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Cavity type problems ruled by infinity Laplacian operator

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Abstract

We study a singularly perturbed problem related to infinity Laplacian operator with prescribed boundary values in a region. We prove that solutions are locally (uniformly) Lipschitz continuous, they grow as a linear function, are strongly non-degenerate and have porous level surfaces. Moreover, for some restricted cases we show the finiteness of the $(n-1)$-dimensional Hausdorff measure of level sets. The analysis of the asymptotic limits is carried out as well.

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1. Introduction

In this paper we study inhomogeneous singularly perturbed problems ruled by the *Infinity Laplacian*, which is defined as follows:

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$$\Delta_\infty u := (Du)^T D^2 u Du = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j}.$$

More precisely, we study *weak* solutions to

$$\begin{cases} \Delta_\infty u^\varepsilon(x) = \zeta_\varepsilon(x, u^\varepsilon) & \text{in} \quad \Omega, \\ u^\varepsilon(x) = \varphi^\varepsilon(x) & \text{on} \quad \partial\Omega, \end{cases} \tag{E_ε}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary, and $0 \le \varphi^\varepsilon \in C(\overline{\Omega})$ with $\|\varphi^\varepsilon\|_{L^\infty(\Omega)} \le \mathcal{A}$, for some constant $\mathcal{A} > 0$. The reaction term ζ_ε represents the singular perturbation of the model. We are interested in singular behaviors of order $\mathrm{O}\left(\frac{1}{\varepsilon}\right)$ along ε-level layers $\{u_\varepsilon \sim \varepsilon\}$, hence we consider (smooth) singular reaction terms $\zeta_\varepsilon \colon \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$0 \le \zeta_\varepsilon(x,t) \le \frac{\mathcal{B}}{\varepsilon}\chi_{(0,\varepsilon)}(t) + \mathcal{C}, \quad \forall\, (x,t) \in \Omega \times \mathbb{R}_+, \tag{1.1}$$

for some constants $\mathcal{B}$, $\mathcal{C} \ge 0$. Clearly $\zeta_\varepsilon \equiv 0$ satisfies (1.1), therefore, to insure that the reaction term is genuinely singular, we will assume also that

$$\mathfrak{R} := \inf_{\Omega\times[a,b]} \varepsilon\zeta_\varepsilon(x, \varepsilon t) > 0, \tag{1.2}$$

for some $0 \le a < b$, and $\mathfrak{R}$ does not depend on ε. Heuristically, (1.2) says that the singular term behaves asymptotically as $\sim \varepsilon^{-1}\chi_{(0,\varepsilon)}$ plus a nonnegative noise that stays uniformly bounded away from infinity. Singular reaction terms built up as approximation of unity

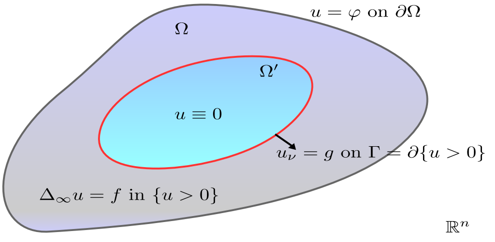
$$\zeta_\varepsilon(x,t) := \frac{1}{\varepsilon}\beta\left(\frac{t}{\varepsilon}\right) + g_\varepsilon(x), \tag{1.3}$$

are particular (simpler) cases covered by analysis to be developed herein (usually β is a nonnegative smooth real function with supp $\beta = [0, 1]$, and $0 \le c_0 \le g_\varepsilon(x) \le c_1 < \infty$). It is easy to check that the reaction term written in (1.3) satisfies (1.1) and (1.2).

We were motivated by the study of the following over-determined problem: given $\Omega \subset \mathbb{R}^n$ a domain, functions $0 \le f, \varphi \in C(\overline{\Omega})$ and $0 < g \in C(\overline{\Omega})$, we would like to find a compact "hypersurface" $\Gamma := \partial\Omega' \subset \Omega$ such that the boundary value problem

$$\begin{cases} \Delta_\infty u(x) = f(x) & \text{in} \quad \Omega\setminus\Omega' \\ u(x) = \varphi(x) & \text{on} \quad \partial\Omega \\ u(x) = 0 & \text{on} \quad \Omega' \\ \frac{\partial u}{\partial \nu}(x) = g(x) & \text{on} \quad \Gamma \end{cases} \tag{1.4}$$

has a solution. Possible limiting functions coming from (E_ε) are natural choices to solve the above problem with $\Gamma = \partial\{u > 0\}$ (the free boundary).



It is important to highlight that, unlike [2] and [11], we can not study (E_ε) as a limit of "variational solutions" of the corresponding inhomogeneous problem with p-Laplacian on the left hand side of (E_ε), because several geometric properties and estimates deteriorate, when $p \to +\infty$, since they depend on p (see, for example, [4,8,12]). This indicates the importance of the non-variational approach.

Viscosity solutions of (E_ε) exhibit two "distinct" free boundaries: the first one is the set of critical points $\mathcal{C}(u^\varepsilon) := \{x \in \Omega \mid \nabla u^\varepsilon(x) = 0\}$, and the second one is the "physical" free boundary, $\Gamma_\varepsilon = \{u^\varepsilon \sim \varepsilon\}$ (ε-level surfaces). We are able to control u^ε in terms of $\mathrm{dist}(x, \Gamma_\varepsilon)$ and see that these two free boundaries do not intersect.

A problem similar to (E_ε) for a fully nonlinear operators in the left hand side was studied in recent years. In fact, in [15] the authors study fully nonlinear uniformly elliptic equations of the form

$$F(x, D^2u^\varepsilon) = \zeta_\varepsilon(u^\varepsilon) \quad \text{in} \quad \Omega,$$

where $\zeta_\varepsilon \sim \frac{1}{\varepsilon}\chi_{(0,\varepsilon)}$. They prove several analytical and geometrical properties of solutions (see also [14] for global regularity character and [12] for an approach with inhomogeneous forcing term). A non-variational setting of the problem was studied in [1], where the authors obtain existence and optimal regularity results for the class of fully nonlinear, anisotropic degenerate elliptic problems

$$|\nabla u^\varepsilon|^\gamma F(D^2u^\varepsilon) = \zeta_\varepsilon(x, u^\varepsilon) \quad \text{in} \quad \Omega, \text{ with } \gamma \geq 0.$$

These summarize current results for singularly perturbed non-variational problems.

We also remark that although regularity of infinity harmonic functions is well studied (see [6,7,16]), regularity results for the inhomogeneous problem $\Delta_\infty u = f$ in Ω, are relatively recent and less developed. In this direction it was shown in [9] that blow-ups are linear, if $f \in C(\Omega) \cap L^\infty(\Omega)$. As a consequence, viscosity solutions of the inhomogeneous problem are Lipschitz continuous and also everywhere differentiable, if $f \in C^1(\Omega) \cap L^\infty(\Omega)$. In [3] Lipschitz regularity was proved for a more general right hand side $f : \Omega \times \mathbb{R} \to \mathbb{R}$ provided $f \in C(\Omega \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R})$.

