HECKE AND STURM BOUNDS FOR HILBERT MODULAR FORMS OVER REAL QUADRATIC FIELDS

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Abstract. Let K be a real quadratic field and $\mathcal{O}_K$ its ring of integers. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$ and $M_{(k_1,k_2)}(\Gamma)$ be the finite dimensional space of Hilbert modular forms of weight (k_1, k_2) for Γ. Given a form $f(z) \in M_{(k_1,k_2)}(\Gamma)$, how many Fourier coefficients determine it uniquely in such space? This problem was solved by Hecke for classical forms, and Sturm proved its analogue for congruences modulo a prime ideal. The present article solves the same problem for Hilbert modular forms over K. We construct a finite set of indices (which depends on the cusps desingularization of the modular surface attached to Γ) such that the Fourier coefficients of any form in such set determines it uniquely.

Introduction

It is a classical result that the space of modular forms of a fixed weight and level is finite dimensional. Since modular forms admit a Fourier expansion, this implies that a few Fourier coefficients should be enough to determine the form uniquely, but how many coefficients are needed?

For classical modular forms, this was already known by Hecke (see [Hec70], page 811, Satz 1 and Satz 2). Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Write $\mathrm{P}\Gamma$ for the image of Γ in $\mathrm{PSL}_2(\mathbb{Z})$ and $d = [\mathrm{PSL}_2 : \mathrm{P}\Gamma]$ for the degree of the map $X(\Gamma) \to X(1)$. Let $f(z) \in M_{2k}(\Gamma)$ be a weight $2k$ modular form for Γ and $f(z) = \sum_{n\geq 0} a_n(f)q^n$ its Fourier expansion at a cusp, where $q = e^{\frac{2\pi i z}{N}}$ is a local uniformizer. Recall that the order of f at the cusp is defined as

$$\mathrm{ord}(f) = \inf\{n \mid a_n(f) \neq 0\}.$$

Theorem (Hecke)**.** *Let $f(z) \in M_{2k}(\Gamma)$ be a weight $2k$ modular form for Γ. If $\mathrm{ord}(f) > dk/6$, then $f = 0$.*

This bound is optimal for $\mathrm{SL}_2(\mathbb{Z})$ when $k \not\equiv 2 \pmod{12}$, since $\lfloor \frac{2k}{12} \rfloor + 1$ is exactly the dimension of the space $M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$. When $k \equiv 2 \pmod{12}$, the order of vanishing is the dimension plus one.

One can consider the same problem with congruence conditions instead of vanishing conditions. Let $\mathcal{O}$ be the ring of integers of a number field F, and $\mathfrak{m}$ a maximal ideal of $\mathcal{O}$. We fix an embedding $F \subset \mathbb{C}$. As before, let $f(z) = \sum_{n\geq 0} a_n(f)q^n \in M_{2k}(\Gamma)$ be a modular form such that $a_n(f) \in \mathcal{O}$ for all $n \geq 0$. Then we define

$$\mathrm{ord}_{\mathfrak{m}}(f) = \min\{n \mid a_n(f) \notin \mathfrak{m}\},$$

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with the convention $\mathrm{ord}_{\mathfrak{m}}(f) = \infty$ if $a_n(f) \in \mathfrak{m}$ for all n.

Theorem (Sturm). *If* $\mathrm{ord}_{\mathfrak{m}}(f) > dk/6$, *then* $\mathrm{ord}_{\mathfrak{m}}(f) = \infty$.

Both results have many applications in computational and theoretical problems. For example, such results are used to compute generators for the Hecke algebra and to compute the discriminant of the integral Hecke algebra, whose divisors are the primes of fusion, giving congruences between modular forms. The main results of the present article are generalizations of the previous results to Hilbert modular forms. For arbitrary totally real number fields, there are some partial results to Hecke's problem. For example, in [BCP02] the authors prove that the number of coefficients that determine a form is less than an unexplicit constant times the sum of the weights using techniques from analysis that do not work over finite fields (see also the references therein for other previous results). In this article we will focus on real quadratic fields only, and prove such a result with an explicit computable constant. Furthermore, our method generalizes easily to fields of positive characteristic, which allows us to solve also Sturm's problem. We also compare in some examples our result with the dimension of the space of modular forms to show how effective it is.

Let K be a real quadratic field and Γ a congruence subgroup of $\mathrm{SL}_2(K)$. Then Γ acts on $\mathfrak{H}^2$, the product of the upper half plane with itself. The quotient $\Gamma\backslash\mathfrak{H}$ is a quasi-projective variety that can be compactified by adding a finite number of cusps. The theory of modular forms can be extended to this variety as recalled in sections 2 and 3. The first main result of the paper is the following.

Theorem 4.1 (Hecke bound). *Let G be a Hilbert modular form of parallel weight $2k$ for the congruence subgroup $\Gamma = \Gamma(\mathcal{O}_K, \mathfrak{a})$. Let c_i, $i = 1, \dots, h$ be the cusps of $\Gamma\backslash\mathfrak{H}$ and suppose that $\mathrm{ord}_{c_i} G \geq s$ for $i = 1, \dots, h$ and $\mathrm{ord}_{c_{i_0}} G \geq r + s$, for some $1 \leq i_0 \leq h$, with*

$$r > \frac{4kn\zeta_K(-1)}{\sum_j (b_{i_0,j} - 2)} - s\left(\frac{\sum_{i=1}^{h} \sum_j (b_{i,j} - 2)}{\sum_j (b_{i_0,j} - 2)}\right).$$

Then G is zero.

In this theorem the numbers $b_{i,j}$ are the self intersection of the components of the cusp desingularization divisors (we recall how to compute such numbers in Appendix A) and the number n needs to be chosen so that surface with level n is minimal and of general type (see Section 2.1 and Summary 2.13). The translation of this result from vanishing orders to vanishing of Fourier coefficients involves a complete knowledge of the cusp desingularization (not only the intersection numbers, but the vertices of the convex hull involved), see Corollary 4.3. The Sturm version of this corollary reads as follows.

Theorem 5.4 (Sturm bound). *Let $\mathcal{O} \subset \mathbb{C}$ be a ring of integers of a number field. Let G be a Hilbert modular form of parallel weight $2k$ for $\Gamma(\mathcal{O}_K, \mathfrak{a})$, which vanishes with order s at all cusps. Suppose that the Fourier expansion of G at the infinity cusp c_1 is*

$$G = \sum_{\xi \in (\mathfrak{a}^{-1})^{\vee}_{+} \cup \{0\}} a_\xi \exp(\xi z_1 + \xi' z_2),$$

with $a_\xi \in \mathcal{O}$ *for all* $\xi \in (\mathfrak{a}^{-1})_+^\vee \cup \{0\}$. *Let* $\mathfrak{p} \subset \mathcal{O}$ *be a prime ideal such that* $\mathfrak{p} \nmid Dn$ *and let* r *be an integer with*

$$r > \frac{4kn\zeta_K(-1)}{\sum_j (b_{1,j}-2)} - s\left(\frac{\sum_{i=1}^h \sum_j (b_{i,j}-2)}{\sum_j (b_{1,j}-2)}\right).$$

If $a_\xi \in \mathfrak{p}$ *for all* $\xi \in (\mathfrak{a}^{-1})_+^\vee \cup \{0\}$ *such that there is a* $j \in J$ *with* $\mathrm{Tr}(\xi A_j) < r+s$, *then* $a_\xi \in \mathfrak{p}$ *for all* $\xi \in (\mathfrak{a}^{-1})_+^\vee \cup \{0\}$.

In this theorem the elements A_j are vertexes representatives of the infinity cusp desingularization.

Once one has a bound for modular forms of parallel weights a general bound follows just by multiplying a form with its conjugate, which is done in Theorem 6.1.

For Hilbert modular forms in $M_{\mathbf{k}}(\Gamma_0(\mathfrak{c},\mathfrak{a}),\chi)$ (i.e. a form for Γ_0 with Nebentypus) one can use Buzzard's trick (as for classical forms) and get the same bound as for $M_{\mathbf{k}}(\Gamma_0(\mathfrak{c},\mathfrak{a}))$ (see Remark 6.3).

Our strategy to prove the Hecke and Sturm type bounds for Hilbert modular forms is an algebro-geometric one. A Hilbert modular form f of parallel weight determines a global section on the Hilbert modular surface X_Γ of a suitable line bundle and, therefore an effective divisor. We then consider a numerically effective (NEF) divisor on X_Γ (which plays the role of the degree function in the proof of the classical Hecke/Sturm theorems given in Section 1) and compute its intersection number with the divisor attached to f. If we prove that this intersection is non positive, then the divisor associated to f is zero and the modular form vanishes.

If many Fourier coefficients of f at a cusp are zero, then the effective divisor associated to f includes many copies of the resolution divisor of such cusp, and at some point its intersection number with the NEF divisor becomes negative and hence the modular form is zero.

The difficulty to follow such strategy is to find a NEF divisor and compute explicitly the intersection numbers of it with different cycles on the Hilbert modular surface. The natural candidate for a NEF divisor is the canonical divisor in the cases where the surface in question is minimal and of general type (hence the assumption of Theorem 4.1).

A particular case of this result was presented as an appendix in [DPS12], where using a similar approach we gave a Sturm/Hecke bound for $K = \mathbb{Q}(\sqrt{5})$, level $\Gamma_0(12\sqrt{5})$ and parallel weight 2.

The proof of our main results is constructive, and depends on the cusp desingularization of the surface X_Γ (which we recall in Appendix A). The result can be thought as a procedure which given K and Γ, constructs the finite set of points to check equality/congruence. We do not include any general bound on the size of such set, since the presently known general bounds for the numbers involved (like class numbers for real quadratic fields and continued fraction lengths) are far from being optimal and would give a cardinality which is far from the real one.

Although we do not compare the number of Fourier coefficients given by the main theorems to the dimension of the space $M_{(k_1,k_2)}$ (since there are no explicit formulas for any of these two quantities) it is important to note that once we fix the congruence subgroup Γ and vary the weights, the rate of growth of both quantities is quadratic, so we do get the correct amount of elements up to a constant (which depends on Γ). See Remark 6.4 and the examples in Section 7.

The article is organized as follows: in the first section we give a proof of the classical Hecke and Sturm theorems that although is not the original one, motivates our generalization to Hilbert modular forms.

In the second section, we recall the main properties and definitions of Hilbert modular surfaces (needed to state clearly the main theorems), their desingularization and their classification. We also give criterions to decide for a particular level, if the given surface is in minimal model and is of general type. Section 2.3 is more technical and can be skipped in a first reading. For this purpose we added a Summary of it at the end which contains all the important facts.

In the third section, we recall the main properties of Hilbert modular forms over real quadratic fields, and we prove the relation between the order of vanishing at a cusp and vanishing of Fourier expansion coefficients.

In the fourth section we state and prove the analogue of Hecke's Theorem for parallel weight $2k$ Hilbert modular forms over real quadratic fields with maximal level structure. The statement is self contained (so there is no need to read the previous sections to understand the statement) but the proof uses the results of the previous sections.

In the fifth section we adapt the proof of the previous section to prove the analogue of Sturm's Theorem for parallel weight $2k$ Hilbert modular forms over real quadratic fields with maximal level structure. The statement is the same in both cases, but the proof in this case uses the integral structure of the modular surfaces.

The sixth section contains statements and proofs for arbitrary weights and levels and some remarks about its effectiveness. The last section contains examples of the method as well as some tables comparing the dimension of the spaces involved and the number of Fourier coefficients needed using our results in each case.

We end the article with two appendices, the first one recalls the "Hirzebruch-Jung continued fraction" method used to compute the cusp desingularization and the second one treats the real quadratic fields not covered by the method described in the previous sections.

1. A geometric proof of Hecke and Sturm theorems

We want to sketch well known proofs of Hecke and Sturm theorems that, although are different than the original proofs of Hecke and Sturm, are generalizable to higher dimensions.

Recall the following facts about divisors. Let C be a curve defined over a field F.

- The group of divisors of C is the free abelian group generated by the closed points of C (so elements are of the form $D = \sum n_p[P]$).
- The divisor D is called effective if $n_P \geq 0$ for all P.
- Let $\mathrm{K}(C)$ be the field of rational functions on C. To a divisor D of C we can associate the (finite dimensional) vector space

$$\mathcal{L}(D) = \{f \in \mathrm{K}(C) \mid \operatorname{div}(f) \geq -D\} \cup \{0\}.$$

- The degree of the divisor $D = \sum n_p[P]$ is defined as

$$\deg(D) = \sum n_P[k(P) : F],$$

where $k(P)$ is the residue field at P. If $\deg(D) < 0$ then $\mathcal{L}(D) = \{0\}$.

We start by proving the Hecke bound for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Choose $N \geq 3$. Then the modular curve $X(\Gamma(N))$ (the compactification of the open curve $\Gamma(N)\backslash\mathfrak{H}$) is a smooth compact complex curve.

Let g be the genus of $X(\Gamma(N))$ and c the number of cusps. Denote the different cusps of $X(\Gamma(N))$ by $\sigma_1, \ldots, \sigma_c$.

