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New compact and singularity free formulations for the magnetic field produced by a finite cylinder considering linearly varying current density

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Abstract. This paper presents new compact analytical expressions for the magnetic field calculation produced by a finite cylindrical sheet. A linearly varying surface current density between the ends of the cylindrical sheet has been assumed as current source. The expressions presented in this work are substantially more compact if compared with the ones currently available in the literature. Since the solutions are given in terms of complete elliptic integrals of the first, second and third kind, and the last two ones diverge at certain arguments, the field expressions also diverge at these critical points, even though the field solution is finite. These singular points are located at the ends of the current sheet cylinder. New analytical expressions are also presented for these critical points avoiding the singularities in an elegant way. The radial component of the magnetic field strength is separately discussed, and it has been proven that its solution at singular points diverges. To improve the computational performance on the evaluation of magnetic field at critical points, an alternative formulation is presented, where complete elliptic integrals of the second and third kind are replaced with alternative functions. Numerical algorithms for the computation of these functions are also presented.

Keywords: Current carrying cylinder, elliptic integrals, magnetic field, singularity

1. Introduction

In a recent article [1], the authors present analytical expressions for the magnetic vector potential and the radial and axial components of the magnetic field strength produced by a finite cylindrical sheet with a linearly varying current density between its ends. This contribution is part of the development of comprehensive semi-analytical integral methodology (SAIM) for fast numerical calculation of magnetic field in electrical machinery.

SAIM allows the rapid calculation of the magnetic field in electrical machinery if compared with traditional methods such as the finite element method (FEM). SAIM was initially conceived for the magnetic field calculation in power transformers [2], however, it can be extended for magnetic modelling of virtually any type of electrical equipment such as motors or generators [6,7].

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One of the main advantages of using techniques based on integral equations (such as SAIM) is that these methods can be considerably faster than techniques based on differential equations (such as FEM). Despite this significant advantage, one of the major challenges of using integral based techniques is the handling of singularities. These singularities make difficult or in some cases impossible the evaluation of the magnetic field at certain points in space [8,9].

Because it is required SAIM to be as fast and reliable as possible, it is very important that the speed on the calculation of the expressions related to the base elements is computationally efficient and free of singularities; both issues have been addressed in this article.

The expressions presented in [1] are the cornerstone of the SAIM methodology, since the linear variation of the surface current density on the elements enables the continuity of the global current density function, avoiding discrete jumps at the junction of two consecutive finite cylindrical current sheets, thus considerably improving the accuracy of the calculations.

While the expressions presented in [1] are accurate, they are liable to be improved through algebraic manipulations leading to more compact and efficient versions, which will be presented in this article. Since the expressions of magnetic field quantities are given in terms of complete elliptic integrals of the first, second and third kinds, there are arguments for which these functions diverge, even when the field solution does not, since these are removable singularities. Those arguments correspond to field points located on the cylindrical current sheet, and especially at its ends. As it can be seen in this article, the expressions can be modified using mathematical manipulations so that the singularities at critical points disappear. The behavior of the magnetic field magnitudes have been investigated at the critical points and their values have been calculated in the case of removable singularities, and also the cases for which there are real unavoidable singularities were verified. Compact mathematical expressions have been found for magnetic field magnitudes even at critical points. This has been achieved by replacing elliptic integrals of the first and third kind with alternative functions. Finally, it has been considered the fact that the values of the alternate functions at critical points cannot be calculated through relationships containing elliptic integrals of the first and third kind, but by means of algorithms that calculate them directly. Consequently, specific calculation algorithms for the alternative functions based on the Arithmetic Geometric Mean (AGM) algorithm [3] are also proposed.

2. Magnetic vector potential

2.1. Compact expression of the general solution

Assume a cylindrical surface current distribution where the current flows in tangential direction as shown in Fig. 1. The surface current distribution K_ϕ varies linearly between the lower end of the cylinder, where its value is $K_\phi = K_1$ [A/m] and the upper end, where $K_\phi = K_2$ [A/m].

The expression for the magnetic vector potential due to the described finite cylindrical current sheet has already been presented in [1] as follows:

$$A_\phi(r,z) = \frac{\mu_0}{2\pi}\int_0^\pi\int_{z_1}^{z_2}\frac{K_\phi r'\cos\phi'}{\sqrt{r^2+r'^2+(z-z')^2-2rr'\cos\phi'}}dz'd\phi' \tag{1}$$

where (r,z,ϕ) are the coordinates of the observation point and (r',z',ϕ') are the coordinates of a generic point on the current sheet. The surface current distribution is assumed to be linearly variable with the z coordinate and can be expressed as

$$K_\phi = g_1K_1 + g_2K_2 \tag{2}$$

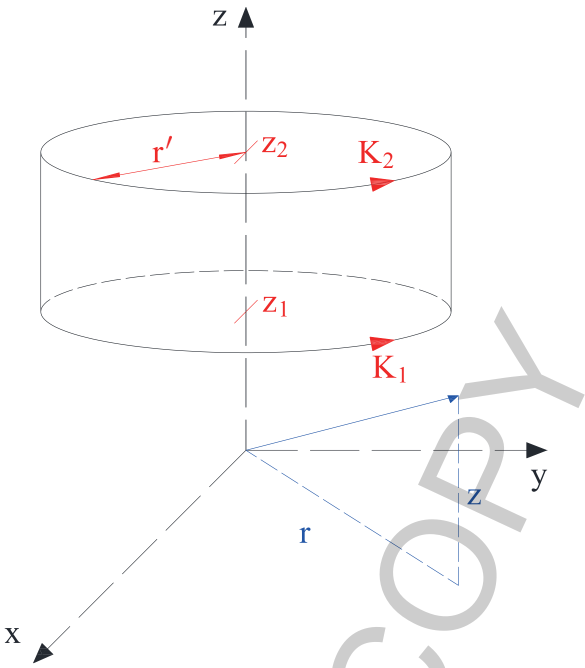

Fig. 1. Cylindrical current sheet with linearly varying current density.

where

$$g_1 = \frac{(z' - z_2)}{(z_1 - z_2)} \text{ and } g_2 = -\frac{(z' - z_1)}{(z_1 - z_2)} \tag{3}$$