This paper is organized as follows: in section 2 we state some preliminary results, which we use later. In section 3 we prove optimal Lipschitz regularity (uniformly in ε). In section 4 we prove geometric non-degeneracy properties of solutions. As a consequence a Harnack type inequality and porosity of level surfaces are proved. In section 5 we show that for some restricted cases the $(n-1)$-dimensional Hausdorff measure of the free boundary is finite. The corresponding asymptotic limit as $\varepsilon \to 0^+$ in (E_ε) is studied in the Section 6. We finish the paper analyzing the one-dimensional profile for the limiting free boundary problem in section 7.

2. Preliminary results

We start with the definition of the solution.

Definition 2.1. A function $u \in C(\Omega)$ is called a viscosity sub-solution (super-solution) of

$$\Delta_\infty u = f(x, u(x)) \quad \text{in} \quad \Omega,$$

if whenever $\phi \in C^2(\Omega)$ and $u - \phi$ has a local maximum (minimum) at $x_0 \in \Omega$ there holds

$$\Delta_\infty \phi(x_0) \geq f(x_0, \phi(x_0)) \quad (\text{resp.} \leq f(x_0, \phi(x_0))).$$

A function u is a viscosity solution when it is a viscosity sub and super-solution at the same time.

As it was shown in [10], the Dirichlet problem

$$\begin{cases} \Delta_\infty v(x) = f(x) & \text{in} \quad \Omega \\ v(x) = g(x) & \text{on} \quad \partial\Omega \end{cases}$$

has a unique viscosity solution for $\Omega \subset \mathbb{R}^n$ bounded, provided $g \in C(\partial\Omega)$ and either $\sup_\Omega f < 0$ or $\inf_\Omega f > 0$. However, the uniqueness may fail, if f changes the sign (see the counter-example in [10, Appendix A]).

We recall a comparison principle result:

Proposition 2.1 *(Comparison principle, see [3,10]). Let $f \in C(\Omega)$ such that $f > 0$, $f < 0$ or $f = 0$ in Ω. If $u, v \in C(\overline{\Omega})$ satisfy*

$$\Delta_\infty u(x) \geq f(x) \geq \Delta_\infty v(x) \;\; \text{in } \Omega, \tag{2.1}$$

then

$$\sup_\Omega (u - v) = \sup_{\partial\Omega} (u - v). \tag{2.2}$$

We construct solutions by Perron's method. We state the following theorem independently of the (E_ε) context, since it may be of independent interest. For the proof we refer to [15] (see also [1]).

Theorem 2.1. *Let $f \in C^{0,1}(\Omega \times [0, \infty))$ be a bounded real function. Suppose that there exist a viscosity sub-solution $\underline{u} \in C(\overline{\Omega}) \cap C^{0,1}(\Omega)$ and super-solution $\overline{u} \in C(\overline{\Omega}) \cap C^{0,1}(\Omega)$ to $\Delta_\infty u = f(x, u)$ satisfying $\underline{u} = \overline{u} = \varphi \in C(\partial\Omega)$. Define the class of functions*

$$\mathcal{S}_\varphi^f := \left\{ w \in C(\overline{\Omega}) \;\middle|\; \begin{array}{c} w \text{ is a viscosity super-solution to} \\ \Delta_\infty u(x) = f(x, u) \text{ in } \Omega \text{ such that } \underline{u} \leq w \leq \overline{u} \\ \text{and } w = \varphi \text{ on } \partial\Omega \end{array} \right\}.$$

Then,

$$u(x) := \inf_{w \in \mathcal{S}_\varphi^f} w(x), \text{ for } x \in \overline{\Omega} \tag{2.3}$$

is a continuous viscosity solution to $\Delta_\infty u(x) = f(x,u)$ *in* Ω *with* $u = \varphi$ *continuously on* $\partial\Omega$.

Existence of the solution to problem (E_ε) follows by choosing $\underline{u} := \underline{u}^\varepsilon$ and $\overline{u} := \overline{u}^\varepsilon$ respectively as solutions to the following boundary value problems:

$$\begin{cases} \Delta_\infty \underline{u}^\varepsilon = \sup\limits_{\Omega\times[0,\infty)} \zeta_\varepsilon & \text{in} \quad \Omega \\ \underline{u}^\varepsilon = \varphi^\varepsilon & \text{on} \quad \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \Delta_\infty \overline{u}^\varepsilon = 0 & \text{in} \quad \Omega \\ \overline{u}^\varepsilon = \varphi^\varepsilon & \text{on} \quad \partial\Omega. \end{cases}$$

Then $\underline{u}^\varepsilon \in C(\overline{\Omega}) \cap C^{0,1}(\Omega)$ and $\overline{u}^\varepsilon \in C(\overline{\Omega}) \cap C^{0,1}(\Omega)$ (see [3,9,10]) are respectively a viscosity sub and super-solutions of (E_ε). We state this as a theorem:

Theorem 2.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain and let* $\varphi^\varepsilon \in C(\partial\Omega)$ *be a nonnegative boundary datum. Then for each fixed* $\varepsilon > 0$ *there exists a (nonnegative) viscosity solution* $u^\varepsilon \in C(\overline{\Omega})$ *to* (E_ε).

As a consequence of Proposition 2.1, we get (uniform) boundness of any family of viscosity solutions.

Lemma 2.1. *Let* u^ε *be a viscosity solution to* (E_ε). *Then there exists a constant* $C > 0$ *independent of* ε *such that*

$$0 \le u^\varepsilon(x) \le C \quad \text{in} \quad \Omega.$$

Next, we recall (see [14]) a Hopf's type lemma below for a future reference.

Lemma 2.2. *Let* u *be a viscosity solution to*

$$\begin{cases} \Delta_\infty u = f & \text{in} \quad B_r(z) \\ u \ge 0 & \text{in} \quad B_r(z). \end{cases}$$

If for some $x_0 \in \partial B_r(z)$,

$$u(x_0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu}(x_0) \le \theta,$$

where ν *is the inward normal vector at* x_0, *then there exists a universal constant* $C > 0$ *such that*

$$u(z) \le C\theta r.$$

Notations. We finish this section by introducing some notations which we shall use in the paper.

- ✓ $\Omega_\varepsilon := \{x \in \Omega \mid 0 \le u^\varepsilon \le \varepsilon\}$ means the ε-level region.
- ✓ $\Gamma_\varepsilon := \{x \in \Omega \mid u^\varepsilon = \varepsilon\}$ means the ε-level surfaces.

