 = \sum_{|\gamma_i|>0} \lambda_i(\rho_i, \gamma_i) \in \text{Ker } F_{n+1}^1$$

Moreover from Lemmas 1 and 3 we have that $f_1 \in k(\mathcal{E}_n \sqcup \mathcal{G}_n)$ and

$$f_2 \in k(\bar{-(0,0)_n} \sqcup (1,0)_n \sqcup \bar{-(0,1)_n} \sqcup (1,1)_n \sqcup (id + (-1)^n \phi_n)((1,0)_n^+)).$$

Proposition 5 *Let $A = kQ/I$ be a quadratic string algebra, $n, m > 0$. If $\mathcal{G}_n = \emptyset = \mathcal{G}_m$ then $\text{HH}^n(A) \cup \text{HH}^m(A) = 0$.*

Proof Let $\bar{f} = \bar{f}_1 + \bar{f}_2 \in \text{HH}^n(A)$, $\bar{g} = \bar{g}_1 + \bar{g}_2 \in \text{HH}^m(A)$ with $f_1 \in \text{Ker } F_{n+1}^0, f_2 \in \text{Ker } F_{n+1}^1, g_1 \in \text{Ker } F_{m+1}^0, g_2 \in \text{Ker } F_{m+1}^1$. We will show that, for any $i, j = 1, 2$,

$$\bar{f}_i \cup \bar{g}_j = \overline{f_i \cup g_j} = 0.$$

The assumption $\mathcal{G}_n = \emptyset$ and Lemma 3 imply that $f_1 = \sum_i \lambda_i(\rho_i, \gamma_i) \in k(\mathcal{E}_n)$, so for any $(\rho', \gamma') \in (AP_m // \mathcal{P})$ we have that

$$(\rho_i, \gamma_i) \cup (\rho', \gamma') = 0$$

because $\rho_i \rho' \notin AP_{n+m}$. Hence $f_1 \cup g_j = 0$. Similarly, $f_i \cup g_1 = 0$. Finally consider $f_2 \cup g_2$ where

$$f_2 = \sum_i \mu_i(\rho_i, \gamma_i) \in \text{Ker } F_{n+1}^1, \quad g_2 = \sum_j \mu'_j(\rho'_j, \gamma'_j) \in \text{Ker } F_{m+1}^1.$$

Recall that

$$\bar{-(0,0)_n} \sqcup (1,0)_n \sqcup \bar{-(0,1)_n} \sqcup (1,1)_n \sqcup (id + (-1)^n \phi_n)((1,0)_n^+)$$

is a basis of $\text{Ker } F_{n+1}^1$ and observe that $f_2 \cup g_2 \in \text{Ker } F_{n+m+1}^1 \cap k(AP_{n+m} // \mathcal{P}_2)$. If $(\rho_i, \gamma_i) \in \bar{-(0,0)_n}$ or $(\rho'_j, \gamma'_j) \in \bar{-(0,0)_m}$ then $\gamma_i \gamma'_j \in I$ and hence $(\rho_i \rho'_j, \gamma_i \gamma'_j) = 0$. If $(\rho_i, \gamma_i) \in \bar{-(0,1)_n}$ and $(\rho'_j, \gamma'_j) \in (1,0)_m$ we have that

$$(\rho_i, \gamma_i) = (\hat{\rho}_i \alpha_1, \hat{\gamma}_i \alpha_1) \quad \text{and} \quad (\rho'_j, \gamma'_j) = (\alpha_2 \hat{\rho}'_j, \alpha_2 \hat{\gamma}'_j)$$

with $\alpha_1, \alpha_2 \in Q_1$. Then $(\rho_i \rho'_j, \gamma_i \gamma'_j) = (\hat{\rho}_i \alpha_1 \alpha_2 \hat{\rho}'_j, \hat{\gamma}_i \alpha_1 \alpha_2 \hat{\gamma}'_j) = 0$ because $\alpha_1 \alpha_2 \in I$ or $\hat{\rho}_i \alpha_1 \alpha_2 \hat{\rho}'_j \notin AP_{n+m}$. In all the remaining cases $(\rho_i \rho'_j, \gamma_i \gamma'_j) \notin \bar{-(0,0)_{n+m}}$, and the desired result follows from Proposition 2. \square