Choose a rational differential form ω in $X(\Gamma(N))$ and let $K = \mathrm{div}(\omega)$ be the corresponding canonical divisor. If $f(z) \in M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ is a modular form, then $g = \frac{f(dz)^{\otimes k}}{\omega^{\otimes k}}$ is a well defined rational function on $X(\Gamma(N))$. Since f vanishes with order $\mathrm{ord}(f)$ at the cusp ∞, the function g vanishes with order $N \mathrm{ord}(f)$ at each cusp of $X(\Gamma(N))$. More precisely, g belongs to the space

$$\mathcal{L}\left(k(K + \sum_{i=1}^{c}[\sigma_i]) - \mathrm{ord}(f)N\sum_{i=1}^{c}[\sigma_i]\right).$$

The degree of the divisor $D := k(K + \sum_{i=1}^{c}[\sigma_i]) - \mathrm{ord}(f)N\sum_{i=1}^{c}[\sigma_i]$ is given by $k(2g-2+c) - \mathrm{ord}(f)Nc$. Since, $2g-2+c = Nc/6$, if $\mathrm{ord}(f) > k/6$, then we conclude that $\deg(D) < 0$, hence $f = 0$.

We next prove the Sturm bound for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Let $p = \mathfrak{m} \cap \mathbb{Z}$. Choose N such that $N \geq 3$ and $p \nmid N$. Let ζ_N be a primitive N-th root of unity. Let $F' = F[\zeta_N]$ and $\mathcal{O}'$ the ring of integers of F'. Since $\mathcal{O}'$ is integral over $\mathcal{O}$, there exists a prime ideal $\mathfrak{m}'$ of $\mathcal{O}'$ such that $\mathfrak{m}' \cap \mathcal{O} = \mathfrak{m}$. Hence, if $\mathrm{ord}_{\mathfrak{m}'}(f) = \infty$, then $\mathrm{ord}_{\mathfrak{m}}(f) = \infty$. Thus, replacing $\mathcal{O}$ by $\mathcal{O}'$, we may assume without loss of generality that $\zeta_N \in \mathcal{O}$.

Since $\zeta_N \in \mathcal{O}$, the curve $X(\Gamma(N))$ has an integral smooth model over $S = \mathrm{Spec}(\mathcal{O}[1/N])$, denoted $\mathcal{X}(\Gamma(N))$ and each cusp σ_i of $X(\Gamma(N))$ determines a section $\overline{\sigma}_i \colon S \to \mathcal{X}(\Gamma(N))$, hence a horizontal divisor, also denoted by $\overline{\sigma}_i$. Let $\mathcal{K}$ be the relative canonical divisor of $\mathcal{X}(\Gamma(N))/S$. The q-expansion principle [Kat73, Corollary 1.6.2], implies that f determines a section, also denoted f, of $\mathcal{O}_{\mathcal{X}(\Gamma(N))}(k(\mathcal{K} + \sum \overline{\sigma}_i))$. Let $\mathcal{X}(\Gamma(N))_{\mathfrak{m}}$ be the fiber of $\mathcal{X}(\Gamma(N))$ over $\mathfrak{m}$. It is a smooth curve over the field $k(\mathfrak{m})$. The restriction of $\mathcal{K}$ to this curve agrees with its canonical divisor, denoted $\mathcal{K}_{\mathfrak{m}}$. We denote by $\overline{\sigma}_{i,\mathfrak{m}}$ the restriction of the horizontal divisor $\overline{\sigma}_i$ to $\mathcal{X}(\Gamma)_{\mathfrak{m}}$. Since $\overline{\sigma}_i$ is given by a section, the divisor $\overline{\sigma}_{i,\mathfrak{m}}$ is prime and satisfies $k(\overline{\sigma}_{i,\mathfrak{m}}) = k(\mathfrak{m})$. The hypothesis of the theorem imply that the restriction of f to $\mathcal{X}(\Gamma)_{\mathfrak{m}}$ determines an element of

$$\mathcal{L}\left(k(\mathcal{K}_{\mathfrak{m}} + \sum_{i=1}^{c}[\overline{\sigma}_{i,\mathfrak{m}}]) - \mathrm{ord}_{\mathfrak{m}}(f)N\sum_{i=1}^{c}[\overline{\sigma}_{1,\mathfrak{m}}]\right).$$

By the same argument as before this restriction is zero, thus $\mathrm{ord}_{\mathfrak{m}}(f) = \infty$.

Let now Γ be a congruence subgroup. Any element $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ acts on $M_{2k}(\Gamma)$ by $f \mapsto f|_{2k}[\gamma]$ and the elements $\gamma \in \mathrm{P}\Gamma$ act trivially. Let $f(z)$ be as in Hecke's Theorem. Write

$$g = \prod_{\gamma \in \mathrm{P}\Gamma\backslash\mathrm{PSL}_2(\mathbb{Z})} f|_{2k}[\gamma].$$

Then $g \in M_{2kd}(\mathrm{SL}_2(\mathbb{Z}))$ and $\mathrm{ord}(g) \geq \mathrm{ord}(f)$. Thus, if $\mathrm{ord}(f) > kd/6$ we deduce that $g = 0$ and a fortiori $f = 0$. The same argument proves the Sturm bound.

We want to mimic this geometric proof in the Hilbert setting. For that purpose we need something which looks like the degree function, whose role will be played by a numerically effective divisor (NEF) in our surface, whose intersection number with the cusp resolutions is non-zero.

2. Hilbert modular surfaces

2.1. **Basic definitions and notations.** Let $D > 0$ be a fundamental discriminant, $K = \mathbb{Q}(\sqrt{D})$ the real quadratic field of discriminant D (which we think of inside the real numbers), $\mathcal{O}_K$ its ring of integers and δ the different of $\mathcal{O}_K$. If $\alpha \in K$, we denote by α' its conjugate under the action of the generator of $\mathrm{Gal}(K/\mathbb{Q})$. An element $\alpha \in K$ is called *totally positive* (and denoted $\alpha \gg 0$) if $\alpha > 0$ and $\alpha' > 0$.

If $\mathfrak{a} \subset K$ is a fractional ideal, we denote by $\Gamma(\mathcal{O}_K, \mathfrak{a})$ the image in $\mathrm{PGL}_2^+(K)$ of the group

$$\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a}) = \left\{ m \in \begin{pmatrix} \mathcal{O}_K & \mathfrak{a}^{-1} \\ \mathfrak{a} & \mathcal{O}_K \end{pmatrix} \ : \ \det(m) = 1 \right\}.$$

If $\mathfrak{c}$ is an integral ideal in K, we denote by $\Gamma(\mathfrak{c}, \mathfrak{a})$ the image in $\Gamma(\mathcal{O}_K, \mathfrak{a})$ of the group

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a}) \ : \ \alpha \equiv \delta \equiv 1 \pmod{\mathfrak{c}}, \beta \in \mathfrak{c}\mathfrak{a}^{-1}, \gamma \in \mathfrak{c}\mathfrak{a} \right\}.$$

A *congruence subgroup* $\Gamma_{\mathfrak{a}} \subset \Gamma(\mathcal{O}_K, \mathfrak{a})$ is a subgroup which contains $\Gamma(\mathfrak{c}, \mathfrak{a})$ for some ideal $\mathfrak{c}$.

The group $\mathrm{GL}_2^+(K)$ acts on $\mathfrak{H}^2$ via

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (z_1, z_2) = \left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'} \right).$$

Since the center acts trivially, we can consider the action of $\mathrm{PGL}_2^+(K)$.

If Γ is a congruence subgroup, the quotient $\Gamma\backslash\mathfrak{H}^2$ is a quasi-projective variety with at most quotient singularities. The Baily-Borel compactification of such quotient, which we denote X_Γ, is obtained as in the classical case by adding the cusps $\mathbb{P}^1(K)$ to the product of two copies of the upper half plane, i.e. $X_\Gamma = \Gamma\backslash(\mathfrak{H}^2 \cup \mathbb{P}^1(K))$. It is a projective variety.

We denote by Y_Γ the minimal desingularization of X_Γ and by Z_Γ the surface obtained by resolving only the cusp singularities of X_Γ which we study in the next sections.

2.2. **On Cusp Resolution.** We briefly recall the cusp desingularization at infinity. For this section we follow closely the exposition of [vdG88]. If M is a lattice in K, we denote by U_M^+ the group (under multiplication) of totally positive elements $\epsilon \in K$ such that $\epsilon M = M$. Let $V \subset U_M^+$ be a subgroup of finite index. We define

$$G(M, V) = \left\{ \begin{pmatrix} \epsilon & m \\ 0 & 1 \end{pmatrix} \ : \ \epsilon \in V \ , \ m \in M \right\} = M \rtimes V.$$

If we denote by $U_{\mathcal{O}_K,\mathfrak{c}}$ the set of units of $\mathcal{O}_K$ that are congruent to 1 modulo $\mathfrak{c}$, for the particular congruence subgroups we will consider, we have the following result.

Lemma 2.1. *The isotropy group of the cusp corresponding to* $(\alpha : \beta) \in \mathbb{P}^1(K)$ *in* $\Gamma(\mathfrak{c}, \mathfrak{a})$ *is conjugate to the image in* $\mathrm{PGL}_2^+(K)$ *of*

$$G(\mathfrak{a}^{-1}\mathfrak{b}^{-2}\mathfrak{c}, U_{\mathcal{O}_K,\mathfrak{c}}^2),$$

where $\mathfrak{b} = \alpha\mathcal{O}_K + \beta\mathfrak{a}^{-1}$.

Proof. See the proof of Lemma 5.2 in [vdG88] (p. 78). □

In particular the isotropy of the infinity cusp (corresponding to $(1:0)$ which we denote ∞) equals the image in $\mathrm{PGL}_2^+(K)$ of the group $G(\mathfrak{a}^{-1}\mathfrak{c}, U^2_{\mathcal{O}_K,\mathfrak{c}})$.

We consider the group $\Gamma(\mathfrak{c},\mathfrak{a})$. Let $M = \mathfrak{a}^{-1}\mathfrak{c} \subset K \subset \mathbb{R}$ be the lattice corresponding to the stabilizer of the ∞-cusp. It acts on $\mathbb{C}^2$ by translation, i.e. $m\cdot(z_1,z_2) = (z_1+m, z_2+m')$. A choice of basis $\{\mu_1,\mu_2\}$ of M determines an isomorphism

$$\phi_{\mu_1,\mu_2}\colon M\backslash\mathbb{C}^2 \to \mathbb{C}^\times \times \mathbb{C}^\times, \qquad (z_1,z_2)\mapsto(u,v),$$

where $\exp(2\pi i z_1) = u^{\mu_1}v^{\mu_2}$ and $\exp(2\pi i z_2) = u^{\mu_1'}v^{\mu_2'}$. A different choice of a basis is given by a matrix $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ which induces the biholomorphic map $\psi\colon \mathbb{C}^\times\times\mathbb{C}^\times \to \mathbb{C}^\times\times\mathbb{C}^\times$ given by

$$(u,v)\mapsto(u^a v^b, u^c v^d).$$

We can always choose a basis of M formed by totally positive elements $\mu_1,\mu_2 \gg 0$. In this case, if $\mathrm{Im}(z_1)$ and $\mathrm{Im}(z_2)$ tends to infinity (that is (z_1,z_2) approaches the infinity cusp) then at least one of u or v approaches 0. Thus it is natural to consider the embedding $\mathbb{C}^\times\times\mathbb{C}^\times\subset\mathbb{C}^2$.

The map ψ can be extended to an open subset of $\mathbb{C}^2$ such that its graph inside $\mathbb{C}^2\times\mathbb{C}^2$ is closed. Therefore if we use it to glue together two copies of $\mathbb{C}^2$ we obtain a Hausdorff space.

Let M_+ denote the elements of M which are totally positive, and consider the embedding of M_+ in $(\mathbb{R}_+)^2$, given by

$$m\mapsto(m,m').$$

Denote by $A_j = (A_j^1, A_j^2)$, $j\in\mathbb{Z}$ the vertices of the boundary of the convex hull of the image of M_+, ordered with the condition $A^1_{j+1} < A^1_j$ for all j. Any pair (A_{j-1},A_j) is a basis for M as $\mathbb{Z}$-module (see [vdG88] Lemma 2.1). In Appendix A we describe the how to compute such bases.

Let σ_j denote the cone spanned by A_{j-1} and A_j, i.e.

$$\sigma_j = \{sA_{j-1}+tA_j \;:\; s,t\in\mathbb{R}_+\}.$$

We obtain a partial compactification of $M\backslash\mathbb{C}^2$ by taking a copy of $\mathbb{C}^2$ for each element σ_j and gluing them together in terms of the change of basis matrix (see [vdG88] page 31). By the above comment we obtain a Hausdorff space. Hence we obtain a partial compactification of $M\backslash\mathfrak{H}^2$ denoted Y^+. Then $Y^+ = M\backslash\mathfrak{H}^2 \cup \bigcup_{j\in\mathbb{Z}} S'_{\infty,j}$, where each $S'_{\infty,j}$ is a rational curve. The space Y^+ is a Hausdorff space. The group of units $U^2_{\mathcal{O}_K,\mathfrak{c}}$ acts freely and properly discontinuously on Y^+ ([vdG88] Lemma 3.1 page 34). A local description of the desingularization of the infinity cusp is obtained by taking the quotient of Y^+ by $U^2_{\mathcal{O}_K,\mathfrak{c}}$. Let S_∞ denote the resolution divisor of the infinite cusp and let $\{S_{\infty,j}\}_j$ be its irreducible components. Then there is a one to one correspondence between the set of classes of vertices A_j under the action of $U^2_{\mathcal{O}_K,\mathfrak{c}}$ and the set of irreducible components of the resolution divisor of the infinity cusp, and each irreducible component is a rational curve.