✓ $\mathfrak{P}(u_0, \Omega') := \{u_0 > 0\} \cap \Omega'$.
✓ $\mathfrak{F}(u_0, \Omega') := \partial\{u_0 > 0\} \cap \Omega'$ shall mean the free boundary.
✓ $d_\varepsilon(x_0) := \text{dist}(x_0, \Omega_\varepsilon)$.
✓ $\mathcal{N}_\delta(G) := \{x \in \mathbb{R}^n \mid \text{dist}(x, G) < \delta\}$ with $G \subset \mathbb{R}^n$.
✓ $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure.
✓ $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.
✓ $\Omega' \Subset \Omega$ means that $\Omega' \subset \overline{\Omega'} \subset \Omega$, and $\overline{\Omega'}$ is compact (Ω' is compactly contained in Ω).
✓ $\mathfrak{D}(u, B_r(x_0)) := \frac{\mathcal{L}^n(\{u>0\}\cap B_r(x_0))}{\mathcal{L}^n(B_r(x_0))}$ indicates the positive density.

Remark 2.1. Throughout this paper universal constants are the ones depending only on physical parameters: dimension and structural properties of the problem, i.e. on n, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

3. Uniform Lipschitz regularity

In this section we prove that viscosity solutions to (E_ε) are (uniformly) locally Lipschitz continuous (which, in view of Theorem 4.1 below (see also Remark 6.1), is optimal).

Theorem 3.1. *Let u^ε be a viscosity solution to (E_ε). For every $\Omega' \Subset \Omega$, there exists a positive constant C_0, independent of ε, such that*

$$\|\nabla u^\varepsilon\|_{L^\infty(\Omega')} \leq C_0(\mathcal{A}, \mathcal{B}, \mathcal{C}, \text{dist}(\Omega', \partial\Omega), \text{diam}(\Omega)).$$

Proof. At first we analyze the closed region $\Omega_\varepsilon := \{0 \leq u^\varepsilon \leq \varepsilon\} \cap \Omega'$. Let $\varepsilon \ll \frac{1}{3}\text{dist}(\Omega', \partial\Omega)$. We fix $x_0 \in \Omega_\varepsilon$ and define $v : B_1 \to \mathbb{R}$ by

$$v(y) := \frac{u^\varepsilon(x_0 + \varepsilon y)}{\varepsilon}.$$

Then one has

$$\Delta_\infty v = \varepsilon\zeta_\varepsilon(x_0 + \varepsilon y, \varepsilon v(y)) := f_\varepsilon(y) \quad \text{in} \quad B_1$$

in the viscosity sense. From (1.1) we have that

$$0 \leq f_\varepsilon(y) \leq \mathcal{B} + \varepsilon\mathcal{C} \leq C_\star(\mathcal{B}, \mathcal{C}, \text{dist}(\Omega', \partial\Omega)).$$

Since $f_\varepsilon \in C^1$, then v is locally differentiable and moreover (see Theorem 2 and Corollary 2 of [9]),

$$|\nabla v(0)| \leq 4 \sup_{B_1} v + \frac{1}{2}4^{\frac{1}{3}}\|f_\varepsilon\|^{\frac{1}{3}}_{L^\infty(B_1)}. \tag{3.1}$$

Since

$$v(0) = \frac{u^\varepsilon(x_0)}{\varepsilon} \leq 1,$$

Lemma 2.1 and the Harnack inequality (see Theorem 7.1 of [3]) imply

$$\|v\|_{L^\infty(B_1)} \le C(\mathcal{A}, \mathcal{B}, \mathcal{C}). \tag{3.2}$$

Combining (3.1) and (3.2), we get

$$|\nabla u^\varepsilon(x_0)| = |\nabla v(0)| \le C_0, \tag{3.3}$$

for some $C_0 = C_0(\mathcal{A}, \mathcal{B}, \mathcal{C}, \operatorname{dist}(\Omega', \partial\Omega), \operatorname{diam}(\Omega)) > 0$ independent of ε.

Now we turn our attention to the case of open region $\{u^\varepsilon > \varepsilon\} \cap \Omega'$. Let

$$\Gamma_\varepsilon := \{x \in \Omega' \mid u^\varepsilon(x) = \varepsilon\}.$$

For a fixed $x_1 \in \{u^\varepsilon > \varepsilon\} \cap \Omega'$, define $r := \operatorname{dist}(x_1, \Gamma_\varepsilon)$. We define also a function $v_r \colon B_1 \to \mathbb{R}$ by

$$v_r(y) := \frac{u^\varepsilon(x_1 + ry) - \varepsilon}{r},$$

and note that

$$\Delta_\infty v_r = r\zeta_\varepsilon(x_1 + ry, rv_r(y) + \varepsilon) := \mathfrak{g}(y),$$

in the viscosity sense. The choice of r implies that $u^\varepsilon(x_1 + ry) > \varepsilon$, for every $y \in B_1$, thus, it follows from (1.1) that $\mathfrak{g}$ is smooth enough and bounded, independently of ε, i.e.,

$$\|\mathfrak{g}\|_{L^\infty(B_1)} \le K_0(\mathcal{B}, \mathcal{C}, \operatorname{diam}(\Omega)).$$

Now let $z_0 \in \Gamma_\varepsilon$ be such that $r = |x_1 - z_0|$. As in the previous case from (3.3) one has

$$|\nabla u^\varepsilon(z_0)| \le C_0(\mathcal{A}, \mathcal{B}, \mathcal{C}, \operatorname{dist}(\Omega', \partial\Omega), \operatorname{diam}(\Omega)). \tag{3.4}$$

Moreover, for $y_0 := \frac{z_0 - x_1}{|z_0 - x_1|} \in \partial B_1$ we have

$$v_r(y_0) = 0 \quad \text{and} \quad \frac{\partial v_r}{\partial \nu}(y_0) \le |\nabla v_r(y_0)| \le C_0.$$

Therefore, by the Lemma 2.2

$$v_r(0) \le C(\mathcal{A}, \mathcal{B}, \mathcal{C}, \operatorname{dist}(\Omega', \partial\Omega), \operatorname{diam}(\Omega)),$$

and this finishes the proof. □

4. Further properties of solutions

In this section we prove several properties of solutions. In particular, we show that solutions grow as a linear function out of ε-level surfaces, inside $\{u^\varepsilon > \varepsilon\}$. This is an optimal estimate, when considered uniform in ε. The proof is based on building an appropriate barrier function. We consider degenerate elliptic equations of the form

$$\Delta_\infty u = \zeta(x,u) \text{ in } \mathbb{R}^n,$$

where the reaction term satisfies the non-degeneracy assumption:

$$\inf_{\mathbb{R}^n \times [a,b]} \zeta(x,t) > 0. \tag{4.1}$$

Proposition 4.1 *(Infinity Laplacian's barrier). Let $0 < a < b < 1$ be fixed. For α and A_0 positive numbers (to be chosen) a posteriori, there exists a radially symmetric function $\Theta_L : \mathbb{R}^n \to \mathbb{R}$ satisfying*

✓ $\Theta_L \in W^{2,\infty}(\mathbb{R}^n) \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n)$,

✓

$$\Delta_\infty \Theta_L(x) \le \zeta(x, \Theta_L(x)) \quad \textit{in} \quad \mathbb{R}^n, \tag{4.2}$$

✓ *there exists a universal $\kappa_0 > 0$ constant such that*

$$\Theta_L(x) \ge 4\kappa_0 L \quad \textit{for} \quad |x| \ge 4L, \tag{4.3}$$

where $L \ge L_0 := \sqrt{\frac{b-a}{A_0}}$.