Corollary 3 *Let $A = kQ/I$ be a quadratic string algebra, $\text{char } k \neq 2$. Then $\text{HH}^n(A) \cup \text{HH}^m(A) = 0$ for any $n, m > 0$ odd natural numbers.*

Proof The assertion follows from the previous proof since for n odd we have that $\text{Ker } F_{n+1}^0 = k\mathcal{E}_n$ and hence the hypothesis $\mathcal{G}_n = \emptyset$ is superfluous in this case. \square

Now we will describe the Lie bracket for gentle algebras.

Lemma 5 *If $A = kQ/I$ is a gentle algebra, $n, m > 1$ and $g \in {}^-(0, 0)_m^-$. Then $f \circ g = 0$ for any $f \in (AP_n//\mathcal{P})$.*

Proof Since $f \circ g = \sum_{i=1}^n (-1)^{(i-1)(m-1)} f \circ_i g$, it suffices to compute $f \circ_i g$ for each i with $1 \leq i \leq n$. Let $g = (\rho, \gamma) = (\beta_1 \cdots \beta_m, \gamma)$, with $\gamma = \mu\gamma' = \gamma''\nu$, $\mu, \nu \in Q_1$, $\beta_1 \neq \mu$ and $\beta_m \neq \nu$ and let $f = (\alpha_1 \cdots \alpha_n, \delta) \in (AP_n//\mathcal{P})$. The non vanishing of $f \circ_i g$ would imply that

$$\alpha_1 \cdots \alpha_{i-1} \beta_1 \cdots \beta_m \alpha_{i+1} \cdots \alpha_n \in AP_{n+m-1} \text{ and } \alpha_i = \mu$$

or

$$\alpha_1 \cdots \alpha_{i-1} \beta_1 \cdots \beta_m \alpha_{i+1} \cdots \alpha_n \in AP_{n+m-1} \text{ and } \alpha_i = \nu.$$

In this case

$$\alpha_{i-1} \beta_1 \in I \text{ and } \alpha_{i-1} \mu \in I$$

or

$$\beta_m \alpha_{i+1} \in I \text{ and } \nu \alpha_{i+1} \in I.$$

But $\beta_1 \neq \mu$, $\beta_m \neq \nu$ and by hypothesis A is gentle. This is a contradiction. \square

Proposition 6 *If $A = kQ/I$ is a gentle algebra, $n, m > 1$, and $\mathcal{G}_{n-1} = \emptyset = \mathcal{G}_{m-1}$ then $[\text{HH}^n(A), \text{HH}^m(A)] = 0$.*

Proof Let $\bar{f} = \bar{f}_1 + \bar{f}_2 \in \text{HH}^n(A)$, $\bar{g} = \bar{g}_1 + \bar{g}_2 \in \text{HH}^m(A)$, with $f_1 \in \text{Ker } F_{n+1}^0$, $f_2 \in \text{Ker } F_{n+1}^1$, $g_1 \in \text{Ker } F_{m+1}^0$, $g_2 \in \text{Ker } F_{m+1}^1$. We will show that $[\bar{f}_i, \bar{g}_j] = 0$ for any i, j . Since $f_1 \in k(AP_n//Q_0)$ and $g_1 \in k(AP_m//Q_0)$, it is clear that $f_i \circ g_1 = 0 = g_i \circ f_1$ for any i . The statement is clear if $\bar{f}_2 = 0 = \bar{g}_2$. If $\bar{f}_2 \neq 0$, by Corollary 2 we may assume that f_2 belongs to $k({}^-(0, 0)_n^-)$. In this case Lemma 5 implies that $g_i \circ f_2 = 0$ for any i . The case $\bar{g}_2 \neq 0$ is analogous, and hence we are done. \square