Recall that we denote Z_Γ the desingularization obtained by applying this process to each cusp of X_Γ.

If we apply the previous process to $M=\mathfrak{a}^{-1}$ and $M=\mathfrak{a}^{-1}n$, where n is a positive integer, since the two lattices are homothetic, each choice of basis for $\mathfrak{a}^{-1}$ gives a basis for $\mathfrak{a}^{-1}n$ and we get an holomorphic map between the respective affine spaces given by sending (u,v) to (u^n,v^n). This map is well behaved under gluing which gives a map

$$\pi\colon Z_{\Gamma((n),\mathfrak{a})} \to Z_{\Gamma(\mathcal{O}_K,\mathfrak{a})}.$$

Remark 2.2. Let E be a component of a cusp resolution of $Z_{\Gamma((n),\mathfrak{a})}$ and E' its image under π. It is clear from the above description that the map between E and E' induced by π has degree n and is totally ramified.

2.3. **Algebraic Surfaces.** Projective non-singular algebraic surfaces X with vanishing irregularity, namely with $H^1(X,\mathcal{O}_X)=0$, are divided in four types, one of them being of general type. For reasons that will become clear later, it is this kind of surfaces the ones we need to work with.

Remark 2.3. If $\mathfrak{c} \subsetneq \mathcal{O}_K$ is an integral ideal in $\mathcal{O}_K$ with $\mathfrak{c}^2 \neq (2)$ and $\mathfrak{c}^2 \neq (3)$ then $X_{\Gamma(\mathfrak{c},\mathfrak{a})}$ has no elliptic points (see [vdG88] page 109). In particular, in these cases, the surfaces $Z_{\Gamma(\mathfrak{c},\mathfrak{a})}$ and $Y_{\Gamma(\mathfrak{c},\mathfrak{a})}$ are the same.

Before stating the surface classification, let us give a quick review of genus theory. Inside the narrow class group of K (or strict class group), there is a particular subgroup, called *principal genus*, made of squares (which is clearly well defined since all squares are totally positive). The genus of an ideal can be thought as the class the ideal represents in the quotient of the narrow class group by the principal genus.

The original formulation of genus theory (due to Legendre and Gauss) is the following: there is a bijection (actually an isomorphism) between the narrow class group of $\mathbb{Q}(\sqrt{D})$ and equivalence classes of binary quadratic forms of discriminant D. The genus of a binary quadratic form is the set of values it represents in $(\mathbb{Z}/D)^\times$. Using the Chinese Reminder Theorem, it is enough to understand the elements represented modulo each prime (or prime power) divisor p of D. There is a sign attached to p which tells (when p is odd) whether the form represents squares modulo p or not, so each genus can be represented by a list of signs indexed by primes. See for example [Cox89] for a nice introduction and exposition of the subject.

Theorem 2.4. *The Hilbert modular surface $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ is rational for*

- $D = 5, 8, 12, 13, 17, 21, 24, 28, 33, 60$ *if $\mathfrak{a}$ is in the principal genus.*
- $D = 12$ *if $\mathfrak{a}$ is not in the principal genus.*

Proof. This is Theorem 3.3 of [vdG88], Chapter VII p. 166. □

Theorem 2.5. *The Hilbert modular surface $Y_{\Gamma(\mathfrak{c},\mathfrak{a})}$, with $\mathfrak{c} \neq \mathcal{O}_K$ and $\mathfrak{a}$ in the genus γ is of general type except in the following cases:*

D	$\mathcal{N}\,\mathfrak{c}$	γ	D	$\mathcal{N}\,\mathfrak{c}$	γ
5	$\{4,5\}$	$+$	8	$\{2,4\}$	$+$
12	$\{2,3,4,6\}$	$+,+$	12	$\{2,3\}$	$-,-$
13	$\{3\}$	$+$	17	$\{2\}$	$+$
21	$\{3\}$	$-,-$	24	$\{2\}$	$-,-$
24	$\{3\}$	$+,+$	28	$\{2\}$	$+,+$
28	$\{3\}$	$-,-$	33	$\{2\}$	$-,-$

Furthermore, if $D > 500$, then $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ is of general type as well.

Proof. This is just part of Theorem 3.4 of [vdG88], p. 167, where a general classifications is given. □

Recall the following definition.

Definition 2.6. A smooth surface S is called a *minimal surface* if for any smooth surface S', any morphism $S \to S'$ that is birational is an isomorphism.

From Castelnuovo's contractibility theorem, a minimal model of a smooth surface can be obtained by contracting exceptional curves, i.e. rational curves with self intersection number -1. We have the following result.

Proposition 2.7. *Assume that $Y_{\Gamma(\mathfrak{c},\mathfrak{a})}$ is of general type and that $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ is not rational. If $\mathcal{N}\,\mathfrak{c} \geq C$, with*

$$C = 3\left(\sum_{i=1}^{h}\sum_{j}(b_{i,j}-2)\right),$$

where the first sum is over ideal class representatives $[\mathfrak{b}_i]$ of $\mathcal{O}_K$ and the $b_{i,j}$ are the self-intersection numbers of the components of the cusp desingularization at $\mathfrak{b}_i$ (see Appendix A), then $Y_{\Gamma(\mathfrak{c},\mathfrak{a})}$ is minimal.

Proof. The statement corresponds to the first case of Theorem 7.19 of [vdG88], p. 184. □

Remark 2.8. This gives an effective bound for the level $\mathfrak{c}$ needed for the Hecke/Sturm bounds (of sections 4 and 5). It will become clear that the smaller $\mathcal{N}\,\mathfrak{c}$ we take, the better the bound gets, so we will say a few more words on how to improve this norm.

Recall the definition of the Hirzebruch-Zagier cycles (which correspond to the modular curves inside the Hilbert modular surfaces). A matrix B in $M_2(K)$ is called skew-hermitian if $B^t = -B'$, where the superscript t means the transpose. Let $\mathfrak{a} \in \mathcal{O}_K$ be an ideal of norm A. A skew-hermitian form B is called integral with respect to $\mathfrak{a}$ if it is of the form

$$B = \begin{pmatrix} a\sqrt{D} & \lambda \\ -\lambda' & \frac{b}{A}\sqrt{D} \end{pmatrix},$$

with $a, b \in \mathbb{Z}$ and $\lambda \in \mathfrak{a}^{-1}$. The integral form B is called primitive if it is not divisible by a natural number greater than 1, i.e. if B is not of the form $m\tilde{B}$, with $\tilde{B}$ integral with respect to $\mathfrak{a}$ and $m > 1$. If we denote by $\mathcal{C}(N)$ the set of skew-hermitian, integral with respect to $\mathfrak{a}$, primitive matrices of determinant N/A, then the cycle F_N is defined by

$$F_N = \bigcup_{B \in \mathcal{C}(N)} \left\{(z_1, z_2) \in \mathfrak{H}^2 \cup \mathbb{P}^1(K) : (z_2 \quad 1)B\begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0\right\}. \tag{1}$$

Abusing the notation, we will also denote by F_N the divisor on any modular surface obtained as the closure of the image of F_N. It will be clear from the context in which surface we consider the curves F_N.

The following conjecture is stated as Conjecture (7.13) in [vdG88].

Conjecture 2.9. *If $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ is not rational, then the canonical divisor can be written as a rational positive linear combination of resolution curves and the divisors F_N.*

Remark 2.10. When $\mathfrak{a}$ is in the genus of $\mathcal{O}_K$ or $(\sqrt{D})$, this conjecture is known in the following cases

(1) When $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ is not of general type.
(2) [Her87, Her89] When $D \equiv 1 \pmod 8$ and either
   (a) there is a divisor a of D with $a \not\equiv 1 \pmod 8$;
   (b) there are two integers $n, m > 0$ with $m \equiv 7 \pmod 8$ and $D = (m^2 - 8)/n^2$.
(3) [Fre03] When $D \not\equiv 1 \pmod 8$.

Assume now that $Y_{\Gamma(\mathfrak{c},\mathfrak{a})}$ is of general type and $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ is not rational. We want to improve our criterion for minimallity of $Y_{\Gamma(\mathfrak{c},\mathfrak{a})}$ assuming that Conjecture 2.9 is true for $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$. By Proposition 7.18 of [vdG88] (p. 183), if E is an exceptional curve in $Y_{\Gamma(\mathfrak{c},\mathfrak{a})}$, then its image in $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ is also exceptional. If Conjecture 2.9 is true for $Y_{\Gamma(\mathfrak{c},\mathfrak{a})}$, the exceptional curves in this surface are components of the divisors F_N. Therefore any exceptional curve in $Y_{\Gamma(\mathfrak{c},\mathfrak{a})}$ is also a component of a divisor F_N. If for example $6 \mid \mathfrak{c}$, then the components of the curves F_N have genus greater than 1 (see for example [Shi94], formula (1.6.4), page 23) and are therefore not exceptional, hence the surface $Y_{\Gamma(\mathfrak{c},\mathfrak{a})}$ is minimal for this level. Actually we can do a little better.

Theorem 2.11. *Assume that $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ is not rational and Conjecture 2.9 is true for this surface. If $n \geq 3$ is an integer and $Y_{\Gamma((n),\mathfrak{a})}$ is of general type, then $Y_{\Gamma((n),\mathfrak{a})}$ is minimal.*

Proof. Recall that $Z_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ is the resolution of the cusps of $X_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ but without resolving the elliptic points which is a $\mathbb{Q}$-variety. By Remark 2.3, $Y_{\Gamma((n),\mathfrak{a})}$ agrees with $Z_{\Gamma((n),\mathfrak{a})}$ and hence we get the following diagram

$$\begin{array}{ccc} & & Y_{\Gamma((n),\mathfrak{a})} \\ & & \big\downarrow{\scriptstyle \pi} \\ Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})} & \xrightarrow{\ f\ } & Z_{\Gamma(\mathcal{O}_K,\mathfrak{a})}, \end{array}$$

where f is the resolution at the elliptic points.

We need to show that there are no exceptional curves on $Y_{\Gamma((n),\mathfrak{a})}$. Assume that there is such an exceptional curve A. Let C' be its image in $Z_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ and C the strict transform of C' in $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$. As we mentioned previously, by Proposition 7.18 of [vdG88], the curve C is exceptional. By Theorem 7.11 of [vdG88] (p. 181), C (hence A) is a component of a divisor F_N for $N = 1, 2, 3$ or 4 (and 9 if $3 \mid D$).

We will show that $A \cdot A < -1$ contradicting the assumption. To this end we start by computing the self-intersection of C'. We have the relation

$$C' \cdot C' = f^*(C') \cdot f^*(C').$$

Using the desingularization of the components of F_N given in [vdG88] (page 169), we get the following cases:

• **The case** $N = 1$**:** The curve C' goes through an elliptic point of order 2 and an elliptic point of order 3. While computing the desingularization at the order 2 point, we get a $\mathbb{P}^1$ with self-intersection -2 and while computing the desingularization of elliptic point of the order 3 we get a $\mathbb{P}^1$ with self-intersection -3 (see Figure (2) in [vdG88], page 169). Let E_2 and E_3 be these two exceptional divisors. We can write $f^*(C') = C + aE_2 + bE_3$. Since $f^*(C') \cdot E_2 = f^*(C') \cdot E_3 = 0$, we get

$$f^*(C') = C + \frac{1}{2}E_2 + \frac{1}{3}E_3.$$

Therefore

$$f^*(C') \cdot f^*(C') = C \cdot C + C \cdot E_2 + \frac{2}{3}C \cdot E_3 + \frac{1}{4}E_2 \cdot E_2 + \frac{1}{9}E_3 \cdot E_3 = -\frac{1}{6}.$$

• **The case** $N = 2$**:** The curve C' goes through an elliptic point of order 2. While computing the desingularization at the order 2 point, we get a $\mathbb{P}^1$ with self-intersection -2 (see Figure (3) in [vdG88], page 169). Let E_2 be the exceptional divisor, so $f^*(C') = C + aE_2$. Since $f^*(C') \cdot E_2 = 0$, we get that $a = \frac{1}{2}$, and

$$f^*(C') \cdot f^*(C') = C \cdot C + C \cdot E_2 + \frac{1}{4}E_2 \cdot E_2 = -\frac{1}{2}.$$

• **The case** $N = 3$**:** Since $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ is not rational, $D \neq 12$. Then the curve C' goes through an elliptic point of order 3. While computing the desingularization at the order 3 point, we get a $\mathbb{P}^1$ with self-intersection -3. Let E_3 be the exceptional divisor, then $f^*(C') = C + bE_3$. Since $f^*(C') \cdot E_3 = 0$, we get that $b = \frac{1}{3}$, and

$$f^*(C') \cdot f^*(C') = C \cdot C + \frac{2}{3}C \cdot E_3 + \frac{1}{9}E_3 \cdot E_3 = -\frac{2}{3}.$$

• **The case** $N = 4$**:** Since $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ is not rational, $D \neq 8$. If $2 \mid D$, then the situation is the same as the case $N = 2$. If $D \equiv 1 \pmod 8$ then C' does not go through any elliptic point, hence the self intersection is -1. If $D \equiv 5 \pmod 8$ then the curve C' goes through two elliptic points of order 3. While computing the desingularization at the two order 3 points, we get two copies of $\mathbb{P}^1$ with self-intersection -3. Let E_3 and E_3' be the exceptional divisors. Then $f^*(C') = C + \frac{1}{3}E_3 + \frac{1}{3}E_3'$, and

$$f^*(C') \cdot f^*(C') = C \cdot C + \frac{2}{3}C \cdot (E_3 + E_3') + \frac{1}{9}(E_3 \cdot E_3 + E_3' \cdot E_3') = -\frac{1}{3}.$$

• **The case** $N = 9$**:** Again we use $D \neq 12$. If $3 \nmid D$, then the curve C' does not go through any elliptic point. If $3 \mid D$, and $D \neq 105$, then C' goes through an elliptic point of order 3, so the blow up gives a $\mathbb{P}^1$ with self intersection number -3 (see the first Figure of [vdG88] page 170), so we are in the same situation as the case $N = 3$.