Proof. Define

$$\Theta_L(x) := \begin{cases} a & \text{for} \quad 0 \le |x| < L; \\ A_0(|x|-L)^2 + a & \text{for} \quad L \le |x| < L + L_0; \\ \psi(L) - \phi(L)|x|^{-\alpha} & \text{for} \quad |x| \ge L + L_0. \end{cases} \tag{4.4}$$

where

$$\phi(L) = \frac{2}{\alpha}\sqrt{(b-a)A_0}\,(L+L_0)^{1+\alpha} \text{ and } \psi(L) = b + \phi(L)\,(L+L_0)^{-\alpha}. \tag{4.5}$$

Clearly $\Theta_L \in W^{2,\infty}(\mathbb{R}^n) \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n)$. Moreover, for $0 \le |x| < L$ the inequality (4.2) is true. In the region $L \le |x| < L + L_0$, we have

$$D_i \Theta_L(x) = 2A_0 \frac{(|x|-L)}{|x|} x_i$$

and

$$D_{ij}\Theta_L(x) = 2A_0\left[\left(\frac{1}{|x|^2} - \frac{(|x|-L)}{|x|^3}\right)x_i \cdot x_j + \frac{(|x|-L)}{|x|}\delta_{ij}\right].$$

Therefore, we obtain

$$\begin{aligned}
\Delta_\infty\Theta_L(x) &= \sum_{i,j=1}^{n} D_i\Theta_L \cdot D_j\Theta_L \cdot D_{ij}\Theta_L \\
&= 8A_0^3\frac{(|x|-L)^2}{|x|^2}\sum_{i,j=1}^{n}\left[\left(\frac{1}{|x|^2} - \frac{(|x|-L)}{|x|}\right)x_i^2x_j^2 + \frac{|x|-L}{|x|}x_i \cdot x_j\delta_{ij}\right] \\
&= 8A_0^3\frac{(|x|-L)^2}{|x|^2}\left[\left(\frac{1}{|x|^2} - \frac{(|x|-L)}{|x|}\right)|x|^4 + \frac{(|x|-L)}{|x|}|x|^2\right] \\
&= 8A_0^3\frac{(|x|-L)^2}{|x|^2}|x|^2 = 8A_0^3(|x|-L)^3 \le 8A_0^3L_0^3 \\
&= (2\sqrt{A_0(b-a)})^3.
\end{aligned}$$

By construction

$$a \le \Theta_L(x) \le b$$

and so, for A_0 sufficiently small, we get

$$\Delta_\infty\Theta_L(x) \le \inf_{\mathbb{R}^n\times[a,b]}\zeta(x,t) \le \zeta(x,\Theta_L(x)).$$

Now, let us turn our attention to the set $|x| \ge L + L_0$. Direct computation shows that

$$D_i\Theta_L(x) = \alpha\phi(L)\frac{x_i}{|x|^{\alpha+2}}$$

and

$$D_{ij}\Theta_L(x) = \alpha\phi(L)|x|^{-(\alpha+2)}\left(-\frac{(\alpha+2)}{|x|^2}x_ix_j + \delta_{ij}\right),$$

hence

$$\Delta_\infty\Theta_L(x) = -\alpha^3\phi^3(L)(\alpha+1)\frac{1}{|x|^{3\alpha+4}}.$$

Finally, for $\alpha > 0$ we get

$$\Delta_\infty\Theta_L(x) \le 0 \le \zeta(x,\Theta_L(x)).$$

Therefore, Θ_L satisfies (4.2). Finally, by (4.5)

$$|x| \geq 4L \geq 2(L+L_0) = 2\left(\frac{\phi(L)}{\psi(L)-b}\right)^{\frac{1}{\alpha}}$$

and hence

$$\Theta_L(x) = \psi(L) - \phi(L)|x|^{-\alpha} \geq \psi(L) - 2^{-\alpha}(\psi(L)-b) \geq C_\alpha \psi(L),$$

for $\alpha > 1$. Therefore,

$$\Theta_L(x) \geq 4\kappa_0 L,$$

where $\kappa_0 > 0$ depends on n and $(b-a)$. □

4.1. Linear growth

In order to establish lower bounds on the growth speed of the solution to (E_ε) inside the set $\{u^\varepsilon > \varepsilon\}$, the strategy now is to consider appropriate scaling versions of the universal barrier Θ_L.

Theorem 4.1. *Let u^ε be a solution of (E_ε). There exists a universal constant $c > 0$ such that for any $x_0 \in \{u^\varepsilon > \varepsilon\}$ and $0 < \varepsilon \leq d_\varepsilon(x_0) \ll 1$ one has*

$$u^\varepsilon(x_0) \geq c d_\varepsilon(x_0).$$

Proof. Without loss of generality we assume that $x_0 = 0$. Set $\eta = \dfrac{d_\varepsilon(0)}{3}$ and define

$$\Theta_\varepsilon(x) := \varepsilon \Theta_{\frac{\eta}{4\varepsilon}}\left(\frac{x}{\varepsilon}\right).$$

Using (4.3) and (4.4) we verify that for $4L_0\varepsilon \leq \eta$,

$$\Theta_\varepsilon(0) = a\varepsilon \quad \text{and} \quad \Theta_{\varepsilon|\partial B_\eta} \geq \kappa_0 \eta. \tag{4.6}$$

Now, we claim that there exists a $z_0 \in \partial B_\eta$ such that

$$\Theta_\varepsilon(z_0) \leq u^\varepsilon(z_0). \tag{4.7}$$

In fact, if

$$\Theta_\varepsilon(x) > u^\varepsilon(x) \quad \text{in} \quad \partial B_\eta,$$

then the auxiliary function

$$v^\varepsilon := \min\{\Theta_\varepsilon, u^\varepsilon\}$$

would be a super-solution to (E_ε), but v^ε is strictly below u^ε, which contradicts the minimality of u^ε. Therefore, by (4.6) and (4.7), we obtain

$$\kappa_0\eta \le \Theta_\varepsilon(z_0) \le u^\varepsilon(z_0) \le \sup_{B_\eta} u^\varepsilon. \tag{4.8}$$

Furthermore, u^ε satisfies (in the viscosity sense)

$$c_0 \le \Delta_\infty u^\varepsilon \le c_1 \quad \text{in} \quad B_{3\eta}.$$

Hence, by Harnack inequality (see Theorem 7.1 of [3]), we get

$$\sup_{B_\eta} u^\varepsilon \le 9u^\varepsilon(0) + 12\sigma\left(\left(\frac{3\eta}{2}\right)^4 c_1\right)^{1/3}.$$

Thus, by (4.8)

$$u^\varepsilon(0) \ge \frac{1}{9}\left(\kappa_0 - C\eta^{1/3}\right)\eta.$$

Finally, by taking $\eta > 0$ small enough we conclude

$$u^\varepsilon(0) \ge c\,\eta,$$

for some $0 < c < 1$ (independent of ε). □

As a consequence of the Lipschitz regularity, Theorem 3.1 and Theorem 4.1, we are able to completely control u^ε in terms of $d_\varepsilon(x_0)$.