From the integrand of Eq. (1) the following function can be defined

$$f_A(r, z) = \frac{r' \cos \phi'}{\sqrt{r^2 + r'^2 + (z - z')^2 - 2rr' \cos \phi'}} \tag{4}$$

so that Eq. (1) becomes

$$\begin{aligned} A_\phi(r, z) &= \frac{\mu_0}{2\pi} \int_0^\pi \int_{z_1}^{z_2} K_\phi f_A dz' d\phi' \\ &= \frac{\mu_0}{2\pi} \left[K_1 \int_0^\pi \int_{z_1}^{z_2} g_1 f_A dz' d\phi' + K_2 \int_0^\pi \int_{z_1}^{z_2} g_2 f_A dz' d\phi' \right] \end{aligned} \tag{5}$$

The goal is to express the magnetic vector potential as a sum of products of two factors, where the first factor is the value of the sheet current density at one end and the second factor is a function of the geometry only. The desired expression is

$$A_\phi(r, z) = f_{\phi 1}(r, z) K_1 + f_{\phi 2}(r, z) K_2 \tag{6}$$

Comparing Eqs (5) and (6) it follows that the geometric factors are

$$f_{\phi 1}(r, z) = \frac{\mu_0}{2\pi} \left. k_{\phi i} \right|_{z_i = z_2} \Big|_{z' = z_1}^{z' = z_2} \tag{7}$$

$$f_{\phi 2}(r,z) = -\frac{\mu_0}{2\pi}\; k_{\phi i}\big|_{z_i=z_1}\Big|_{z'=z_1}^{z'=z_2} \tag{8}$$

where $k_{\phi i}$ is the following generalized coefficient

$$k_{\phi i}(r,z) = \frac{1}{(z_1 - z_2)} \int_0^{\pi} \int \left(z' - z_i\right) f_A(r,z) dz' d\phi' \tag{9}$$

where $z_i = z_1$ in the case of $f_{\phi 2}$ and $z_i = z_2$ in case $f_{\phi 1}$. Note that the integral over z' in Eq. (9) is an indefinite integral as it has been chosen the strategy to defer the application of the limits of z' to the end, which is expressed in Eqs (7) and (8).

Thus, the expression of the magnetic vector potential at a point of coordinates (r, z) is

$$A_\phi(r,z) = \frac{\mu_0}{2\pi}\left[K_1 k_{\phi i}(r,z)\big|_{z_i=z_2}\Big|_{z'=z_1}^{z'=z_2} - K_2 k_{\phi i}(r,z)\big|_{z_i=z_1}\Big|_{z'=z_1}^{z'=z_2}\right] \tag{10}$$

After making the corresponding integrations, the following expression for the generalized coefficient $k_{\phi i}$ is obtained

$$k_{\phi i}(r,z) = a_1 E(k) + a_2 K(k) + (z - z_i)\left(a_3 E(k) + a_4 K(k) + a_5 \Pi(\alpha^2, k)\right) \tag{11}$$

where $K(k)$, $E(k)$, and $\Pi(\alpha^2, k)$ are the elliptic integrals of the first, second and third kind respectively.

The arguments of the elliptic integrals are in general

$$k^2 = \frac{4rr'}{(r + r')^2 + (z - z')^2} \quad \text{and} \quad \alpha^2 = \frac{4rr'}{(r + r')^2} \tag{12}$$

Notice that the condition $0 \leqslant k^2 \leqslant \alpha^2 \leqslant 1$ always holds.

The coefficients in Eq. (11) are

$$\begin{aligned} a_1 &= -\frac{\left(r^2 + r'^2 + (z - z')^2\right)\gamma_1}{3r\gamma_2} \quad a_2 = \frac{\gamma_1 {\gamma'}_1^2}{3r\gamma_2} \quad a_3 = \frac{\gamma_1 (z - z')}{2r\gamma_2} \\ a_4 &= -\frac{\left(2r^2 + 2r'^2 + (z - z')^2\right)(z - z')}{2r\gamma_2\gamma_1} \quad a_5 = \frac{(r - r')^2 (z - z')}{2r\gamma_1\gamma_2} \end{aligned} \tag{13}$$

where

$$\begin{aligned} \gamma_1 &= \sqrt{(r + r')^2 + (z - z')^2} \\ \gamma'_1 &= \sqrt{(r - r')^2 + (z - z')^2} \\ \gamma_2 &= z_1 - z_2 \end{aligned} \tag{14}$$

Note that the integration in Eq. (9) was performed for a generic field point with coordinates (r, z), resulting in a solution based on complete elliptic integrals. For special field points, such as the ends of the current sheet, the solutions may take a different form, as will be seen later.

2.2. Analysis of the singularities of the solution

Although the magnetic vector potential due to the described cylindrical current distribution has no real singularities, it is evident that the expression of the solution according to Eqs (11) and (12) has difficulties when is evaluated at field points where the complete elliptic integrals K and Π take unlimited values. This is not true for the elliptic integral of second kind E. In both cases infinite values are obtained when the argument k is 1, which happens for a field point at the lower end ($r = r', z = z_1$) when $k_{\phi i}$ is evaluated at $z' = z_1$, and also for a field point at the upper end ($r = r', z = z_2$) when $k_{\phi i}$ is evaluated at $z' = z_2$.

Furthermore, the function Π, which has the additional argument α, takes an infinite value when $\alpha = 1$, which occurs over the entire surface of the cylinder, i.e., $r = r'$. While these singularities are removable, Eq. (11) does not allow the calculation at these points.

The values of the magnetic vector potential at these points is to be investigated first and then Eq. (11) is to be modified so as to calculate the magnetic vector potential even at these problematic points.

2.3. Magnetic vector potential at the lower end of the cylinder

The symmetry of the cylinder indicates that the situation at the lower end of the cylinder is similar to that at the upper end. Considering now a field point at the lower end of the cylinder with coordinates $z = z_1$ and $r = r'$. In these conditions it follows that

$$k^2 = \frac{4r'^2}{4r'^2 + (z_1 - z')^2} \quad \text{and} \quad \alpha^2 = 1 \tag{15}$$

Then Eq. (9) becomes

$$k_{\phi i}(r', z_1) = \frac{r'}{(z_1 - z_2)} \int_0^{\pi} \int \frac{(z' - z_i) \cos \phi'}{\sqrt{(z_1 - z')^2 + 2r'^2 (1 - \cos \phi')}} dz' d\phi' \tag{16}$$