If $D = 105$, the picture is similar, but in this case some components are not disjoint any more. Even though, the same computation applies.

Let g denote the degree of π and d the degree of the morphism induced by π between the modular curve A and its image C'. Since the morphism π is not ramified over C', the preimage of C' consists on $c = g/d$ curves which are translates of A,

$$\pi^*(C') \cdot \pi^*(C') = gC' \cdot C' \qquad \text{and} \qquad \pi^*(C') \cdot \pi^*(C') \geq cA \cdot A.$$

Therefore

$$A \cdot A \leq dC' \cdot C'.$$

Note that $d = [\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(n)]$, where $\Gamma(n)$ is the classical congruence subgroup. Since $n \geq 3$, $d > 6$ and $A \cdot A < -1$. Thus A is not exceptional. □

Remark 2.12. It is clear that if $Y_{\Gamma((n),\mathfrak{a})}$ is a minimal surface of general type and m is a positive integer, then $Y_{\Gamma((mn),\mathfrak{a})}$ is also a minimal surface of general type.

Summary 2.13. *The following holds:*

- *If* $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ *is not rational and*
$$n \geq \sqrt{3\left(\sum_{i=1}^{h}\sum_{j}(b_{i,j}-2)\right)},$$
then $Y_{\Gamma((n),\mathfrak{a})}$ *is a minimal surface of general type.*
- *If* $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ *is not rational and satisfies Conjecture 2.9 (see Remark 2.10) then* $Y_{\Gamma((n),\mathfrak{a})}$ *is a minimal surface of general type for* $n \geq 3$.
- *If* $D = 5, 8, 12, 13, 17, 21, 24, 28, 33, 60$ *and* $\mathfrak{a}$ *is in the principal genus or* $D = 12$ *and* $\mathfrak{a}$ *is not in the principal genus, then* $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ *is rational so this section results do not apply. These cases will be treated separately in Appendix B.*

3. Hilbert modular forms

In this section we recall the definition and basic properties of Hilbert modular forms.

Definition 3.1. Let $\Gamma_{\mathfrak{a}}$ be a congruence subgroup, and k_1 and k_2 be integers such that $k_1 \equiv k_2 \pmod 2$. A holomorphic function $G\colon \mathfrak{H}^2 \to \mathbb{C}$ is called a *Hilbert modular form* of weight $\mathbf{k} = (k_1, k_2)$ for the group $\Gamma_{\mathfrak{a}}$ if for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\mathfrak{a}}$ one has, for each $\mathbf{z} = (z_1, z_2) \in \mathfrak{H}^2$,

$$G(\gamma\mathbf{z}) = (cz_1 + d)^{k_1}(c'z_2 + d')^{k_2}G(\mathbf{z}). \tag{2}$$

If k is an integer and G is a modular form of weight $\mathbf{k} = (k, k)$, we will call it a modular form of parallel weight k. We will denote by $M_{\mathbf{k}}(\Gamma_{\mathfrak{a}})$ the space of all modular forms of weight $\mathbf{k}$ and by $M_k(\Gamma_{\mathfrak{a}})$ the space of all modular forms of parallel weight k.

Let G be a Hilbert modular form of weight (k_1, k_2). It admits a Fourier expansion in each cusp. Since all the cusps are conjugate to the infinity cusp (possibly altering the ideals) by an element of $\mathrm{PSL}_2(K)$, we will just recall the case of the infinity cusp. Since $\Gamma_{\mathfrak{a}}$ is a congruence group, the isotropy group of the cusp $(1 : 0)$ contains some $G(M, V)$ (since for example for $\Gamma(\mathfrak{c}, \mathfrak{a})$ it equals $G(\mathfrak{a}^{-1}\mathfrak{c}, U^2_{\mathcal{O}_K,\mathfrak{c}})$). The modularity condition implies that, if $m \in M$ and $\epsilon \in U_{\mathcal{O}_K,\mathfrak{c}}$ then

$$G(z_1 + m, z_2 + m') = G(z_1, z_2), \tag{3}$$

$$G(\epsilon^2 z_1, {\epsilon'}^2 z_2) = \epsilon^{-k_1}{\epsilon'}^{-k_2}G(z_1, z_2). \tag{4}$$

The periodicity condition (3) implies that G admits the Fourier expansion

$$G = \sum_{\xi \in M^\vee} a_\xi \exp(2\pi i(\xi z_1 + \xi' z_2)),$$

where $M^\vee$ is the set of $\xi \in K$ such that $\mathrm{Tr}(m\xi) \in \mathbb{Z}$ for all $m \in M$. Let $M^\vee_+$ denote the set of totally positive elements of $M^\vee$. Then the holomorphicity of G implies that the only non-zero coefficients a_ξ of the above expansion are a_0 and a_ξ with $\xi \in M^\vee_+$. Hence

$$G = \sum_{\xi \in M^\vee_+ \cup \{0\}} a_\xi \exp(2\pi i(\xi z_1 + \xi' z_2)).$$

The modularity equation (4) implies that the coefficients of the Fourier expansion satisfy the condition

$$a_{\xi\epsilon^2} = \epsilon^{k_1}\epsilon'^{k_2} a_\xi \quad \text{for all} \quad \epsilon \in U_{\mathcal{O}_K,\mathfrak{c}}. \tag{5}$$

In particular, if G is of parallel weight $2k$ then $a_{\xi\epsilon^2} = a_\xi$.

By means of the Fourier expansion, we see that every modular form determines a holomorphic function in an analytic neighborhood of each cusp.

Definition 3.2. (1) A Hilbert modular form G is called a cusp form if, for each cusp, the coefficient a_0 of the Fourier expansion of G is zero. We denote by $S_{\mathbf{k}}(\Gamma) \subset M_{\mathbf{k}}(\Gamma)$ the space of modular cusp forms of weight $\mathbf{k}$ and by $S_{2k}(\Gamma) \subset M_{2k}(\Gamma)$ the space of modular cusp forms of parallel weight $2k$.

(2) Let G be a Hilbert modular form of parallel weight $2k$ for the group $\Gamma(\mathfrak{c},\mathfrak{a})$ and c_i a cusp of $X_{\Gamma(\mathfrak{c},\mathfrak{a})}$. Let S_i be the resolution divisor of c_i in $Y_{\Gamma(\mathfrak{c},\mathfrak{a})}$. The modular form G determines a holomorphic function f in an analytic neighborhood U_i of S_i (as explained in [vdG88], Chapter II, section 1). We say that G vanishes with order at least a at the cusp c_i if the divisor $\operatorname{div}(f) - aS_i$ is effective in U_i. We will write $\operatorname{ord}_{c_i} G = a$ if G vanishes at the cusp c_i with order a but does not vanish with order $a+1$.

The vanishing of a Hilbert modular form at a cusp can be read from the Fourier expansion. For simplicity we will treat only the case of the infinity cusp. Let $\{A_j\}_{j\in J}$ be a set of representatives under the action of V, of the corners of the convex hull of M_+ (see Section 2.2).

Lemma 3.3. *Let G be a modular form of parallel weight $2k$ for a congruence subgroup $\Gamma_{\mathfrak{a}}$ and*

$$G = \sum_{\xi\in M_+^\vee\cup\{0\}} a_\xi \exp(2\pi i(\xi z_1 + \xi' z_2)),$$

its Fourier expansion at the infinity cusp. Then

$$\operatorname{ord}_{c_1} G = \inf\{\operatorname{Tr}(\xi A_j) \mid j \in J, a_\xi \neq 0\}.$$

Thus, G vanishes with order a at the infinity cusp if and only if $a_\xi = 0$ for all $\xi \in M_+^\vee \cup \{0\}$ such that there is a $j \in J$ with $\operatorname{Tr}(\xi A_j) < a$.

Proof. By (5), the vanishing condition for the coefficients of the Fourier expansion is equivalent to the condition $a_\xi = 0$ for all $\xi \in M_+^\vee \cup \{0\}$ such that there is a $j \in \mathbb{Z}$ with $\operatorname{Tr}(\xi A_j) < a$. Let A_j, A_{j+1} be a totally positive basis of M as in Section 2.2. To this basis, there is associated a local analytic chart of a piece of the cusp resolution. Let u, v be the local coordinates of this chart. The divisors $u = 0$ and $v = 0$ correspond to components of the cusp resolution divisor. With these coordinates, the Fourier expansion of G, is given by

$$G(u,v) = \sum_{\xi\in M_+^\vee\cup\{0\}} a_\xi u^{\operatorname{Tr}(\xi A_j)} v^{\operatorname{Tr}(\xi A_{j+1})}.$$

Thus, the lemma follows directly from the definition of order of vanishing at a cusp. ☐

From the lemma, it is clear that a modular form is a cusp form if and only if it vanishes at each cusp with order one.

Let $G \in M_2(\Gamma)$ be a modular form of parallel weight 2. Then $\omega_G = G dz_1 \wedge dz_2$ is a Γ-invariant differential form on $\mathfrak{H}^2$. Thus, it defines a differential form on $\Gamma\backslash\mathfrak{H}^2$, hence on an open subset of Y_Γ. It can be seen ([vdG88, Ch 3. §3]) that ω_G can be extended to a differential form on Y_Γ that is regular on the resolution divisors of the elliptic fixed points and has at most logarithmic poles at the resolution divisors of the cusps. This gives us the identifications

$$S_2(\Gamma) \xrightarrow{\simeq} H^0(Y_\Gamma, \mathcal{O}(K_{Y_\Gamma})), \qquad M_2(\Gamma) \xrightarrow{\simeq} H^0(Y_\Gamma, \mathcal{O}(K_{Y_\Gamma} + S)),$$

where K_{Y_Γ} is the canonical divisor of Y_Γ and $S = \sum S_i$ is the sum of the resolution divisors of all the cusps.

From the above identifications one can derive the following result.

Proposition 3.4. *Let Γ be a congruence subgroup, $\{c_1, \dots, c_h\}$ the set of cusps of X_Γ, S_i the resolution divisor of c_i on Y_Γ and $S = \sum S_i$. Fix integers $1 \le i_0 \le h$, $a, s \ge 0$. Then we can identify the space of all modular forms for Γ of parallel weight $2k$, vanishing order at least s at all the cusps and at least $a + s$ at the cusp i_0 with the space of global sections $H^0(Y_\Gamma, \mathcal{O}(kK_{Y_\Gamma} + (k - s)S - aS_{i_0}))$.*

4. Hecke Bound

In this section we will derive a Hecke type bound for Hilbert modular forms for the group $\Gamma(\mathcal{O}_K, \mathfrak{a})$. We will assume that $D > 0$ is such that $Z := Z_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ is not rational. Choose $n \ge 3$ satisfying the conditions of Summary 2.13, so $Z_n := Z_{\Gamma((n),\mathfrak{a})}$ is a minimal surface of general type.

Let S be the cusp resolution divisor on Z. We order the cusps of $X_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ as c_i, $i = 1, \dots, h$ and we decompose S as

$$S = \sum_{i=1}^{h} S_i,$$

where S_i is the resolution divisor over the cusp c_i.

For each $i = 1, \dots, h$, let $b_{i,j} \ge 2$ be the integers that appear while computing the desingularization of the cusp c_i of $X_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ as explained in Appendix A where, following the Appendix notation, $b_{i,j}$ is the b_j obtained while computing the desingularization of the cusp c_i. Let $1 \le i_0 \le h$.

Theorem 4.1 (Hecke bound)**.** *With the previous hypothesis on D and n, let G be a Hilbert modular form of parallel weight $2k$ for $\Gamma(\mathcal{O}_K, \mathfrak{a})$ and suppose that $\operatorname{ord}_{c_i} G \ge s$ for $i = 1, \dots, h$ and $\operatorname{ord}_{c_{i_0}} G \ge r + s$, with*

$$r > \frac{4kn\zeta_K(-1)}{\sum_j (b_{i_0,j} - 2)} - s\left(\frac{\sum_{i=1}^{h}\sum_j (b_{i,j} - 2)}{\sum_j (b_{i_0,j} - 2)}\right). \tag{6}$$

Then G is zero.