Corollary 4.1. *For a sub-domain $\Omega' \Subset \Omega$, there exists $C > 0$, depending on universal parameters and Ω', such that for $x_0 \in \mathfrak{P}(u^\varepsilon - \varepsilon, \Omega')$ and $\varepsilon \le d_\varepsilon(x)$, there holds*

$$C^{-1}d_\varepsilon(x_0) \le u^\varepsilon(x_0) \le C\,d_\varepsilon(x_0).$$

Proof. The inequality from below is exactly the Theorem 4.1. Now take $y_0 \in \mathfrak{F}(u^\varepsilon - \varepsilon, \Omega')$, such that $|y_0 - x_0| = d_\varepsilon(x_0)$. From Theorem 3.1,

$$u^\varepsilon(x_0) \le C\,d_\varepsilon(x_0) + u^\varepsilon(y_0) \le C\ d_\varepsilon(x_0),$$

and the corollary is proved. □

4.2. Strong non-degeneracy

Next we see that solutions are strongly non-degenerate close to ε-level sets. This means that the maximum of u^ε on the boundary of a ball B_r centered in $\{u^\varepsilon > \varepsilon\}$ is of order r.

Theorem 4.2. *Let $\Omega' \Subset \Omega$. There exists a universal constant $c > 0$ such that for $x_0 \in \mathfrak{P}(u^\varepsilon - \varepsilon, \Omega')$, $\varepsilon \le \rho \ll 1$, there holds*

$$c\,\rho < \sup_{B_\rho(x_0)} u^\varepsilon \le c^{-1}(\rho + u^\varepsilon(x_0)).$$

Proof. By taking $\Theta_\varepsilon(x) = \varepsilon\Theta_{\frac{\rho}{4\varepsilon}}(x)$ we have

$$u^\varepsilon(z) > \Theta_\varepsilon(z),$$

for some point $z \in \partial B_\rho(x_0)$. Note that

$$\kappa_0\rho \le \Theta_\varepsilon(z) < u^\varepsilon(z) \le \sup_{B_\rho(x_0)} u^\varepsilon,$$

where κ_0 is as in Proposition 4.1. The upper estimate is a direct consequence of the Lipschitz regularity. □

As a consequence we get a positive density result.

Corollary 4.2. *Let $x_0 \in \{u^\varepsilon > \varepsilon\}$ and $\varepsilon \le \rho \ll 1$. There exists a universal constant $c_0 \in (0, 1)$ such that*

$$\mathfrak{D}(u^\varepsilon - \varepsilon, B_\rho(x_0)) \ge c_0.$$

Proof. As we saw in the previous theorem, there exists $y_0 \in B_\rho(x_0)$ such that

$$u^\varepsilon(y_0) \ge c_0\rho.$$

On the other hand, by Lipschitz regularity, for $z \in B_{\kappa\rho}(y_0)$, we have

$$u^\varepsilon(z) + C\kappa\rho \ge u^\varepsilon(y_0).$$

Thus, by using the estimates from above, we are able to choose $\kappa > 0$ small enough in order to have

$$z \in B_{\kappa\rho}(y_0) \cap B_\rho(x_0) \quad \text{and} \quad u^\varepsilon(z) > \varepsilon.$$

So we conclude that there exists a portion of $B_\rho(x_0)$ with volume of order $\sim \rho^n$ within $\{u^\varepsilon > \varepsilon\}$. Therefore, we have a uniform positive density result for the solution of (E_ε). More precisely,

$$\mathcal{L}^n(B_\rho(x_0) \cap \{u^\varepsilon > \varepsilon\}) \ge \mathcal{L}^n(B_\rho(x_0) \cap B_{\kappa\rho}(y_0)) = c_0\,\mathcal{L}^n(B_\rho(x_0)),$$

for some constant $c_0 > 0$ independent of ε. □

4.3. Harnack type inequality

For solutions of (E_ε) the Harnack inequality is valid for balls that touch the free boundary along the ε-layers, i.e., $\partial\{u^\varepsilon > \varepsilon\}$.

Theorem 4.3. *Let u^ε be a solution of* (E_ε)*. Let also $x_0 \in \{u^\varepsilon > \varepsilon\}$ and $\varepsilon \le d := d_\varepsilon(x_0)$. Then,*

$$\sup_{B_{\frac{d}{2}}(x_0)} u^\varepsilon(x) \le C \inf_{B_{\frac{d}{2}}(x_0)} u^\varepsilon(x)$$

for a universal constant $C > 0$ independent of ε.

Proof. Let z_1, z_2 be extremal points for u^ε in $\overline{B_{\frac{d}{2}}(x_0)}$, i.e.,

$$\inf_{B_{\frac{d}{2}}(x_0)} u^\varepsilon(x) = u^\varepsilon(z_1) \quad \text{and} \quad \sup_{B_{\frac{d}{2}}(x_0)} u^\varepsilon(x) = u^\varepsilon(z_2).$$

Since $d_\varepsilon(z_1) \ge \frac{d}{2}$, by Corollary 4.1

$$u^\varepsilon(z_1) \ge C_1 d. \tag{4.9}$$

Moreover, by Theorem 4.2

$$u^\varepsilon(z_2) \le C_2\left(\frac{d}{2} + u^\varepsilon(x_0)\right). \tag{4.10}$$

Taking $y \in \partial\{u^\varepsilon > \varepsilon\}$ such that $d = |x_0 - y|$ and $z \in \overline{B_d(y)} \cap \{u^\varepsilon > \varepsilon\}$, we get from Corollary 4.1 and Theorem 4.2

$$u^\varepsilon(x_0) \le \sup_{B_d(z)} u^\varepsilon \le C_2(d + u^\varepsilon(z)) \le C_3 d. \tag{4.11}$$

Combining (4.9), (4.10) and (4.11), we conclude

$$\sup_{B_{\frac{d}{2}}(x_0)} u^\varepsilon(x) \le C \inf_{B_{\frac{d}{2}}(x_0)} u^\varepsilon(x). \quad \square$$

4.4. Porosity of the level surfaces

As a consequence of the growth rate and the non-degeneracy property, we get porosity of level sets.