The integrals in Eq. (16) must be solved for this particular case, which leads to the following result

$$k_{\phi i}(r', z_1) = \frac{r'}{(z_1 - z_2)} \left(\xi_{01} + \xi_1 E(k) + \xi_2 K(k)\right) \tag{17}$$

where

$$\begin{aligned} \xi_{01} &= \frac{\pi (z_1 - z_i)}{2} \\ \xi_1 &= \frac{\sqrt{4r'^2 + (z_1 - z')^2}}{6r'^2} \left(-4r'^2 + (z_1 - z')(z_1 - 3z_i + 2z')\right) \\ \xi_2 &= -\frac{\sqrt{4r'^2 + (z_1 - z')^2}}{6r'^2} (z_1 - z')(z_1 - 3z_i + 2z') \end{aligned} \tag{18}$$

Note that due to the strategy to defer the application of the limits of the integral with respect to z', the constant ξ_{01} appears in the solution. This constant appears in both terms $z' = z_1$ and $z' = z_2$ in applying the fundamental theorem of calculus in the evaluation of Eqs (7) and (8), so both instances cancel each other. Therefore Eq. (17) can be simplified as

$$k_{\phi i}^{*}(r', z_1) = \frac{r'}{(z_1 - z_2)} \left(\xi_1 K(k) + \xi_2 E(k)\right) \tag{19}$$

Note that Eq. (19) is similar to Eq. (11) but the difference is that the term containing the function. Π disappeared. It can be seen, except for the term mentioned, that both equations are fully equivalent for the field point (r', z_1). The disappearance of the. Π term in Eq. (11) indicates that in this case the indeterminacy of the term $a_5\Pi$ (a_5 is zero and Π is infinity) should be resolved in favor of a_5 because the specific integration yields a result compatible with the vanishing of this term.

The expression Eq. (19) can then be used to calculate these terms in Eqs (7) and (8) where $z' = z_2$. However, when $z' = z_1$, then $k = 1$ and the function $K(k)$ tends to infinite. The use of Eq. (19) in this case is obviously inconvenient, and it is necessary to integrate this specific case where $z' = z_1$. The result obtained is

$$k_{\phi i}(r', z_1)\big|_{z'=z_1} = -\frac{4r'^2}{3\,(z_1 - z_2)} + (z_1 - z_i)\,\frac{\pi}{2}\frac{r'}{(z_1 - z_2)} \tag{20}$$

Note that the second term of Eq. (20) is equal to the first term in Eq. (17) and, as stated before, there are two such terms that cancel each other when replacing the limits of z', so that Eq. (20) simplifies to

$$k_{\phi i}^{*}(r', z_1)\big|_{z'=z_1} = -\frac{4r'^2}{3\,(z_1 - z_2)} \tag{21}$$

The use of this result in Eqs (7) and (8) at the lower end of the cylinder leads to

$$f_{\phi 1}(r', z_1) = \frac{\mu_0}{2\pi}\left(k_{\phi i}^{*}(r', z_1)\big|_{\substack{z'=z_2\\ z_i=z_2}} + \frac{4r'^2}{3\,(z_1 - z_2)}\right) \tag{22}$$

$$f_{\phi 2}(r', z_1) = -\frac{\mu_0}{2\pi}\left(k_{\phi i}^{*}(r', z_1)\big|_{\substack{z'=z_2\\ z_i=z_1}} + \frac{4r'^2}{3\,(z_1 - z_2)}\right) \tag{23}$$

thus the singularity has been removed.

2.4. Evaluation of the magnetic vector potential at the upper end of the cylinder

Similarly to the case of the evaluation at the bottom of the cylinder, the corresponding coordinates are $z = z_2$ and $r = r'$. By performing a procedure similar to that of the previous section, the following expression can be derived for the case $z' \neq z_2$,

$$k_{\phi i}^{*}(r', z_2) = \frac{r'}{(z_1 - z_2)} \left(\xi_3 E(k) + \xi_4 K(k)\right) \tag{24}$$

where

$$\xi_3 = \frac{\sqrt{4r'^2 + (z_2 - z')^2}}{6r'^2}\left(-4r'^2 + \left(z_2 - z'\right)\left(z_2 - 3z_i + 2z'\right)\right) \tag{25}$$

$$\xi_4 = -\frac{\sqrt{4r'^2 + (z_2 - z')^2}}{6r'^2}\left(z_2 - z'\right)\left(z_2 - 3z_i + 2z'\right)$$

For the case $z' = z_2$ the integral must be solved for this specific value. The result is

$$k^*_{\phi i}(r', z_2)\big|_{z'=z_2} = -\frac{4r'^2}{3\,(z_1 - z_2)} \tag{26}$$

which is the same result as in Eq. (21). Consequently,

$$f_{\phi 1}(r', z_2) = \frac{\mu_0}{2\pi}\left(-\frac{4r'^2}{3\,(z_1 - z_2)} - k^*_{\phi i}(r', z_2)\big|_{\substack{z'=z_1\\ z_i=z_2}}\right) \tag{27}$$

$$f_{\phi 2}(r', z_2) = -\frac{\mu_0}{2\pi}\left(-\frac{4r'^2}{3\,(z_1 - z_2)} - k^*_{\phi i}(r', z_2)\big|_{\substack{z'=z_1\\ z_i=z_1}}\right) \tag{28}$$

These equations are very similar to Eqs (22) and (23). It has been proven that also in this case the singularity is removable.

2.5. *Modified expression of the general solution in order to avoid singularities*

Equation (11) can be used without difficulty at the general field point $P(r, z)$, provided that it is not at one end of the current sheet cylinder, i.e., fields points where $r = r'$ and $z = z_1$ or $z = z_2$. For these critical points the arguments of the elliptic integrals are $k = 1$ and $\alpha = 1$, for which K and Π diverge, so that the expression presented does not allow the numerical computation of the magnetic vector potential, although the corresponding values on these points are finite. In this case, as shown in the previous sections, the singularities of the magnetic vector potential are removable. This suggests the possibility of formulating an alternative expression to Eq. (11) so that to express the solution in terms of functions other than K and Π. This possibility has been investigated by the authors, and the following alternative expression has been obtained

$$k_{\phi i}(r, z) = a_1 E(k) + a'_2\left[k' K(k)\right] + (z - z_i)\left(a_3 E(k) + a'_4\left[k' K(k)\right] + a'_5 \Lambda_0(\xi, k)\right) \tag{29}$$

where

$$k'^2 = 1 - k^2 \text{ or } k' = \frac{\gamma'_1}{\gamma_1} \tag{30}$$

$$a'_2 = \frac{a_2}{k'} = \frac{\gamma_1^2 \gamma'_1}{3r\gamma_2} \quad a'_4 = \frac{a_4}{k'} = -\frac{\left(2r^2 + 2r'^2 + (z - z')^2\right)(z - z')}{2r\gamma_2\gamma'_1} \tag{31}$$

$$a'_5 = \frac{\pi}{4r\gamma_2}\,\text{sgn}\,\left(z - z'\right)\left|r^2 - r'^2\right|$$