Before proving the theorem we recall some known results. Since, by hypothesis n is big enough (see Summary 2.13), Z_n does not have elliptic points, hence it is already smooth. Let $\pi\colon Z_n \to Z$ be the projection and d its degree. Let c' be the number of cusps of $X_{\Gamma((n),\mathfrak{a})}$ that are over a cusp of $X_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$. By [vdG88, Lemma 5.2, Chapter IV] and its proof

$$d = n^2 c' [U^2_{\mathcal{O}_K} : U^2_{\mathcal{O}_K,(n)}]. \tag{7}$$

For each $i' = 1, \cdots, hc'$, let $b'_{i',j}$ be the integers that appear in the cusp desingularization process of $X_{\Gamma((n),\mathfrak{a})}$. Let $c_{i'}$ be a cusp of $X_{\Gamma((n),\mathfrak{a})}$ over a cusp c_i of

$X_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$. Then the sequence $(b'_{i',j})_j$ is a repetition of $[U^2_{\mathcal{O}_K} : U^2_{\mathcal{O}_K,(n)}]$ times the sequence $(b_{i,j})_j$. Therefore

$$\sum_{i'=1}^{c'h}\sum_j (2-b'_{i',j}) = \frac{d}{n^2}\sum_{i=1}^{h}\sum_j (2-b_{i,j}). \tag{8}$$

Let S' be the cusp resolution divisor of Z_n and, for $i=1,\dots h$, let S'_i be the sum of the resolution divisor of all the cusps on Z_n over c_i. By Remark 2.2 we have

$$\pi^*(S) = nS' \qquad \text{and} \qquad \pi^*(S_i) = nS'_i. \tag{9}$$

By the geometry of the cusp resolutions (see Appendix A) we have

$$S_i \cdot S_l = \begin{cases} \sum_j (2-b_{i,j}) & \text{if } i = l, \\ 0 & \text{if } i \neq l. \end{cases}$$

From this, using equation (9), we deduce

$$S'_i \cdot S'_l = \begin{cases} \frac{d}{n^2}\sum_j (2-b_{i,j}) & \text{if } i = l, \\ 0 & \text{if } i \neq l. \end{cases} \tag{10}$$

Proposition 4.2. *For each i there is a j with $b_{i,j} > 2$. In particular the denominators in Theorem 4.1 are different from zero.*

Proof. We are led to prove that $S_i \cdot S_i < 0$. Let X be a non-singular projective surface and let r be the rank of its Neron-Severi group. The intersection product gives a pairing on the Neron-Severi group with signature $(1, r-1)$ by the Hodge index theorem. The claim follows from the following.

Claim: let Y be a projective surface and $f : X \to Y$ be a dominant morphism with X a non-singular surface. Let $p \in Y$ and let $D = f^{-1}(p)$. If E is a divisor on X whose support is contained in D then $E \cdot E < 0$.

Let H' be an ample divisor of Y not going through p. Let H be the inverse image of H' under f. Then

$$H \cdot H > 0 \qquad \text{and} \qquad H \cdot E = 0. \tag{11}$$

In particular H and E are independent as elements in the Neron-Severi group. If $E \cdot E \geq 0$ then on the 2-dimensional subspace $\langle H, E\rangle$ the intersection product is not negative, but a form of signature $(1, r-1)$ cannot have such a subspace. □

Let K_{Z_n} be the canonical divisor of Z_n. Since each divisor S_i is a cycle of rational curves, it has arithmetic genus 1. Then the adjunction formula implies that

$$(K_{Z_n} + S'_i) \cdot S'_i = 0, \quad (K_{Z_n} + S') \cdot S' = 0. \tag{12}$$

Therefore

$$K_{Z_n} \cdot S'_i = \frac{d}{n^2}\sum_j (b_{i,j} - 2), \qquad K_{Z_n} \cdot S' = \frac{d}{n^2}\sum_{i=1}^{h}\sum_j (b_{i,j} - 2). \tag{13}$$

Moreover, by [vdG88, Chapter IV, Theorem 2.5] (page 64), [vdG88, Chapter IV, Theorem 1.1], (pp. 59) and equation (8),

$$K_{Z_n} \cdot K_{Z_n} = 2\operatorname{Vol}(Z_n) + \frac{d}{n^2}\sum_{i=1}^{h}\sum_j (2-b_{i,j}) = 4d\zeta_K(-1) + \frac{d}{n^2}\sum_{i=1}^{h}\sum_j (2-b_{i,j}). \tag{14}$$

Proof of Theorem 4.1. Since G is a Hilbert modular form of parallel weight $2k$ that vanishes with order s at every cusp and with order $r+s$ at the cusp c_{i_0}, by Proposition 3.4, it determines a global section of $\mathcal{O}(k(K_{Z_n}+S')-snS'-rnS'_{i_0})$. Since Z_n is a minimal surface of general type, K_{Z_n} is NEF. Hence, if $G\neq 0$, the intersection number $K_{Z_n}\cdot(k(K_{Z_n}+S')-snS'-rnS'_{i_0})$ must be non-negative. If we prove that this number is negative, we are done. Using equations (13) and (14), we obtain

$$(15)\quad K_{Z_n}\cdot(k(K_{Z_n}+S')-snS'-rnS'_{i_0}) = d\left(4k\zeta_k(-1)+\frac{s}{n}\sum_{i=1}^{h}\sum_{j}(2-b_{i,j})+\frac{r}{n}\sum_{j}(2-b_{i_0,j})\right)$$

proving the Theorem. □

By virtue of Lemma 3.3, we can state the same result in terms of Fourier expansions. For simplicity we will treat only the case of the infinity cusp. Assume that we have numbered the cusps in such a way that the infinity cusp is c_1. The lattice corresponding to the isotropy group of the infinity cusp is $M=\mathfrak{a}^{-1}$ and the group of units V equals $U^2_{\mathcal{O}_K}$. Let $\{A_j\}_{j\in J}$ be a set of representatives under the action of $U^2_{\mathcal{O}_K}$ of the corners of the convex hull of $(\mathfrak{a}^{-1})_+$.

Corollary 4.3. *With the same hypothesis on D and n, let G be a Hilbert modular form of parallel weight $2k$ for $\Gamma(\mathcal{O}_K,\mathfrak{a})$ which vanishes with order s at all the cusps. Let r be an integer with*

$$r>\frac{4kn\zeta_K(-1)}{\sum_j(b_{1,j}-2)}-s\left(\frac{\sum_{i=1}^{h}\sum_j(b_{i,j}-2)}{\sum_j(b_{1,j}-2)}\right).$$

Suppose that the Fourier expansion of G at the infinity cusp is

$$G=\sum_{\xi\in(\mathfrak{a}^{-1})^\vee_+\cup\{0\}} a_\xi\exp(2\pi i(\xi z_1+\xi' z_2)).$$

If $a_\xi=0$ for all $\xi\in(\mathfrak{a}^{-1})^\vee_+\cup\{0\}$ such that there is a $j\in J$ with $\operatorname{Tr}(\xi A_j)<r+s$, then $G=0$.

Remark 4.4. Although both Theorem 4.1 and Corollary 4.3 are stated for forms vanishing with order s at all cusps, the two usual cases are $s=0$ for a general Hilbert modular form and $s=1$ for a cusp form.

Remark 4.5. The bound we got in Theorem 4.1 relies on the choice of an auxiliary positive integer n such that Z_n is a minimal surface of general type, and there is a dependence of n in the formula. We can think of this dependence in a somehow different way. We need to construct a NEF divisor in some surface. What we did was to start with a parallel weight $2k$ Hilbert modular form G for $\Gamma(\mathcal{O}_K,\mathfrak{a})$ and considered its pullback to Z_n, where we can identify a NEF divisor, namely the canonical divisor. But we can do the opposite, recall the following result concerning NEF divisors under maps.

Lemma 4.6. *Let $\pi: X\to Y$ be a surjective generically finite map between surfaces. Let $\mathcal{D}\subset Y$ be a Cartier divisor. Then $\mathcal{D}$ is a NEF divisor if and only if $\pi^*(\mathcal{D})$ is a NEF divisor.*

This implies that we can do the computations in "level 1". Take any (rational) divisor $\mathcal{D}$ in Z_1 whose pullback to Z_n is the canonical divisor and compute the intersection numbers with it (which of course gives the same bound). Thus, the dependence on n does not come from where we compute the intersection numbers but from where we can identify a NEF divisor.

Thus, there are two ways for getting a better bound in some particular cases:

(1) If one can compute the cone of NEF divisors, one can make the same computations for each generator of the NEF cone to get the best bound.
(2) If $Y_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ is of general type (which happens for example if $D > 500$), one can compute its minimal model, and take as NEF divisor any divisor $\mathcal{D}$ in $Z_{\Gamma(\mathcal{O}_K,\mathfrak{a})}$ whose pullback to the minimal model is the canonical divisor to get a bound with "$n = 1$".

5. Sturm bound

To make the computation of the previous section work over a finite field, we need to use the integral structure of the Hilbert modular surface. Such structure comes from their moduli interpretation and has been developed in [Rap78], [Cha90] and [Pap95], see also the book [Gor02].

Let $D > 0$ be a fundamental discriminant. Let $\mathfrak{a}$ be a fractional ideal, $n \geq 3$ a positive integer and ζ_n a primitive n-th root of unity. Consider the modular surface $Y_{\Gamma((n),\mathfrak{a})}$ and let S' be the cusp resolution. The first input we need is the existence of a nice regular model of $Y_{\Gamma((n),\mathfrak{a})}$.

Theorem 5.1. *There exist a smooth, proper scheme $\mathcal{Y}_{\Gamma((n),\mathfrak{a})}$ over $\mathbb{Z}[1/(Dn),\zeta_n]$, such that*

$$\mathcal{Y}_{\Gamma((n),\mathfrak{a})} \underset{\mathbb{Z}[1/(Dn),\zeta_n]}{\times} \operatorname{Spec}(\mathbb{C}) = Y_{\Gamma((n),\mathfrak{a})}.$$

In particular $\mathcal{Y}_{\Gamma((n),\mathfrak{a})}$ is regular and flat over $\mathbb{Z}[1/(Dn),\zeta_n]$. Moreover, there is a relative normal crossing divisor $\mathcal{S}'$ of $\mathcal{Y}_{\Gamma((n),\mathfrak{a})}$ whose restriction to $Y_{\Gamma((n),\mathfrak{a})}$ is S'.

Proof. See [Cha90] Theorem 3.6, [Rap78] Théorème 5.1 and Corollaire 5.3. and [Pap95] Theorem 2.1.2. □

The second input we need is the q-expansion principle. Let K be the canonical divisor of $Y_{\Gamma((n),\mathfrak{a})}$ and let $\mathcal{K}$ be the relative canonical divisor of $\mathcal{Y}_{\Gamma((n),\mathfrak{a})}$. Let R be a subalgebra of $\mathbb{C}$ that contains $\mathbb{Z}[1/(Dn),\zeta_n]$. We will denote by $\mathcal{Y}_{\Gamma((n),\mathfrak{a}),R}$, $\mathcal{K}_R$ and $\mathcal{S}'_R$ the objects obtained after extending scalars to R. We know that a modular form of parallel weight $2k$ determines a section of $\mathcal{O}_{Y_{\Gamma((n),\mathfrak{a})}}(k(K+S'))$.

Theorem 5.2. *Let G be a Hilbert modular form of parallel weight $2k$ for $\Gamma((n),\mathfrak{a})$, and let*

$$G = \sum_{\xi \in (n\mathfrak{a}^{-1})^{\vee}_{+} \cup \{0\}} a_\xi \exp(2\pi i(\xi z_1 + \xi' z_2)),$$

be its Fourier expansion at a cusp. Then the form G determines a section of $\mathcal{O}_{\mathcal{Y}_{\Gamma((n),\mathfrak{a})},R}(k(\mathcal{K}_R + \mathcal{S}'_R))$ if and only if $a_\xi \in R$ for all $\xi \in M^{\vee}_{+} \cup \{0\}$.

Proof. See [Cha90] Theorem 4.3 and [Rap78] Théorème 6.7. □

Finally we need to know that the fibers of $\mathcal{Y}_{\Gamma((n),\mathfrak{a})}$ are also minimal surfaces.

Proposition 5.3. *Let $\mathcal{O}$ be a Dedekind domain contained in $\mathbb{C}$ that contains $\mathbb{Z}[1/(Dn), \zeta_n]$. Let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal and let $\overline{k(\mathfrak{p})}$ be an algebraic closure of the residue field $k(\mathfrak{p})$. Denote $\mathcal{Y}_{\Gamma((n),\mathfrak{a}),\overline{k(\mathfrak{p})}} = \mathcal{Y}_{\Gamma((n),\mathfrak{a})} \underset{\mathbb{Z}[1/(Dn),\zeta_n]}{\times} \overline{k(\mathfrak{p})}$. If $Y_{\Gamma((n),\mathfrak{a})}$ is a minimal surface of general type then the same is true for $\mathcal{Y}_{\Gamma((n),\mathfrak{a}),\overline{k(\mathfrak{p})}}$.*

Proof. This follows from [KU85] Theorem 9.1 and Lemma 9.6. We would like to thank Professor Qing Liu for answering our question in mathoverflow. (`http://mathoverflow.net/questions/70942`) □

We now assume that D and n satisfy furthermore the hypothesis of the previous section and we use the notations of that section. Again, for simplicity we state the result for the infinity cusp.