Definition 4.1. A set $E \subset \mathbb{R}^n$ is called porous with porosity $\delta > 0$, if $\exists\, R > 0$ such that

$$\forall x \in E,\ \ \forall r \in (0, R),\ \ \exists y \in \mathbb{R}^n \ \text{such that}\ B_{\delta r}(y) \subset B_r(x) \setminus E.$$

A porous set of porosity δ has Hausdorff dimension not exceeding $n - c\delta^n$, where $c = c(n) > 0$ is a constant depending only on n. In particular, a porous set has Lebesgue measure zero (see, for example, [17]).

Theorem 4.4. *Let u^ε be a solution of* (E_ε)*. Then the level sets $\partial\{u^\varepsilon > \varepsilon\}$ are porous with porosity constant independent of ε.*

Proof. Let $R > 0$ and $x_0 \in \Omega$ be such that $\overline{B_{4R}(x_0)} \subset \Omega$.

We aim to prove the set $\mathfrak{F}(u^\varepsilon - \varepsilon, B_R(x_0))$ is porous.

Let $x \in \mathfrak{F}(u^\varepsilon - \varepsilon, B_R(x_0))$. For each $r \in (0, R)$ we have $\overline{B_r(x)} \subset B_{2R}(x_0) \subset \Omega$. Let $y \in \partial B_r(x)$ such that $u^\varepsilon(y) = \sup\limits_{\partial B_r(x)} u^\varepsilon$. By non-degeneracy

$$u^\varepsilon(y) \geq cr, \tag{4.12}$$

where $c > 0$ is a constant. On the other hand, we know that near the free boundary

$$u^\varepsilon(y) \leq C d_\varepsilon(y), \tag{4.13}$$

where $C > 0$ is a constant, and $d_\varepsilon(y)$ is the distance of y from the set $\overline{B_{2R}(x_0)} \cap \Gamma_\varepsilon$. Now, from (4.12) and (4.13) we get

$$d_\varepsilon(y) \geq \delta r \tag{4.14}$$

for a positive constant $\delta < 1$.

Let now $y^* \in [x, y]$ be such that $|y - y^*| = \frac{\delta r}{2}$, then it is not hard to see that

$$B_{\frac{\delta}{2}r}(y^*) \subset B_{\delta r}(y) \cap B_r(x). \tag{4.15}$$

Indeed, for each $z \in B_{\frac{\delta}{2}r}(y^*)$

$$|z - y| \leq |z - y^*| + |y - y^*| < \frac{\delta r}{2} + \frac{\delta r}{2} = \delta r,$$

and

$$|z - x| \leq |z - y^*| + \big(|x - y| - |y^* - y|\big) < \frac{\delta r}{2} + \left(r - \frac{\delta r}{2}\right) = r,$$

and (4.15) follows.

Since by (4.14) $B_{\delta r}(y) \subset B_{d_\varepsilon(y)}(y) \subset \{u^\varepsilon > \varepsilon\}$, then

$$B_{\delta r}(y) \cap B_r(x) \subset \{u^\varepsilon > \varepsilon\},$$

which together with (4.15) provides

$$B_{\frac{\delta}{2}r}(y^*) \subset B_{\delta r}(y) \cap B_r(x) \subset B_r(x) \setminus \partial\{u_\varepsilon > \varepsilon\} \subset B_r(x) \setminus \mathfrak{F}(u^\varepsilon - \varepsilon, B_R(x_0)). \quad \square$$

5. Hausdorff measure estimates

In this section we prove the finiteness of the $(n-1)$-dimensional Hausdorff measure of level surfaces. For that we restrict ourselves to the case when the reaction term, which propagates up to the free boundary, is non-degenerate. Suppose that $a = 0$ in (1.2) and for some $b > 0$

$$\mathfrak{R}_0 := \inf_{\Omega\times[0,b]} \varepsilon\zeta_\varepsilon(x, \varepsilon t) > 0. \tag{5.1}$$

Definition 5.1 *(Asymptotic concavity property).* We say that an operator $F : \Omega \times Sym(n) \to \mathbb{R}$ is *asymptotically concave*, if there exists

$$\mathfrak{A} \in \mathcal{A}_{\lambda,\Lambda} := \left\{ A \in \mathrm{Sym}(n) \;\middle|\; \lambda\|\xi\|^2 \le \sum_{i,j=1}^{n} A_{ij}\xi_i\xi_j \le \Lambda\|\xi\|^2,\ \forall \xi \in \mathbb{R}^n \right\}$$

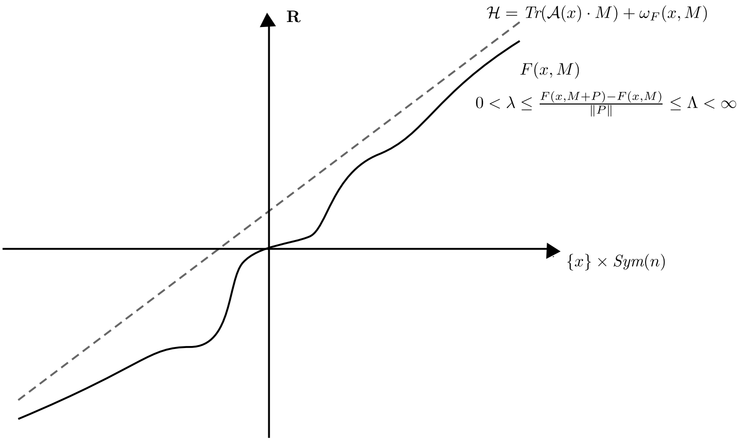
and a continuous function $\omega_F : \Omega \times Sym(n) \to \mathbb{R}$ such that

$$F(x, M) \le \mathrm{Tr}(\mathfrak{A}(x)\cdot M) + \omega_F(x, M),\ \forall\ (x, M) \in \Omega \times Sym(n), \tag{\textbf{ACP}}$$

with

$$\lim_{\|M\|\to\infty} |\omega_F(x, M)| := \mathcal{K} < \infty, \quad \forall\ x \in \Omega. \tag{5.2}$$

Remark 5.1. The (ACP) condition is weaker than concavity assumption. Geometrically, it means that for each $x \in \Omega$ fixed, there exists a hyperplane which decomposes $\mathbb{R} \times Sym(n)$ in two semi-spaces such that the graph of $F(x, \cdot)$ is always below this hyperplane. Moreover, by assuming $F(x, 0) = 0$, the assumption (5.2) means that the distance from the hyperplane to the graph of F goes to infinity for matrices with big enough norms (see [1] and [13]).



Definition 5.2. Let v be the solution of (E_ε). We write $v \in \mathcal{S}(F, G, H)$, if

$$\Delta_\infty v \le G(|Dv|)F(x, D^2v) + H(x, |Dv|),$$

where

- ✓ $F : \Omega \times Sym(n) \to \mathbb{R}$ is a fully nonlinear uniformly elliptic operator with $F(x, 0) = 0$;
- ✓ $G : \mathbb{R}_+ \to \mathbb{R}$ is a non-negative continuous function and injective;
- ✓ $H : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is a bounded continuous function.