Λ_0 is the Lambda Heuman function, whose first argument is

$$\xi = arcsen\sqrt{\frac{\alpha^2 - k^2}{\alpha^2 k'^2}} \quad sen\,\xi = \sqrt{\frac{(z - z')^2}{(r - r')^2 + (z - z')^2}} \tag{32}$$

The functions $K(k)$ and $\Pi(\alpha, k)$ have been respectively replaced by $k'K(k)$ and $\Lambda_0(\xi, k)$ in Eq. (29). The functions $k'K(k)$ and $\Lambda_0(\xi, k)$ are bounded, so that they do not diverge at critical points. Note that the coefficient a'_5 vanishes for $\alpha = 1$ and furthermore if $\alpha = 1$ then $\xi = \pi/2$ and $\Lambda_0 = 1$ and consequently the last term vanishes. This result also holds if $k = 1$. On the other hand, the function $k'K(k)$ vanishes for $k = 1$.

A fair question would be how the functions $k'K(k)$ and $\Lambda_0(\xi, k)$ are to be finally calculated in order to avoid numerical problems when calculating them at critical points. A contribution to the solution of this problem is given in Section 5.

It can be concluded that Eq. (29) is suitable for calculating the magnetic vector potential in general points of space, including the critical points. Finally, it can be also concluded that the magnetic vector potential has no real singularities.

3. Axial magnetic field strength

3.1. Compact expression of the general solution

The expression for the axial component of the magnetic field strength due to a finite cylindrical current sheet is [1]:

$$H_z(r,z) = \frac{1}{2\pi r}\int_0^{\pi}\int_{z_1}^{z_2} K_\phi \frac{\partial\,(r f_A)}{\partial r} dz' d\phi' \tag{33}$$

Substituting Eq. (2) into Eq. (33) gives

$$H_z(r,z) = \frac{1}{2\pi r}\left[K_1\int_0^{\pi}\int_{z_1}^{z_2} g_1 \frac{\partial\,(r f_A)}{\partial r} dz' d\phi' + K_2\int_0^{\pi}\int_{z_1}^{z_2} g_2 \frac{\partial\,(r f_A)}{\partial r} dz' d\phi'\right] \tag{34}$$

As in the case of the magnetic vector potential, the solution is to be expressed as a sum of products of current distribution factors and geometrical factors, namely

$$H_z(r,z) = f_{z1}(r,z)K_1 + f_{z2}(r,z)K_2 \tag{35}$$

Comparing Eqs (35) and (36), it can be seen that the geometrical factors are

$$f_{z1}(r,z) = \frac{1}{2\pi}\; k_{zi}\big|_{z_i=z_2}\Big|_{z'=z_1}^{z'=z_2} \tag{36}$$

$$f_{z2}(r,z) = -\frac{1}{2\pi}\; k_{zi}\big|_{z_i=z_1}\Big|_{z'=z_1}^{z'=z_2} \tag{37}$$

where

$$k_{zi}(r,z) = \frac{1}{r\,(z_1 - z_2)}\int_0^{\pi}\int \left(z' - z_i\right)\frac{\partial\,(r f_A)}{\partial r} dz' d\phi' \tag{38}$$

After making the corresponding integrations, the following expression for k_{zi} can be derived

$$k_{zi}(r,z) = b_1 E(k) + b_2 K(k) + (z - z_i)\left(b_3 K(k) + b_4 \Pi(\alpha^2, k)\right) \tag{39}$$

$$b_1 = -\frac{\gamma_1}{\gamma_2} \qquad b_2 = \frac{r^2 - r'^2 + (z - z')^2}{\gamma_2\gamma_1} \tag{40}$$

$$b_3 = -\frac{(z - z')}{\gamma_2\gamma_1} \qquad b_4 = \frac{(r - r')\,(z - z')}{\gamma_2\gamma_1\,(r + r')}$$

3.2. *Analysis of the singularities of the solution*

The presented H_z solution has the same disadvantages as the solution initially formulated for A, as it will present problems at observation points located at the end of the cylindrical current sheet ($r = r'$, $z = z_1; r = r', z = z_2$).

The first task is to investigate the values of H_z at these points and then modify Eq. (39) so as to enable the calculation of H_z even at these points.

3.3. *Evaluation of the axial magnetic field intensity at the lower end of the cylinder*

Assume now that the observation point is located at the lower end of the cylinder, so that $z = z_1$ and $r = r'$, therefore Eq. (15) holds. In these conditions, the following equation holds

$$k_{zi}(r', z_1) = \frac{r'}{(z_1 - z_2)} \int_0^{\pi} \int \frac{(z' - z_i)\cos\phi' \left((z_1 - z')^2 + r'^2 (1 - \cos\phi')\right)}{\left((z_1 - z')^2 + 2r'^2 (1 - \cos\phi')\right)^{3/2}} dz' d\phi' \quad (41)$$

The integrals in Eq. (41) must be solved for this particular case, which leads to the following result

$$k_{zi}(r', z_1) = \frac{r'}{(z_1 - z_2)} \left(\beta_{01} + \beta_1 E(k) + \beta_2 K(k)\right) \quad (42)$$

where

$$\begin{aligned} \beta_{01} &= \frac{\pi (z_1 - z_i)}{2} \\ \beta_1 &= -\sqrt{4r'^2 + (z_1 - z')^2} \\ \beta_2 &= -\frac{(z_1 - z')(z' - z_i)}{\sqrt{4r'^2 + (z_1 - z')^2}} \end{aligned} \quad (43)$$

The elimination of the terms that will cancel each other when evaluating the expression at the limits of z' gives

$$k^*_{zi}(r', z_1) = \frac{r'}{(z_1 - z_2)} \left(\beta_1 E(k) + \beta_2 K(k)\right) \quad (44)$$

Equation (44) for k_{zi} can then be used to calculate the terms in Eqs (36) and (37) where $z' = z_2$. However, when $z' = z_1$, then $k = 1$ and the function $K(k)$ is infinite. The use of Eq. (44) is obviously inconvenient in this case, so that the integration must be performed for this specific case where $z' = z_1$. By doing this, the following result is obtained

$$k_{zi}(r', z_1)\big|_{z'=z_1} = \frac{r'}{(z_1 - z_2)} \left(-2r' + \frac{\pi}{2}(z_1 - z_i)\right) \quad (45)$$