Theorem 5.4 (Sturm bound)**.** *Let $\mathcal{O} \subset \mathbb{C}$ be a ring of integers of a number field. Let G be a Hilbert modular form of parallel weight $2k$ for $\Gamma(\mathcal{O}_K, \mathfrak{a})$, which vanishes with order s at all cusps. Suppose that the Fourier expansion of G at the infinity cusp c_1 is*

$$G = \sum_{\xi \in (\mathfrak{a}^{-1})^\vee_+ \cup \{0\}} a_\xi \exp(\xi z_1 + \xi' z_2),$$

with $a_\xi \in \mathcal{O}$ for all $\xi \in (\mathfrak{a}^{-1})^\vee_+ \cup \{0\}$. Let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal such that $\mathfrak{p} \nmid Dn$ and let r be an integer with

$$r > \frac{4kn\zeta_K(-1)}{\sum_j (b_{1,j} - 2)} - s\left(\frac{\sum_{i=1}^h \sum_j (b_{i,j} - 2)}{\sum_j (b_{1,j} - 2)}\right).$$

If $a_\xi \in \mathfrak{p}$ for all $\xi \in (\mathfrak{a}^{-1})^\vee_+ \cup \{0\}$ such that there is a $j \in J$ with $\operatorname{Tr}(\xi A_j) < r + s$, then $a_\xi \in \mathfrak{p}$ for all $\xi \in (\mathfrak{a}^{-1})^\vee_+ \cup \{0\}$.

Proof. With the same argument as in the proof of the classical Sturm theorem, we can assume without loss of generality that $\mathbb{Z}[1/(Dn), \zeta_n] \subset \mathcal{O}$. We consider the regular model $\mathcal{Y}_{\Gamma((n),\mathfrak{a})}$ of $Y_{\Gamma((n),\mathfrak{a})}$ provided by Theorem 5.1. As before, we denote by $\mathcal{Y}_{\Gamma((n),\mathfrak{a}),\mathcal{O}}$ the model over $\operatorname{Spec}(\mathcal{O})$ obtained after base change. Since G is a modular form for $\Gamma(\mathcal{O}_K, \mathfrak{a})$ it is also a modular form for $\Gamma((n), \mathfrak{a})$. By the q-expansion principle (Theorem 5.2) the modular form G determines a section of $\mathcal{O}_{\mathcal{Y}_{\Gamma((n),\mathfrak{a}),\mathcal{O}}}(k(\mathcal{K}_\mathcal{O} + \mathcal{S}'_\mathcal{O}))$, that we denote also by G. The vanishing hypothesis imply that, when we restrict G to $\mathcal{Y}_{\Gamma((n),\mathfrak{a}),\overline{k(\mathfrak{p})}}$ we obtain a global section of

$$\mathcal{O}_{\mathcal{Y}_{\Gamma((n),\mathfrak{a}),\overline{k(\mathfrak{p})}}}(k(\mathcal{K}_{\overline{k(\mathfrak{p})}} + \mathcal{S}'_{\overline{k(\mathfrak{p})}}) - sn\mathcal{S}'_{\overline{k(\mathfrak{p})}} - rn\mathcal{S}'_{i_0,\overline{k(\mathfrak{p})}}).$$

By Proposition 5.3 the canonical divisor $\mathcal{K}_{\overline{k(\mathfrak{p})}}$ is NEF. Since intersection numbers are preserved by specialization, from equation (15) we deduce that

$$\mathcal{K}_{\overline{k(\mathfrak{p})}} \cdot (k(\mathcal{K}_{\overline{k(\mathfrak{p})}} + \mathcal{S}'_{\overline{k(\mathfrak{p})}}) - sn\mathcal{S}'_{\overline{k(\mathfrak{p})}} - rn\mathcal{S}'_{i_0,\overline{k(\mathfrak{p})}}) < 0$$

Therefore the restriction of G to $\mathcal{Y}_{\Gamma((n),\mathfrak{a}),\overline{k(\mathfrak{p})}}$ is zero, proving the result. □

6. General weights and levels.

Although the main results of the previous sections are stated only for modular forms of level $\Gamma(\mathcal{O}_K, \mathfrak{a})$ and parallel weight $(2k, 2k)$, they can be generalized to any congruence subgroup $\Gamma_\mathfrak{a}$ and any weight (k_1, k_2) satisfying the parity condition

$k_1 \equiv k_2 \pmod 2$ using exactly the same tricks as for classical modular forms. Assume that n satisfies the hypothesis of Theorem 4.1.

Let $\Gamma_{\mathfrak{a}}$ be a congruence subgroup, (k_1, k_2) a weight satisfying the previous parity condition. Let $\{A_j\}_{j\in J}$ be a set of representatives under the action of $U^2_{\mathcal{O}_K}$, of the corners of the convex hull of $(\mathfrak{a}^{-1})_+$ as in Section 2.2.

Theorem 6.1. *Let G be a modular form of weight (k_1, k_2) for $\Gamma_{\mathfrak{a}}$ which vanishes with order s at all the cusps. Suppose that the Fourier expansion of G at the infinity cusp is*

$$G = \sum_{\xi \in M_+^\vee \cup \{0\}} a_\xi \exp(2\pi i(\xi z_1 + \xi' z_2)).$$

for an appropriate lattice $M \subset \mathfrak{a}^{-1}$. Let

$$r > \frac{(k_1+k_2)n[\Gamma(\mathcal{O}_K, \mathfrak{a}) : \Gamma_{\mathfrak{a}}]\zeta_K(-1)}{\sum_j (b_{1,j}-2)} - s\left(\frac{\sum_{i=1}^h \sum_j (b_{i,j}-2)}{\sum_j (b_{1,j}-2)}\right)$$

be an integer. If $a_\xi = 0$ for all $\xi \in M_+^\vee \cup \{0\}$ such that there is a $j \in J$ with $\mathrm{Tr}(\xi A_j) < r+s$, then $G = 0$.

Proof. Assume first that $k_1 = k_2 = 2k$. Let $H(z_1, z_2)$ be the Hilbert modular form given by

$$H(z_1, z_2) = \prod_{\substack{\alpha \in \Gamma \backslash \Gamma(\mathcal{O}_K, \mathfrak{a}) \\ \alpha \notin \Gamma}} G(z_1, z_2)|_{2k}[\alpha],$$

where the product is taken over coset representatives of $\Gamma(\mathcal{O}_K, \mathfrak{a})$ modulo $\Gamma_{\mathfrak{a}}$ (acting on the left) not in the trivial class.

The form $G(z_1, z_2)H(z_1, z_2)$ is a form of weight $2k[\Gamma(\mathcal{O}_K, \mathfrak{a}) : \Gamma_{\mathfrak{a}}]$ for $\Gamma(\mathcal{O}_K, \mathfrak{a})$, so we can apply the Hecke bound of section 4 to it. There is an integer N such that $\Gamma((N), \mathfrak{a}) \subset \Gamma_{\mathfrak{a}}$ and $N(\mathfrak{a}^{-1}) \subset M$, thus we can write the Fourier expansion of G as

$$G(z_1, z_2) = \sum_{\xi \in \frac{1}{N}(\mathfrak{a}^{-1})_+^\vee \cup \{0\}} a_\xi \exp(2\pi i(\xi z_1 + \xi' z_2)).$$

Since $\Gamma((N), \mathfrak{a})$ is a normal subgroup of $\Gamma(\mathcal{O}_K, \mathfrak{a})$, the function $H(z_1, z_2)$ is a modular form for it. Thus it has a Fourier expansion

$$H(z_1, z_2) = \sum_{\xi \in \frac{1}{N}(\mathfrak{a}^{-1})_+^\vee \cup \{0\}} b_\xi \exp(2\pi i(\xi z_1 + \xi' z_2)).$$

The product of this two Fourier expansions is

$$\sum_{\eta \in (\mathfrak{a}^{-1})_+^\vee} \left(\sum_{\xi, \eta-\xi \in \frac{1}{N}(\mathfrak{a}^{-1})_+^\vee \cup \{0\}} a_\xi b_{\eta-\xi} \right) \exp(2\pi i(\eta z_1 + \eta' z_2)).$$

In principle, the exterior sum should run over elements in $\frac{1}{N}(\mathfrak{a})_+^\vee$, but since we know that GH is a modular form for $\Gamma(\mathcal{O}_K, \mathfrak{a})$, all the other terms are zero.

Note that since $\eta - \xi \gg 0$ (or zero), $\eta - \xi \geq 0$ and $\eta' - \xi' \geq 0$, so $\mathrm{Tr}(\xi m) \leq \mathrm{Tr}(\eta m)$ for $m \in \mathfrak{a}_+^{-1}$. In particular, if $a_\xi = 0$ for all the elements in the hypothesis, the coefficients of $G(z_1, z_2)H(z_1, z_2)$ are all zero for all η with $\mathrm{Tr}(\eta A_j) \leq r + s$ for some $j \in J$ and the result follows from Corollary 4.3.

For general weights (k_1, k_2), it is enough to apply the previous case to the form $G(z_1, z_2)G(z_2, z_1)$, which has parallel weight $k_1 + k_2$ (even) and vanishes with order $2s$ at all the cusps and with order $2r + 2s$ at the infinity cusp. □

Remark 6.2. The same statement and proof gives Sturm bound holds for general weights and level.

Remark 6.3. As in the classical case, one can obtain for forms in $M_{\mathbf{k}}(\Gamma_0(\mathfrak{c}, \mathfrak{a}), \chi)$ (i.e. forms with a character) the same bound as the one for forms in $M_{\mathbf{k}}(\Gamma_0(\mathfrak{c}, \mathfrak{a}))$, by using Buzzard's trick. If $\operatorname{ord}(\chi)$ denotes the order of χ, then we consider $G(z_1, z_2)^{\operatorname{ord}(\chi)}$, which vanishes with order $\operatorname{ord}(\chi)s$ at all cusps and $\operatorname{ord}(\chi)s+\operatorname{ord}(\chi)r$ at the infinity cusp, but is a form for $\Gamma_0(\mathfrak{c}, \mathfrak{a})$, so the values of $\operatorname{ord}(\chi)$ cancels in the formula.

Remark 6.4. If in the Hecke/Sturm bound we fix the level and let the weight grow, the number of elements of the Fourier expansion to check equality/congruence grows quadratically with the weight since we have to search for elements in a cone whose trace grows linearly in the weight. If we stick to parallel weight forms, it is known that the same happens with the dimension of such modular forms spaces. This implies that the bound we got is the best possible up to a constant (depending only on the level and the base field).

Remark 6.5. When the narrow class number is greater than 1, one can relate modular forms for the different subgroups $\mathrm{PGL}_2^+(\mathcal{O}_K, \mathfrak{a})$ (varying $\mathfrak{a}$) using the action of the Hecke operators. This allows to take the number of coefficients needed to check congruences/equality of modular forms to be the minimum between all the ideals, but they need not be the ones with smaller trace. See the Remark 7.1.

7. Examples

We apply the main results of this article to different examples of real quadratic fields. All computation were done using the mathematical software [PAR12] and some code written by ourselves. We chose fields that have: trivial narrow class group, trivial class group but non-trivial narrow class group and non-trivial class group respectively.

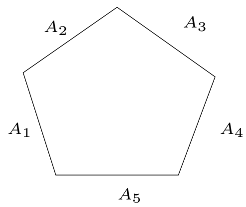
7.1. **The case $\mathbb{Q}(\sqrt{29})$.** The class number and the narrow class number of $\mathbb{Q}(\sqrt{29})$ are both 1. The discriminant is $29 \not\equiv 1 \pmod 8$, hence Conjecture 2.9 holds and we can take $n = 3$ for the Hecke/Sturm bound. Table 7.1 contains the information of the desingularization process applied to the principal ideal as explained in Appendix A (specially RemarkA.1). The abbreviation S.I. of the table stands for self intersection number. With this information, equation (6) of Theorem 4.1 reads

$$r > \frac{2 \cdot 2k \cdot 3 \cdot 1}{5 \cdot 2} - s = \frac{6k}{5} - s.$$

Let $G(z_1, z_2) \in M_{2k}(\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{29})}))$ with Fourier expansion

$$G(z_1, z_2) = \sum_{\xi \in (\frac{1}{\sqrt{29}}\mathbb{Z} + (\frac{1}{2} + \frac{1}{2\sqrt{29}})\mathbb{Z})^+} a_\xi \exp(2\pi i(\xi z_1 + \xi' z_2)).$$

If $a_\xi = 0$ for all ξ that satisfy $\operatorname{Tr}(m\xi) \le \frac{6k}{5}$, with m any of the five vertices then $G(z_1, z_2)$ is the zero form. For cusp forms of parallel weight 2, the vector space of



Label	Point	S.I.
A_1	1	-7
A_2	$\frac{7-\sqrt{29}}{2}$	-2
A_3	$6-\sqrt{29}$	-2
A_4	$\frac{17-3\sqrt{29}}{2}$	-2
A_5	$11-2\sqrt{29}$	-2

TABLE 7.1. Infinity cusp desingularization for $\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{29})}, 1)$

which has dimension 1, the bound says it is enough to check elements with trace 1. The first vertex gives the five non-equivalent points

$$\xi = \frac{1}{2} \pm \frac{1}{2\sqrt{29}}, \frac{1}{2} \pm \frac{3}{2\sqrt{29}}, \frac{1}{2} + \frac{5}{2\sqrt{29}}. \tag{16}$$

The second vertex, the third vertex, the fourth and the fifth vertex give the point $\xi = \frac{1}{2} + \frac{5}{2\sqrt{29}}$. In particular, the five Fourier coefficients indexed by the elements listed in (16) determine whether the form is zero or not.

For different values of k, we computed the number of elements that satisfy $\mathrm{Tr}(m\xi) \le \frac{6k}{5}$ with m any of the five vertices of the desingularization. By Theorem 4.1, such quantity is the number of Fourier coefficients needed to determine whether a form in $S_{2k}(\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{29})}))$ is zero or not. The information is summarized in Table 7.2 which also contains the dimension of $S_{2k}(\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{29})}))$.