Example 1 *(φ-Laplacian operator).* The φ-Laplacian operator in Orlicz–Sobolev spaces can be defined as

$$\Delta_\varphi u = \frac{\varphi(|\nabla u|)}{|\nabla u|}\left[\Delta u + \left\{\frac{\varphi'(|\nabla u|)|\nabla u|}{\varphi(|\nabla u|)} - 1\right\}\frac{\Delta_\infty u}{|\nabla u|^2}\right],$$

for an appropriate increasing function $\varphi : [0, \infty) \to [0, \infty)$ satisfying the generalized Ladyzhenskaya–Ural'tseva condition:

$$0 < g_0 \le \frac{\varphi'(t)t}{\varphi(t)} \le g_1, \quad \text{if} \quad t > 0,$$

where g_0 and g_1 are constants. Therefore, for a φ-harmonic function one has (where $\nabla u \neq 0$)

$$\Delta_\infty u \le \frac{\varphi(|\nabla u|)|\nabla u|^2}{\varphi'(|\nabla u|)|\nabla u| - \varphi(|\nabla u|)}\Delta u.$$

Example 2 *(Convex functions).* For convex functions we have following relation

$$\Delta_\infty u = \langle D^2 u Du, Du\rangle \le |\nabla u|^2 \Delta u,$$

since $\|D^2 u\|$ is controlled by Δu.

The proof of the following proposition is similar to the corresponding result from [1]. We sketch it here for reader's convenience.

Proposition 5.1. *For the every fixed $\Omega' \Subset \Omega$, $\rho < \operatorname{dist}(\Omega', \partial\Omega)$ and $C \gg 1$, there exists a universal ε_0 such that*

$$\int_{B_\rho(x_\varepsilon)} [\zeta_\varepsilon(x, u^\varepsilon(x)) - C]\, dx \ge 0, \tag{5.3}$$

for any $x_\varepsilon \in \mathfrak{F}(u^\varepsilon - \varepsilon, \Omega')$ whenever $\varepsilon \le \varepsilon_0$.

Proof. If (5.3) is not true, then there are $C_0 > 0$ and $\rho < \operatorname{dist}(\Omega', \partial\Omega)$ such that

$$\int_{B_\rho(x_k)} \left(\zeta_{\varepsilon_k}(x, u^{\varepsilon_k}) - C_0\right) dx < 0,$$

for points $x_{\varepsilon_k} \in \mathfrak{F}(u^{\varepsilon_k} - \varepsilon_k, \Omega')$ and a sequence $\varepsilon_k \to 0$ as $k \to \infty$. Define

$$v_k(y) := \frac{bu^{\varepsilon_k}(x_{\varepsilon_k} + \varepsilon_k y)}{\varepsilon_k}.$$

Then

$$\int_{B_{\rho/\varepsilon_k}} \left((\varepsilon_k b^{-1}) \zeta_{\varepsilon_k}(x_{\varepsilon_k} + \varepsilon_k y, \varepsilon_k b^{-1} v_k) - C_0 \varepsilon_k b^{-1} \right) dx < 0. \tag{5.4}$$

Note that

$$\| \Delta_\infty v_k \|_{L^\infty(B_{\rho/\varepsilon_k})} \le \frac{\mathcal{B} + \mathcal{C}}{b},$$

independent of ε.

By the regularity of v_k one has (up to a subsequence) that

$$v_\infty := \lim_{k \to \infty} v_k,$$

in the $C^{0,\alpha}_{\text{loc}}$ topology. Combining (5.1) and (5.4), we deduce that

$$\text{either } v_\infty \equiv 0, \quad \text{or else} \quad v_\infty \ge b, \text{ everywhere in } \mathbb{R}^n.$$

The first case is not possible since $v_\infty(0) = b > 0$. If $v_\infty \ge b$, we have that 0 is a minimum point, which leads to a contradiction, since by non-degeneracy

$$0 = |\nabla v_\infty(0)| = |\nabla u^{\varepsilon_k}(0)| + \mathrm{o}(1) \ge c > 0. \quad \square$$

Thus, combining the **(ACP)** condition and the Proposition 5.1, we obtain:

Lemma 5.1. *Let $u^\varepsilon \in \mathcal{S}(F, G, H)$ with F being asymptotically concave and let $x_\varepsilon \in \mathfrak{F}(u^\varepsilon - \varepsilon, \Omega')$. Then*

$$\int_{B_\rho(x_\varepsilon)} A_{ij} u^\varepsilon_{ij} \, dx \ge 0. \tag{5.5}$$

Proof. Note that

$$F(x, D^2 u^\varepsilon) \ge [\zeta_\varepsilon(x, u^\varepsilon) - H(x, |\nabla u^\varepsilon|)] G(|\nabla u^\varepsilon|)^{-1}$$

in $\{u^\varepsilon > \varepsilon\} \cap \Omega'$, for any $\Omega' \Subset \Omega$. Hence, by Lipschitz regularity and properties of G and H, one has

$$F(x, D^2 u^\varepsilon) \ge [\zeta_\varepsilon(x, u^\varepsilon) - C_H] G(C)^{-1}.$$

Therefore, by **(ACP)** condition

$$\int_{B_\rho(x_\varepsilon)} A_{ij} u^\varepsilon_{ij}\, dx \geq \int_{B_\rho(x_\varepsilon)} \left[(\zeta_\varepsilon(x,u^\varepsilon) - C_H)G(C)^{-1} - \mathcal{K}\right] dx$$
$$\geq G(C)^{-1} \int_{B_\rho(x_\varepsilon)} \left[\zeta_\varepsilon(u^\varepsilon) - (C_H + G(C)\mathcal{K})\right] dx,$$

where $C > 0$ comes from the universal control on the Lipschitz norm in $B_\rho(x_\varepsilon)$. Combining the estimate above and the Proposition 5.1, we obtain (5.5). □

Lemma 5.1 plays a crucial role in the study of regularity of level surfaces, since it leads to the following result (see Theorem 5.6 in [1]):

Theorem 5.1. *Let $\Omega' \Subset \Omega$ and $u^\varepsilon \in \mathcal{S}(F,G,H)$ with F being asymptotically concave. There exists a $C > 0$ constant depending on Ω' such that*

$$\mathcal{H}^{n-1}(\mathfrak{P}(u^\varepsilon - C_1\varepsilon, B_\rho(x_\varepsilon))) \leq C\rho^{n-1}, \tag{5.6}$$

for some $C_1 > 1$ and for all $x_\varepsilon \in \mathfrak{F}\left(u^\varepsilon - C_1\varepsilon, \Omega'\right)$, provided $d_\varepsilon(x_\varepsilon) < dist(\Omega', \partial\Omega)$ and $C_1\varepsilon \leq \rho$.