Note that the second term of Eq. (45) is equal to the first term in Eq. (42) and, as stated before, it is canceled when evaluating at the limits of z', so that Eq. (45) simplifies to

$$k^*_{zi}(r', z_1)\big|_{z'=z_1} = -\frac{2r'^2}{(z_1 - z_2)} \quad (46)$$

The evaluation of the geometric factors at the lower end of the cylinder using Eqs (36) and (37) gives

$$f_{z1}(r', z_1) = \frac{1}{2\pi r'}\left(k^*_{zi}(r', z_1)\big|_{\substack{z'=z_2\\ z_i=z_2}} + \frac{2r'^2}{(z_1 - z_2)}\right) \tag{47}$$

$$f_{z2}(r', z_1) = -\frac{1}{2\pi r'}\left(k^*_{zi}(r', z_1)\big|_{\substack{z'=z_2\\ z_i=z_1}} + \frac{2r'^2}{(z_1 - z_2)}\right) \tag{48}$$

thus the problem of removable singularities has been solved.

3.4. Evaluation of the axial magnetic field intensity at the upper end of the cylinder

Similarly to the case of the evaluation at the bottom of the cylinder, the corresponding coordinates of the observation point are $z = z_2, r = r'$. By proceeding in a similar manner as in the previous section, the following set of equations can be obtained

$$k^*_{zi}(r', z_2) = \frac{r'}{(z_1 - z_2)}\left(\beta_3 E(k) + \beta_4 K(k)\right) \tag{49}$$

where

$$\begin{aligned} \beta_3 &= -\sqrt{4r'^2 + (z_2 - z')^2} \\ \beta_4 &= -\frac{(z_2 - z')(z' - z_i)}{\sqrt{4r'^2 + (z_2 - z')^2}} \\ k^2 &= \frac{4r'^2}{4r'^2 + (z_2 - z')^2} \end{aligned} \tag{50}$$

In the case that $z' = z_2$ the integration must be performed for this special case. The result is

$$k^*_{zi}(r', z_2)\big|_{z'=z_2} = -\frac{2r'^2}{(z_1 - z_2)} \tag{51}$$

which is the same result obtained in Eq. (46). Thus

$$f_{z1}(r', z_2) = \frac{1}{2\pi r'}\left(-\frac{2r'^2}{(z_1 - z_2)} - k^*_{zi}(r', z_2)\big|_{\substack{z'=z_1\\ z_i=z_2}}\right) \tag{52}$$

$$f_{z2}(r', z_2) = -\frac{1}{2\pi r'}\left(-\frac{2r'^2}{(z_1 - z_2)} - k^*_{zi}(r', z_2)\big|_{\substack{z'=z_1\\ z_i=z_1}}\right) \tag{53}$$

As in the case of the lower edge of the cylinder, the singularity is removable.

3.5. Modified expression of the general solution in order to avoid singularities

As in the case of the magnetic vector potential, Eq. (39) can be used without difficulty at a general point $P(r, z)$, provided it is not located at the ends of cylindrical current sheet, i.e., for $r = r'$ and $z = z_1$

or $z = z_2$. For such cases the following alternative expression has been obtained

$$k_{zi}(r,z) = b_1 E(k) + b_2' \left[k' K(k)\right] + (z - z_i)\left(b_3' \left[k' K(k)\right] + b_4' \Lambda_0(\xi, k)\right) \tag{54}$$

where

$$b_1 = -\frac{r\gamma_1}{\gamma_2} \qquad b_2' = \frac{r}{\gamma_2 \gamma_1'}\left(r^2 - r'^2 + \left(z - z'\right)^2\right)$$
$$b_3' = -\frac{r\,(z - z')}{\gamma_2 \gamma_1'} \qquad b_4' = \frac{\pi r}{2\gamma_2}\,\mathrm{sgn}\,\left(r - r'\right)\,\mathrm{sgn}\,\left(z - z'\right) \tag{55}$$

It can be concluded that Eq. (55) is suitable to calculate the axial magnetic field intensity at any point in space, including the critical points. The axial component of the magnetic field has no true singularities, only removable ones.

4. Radial magnetic field intensity

4.1. Compact expression of the general solution

The radial component of the magnetic field intensity due to a cylindrical finite current sheet can be expressed as [1]:

$$H_r(r,z) = -\frac{1}{2\pi}\int_0^{\pi}\int_{z_1}^{z_2} K_\phi \frac{\partial f_A}{\partial z} dz' d\phi' \tag{56}$$

The substitution of Eq. (2) into Eq. (56) gives

$$H_r(r,z) = -\frac{1}{2\pi}\left[K_1 \int_0^{\pi}\int_{z_1}^{z_2} g_1 \frac{\partial f_A}{\partial z} dz' d\phi' + K_2 \int_0^{\pi}\int_{z_1}^{z_2} g_2 \frac{\partial f_A}{\partial z} dz' d\phi'\right] \tag{57}$$

As in the case of the magnetic vector potential, the solution can be expressed as a sum of products of current distribution factors and geometrical factors as follows

$$H_r(r,z) = f_{r1}(r,z) K_1 + f_{r2}(r,z) K_2 \tag{58}$$

The comparison of Eq. (57) with Eq. (58) leads to the following expressions for the geometrical factors

$$f_{r1}(r,z) = \frac{1}{2\pi}\left. \left. k_{ri}\right|_{z_i = z_2}\right|_{z'=z_1}^{z'=z_2} \tag{59}$$

$$f_{r2}(r,z) = -\frac{1}{2\pi}\left. \left. k_{ri}\right|_{z_i = z_1}\right|_{z'=z_1}^{z'=z_2} \tag{60}$$

where

$$k_{ri}(r,z) = -\frac{1}{(z_1 - z_2)}\int_0^{\pi}\int \left(z' - z_i\right)\frac{\partial f_A}{\partial z} dz' d\phi' \tag{61}$$

After performing the corresponding integrations, the following expression for k_{ri} can be found

$$k_{ri} = c_1 E(k) + c_2 K(k) + (z - z_i)\left(c_3 E(k) + c_4 K(k)\right) + c_5 \Pi(\alpha^2, k) \tag{62}$$

where

$$c_1 = \frac{\gamma_1 (z - z')}{2r\gamma_2} \quad c_2 = -\left(z - z'\right) \frac{(z - z')^2}{2r\gamma_2\gamma_1}$$
$$c_3 = -\frac{\gamma_1}{r\gamma_2} \quad c_4 = \frac{\left(r^2 + r'^2 + (z - z')^2\right)}{r\gamma_1\gamma_2} \tag{63}$$
$$c_5 = -a_5 = -\frac{(r - r')^2 (z - z')}{2r\gamma_1\gamma_2}$$

4.2. Analysis of the singularities of the solution

The solution presented for H_r has the same disadvantages as the solution initially formulated for A, as it will diverge at the end points of the cylinder ($r = r', z = z_1; r = r', z = z_2$). It will be seen that in this case, unlike the case of A and H_z, the singularities are not removable.