$2k$	20	30	40	50	100	150	200	300
Number of Elts	390	855	1500	2326	9151	20477	36302	81453
Dimension	92	212	381	602	2451	5552	9902	22352

TABLE 7.2. Number of coefficients versus dimension for $S_{2k}(\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{29})}))$

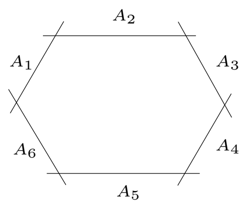
7.2. **The case $\mathbb{Q}(\sqrt{11})$.** The class number of $\mathbb{Q}(\sqrt{11})$ is 1 while the narrow class number is 2. Generators for the narrow class group are the principal ideal and the fractional ideal $(\sqrt{11})^{-1}$. Since $D = 44 \not\equiv 1 \pmod 8$, Conjecture 2.9 holds and we can take $n = 3$. Table 7.3 contains the information of the desingularization process applied to the principal ideal as explained in Appendix A (specially RemarkA.1). The abbreviation S.I. of the table stands for self intersection number. With this information, equation (6) of Theorem 4.1 reads

$$r > \frac{4k \cdot 3 \cdot 7}{12 \cdot 6} - s = \frac{7k}{6} - s.$$

Let $G(z_1, z_2) \in M_{2k}(\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, 1))$ with Fourier expansion

$$G(z_1, z_2) = \sum_{\xi \in (\frac{1}{2}\mathbb{Z} + \frac{1}{2\sqrt{11}}\mathbb{Z})^+} a_\xi \exp(2\pi i(\xi z_1 + \xi' z_2)).$$

If $a_\xi = 0$ for all ξ that satisfy $\mathrm{Tr}(m\xi) \le \frac{7k}{6}$, with m any of the six vertices, then $G(z_1, z_2)$ is the zero form.



Label	Point	S.I.
A_1	1	−8
A_2	$4-\sqrt{11}$	−2
A_3	$7-2\sqrt{11}$	−2
A_4	$10-3\sqrt{11}$	−8
A_5	$73-22\sqrt{11}$	−2
A_6	$136-41\sqrt{11}$	−2

TABLE 7.3. Infinity cusp desingularization for $\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, 1)$

$2k$	20	30	40	50	100	150
Number of Elts	792	1836	3312	5220	20532	45936
Dimension	212	492	888	1402	5718	12952

TABLE 7.4. Number of coefficients versus dimension for $S_{2k}(\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, 1))$

For cusp forms of parallel weight 2, the vector space of which has dimension 2, the bound says it is enough to check elements with trace 1. The first vertex gives the seven non-equivalent points

$$\frac{1}{2} \pm \frac{3}{2\sqrt{11}}, \frac{1}{2} \pm \frac{1}{\sqrt{11}}, \frac{1}{2} \pm \frac{1}{2\sqrt{11}}, \frac{1}{2}.$$

The second and the third vertices give the point $\frac{1}{2} + \frac{3}{2\sqrt{11}}$. The fourth vertex gives the points

$$\frac{1}{2} + \frac{3}{2\sqrt{11}}, 2 + \frac{13}{2\sqrt{11}}, \frac{7}{2} + \frac{23}{2\sqrt{11}}, 5 + \frac{33}{2\sqrt{11}}, \frac{13}{2} + \frac{43}{2\sqrt{11}}, 8 + \frac{53}{2\sqrt{11}}, \frac{19}{2} + \frac{63}{2\sqrt{11}}.$$

The last two vertices give the point $\frac{19}{2}+\frac{63}{2\sqrt{11}}$. Note that since the elements $\frac{19}{2}+\frac{63}{2\sqrt{11}}$ and $\frac{1}{2} - \frac{3}{2\sqrt{11}}$ differ by an even power of the fundamental unit, there are only twelve conditions to check.

For different values of k, we computed the number of elements that satisfy $\mathrm{Tr}(m\xi) \leq \frac{7k}{6}$ with m any of the six vertices of the desingularization. By Theorem 4.1, such quantity is the number of Fourier coefficients needed to determine whether a form in $S_{2k}(\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, 1))$ is zero or not. The information is summarized in Table 7.4 which also contains the dimension of $S_{2k}(\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, 1))$.

7.2.1. *The subgroup* $\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, (\sqrt{11})^{-1})$. Table 7.5 contains the information of the desingularization process applied to the ideal $(\sqrt{11})^{-1}$. With this information, equation (6) of Theorem 4.1 reads

$$r > \frac{7k}{3} - s.$$

Let $G(z_1, z_2) \in M_{2k}(\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, (\sqrt{11})^{-1}))$ with Fourier expansion

$$G(z_1, z_2) = \sum_{\xi \in (\frac{1}{2}\mathbb{Z} + \frac{\sqrt{11}}{2}\mathbb{Z})^+} a_\xi \exp(2\pi i(\xi z_1 + \xi' z_2)).$$

Point	S.I.	Point	S.I.
$\frac{3}{\sqrt{11}}+1$	-5	$-\frac{3}{\sqrt{11}}+1$	-5
$\frac{2}{\sqrt{11}}+1$	-2	$-\frac{13}{\sqrt{11}}+4$	-2
$\frac{1}{\sqrt{11}}+1$	-2	$-\frac{23}{\sqrt{11}}+7$	-2
1	-2	$-3\sqrt{11}+10$	-2
$-\frac{1}{\sqrt{11}}+1$	-2	$-\frac{43}{\sqrt{11}}+13$	-2
$-\frac{2}{\sqrt{11}}+1$	-2	$-\frac{53}{\sqrt{11}}+16$	-2

TABLE 7.5. Infinity cusp desingularization for $\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, (\sqrt{11}))$

$2k$	20	30	40	50	100	150
Number of Elts	1657	3780	6487	10267	40717	92401
Dimension	213	493	889	1403	5719	12953

TABLE 7.6. Number of coefficients versus dimension for $S_{2k}(\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, (\sqrt{11})^{-1}))$

If $a_\xi = 0$ for all ξ that satisfy $\mathrm{Tr}(m\xi) \leq \frac{7k}{3}$, with m any of the twelve vertices then $G(z_1, z_2)$ is the zero form. For cusp forms of parallel weight 2, the vector space of which has dimension 3, the bound says it is enough to check elements with trace 1 or 2. There are 18 such elements which can be easily computed.

For different values of k, we computed the number of elements that satisfy $\mathrm{Tr}(m\xi) \leq \frac{7k}{3}$ with m any of the twelve vertices of the desingularization. By Theorem 4.1, such quantity is the number of Fourier coefficients needed to determine whether a form in $S_{2k}(\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, (\sqrt{11})^{-1}))$ is zero or not. The information is summarized in Table 7.6 which also contains the dimension of $S_{2k}(\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, (\sqrt{11})^{-1}))$.

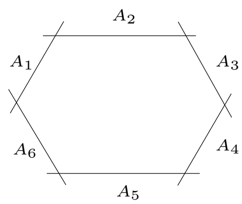
Remark 7.1. Hecke operators do not act on the surface $Y_{\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{11})},1)}$, but do act (as correspondences) on the product $Y_{\mathrm{PGL}_2^+(\mathcal{O}_{\mathbb{Q}(\sqrt{11})},1)} \times Y_{\mathrm{PGL}_2^+(\mathcal{O}_{\mathbb{Q}(\sqrt{11})},(\sqrt{11})^{-1})}$, i.e. they act on pairs of Hilbert modular forms where the first component is invariant under $\mathrm{PGL}_2^+(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, 1)$ and the second one under $\mathrm{PGL}_2^+(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, (\sqrt{11})^{-1})$ (such pairs of forms correspond to automorphic forms over $\mathbb{Q}(\sqrt{11})$; a good introduction to the subject is the book [Gar90], where the relation between automorphic forms and Hilbert modular forms is well explained and the definition and main properties of Hecke operators is given). A form in $M_{\mathbf{k}}(\mathrm{PGL}_2^+(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, (\sqrt{11})^{-1}))$ can be thought as an automorphic form supported only in one component.

Let $\mathfrak{p}_{11}$ denote the prime ideal generated by $\sqrt{11}$. The Hecke operator $T_{\mathfrak{p}_{11}}$ sends a form F supported in $M_{\mathbf{k}}(\mathrm{PGL}_2^+(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, (\sqrt{11})^{-1}))$ to a form supported in $M_{\mathbf{k}}(\mathrm{PGL}_2^+(\mathcal{O}_{\mathbb{Q}(\sqrt{11})}, 1))$. Furthermore, if

$$F(z_1, z_2) = \sum_{\xi \in M_+^\vee \cup \{0\}} a_\xi \exp(2\pi i(\xi z_1 + \xi' z_2)).$$

then

$$T_{\mathfrak{p}_{11}}(F)(z_1, z_2) = \sum_{\xi \in M_+^\vee \cup \{0\}} \left(11 a_\xi + a_{\frac{\xi}{11}}\right) \exp(2\pi i(\xi z_1 + \xi' z_2)).$$



Label	Point	S.I.
A_1	1	-8
A_2	$4-\sqrt{10}$	-2
A_3	$7-2\sqrt{10}$	-2
A_4	$10-3\sqrt{10}$	-2
A_5	$13-4\sqrt{10}$	-2
A_6	$16-5\sqrt{10}$	-2

TABLE 7.7. Infinity cusp desingularization for $\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{10})}, 1)$

Assume that the form $F(z_1, z_2)$ is not in the kernel of the Hecke operator $T_{\mathfrak{p}_{11}}$ (which is usually the case). Then if the Fourier coefficients a_ξ and $a_{\frac{\xi}{11}}$, with ξ in the Hecke/Sturm set for the trivial class are all zero/congruent to zero, then the form $F(z_1, z_2)$ itself is the zero form. In this way one can restrict only to the principal class and deduce from it the other ones. Also one can hope to get better bounds comparing the different narrow class group classes.

It is an interesting problem to study the action of the Hecke operators and see how one can improve our Hecke/Sturm bound for general real quadratic fields.

7.3. **The case** $\mathbb{Q}(\sqrt{10})$**.** This is the first real quadratic field with non-trivial class group. The class group has order 2 and the two representatives can be taken as 1 and $\langle 2, \sqrt{10}\rangle$ (the unique prime ideal dividing 2). The discriminant of such field is $D = 40 \not\equiv 1 \pmod 8$, hence Conjecture 2.9 holds and we can take $n = 3$ for the Hecke/Sturm bound. Table 7.7 contains the information of the desingularization process applied to the principal ideal as explained in Appendix A. With this information, equation (6) of Theorem 4.1 reads

$$r > \frac{2 \cdot 2k \cdot 3 \cdot 7}{6 \cdot 6} - s\frac{6+4}{6} = \frac{7k}{3} - \frac{5s}{3}.$$

Let $G(z_1, z_2) \in M_{2k}(\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{10})}, 1))$ with Fourier expansion

$$G(z_1, z_2) = \sum_{\xi \in (\frac{1}{2}\mathbb{Z} + \frac{1}{2\sqrt{10}}\mathbb{Z})^+} a_\xi \exp(2\pi i(\xi z_1 + \xi' z_2)).$$

If $a_\xi = 0$ for all ξ such that $\operatorname{Tr}(m\xi) \leq \frac{7k}{3}$, with m any of the six vertices then $G(z_1, z_2)$ is the zero form. For cusp forms of parallel weight 2, the vector space of which has dimension 1, the bound says it is enough to check elements with trace 1. The first vertex gives the non-equivalent points

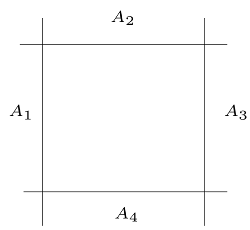
$$\xi = \frac{-2}{2\sqrt{10}} + \frac{1}{2}, \frac{-1}{2\sqrt{10}} + \frac{1}{2}, \frac{1}{2}, \frac{1}{2\sqrt{10}} + \frac{1}{2}, \frac{2}{2\sqrt{10}} + \frac{1}{2}, \frac{3}{2\sqrt{10}} + \frac{1}{2}.$$

All the other ones give the point $\xi = \frac{3}{2\sqrt{10}} + \frac{1}{2}$.

For different values of k, we computed the number of elements that satisfy $\operatorname{Tr}(m\xi) \leq \frac{7k}{3}$ with m any of the six vertices of the desingularization. By Theorem 4.1, such quantity is the number of Fourier coefficients needed to determine whether a form in $S_{2k}(\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{10})}, 1))$ is zero or not. The information is summarized in Table 7.8 which also contains the dimension of $S_{2k}(\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{10})}, 1))$.