Corollary 4 *Let $A = kQ/I$ be a gentle algebra, $\text{char } k \neq 2$. Then $[\text{HH}^n(A), \text{HH}^m(A)] = 0$ for any $n, m \geq 1$ even natural numbers.*

Proof From the previous proof and Proposition 4 we deduce that we only have to consider the case $f_2 \in k(1, 1)_n \cap k(AP_n//Q_1)$ and $g_2 \in k(1, 1)_m \cap k(AP_m//Q_1)$. Since A is gentle we have that $(1, 1)_{2s} \cap (AP_{2s}//Q_1) = (1, 1)_{2s}^{\mathcal{G}}$, and the proof of Proposition 3 shows that $k(1, 1)_{2s}^{\mathcal{G}} = \text{Im } G_{2s}^2 \subset \text{Im } F_{2s}$, so $\bar{f}_2 = 0$ and $\bar{g}_2 = 0$ in this case. \square

4.4 Non-vanishing of the cup product and the Lie bracket

In this section we find conditions on the presentation (Q, I) of a quadratic string algebra in order to get non-trivial structures.

Theorem 3 *Let $A = kQ/I$ be a quadratic string algebra and $\mathcal{G}_n \neq \emptyset$ for some $n > 0$. Then the cup product defined in $\mathrm{HH}^*(A)$ is non-trivial. More precisely,*

- (i) *if n is even and $\mathrm{char} k \neq 2$, $\mathrm{HH}^{s_1 n}(A) \cup \mathrm{HH}^{s_2 n}(A) \neq 0$;*
- (ii) *if n is odd and $\mathrm{char} k \neq 2$, $\mathrm{HH}^{2s_1 n}(A) \cup \mathrm{HH}^{2s_2 n}(A) \neq 0$;*
- (iii) *if $\mathrm{char} k = 2$, $\mathrm{HH}^{s_1 n}(A) \cup \mathrm{HH}^{s_2 n}(A) \neq 0$*

for any $s_1, s_2 \geq 1$.

Proof By hypothesis there exists $\omega = (\alpha_1 \cdots \alpha_n, e_{s(\alpha_1)}) \in \mathcal{G}_n$. If this element has order r and $\mathrm{char} k = 2$ or, $\mathrm{char} k \neq 2$ and n is even, we have seen in the proof of Lemma 3 that

$$N(\omega) = N(\alpha_1 \cdots \alpha_n, e_{s(\alpha_1)}) = \sum_{i=0}^{r-1} \tau^i(\alpha_1 \cdots \alpha_n, e_{s(\alpha_1)}) \in \mathrm{Ker} F_{n+1}^0$$

Moreover, for any $s \geq 1$, the element $\omega^s = ((\alpha_1 \cdots \alpha_n)^s, e_{s(\alpha_1)})$ belongs to \mathcal{G}_{sn} , has order r and

$$N(\omega^s) = N((\alpha_1 \cdots \alpha_n)^s, e_{s(\alpha_1)}) = \sum_{i=0}^{r-1} \tau^i((\alpha_1 \cdots \alpha_n)^s, e_{s(\alpha_1)}) \in \mathrm{Ker} F_{sn+1}^0$$

The element $N(\omega^s)$ is non zero in $\mathrm{HH}^{sn}(A)$ because $\mathrm{Im} F_{sn} \subseteq k(AP_{sn}/\mathcal{P}_1)$. For any $s_1, s_2 \geq 1$ and $0 \leq i, j < r$, we have

$$\tau^i(\omega^{s_1}) \cup \tau^j(\omega^{s_2}) = \delta_{ij} \tau^i(\omega^{s_1+s_2})$$

where δ_{ij} is the Kronecker's delta. Then

$$N(\omega^{s_1}) \cup N(\omega^{s_2}) = N(\omega^{s_1+s_2})$$

and hence

$$\mathrm{HH}^{s_1 n}(A) \cup \mathrm{HH}^{s_2 n}(A) \neq (0).$$

If n is odd and $\mathrm{char} k \neq 2$, we consider the elements ω^{2s} and we get that

$$\mathrm{HH}^{2s_1 n}(A) \cup \mathrm{HH}^{2s_2 n}(A) \neq (0)$$

for any $s_1, s_2 \geq 1$. □

Theorem 4 *Let $\mathrm{char} k = 0$ and let $A = kQ/I$ be a quadratic string algebra such that $\mathcal{G}_n \neq \emptyset$ for some $n > 0$. Then the Lie bracket defined in $\mathrm{HH}^*(A)$ is non-trivial. More precisely,*