4.3. Evaluation of the radial magnetic field intensity at the lower end of the cylinder

Assume now that the observation point is at the lower end of the cylindrical current sheet; then its corresponding coordinates are $z = z_1 \; yr = r'$. Therefore, the following equation holds for this case

$$k_{ri}(r', z_1) = -\frac{r'}{(z_1 - z_2)} \int_0^\pi \int \left(z' - z_i\right) \frac{(z' - z_1)\cos\phi'}{\left((z_1 - z')^2 + 2r'^2 (1 - \cos\phi')\right)^{3/2}} dz' d\phi' \tag{64}$$

By performing the integrations in Eq. (64), the following result can be derived

$$k_{ri}(r', z_1) = -\frac{r'}{(z_1 - z_2)} \left(\eta_0 + \eta_1 E(k) + \eta_2 K(k)\right) \tag{65}$$

where

$$\eta_0 = \frac{\pi}{2}$$
$$\eta_1 = \frac{\sqrt{4r'^2 + (z_1 - z')^2}\,(z_1 + 2z_i + z')}{2r'^2} \tag{66}$$
$$\eta_2 = -\frac{4r'^2 (z_1 + z_i) + (z_1 - z')^2 (z_1 + 2z_i + z')}{2r'^2 \sqrt{4r'^2 + (z_1 - z')^2}}$$

By eliminating the term which disappears by evaluating at the limits of z', Eq. (65) becomes

$$k_{ri}^*(r', z_1) = -\frac{r'}{(z_1 - z_2)} \left(\eta_1 E(k) + \eta_2 K(k)\right) \tag{67}$$

Equation (67) for k_{ri} can then be used without numerical problems to calculate the terms in Eqs (59) and (60) for $z' = z_2$. However, when $z' = z_1, k = 1$ and the function $K(k)$ has an infinite value.

Obviously, the use of Eq. (67) is not possible in this case. The integration is to be made for the specific case of $z' = z_1$.

In turn, two cases can be distinguished. On the one hand, one case is for $z_i = z_1$, for which

$$k_{ri}(r', z_1)\big|_{\substack{z'=z_1 \\ z_i=z_1}} = -\frac{\pi r'}{2\,(z_1 - z_2)} \tag{68}$$

On the other hand, another case is for $z_i = z_2$, for which

$$k_{ri}(r', z_1)\big|_{\substack{z'=z_1 \\ z_i=z_2}} = \frac{1}{\sqrt{2}} I_\infty - \frac{\pi}{2} \frac{r'}{(z_1 - z_2)} \tag{69}$$

where

$$I_\infty = \int_0^\pi \frac{\cos\phi'}{\sqrt{(1 - \cos\phi')}} d\phi' = +\infty \tag{70}$$

Note that the right hand side of Eq. (68) is canceled by the term corresponding to η_0 when the geometric factors are calculated. The same principle applies to the second term of the right hand side of Eq. (69). The evaluation of the geometrical factors at the lower end of the cylindrical current sheet gives

$$f_{r1}(r', z_1) = \frac{1}{2\pi}\left(k^*_{ri}(r', z_1)\big|_{\substack{z'=z_2 \\ z_i=z_2}} - \frac{1}{\sqrt{2}} I_\infty \right) \tag{71}$$

$$f_{r2}(r', z_1) = -\frac{1}{2\pi}\left(k^*_{ri}(r', z_1)\big|_{\substack{z'=z_2 \\ z_i=z_1}} \right) \tag{72}$$

The radial component of the magnetic field intensity is

$$H_r(r', z_1) = \frac{1}{2\pi}\left(k^*_{ri}(r', z_1)\big|_{\substack{z'=z_2 \\ z_i=z_2}} - \frac{1}{\sqrt{2}} I_\infty \right) K_1 - \frac{1}{2\pi}\left(k^*_{ri}(r', z_1)\big|_{\substack{z'=z_2 \\ z_i=z_1}} \right) K_2 \tag{73}$$

$$H_r(r', z_1) \cong -\frac{K_1}{2\pi\sqrt{2}} I_\infty \tag{74}$$

This indicates that the value of the radial component of the magnetic field at the lower end of the cylinder is infinite and is concluded that there is a real singularity in this point.

4.4. *Evaluation of the radial magnetic field at the upper end of the cylinder*

In this case the observation point is located at the upper end of the cylindrical current sheet; then its corresponding coordinates are $z = z_2$ $y r = r'$. The procedure to obtain the radial magnetic field solution is similar to the one used in the previous section, and it leads to the following expression for the radial magnetic field intensity on the cylinder at $z = z_2$

$$H_r(r', z_2) = \frac{1}{2\pi}\left(- k^*_{ri}(r', z_2)\big|_{\substack{z'=z_1 \\ z_i=z_2}} \right) K_1 - \frac{1}{2\pi}\left(-\frac{1}{\sqrt{2}} I_\infty - k^*_{ri}(r', z_2)\big|_{\substack{z'=z_1 \\ z_i=z_1}} \right) K_2 \tag{75}$$

$$H_r(r', z_2) = \frac{K_2}{2\pi\sqrt{2}} I_\infty \tag{76}$$

This indicates that the value of the radial component of the magnetic field intensity at the top of the cylinder is infinite and that there is a real singularity.