$2k$	20	30	40	50	100	150
Number of Elts	1518	3570	6486	9918	40716	91350
Dimension	212	492	888	1402	5718	12952

TABLE 7.8. Number of coefficients versus dimension for $S_{2k}(\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{10})}, 1))$



Label	Point	S.I.
A_1	2	-4
A_2	$4-\sqrt{10}$	-3
A_3	$10-3\sqrt{10}$	-2
A_4	$16-5\sqrt{10}$	-3

TABLE 7.9. Infinity cusp desingularization for $\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{10})}, \langle 2, \sqrt{10}\rangle)$

By the same classical translation, working with the cusp $\langle 2, \sqrt{10}\rangle$ is equivalent to work with the infinity cusp for the group of level $\langle 2, \sqrt{10}\rangle$. Table 7.9 contains the information of the desingularization process applied to the prime ideal $\langle 2, \sqrt{10}\rangle$. With this information, equation (6) of Theorem 4.1 reads

$$r > \frac{2\cdot 2k\cdot 3\cdot 7}{4\cdot 6} - s\frac{6+4}{4} = \frac{7k}{2} - \frac{5s}{2}.$$

Let $G(z_1, z_2) \in M_{2k}(\Gamma(\mathcal{O}_K, \langle 2, \sqrt{10}\rangle))$ with Fourier expansion

$$G(z_1, z_2) = \sum_{\xi \in (\frac{1}{4}\mathbb{Z} + \frac{1}{2\sqrt{10}}\mathbb{Z})^+} a_\xi \exp(2\pi i(\xi z_1 + \xi' z_2)).$$

If $a_\xi = 0$ for all ξ such that $\mathrm{Tr}(m\xi) \leq \frac{7k}{2}$, with m any of the four vertices then $G(z_1, z_2)$ is the zero form. For cusp forms of parallel weight 2, the vector space of which has dimension 1, we need to check the elements with trace 1 or 2. The first vertex gives the non-equivalent points (up to units squared)

$$\xi = \frac{1}{4}, \frac{1}{4} + \frac{1}{2\sqrt{10}}, \frac{1}{4} - \frac{1}{2\sqrt{10}}, \frac{1}{2} - \frac{1}{\sqrt{10}}, \frac{1}{2} - \frac{1}{2\sqrt{10}}, \frac{1}{2}, \frac{1}{2} + \frac{1}{\sqrt{10}}, \frac{1}{2} + \frac{1}{2\sqrt{10}}, \frac{1}{2} + \frac{3}{2\sqrt{10}}.$$

The first three points have trace 1, while the others trace 2. The second vertex gives the points

$$\xi = \frac{1}{4} + \frac{1}{2\sqrt{10}}, \frac{1}{2} + \frac{3}{2\sqrt{10}}, \frac{1}{4}, \frac{1}{2} + \frac{1}{\sqrt{10}}, \frac{3}{4} + \frac{2}{\sqrt{10}}, 1 + \frac{3}{\sqrt{10}},$$

where the first two elements give trace 1 while the others trace 2. The third vertex gives the points

$$\xi = \frac{1}{2} + \frac{3}{2\sqrt{10}}, \frac{1}{4} + \frac{1}{2\sqrt{10}}, 1 + \frac{3}{\sqrt{10}}, \frac{7}{4} + \frac{11}{2\sqrt{10}},$$

$2k$	20	30	40	50	100	150
Number of Elts	2244	5304	9384	14964	60204	135720
Dimension	212	492	888	1402	5718	12952

TABLE 7.10. Number of coefficients versus dimension for $S_{2k}(\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{10})}, \langle 2, \sqrt{10}\rangle))$

where the first one corresponds to trace 1 and the other to trace 2. Note that the last element is equivalent to $\frac{1}{4} - \frac{1}{2\sqrt{10}}$. The last vertex gives the points

$$\xi = \frac{1}{2} + \frac{3}{2\sqrt{10}}, \frac{7}{4} + \frac{11}{2\sqrt{10}}, 1 + \frac{3}{\sqrt{10}}, \frac{9}{4} + \frac{7}{\sqrt{10}}, \frac{7}{2} + \frac{11}{\sqrt{10}}, \frac{19}{4} + \frac{15}{\sqrt{10}},$$

where the first two elements correspond to trace 1 and the others to trace 2. The last two elements are equivalent to the elements $\frac{1}{2} - \frac{1}{\sqrt{10}}$ and $\frac{1}{4}$ respectively, so we need to check 12 coefficients.

For different values of k, we computed the number of elements that satisfy $\mathrm{Tr}(m\xi) \leq \frac{7k}{2}$ with m any of the 4 vertices of the desingularization. By Theorem 4.1, such quantity is the number of Fourier coefficients needed to determine whether a form in $S_{2k}(\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{10})}, \langle 2, \sqrt{10}\rangle))$ is zero or not. The information is summarized in Table 7.10 which also contains the dimension of $S_{2k}(\Gamma(\mathcal{O}_{\mathbb{Q}(\sqrt{10})}, \langle 2, \sqrt{10}\rangle))$.

APPENDIX A. COMPUTING THE CUSP DESINGULARIZATION

In this appendix we recall how to compute the resolution divisor of a cusp of a Hilbert modular surface. The isotropy group of any cusp for $\Gamma(\mathfrak{c}, \mathfrak{a})$ is conjugate to a group of the form $G(M, V)$, where $M \subset K$ is an $\mathcal{O}_K$-module and $V \subset U_K^+$ is a subgroup of finite index. As a transformation group $G(M, V) = M \rtimes V$. The geometry of the resolution divisor only depends on M and V.

To compute the desingularization of the cusp we first need a reduced oriented basis of M.

An oriented basis of M is a $\mathbb{Z}$-basis $M = \langle \alpha, \beta\rangle$ such that $\det \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} > 0$. To an oriented basis we can associate the indefinite binary quadratic form $Q(x, y) = \frac{1}{\mathcal{N}(M)} \mathcal{N}(\alpha x + \beta y)$, where $\mathcal{N}(M)$ indicates the content of the form $\mathcal{N}(\alpha x + \beta y)$, i.e. the rational number which makes $Q(x, y)$ an integral primitive form.

If λ is a totally positive element, multiplication by λ sends oriented bases of M to oriented bases of λM, but clearly $\langle \alpha, \beta\rangle$ and $\langle \lambda\alpha, \lambda\beta\rangle$ have the same quadratic form attached. Choosing a different oriented basis gives an $\mathrm{SL}_2(\mathbb{Z})$-equivalent form, hence we get a bijection between the narrow class group of K and $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of integral primitive indefinite binary quadratic forms of discriminant D.

Following [vdG88], we call a form $ax^2 + bxy + cy^2$ of discriminant D *reduced* if

$$0 < \frac{b - \sqrt{D}}{2a} < 1 < \frac{b + \sqrt{D}}{2a}. \tag{17}$$

Thus, using strict $\mathrm{SL}_2(\mathbb{Z})$ equivalence, one can reduce any indefinite integral binary quadratic form of discriminant D to a reduced one. In other words, starting from an oriented basis of M one gets an oriented basis of the form $\frac{1}{a} M_{red} = \frac{\lambda}{a} M = \left(\frac{b+\sqrt{D}}{2a}\right)\mathbb{Z} + \mathbb{Z}$ (λ is a generator of the quotient of the reduced ideal by M).

Remark A.1. This notion of a reduced form is not universal. For example, in Cohen's book (see ([Coh93] Definition 5.6.2) a reduced indefinite integral binary quadratic form satisfies

$$0 \leq \frac{\sqrt{D}-b}{2|a|} < 1 < \frac{\sqrt{D}+b}{2|a|}.$$

Starting from $Q(x,y)$ one can use Cohen's algorithm ([Coh93] Algorithm 5.6.5 which is for example implemented in [PAR12]) to get a Cohen-reduced form. Note that we can always take as reduced form one with $a > 0$ (by Proposition 5.6.6 of [Coh93]) and remove the previous absolute value. If we apply the change of variables given by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, which sends b to $b+2a$, we get a reduced form in the sense of (17).

Once we have computed the reduced basis for λM, we consider the set of totally positive elements $(\lambda M)_+$ as a subset of $\mathbb{R}^2$ and its convex hull in $\mathbb{R}^2$. The points of $(\lambda M)_+$ that belong to the boundary of its convex hull are A_k, $k \in \mathbb{Z}$, with

$$A_{-1} = w_0 := \frac{b+\sqrt{D}}{2a}, \qquad A_0 = 1, \qquad A_{k+1} := b_k A_k - A_{k-1},$$

where the numbers b_k, for $k \geq 0$, are defined recursively by

$$b_k := \lceil w_k \rceil \qquad \text{and} \qquad w_{k+1} := \frac{1}{b_k - w_k}.$$

The sequence $\{b_k\}$ (and $\{w_k\}$ also) is periodic with some period r. We extend the definition of b_k to $k \in \mathbb{Z}$ using this periodicity.

Now we add the multiplicative structure. Let ε be a generator of $U^2_{\mathcal{O}_K}$. It acts on the sequence $\{A_k\}$ with a finite number of representatives. Then for all $k \in \mathbb{Z}$,

$$A_k = \varepsilon^{\pm\nu} A_{k+r},$$

where $\nu = 1$ if the fundamental unit of K has norm -1 and $\nu = 2$ otherwise.

If we want to compute the boundary of M_+, then we just multiply the previous process by a (the norm of the reduced ideal) times a generator of the quotient M/M_{red}

Let $\tilde{r} = r \cdot \nu \cdot [U^2_{\mathcal{O}_K} : V]$. Then the cusp resolution attached to $G(M,V)$ is completely described by the period $\tilde{r}$ and the numbers b_k. Namely, the resolution divisor consists of $\tilde{r}$ lines S_k, $k \in \mathbb{Z}/\tilde{r}$ (each one isomorphic to $\mathbb{P}^1$) which satisfy:

- $S_k^2 = -b_k$ if $\tilde{r} \geq 2$.
- Let n, m be integers, $n \neq m$ and $\tilde{r} \geq 3$. Then:
  - If $n \not\equiv m \pm 1 \pmod{\tilde{r}}$, $S_n \cap S_m = \emptyset$.
  - If $n \equiv m \pm 1 \pmod{\tilde{r}}$, $S_n \cap S_m$ is one point.
- If $\tilde{r} = 1$, then S_0 is singular and $S_0^2 = -b_0 + 2$.
- If $\tilde{r} = 2$, then S_0 and S_1 are non-singular and intersect in 2 points.

Appendix B. Rational case

Recall that $Y_{\Gamma(\mathcal{O}_K, \mathfrak{a})}$ is rational for $D = 5, 8, 12, 13, 17, 21, 24, 28, 33, 60$ and $\mathfrak{a}$ in the principal genus, or for $D = 12$ and $\mathfrak{a}$ not in the principal genus. The purpose of this appendix is to give a Sturm bound for some of these cases. If $\mathfrak{c}$ is an integral ideal such that $Y_{\Gamma(\mathfrak{c},\mathfrak{a})}$ or a blow down of it, is a minimal surface of general type we still get the Hecke/Sturm bound

$$a > \frac{4k[\Gamma(\mathfrak{c},\mathfrak{a}):\Gamma(\mathcal{O}_K,\mathfrak{a})]\zeta_k(-1)}{\sum_j (b_{i_0,j}-2)} - s\left(\frac{\sum_{i=1}^h \sum_j (b_{i,j}-2)}{\sum_j (b_{i_0,j}-2)}\right),$$

where the numbers $b_{i,j}$ are the ones appearing in the cusp desingularization process of $Y_{\Gamma(\mathfrak{c},\mathfrak{a})}$, or that of its blow down. Here is a summary of the ideals $\mathfrak{c}$ which give a minimal surface of general type for some values of D:

- $D = 5$: $\mathfrak{c} = 3$ ([vdG88] Example 7.5 p. 179). There are ten non-equivalent cusps, each one resolved by a cycle $(3, 3, 3, 3)$.
- $D = 8$: $\mathfrak{c} = \mathfrak{p}_7$ a prime ideal or norm 7 ([vdG88] page 196). There are eight cusps, each one resolved by a cycle $(4, 2, 4, 2, 4, 2)$.
- $D = 13$, $\mathfrak{c} = 2$ (see [vdGZ77] page 197) gives a surface of general type with the components of F_1 as the unique exceptional curves. There are 5 cusps, each one in the minimal model is resolved by a cycle $(2, 2, 3, 2, 2, 3, 2, 2, 3)$.
- $D = 17$: $\mathfrak{c} = 2$ (see [vdG88], page 198) gives a surface of general type with the components of F_1 as the unique exceptional curves. There are 9 cusps, each one resolved in the minimal model by a cycle $(2, 2, 3, 3, 3)$.
- $D = 21$: $\mathfrak{c} = 2$ (Theorem 3 of [vdGZ77]) gives a surface of general type with the components of F_1 as the unique exceptional curves. There are 5 cusps, each one resolved in the minimal model by a cycle $(5, 5, 5, 5, 5, 5)$.
- If $D = 24$: $\mathfrak{c} = \mathfrak{p}_2$, the prime ideal of norm 2 (see [vdG78] page 166) gives a surface of general type with the components of F_1 as the unique exceptional ones. There are 3 non-equivalent cusps, each one resolved in the minimal model by a cycle $(2, 2, 2, 3, 2, 2, 2, 3)$.
- $D = 12$ and $\mathfrak{a}$ not in the principal genus: $\mathfrak{c} = 2$ ([vdG88] page 197) gives a minimal surface of general type. There are 3 cusps, each one resolved by a cycle $(2, 3)$.

With these data, we get the following Hecke bounds for Hilbert modular form of parallel weight k, level $\Gamma(\mathfrak{c},\mathfrak{a})$ and vanishing with order s at all cusps:

D	5	8	12	13	17	21	24
$\mathfrak{a}$	1	1	$\sqrt{3}$	1	1	1	1
$a >$	$48k - 10s$	$\frac{14k}{3} - 8s$	$4k - 3s$	$\frac{40k}{3} - 5s$	$4k - 9s$	$\frac{40k}{9} - 5s$	$12k - 3s$

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