- (i) *if n is even, $[\mathrm{HH}^{s_1 n+1}(A), \mathrm{HH}^{s_2 n+1}(A)] \neq 0$;*
- (ii) *if n is odd, $[\mathrm{HH}^{2s_1 n+1}(A), \mathrm{HH}^{2s_2 n+1}(A)] \neq 0$*

for any $s_1, s_2 \geq 1$, $s_1 \neq s_2$.

Proof By hypothesis there exists $\omega = (\alpha_1 \cdots \alpha_n, e_{s(\alpha_1)}) \in \mathcal{G}_n$ and suppose that it has order r . From Lemma 1 we have that

$$\psi(\omega) = (\alpha_1 \cdots \alpha_n \alpha_1, \alpha_1) \in \text{Ker } F_{n+2}^1$$

because $(\alpha_1 \cdots \alpha_n \alpha_1, \alpha_1) \in (1, 1)_{n+1}$. Moreover, for any $s \geq 1$, the element

$$\omega^s = ((\alpha_1 \cdots \alpha_n)^s, e_{s(\alpha_1)}) \in \mathcal{G}_{sn}$$

also has order r and

$$\psi(\omega^s) = ((\alpha_1 \cdots \alpha_n)^s \alpha_1, \alpha_1) \in \text{Ker } F_{sn+2}^1.$$

If n is even, the element $\psi(\omega^s)$ is non zero in $\text{HH}^{sn+1}(A)$ because we have seen in the proof of Lemma 3 that $\psi(\omega^s) \in \text{Im } F_{sn+1}$ if and only if $w^s \in \text{Im}(1 - \tau)$. But $\text{Im}(1 - \tau) = \text{Ker } N$ and $w^s \notin \text{Ker } N$. For any $s_1, s_2 \geq 1$, we have that

$$\psi(\omega^{s_1}) \circ_i \psi(\omega^{s_2}) = \begin{cases} \psi(\omega^{s_1+s_2}), & \text{if } i = lr + 1, \text{ for } l = 0, \dots, s_1 \frac{n}{r}; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \psi(\omega^{s_1}) \circ \psi(\omega^{s_2}) &= \sum_{i=1}^{s_1 n + 1} (-1)^{(i-1)(s_2 n)} \psi(\omega^{s_1}) \circ_i \psi(\omega^{s_2}) = \sum_{l=0}^{s_1 \frac{n}{r}} (-1)^{(lr)(s_2 n)} \psi(\omega^{s_1+s_2}) \\ &= (s_1 \frac{n}{r} + 1) \psi(\omega^{s_1+s_2}) \end{aligned}$$

and hence

$$\begin{aligned} [\psi(\omega^{s_1}), \psi(\omega^{s_2})] &= \psi(\omega^{s_1}) \circ \psi(\omega^{s_2}) - (-1)^{(s_1 n)(s_2 n)} \psi(\omega^{s_2}) \circ \psi(\omega^{s_1}) \\ &= (s_1 \frac{n}{r} + 1) \psi(\omega^{s_1+s_2}) - (s_2 \frac{n}{r} + 1) \psi(\omega^{s_1+s_2}) \\ &= \frac{n}{r} (s_1 - s_2) \psi(\omega^{s_1+s_2}). \end{aligned}$$

So,

$$[\text{HH}^{ns_1+1}(A), \text{HH}^{ns_2+1}(A)] \neq 0 \text{ if } s_1 \neq s_2.$$

If n is odd, we consider the element

$$\psi(\omega^{2s}) = ((\alpha_1 \cdots \alpha_n)^{2s} \alpha_1, \alpha_1) \in \text{Ker } F_{2sn+2}$$

and we get that

$$[\text{HH}^{2ns_1+1}(A), \text{HH}^{2ns_2+1}(A)] \neq 0, \text{ if } s_1 \neq s_2.$$

□

Remark 2 The previous theorem also holds in any characteristic if we add some hypothesis on the element $\frac{n}{r}$.