4.5. Modified expression of the general solution in order to avoid singularities

As in the case of the magnetic vector potential, Eq. (62) can be used without difficulty at a general observation $P(r,z)$, provided it is not located at the ends of the current sheet cylinder, i.e., for points such that $r = r'$ and $z = z_1$ or $z = z_2$. In this case the singularities at these points are not removable, so one might think that it would be pointless to modify the expression of H_r in this case. However, there is a removable singularity, as before, for points on the cylinder such that $r = r'$ but $z \neq z_1$ or $z \neq z_2$. To overcome this difficulty, the following alternative expression has been obtained

$$k_{ri} = c_1 E(k) + c_2' \left[k' K(k)\right] + (z - z_i)\left(c_3 E(k) + c_4' \left[k' K(k)\right]\right) + c_5' \Lambda_0(\xi, k) \tag{77}$$

where

$$\begin{aligned} c_1 &= \frac{\gamma_1 (z - z')}{2r\gamma_2} \quad c_2' = -\left(z - z'\right) \frac{(z - z')^2}{2r\gamma_2\gamma_1'} \\ c_3 &= -\frac{\gamma_1}{r\gamma_2} \quad c_4' = \frac{\left(r^2 + r'^2 + (z - z')^2\right)}{r\gamma_1'\gamma_2} \\ c_5' &= -\frac{\pi}{4r\gamma_2} \operatorname{sgn}\left(z - z'\right)\left|r^2 - r'^2\right| \end{aligned} \tag{78}$$

It can be concluded that Eq. (78) is suitable for calculating the radial component of the magnetic field intensity at any point in space, including points on the current sheet, unless the endpoints where unavoidable singularities occur.

5. Numerical calculation of the solutions

The functions appearing in the magnetic field solutions are typical for solutions of axisymmetric problems with open boundary. These functions are the complete elliptic integrals of the first, second and third kind, $K(k), E(k)$, and $\Pi(\alpha^2, k)$ respectively. An efficient way to calculate these functions is to apply the arithmetic geometric mean (AGM) method as is described in [3], and more recently in [4]. This is the algorithm used for the numerical calculation of these functions in most commercial mathematical programs. The functions $K(k)$ and $\Pi(\alpha^2, k)$ have the property that they diverge when the argument $k = 1$, and additionally the function $\Pi(\alpha^2, k)$ also diverges when its first argument is $\alpha = 1$. The function $E(k)$ is bounded in the whole useful range of the argument k and it does not represent a calculation problem.

While many magnetic field solutions are described in terms of complete elliptic integrals, some of them diverge for some arguments, but this does not necessarily means that the solution itself diverges for these critical points, as it has been proven above. This is however a drawback, since the solution can not be calculated at critical points by means of expressions containing these functions.

That is why many authors in the literature often replace these functions by others that behave well for the critical arguments. Note that although new functions can theoretically be defined so that they do not diverge at critical points, care should be taken not to use algorithms that do diverge at these points. The following sections deal with the numerical calculation of functions that can replace $K(k)$ and $\Pi(\alpha^2, k)$ in the magnetic field solutions.

5.1. Numerical calculation of $K(k)$ and $[k'K(k)]$

For the calculation of these functions, the AGM [4] is used. Let a_0 and g_0 be positive numbers defined as

$$a_{n+1} = \frac{a_n + g_n}{2} \quad g_{n+1} = \sqrt{a_n g_n} \tag{79}$$

When n tends to ∞, a_n and g_n converge to a common limit $M(a_0, g_0)$ called AGM of a_0 and g_0.

For the case of the function $K(k)$, the following equation holds

$$K(k) = \frac{\pi}{2\,M(1, k')} \quad -\infty < k < 1 \tag{80}$$

It can be seen that $k = 1$ is not a possible value. Actually the range of interest of the values of k is $0 \leqslant k^2 \leqslant 1$. If $k = 1$ in Eq. (80) then $k' = 0$ and M converges to 0, so K diverges.

It has been seen that the proposed solution to avoid this problem was to replace K by the function $k'K(k)$.

The integral representation of the AGM is

$$\frac{1}{M(a_0, g_0)} = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{a_0^2 \cos^2\theta + g_0^2 \sin^2\theta}} \tag{81}$$

which allows deriving that

$$k'K(k) = \frac{\pi}{2M(1/k', 1)} \tag{82}$$

allowing the direct calculation of $k'K(k)$, but the inconvenience of the value $k = 1$ still remains.

For that case the property 164.02 of [5] can be used

$$k'K\left(k\right) = E\left(k_1\right)\left(1 + k'\right) - E\left(k\right) \tag{83}$$

where

$$k_1 = \frac{(1 - k')}{(1 + k')} = 1 \tag{84}$$

and it can be seen that

$$\left[k'K\left(k\right)\right]_{k=1} = 0 \tag{85}$$

5.2. Numerical calculation of $\Pi(\alpha^2, k)$ and $\Lambda_0(\xi, k)$

In the modified expressions to avoid singularities the complete elliptic integral of the third kind $\Pi(\alpha^2, k)$ has been replaced by the Heumann Lambda function $\Lambda_0(\xi, k)$. The relationship between them is (Equation 413.01 of [5])

$$\Pi\left(\alpha^2, k\right) = \frac{\alpha\pi\Lambda_0\left(\xi, k\right)}{2\sqrt{\left(\alpha^2 - k^2\right)\left(1 - \alpha^2\right)}} \tag{86}$$

where

$$\xi = arcsen\sqrt{\frac{\alpha^2 - k^2}{\alpha^2 k'^2}} \text{ and } k^2 < \alpha^2 < 1 \tag{87}$$

The following algorithm based on AGM for calculating $\Pi(\alpha^2, k)$ is given in [4]:

$$\begin{aligned}
\Pi(\alpha^2, k) &= \frac{K}{2}\left(2 + \frac{\alpha^2}{(1-\alpha^2)}\sum_{n=0}^{\infty} Q_n\right) \\
a_0 &= 1;\ \ g_0 = k';\ \ p_0^2 = 1 - \alpha^2;\ \ Q_0 = 1 \\
p_{n+1} &= \frac{p_n^2 + a_n g_n}{2p_n} \\
\varepsilon_n &= \frac{p_n^2 - a_n g_n}{p_n^2 + a_n g_n} \\
Q_{n+1} &= \frac{1}{2} Q_n \varepsilon_n \\
n &= 0, 1, 2, \ldots
\end{aligned} \tag{88}$$

It should be noted that whenever the function $\Pi(\alpha^2, k)$ appears along with the value $k = 1$ as the second argument, the respective terms in the solutions of A, H_z and H_r vanish, since the coefficient that multiplies the function $\Pi(\alpha^2, k)$ goes to zero much faster than it. However the vanishing of the term containing the function $\Pi(\alpha^2, k)$ is not clearly explained at first glance, as it has an indetermination of the type zero multiplied by infinity. In practical terms, it would be enough to remove the term of the corresponding field solution when $k = 1$. This case can be explained in a better way by replacing $\Pi(\alpha^2, k)$ by $\Lambda_0(\xi, k)$, since the latter function remains always bounded and the vanishing of the mentioned term is justified because the coefficient tends to zero.

Furthermore, the case for $k \neq 1$ but $\alpha = 1$ (observation point on the cylinder but not at its ends) remains to be considered. In that case $\Pi(\alpha^2, k)$ diverges even though the solution of the three mentioned field magnitudes have a finite value. It is here also advantageous to use $\Lambda_0(\xi, k)$, because if $\alpha = 1$, then $\xi = \pi/2$ and $\Lambda_0(\pi/2, k) = 1$. In this case the term of the solution expression has a specific value, unlike the case where $k = 1$ where the term is zero. It is important to notice that the field expression yields the correct value, which is achieved using $\Lambda_0(\xi, k)$.

Based on the relationship Eq. (86) between $\Pi(\alpha^2, k)$ and $\Lambda_0(\xi, k)$ and Eq. (88), the following calculation formula can be written

$$\Lambda_0(\xi, k) = \frac{K}{\pi}\frac{\sqrt{\alpha^2 - k^2}}{\alpha\sqrt{1-\alpha^2}}\left(2 + \alpha^2\left(\sum_{n=0}^{\infty} Q_n - 2\right)\right) \tag{89}$$

It should be noticed that this formula is not applicable to the argument $\alpha = 1$, and this has two reasons. The first one is that it contains the factor $\sqrt{1-\alpha^2}$ in the denominator, which tends to zero if $\alpha = 1$. The second reason is that the sum $\sum_{n=0}^{\infty} Q_n$ of Carson algorithm collapses for $\alpha = 1$. It is important to observe that this situation can be avoided simply by assigning the value $\Lambda_0(\pi/2, k) = 1$ when $\alpha = 1$.

However, the authors have considered convenient to describe an alternative treatment to this situation, applicable to the solutions for A and H_r. The authors have proposed a modified version of the Carson algorithm for the sum $\sum_{n=0}^{\infty} Q_n$ so that it does not diverge for $\alpha = 1$. This algorithm can be formulated as follows

$$\begin{aligned} a_0 &= 1;\;\; g_0 = e_0 = k';\;\; d_0^2 = 1 - \alpha^2;\;\; c_0 = 1;\;\; Q_0 = 1 \\ c_n &= 4c_{n-1}d_{n-1}^2 \\ e_n &= a_n g_n c_n \\ d_n &= d_{n-1}^2 + e_{n-1} \\ \varepsilon_n &= \frac{d_n^2 - e_n}{d_n^2 + e_n} \\ Q_{n+1} &= \frac{1}{2} Q_n \varepsilon_n \\ n &= 0, 1, 2, \ldots \end{aligned} \tag{90}$$

On the other hand, for the solution of the magnetic vector potential A of Eq. (29), the last term can be modified as follows

$$a_5' \Lambda_0(\xi, k) = a_5'' \Lambda_1(\xi, k) \tag{91}$$

where

$$a_5'' = \frac{\pi}{4r\gamma_2} \operatorname{sgn}\left(z - z'\right)\left(r + r'\right)^2 \tag{92}$$

and

$$\Lambda_1(\xi, k) = \frac{K}{\pi} \frac{\sqrt{\alpha^2 - k^2}}{\alpha} \left(2 + \alpha^2 \left(\sum_{n=0}^{\infty} Q_n - 2\right)\right) \tag{93}$$

Similarly, the following relationship can be used for the respective term of H_r in Eq. (77)

$$c_5' \Lambda_0(\xi, k) = c_5'' \Lambda_1(\xi, k) \tag{94}$$

where

$$c_5'' = -\frac{\pi}{4r\gamma_2} \operatorname{sgn}\left(z - z'\right)\left(r + r'\right)^2$$

This prevents making special considerations for the points corresponding to $\alpha = 1$ for A and H_r.

For the case of H_z a similar procedure is not possible, as there is a discontinuity in the function for $\alpha = 1$ and there is no choice but to avoid the situation by assigning the value $\Lambda_0(\pi/2, k) = 1$.

6. Performance comparison between new and original formulas [1]

Table 1 shows the results of three experiments designed to measure the performance of the mathematical expressions proposed in this work.

Table 1

Number of cylindrical elements [ns]	Number of evaluation points [nf]	Total number of interactions [ni]	Average time original formulation (s)	Average time new formulation (s)	Acceleration factor (p.u.)
2,000	2,000	4,000,000	7.03	5.83	1.21
3,000	3,000	9,000,000	16.05	13.10	1.22
4,000	4,000	16,000,000	28.71	22.29	1.29

A number of ns cylindrical elements has been considered, for which the dimensions, location and current densities were specified in a random way. The goal is to determine the vector potential, radial and axial field intensity in a number nf of evaluation points due to each cylindrical element. According to the above, it have been a total amount of $\mathrm{ni} = \mathrm{ns} * \mathrm{nf}$ interactions between the sources and the evaluation points.

As shown in Table 1, in the three experiments the performance of the proposed formulations was much better than that of the formulations proposed in [1]. However, it can also be seen that the efficiency is increased when the amount of interactions are higher, reaching an acceleration factor close to 30% for an amount of 16 million interactions.

The results reported in the Table 1 were obtained using a laptop with 16 GB of RAM with a Core i7 2 GHz.

7. Conclusions

Compact analytical expressions for a fast calculation of the magnetic field produced by a finite cylindrical surface current distribution considering a linearly varying current density have been presented. These expressions represent a major improvement of the ones given in [1].

The singularities of the magnitudes A, H_z and H_r appearing on the current sheet surface and at its ends have been investigated. It has been verified that $\mathbf{A}$ and $\mathbf{H_z}$ have removable singularities, and, on the other hand, that $\mathbf{H_r}$ has removable singularities at points on the cylinder surface other than those at the top and bottom ends. On these last locations the singularities are not removable, but real. The values of A, H_z and H_r at critical points were determined, provided the singularities are of removable type.

Alternative compact analytical expressions for the evaluation of magnetic field magnitudes for all points in space have been proposed, including the critical points on the cylinder. This has been accomplished through the use of the alternative functions $k'K(k)$ and $\Lambda_0(\xi, k)$.

The analytical expressions presented in this article were validated satisfactorily by numerical integration.

Finally, calculation algorithms based on AGM have been proposed for the calculation of the alternative functions.

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