Example 6 For the algebra in Example 2 we have that $\mathcal{G}_n \neq \emptyset$ hence our Theorems 4 and 3 say that its Gerstenhaber algebra structure is non-trivial. More precisely, if we consider the following sets of generators:

$$- \{u_1 = (\beta_1, \beta_1), v_1 = (\alpha_1, \alpha_1), w_1 = (\alpha_7, \alpha_7)\} \text{ for } \text{HH}^1(A),$$

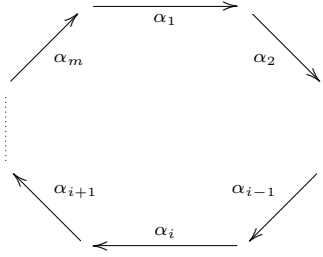
- $\{u_7 = ((\beta_1\beta_2)^3\beta_1, \beta_1), v_7 = (\alpha_1 \dots \alpha_7, e_1)\}$ for $\mathrm{HH}^7(A)$
- $\{u_{2n+1} = ((\beta_1\beta_2)^n\beta_1, \beta_1)\}$ for $\mathrm{HH}^{2n+1}(A)$ when $n > 0, n \neq 3$, and
- $\{u_{2n} = N(((\beta_1\beta_2)^n, e_6))\}$ for $\mathrm{HH}^{2n}(A)$ when $n \geq 1$,

then

$$\begin{aligned}
u_{2i} \cup u_{2j} &= u_{2(i+j)} \\
u_{2i} \cup u_{2j+1} &= u_{2j+1} \cup u_{2i} = u_{2(i+j)+1} \\
[u_{2i+1}, u_{2j+1}] &= (i-j) u_{2(i+j)+1} \\
[u_{2i}, u_{2j+1}] &= -[u_{2j+1}, u_{2i}] = i u_{2(i+j)} \\
[v_7, v_1] &= -[v_1, v_7] = v_7 \\
[v_7, w_1] &= -[w_1, v_7] = v_7
\end{aligned}$$

and all the other cases are zero.

Example 7 Let $\mathrm{char} k \neq 2$ and let $A = kQ/I$ be the radical square zero algebra whose quiver Q consists of an oriented cycle of length $m > 1$



A straightforward computation shows that

$$\mathrm{HH}^{qm}(A) = k, \quad \mathrm{HH}^{qm+1}(A) = k$$

when qm is even and it is zero in all the other cases. More precisely,

$$u_{qm} = N((\alpha_1, \dots, \alpha_m)^q, e_{s(\alpha_1)}) \quad \text{and} \quad u_{qm+1} = ((\alpha_1 \dots, \alpha_m)^q \alpha_1, \alpha_1)$$

are generators of $\mathrm{HH}^{qm}(A)$ and $\mathrm{HH}^{qm+1}(A)$ respectively. From Theorems 4 and 3 we know that its Gerstenhaber algebra structure is non-trivial and the bracket on the generators is given by

$$\begin{aligned}
[u_{q_1 m+1}, u_{q_2 m+1}] &= (q_1 - q_2) u_{(q_1+q_2)m+1} \\
[u_{q_1 m}, u_{q_2 m+1}] &= -[u_{q_2 m+1}, u_{q_1 m}] = q_1 u_{(q_1+q_2)m}
\end{aligned}$$

If we only consider odd degrees, that is,

$$\mathrm{HH}^{odd}(A) = \bigoplus_{\substack{q \geq 0, \\ qm \text{ even}}} \mathrm{HH}^{qm+1}(A),$$

then the Gerstenhaber bracket endows $\mathrm{HH}^{odd}(A)$ with a Lie algebra structure such that it is isomorphic to the infinite dimensional Witt algebra. In this way we recover the results in [29, Proposition 4.3.4].

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