6. The limiting problem

As a consequence of Theorem 3.1 and Lemma 2.1 we obtain the following result:

Theorem 6.1. *If $\{u^\varepsilon\}_{\varepsilon>0}$ is a solution to (E_ε), then for any sequence $\varepsilon_k \to 0^+$ there exist a subsequence $\varepsilon_{k_j} \to 0^+$ and $u_0 \in C^{0,1}_{\text{loc}}(\Omega)$ such that*

(1) $u^{\varepsilon_{k_j}} \to u_0$ locally uniformly in Ω;
(2) $0 \leq u_0(x) \leq K_0$ in $\overline{\Omega}$ for some constant K_0 independent of ε;
(3) $\Delta_\infty u_0(x) = g(x)$ in $\Omega \setminus \mathfrak{F}(u_0, \Omega')$, with g being a bounded and nonnegative continuous function.

Remark 6.1. It follows from (3) (using the corresponding regularity result from [9]) that u_0 is locally differentiable in $\mathfrak{P}(u_0, \Omega')$. However, that property deteriorates as $\text{dist}(\partial\Omega', \partial\{u_0 > 0\}) \to 0$. On the other hand, the gradient remains controlled even when $\text{dist}(x_0, \mathfrak{F}(u_0, \Omega')) \to 0$.

Hereafter we will use the following definition when referring to u_0:

$$u_0(x) := \lim_{j\to\infty} u^{\varepsilon_j}(x).$$

Theorem 6.2. *Let $\Omega' \Subset \Omega$. Fix $x_0 \in \mathfrak{P}(u_0, \Omega')$ such that* $\text{dist}(x_0, \mathfrak{F}(u_0, \Omega')) \leq \text{dist}(\Omega', \partial\Omega)$. *Then there exists a constant $C > 0$ independent of ε such that*

$$C^{-1}\text{dist}(x_0, \mathfrak{F}(u_0, \Omega')) \leq u_0(x_0) \leq C\,\text{dist}(x_0, \mathfrak{F}(u_0, \Omega')). \tag{6.1}$$

Proof. From Corollary 4.1 we know that there exists $y_\varepsilon \in \Omega_\varepsilon$ such that

$$d_\varepsilon(x) = |x - y_\varepsilon| \text{ and } u^\varepsilon(x) \geq c\, d_\varepsilon(x) = c\,|x - y_\varepsilon|,$$

for some constant $c > 0$ independent of ε. Passing to a subsequence, if necessary, we get for $y_\varepsilon \to y_0 \in \mathfrak{F}(u_0, \Omega')$

$$u_0(x) \geq c\,|x_0 - y_0| \geq c\,\mathrm{dist}(x, \mathfrak{F}(u_0, \Omega')).$$

Finally, the upper bound is a consequence of the local Lipschitz estimate for u_0. □

The next theorem is an immediate consequence of Theorem 4.2 as $\varepsilon \to 0^+$.

Theorem 6.3. *Let $\Omega' \Subset \Omega$. For any $x_0 \in \mathfrak{P}(u_0, \Omega')$ such that* $\mathrm{dist}(x_0, \mathfrak{F}(u_0, \Omega')) \leq \mathrm{dist}(\Omega', \partial\Omega)$, *there exist constants $C_0 > 0$ and $r_0 > 0$ independent of ε, such that*

$$C_0^{-1} r \leq \sup_{B_r(x_0)} u_0 \leq C_0(r + u_0(x_0))$$

provided $r \leq r_0$.

The following result shows that, in Hausdorff distance, Ω_ε converges to $\mathfrak{P}(u_0, \Omega')$ as $\varepsilon \to 0^+$.

Theorem 6.4. *Let $\Omega' \Subset \Omega$. Then for a $C_1 > 1$, the following inclusions hold:*

$$\mathfrak{P}(u_0, \Omega') \subset \mathcal{N}_\delta(\{u^{\varepsilon_j} > C_1\varepsilon_j\}) \cap \Omega' \text{ and } \{u^{\varepsilon_j} > C_1\varepsilon_j\} \cap \Omega' \subset \mathcal{N}_\delta(\{u_0 > 0\}) \cap \Omega',$$

provided $\varepsilon_j \leq \delta \ll 1$.

Proof. We prove the first inclusion (the other one can be obtained in a similar way). Suppose that it is not true. Then there exists a $\delta_0 > 0$ such that for every $\varepsilon_j \to 0$ and $\forall x_j \in \mathfrak{P}(u_0, \Omega')$

$$\mathrm{dist}(x_j, \{u^{\varepsilon_j} > C_1\varepsilon_j\}) > \delta_0. \tag{6.2}$$

For some $y \in \overline{B_{\frac{\delta_0}{2}}(x_j)} \cap \{u^{\varepsilon_j} > C_1\varepsilon_j\}$ we have from Theorem 6.3

$$u^{\varepsilon_j}(y) = \sup_{B_{\frac{\delta_0}{2}}(x_j)} u^{\varepsilon_j}(x_j) \geq \frac{1}{2} \sup_{B_{\frac{\delta_0}{2}}(x_j)} u_0(x_j) \geq c\delta_0 \geq C_1\varepsilon_j,$$

which contradicts (6.2). □

Theorem 6.5. *Given $\Omega' \Subset \Omega$, there exist constants $C > 0$ and $\rho_0 > 0$, depending only on Ω' and universal parameters, such that for any $x_0 \in \mathfrak{F}(u_0, \Omega')$ there holds*

$$C^{-1}\rho \leq \fint_{\partial B_\rho(x_0)} u_0(x)\, d\mathcal{H}^{n-1} \leq C\,\rho, \tag{6.3}$$

provided $\rho \leq \rho_0$.

Proof. The upper bound follows from the Lipschitz regularity of u_0. The lower bound is a consequence of the nondegeneracy. □

Remark 6.2. Repeating the steps of the proof of Theorem 4.3 one can show that the Harnack inequality is true for u_0 in touching balls. Furthermore, as a consequence of the non-degeneracy and the growth rate, one can prove (as it was done in Theorem 4.4) that the free boundary $\mathfrak{F}(u_0)$ is a porous set.

Next, we prove several geometric-measure properties for $\mathfrak{F}(u_0)$. The ultimate goal is to prove the local finiteness of the $(n-1)$-dimensional Hausdorff measure of the limiting level surface.

First we see that the set $\{u_0 > 0\}$ has uniform density along $\mathfrak{F}(u_0)$.

Theorem 6.6. *Let $\Omega' \Subset \Omega$. There exists a constant $c_0 > 0$ such that for any $x_0 \in \mathfrak{F}(u_0, \Omega')$ there holds*

$$\mathfrak{D}(u_0, B_\rho(x_0)) \geq c_0 \tag{6.4}$$

provided $\rho \ll 1$. In particular, $\mathcal{L}^n(\mathfrak{F}(u_0)) = 0$.

Proof. The estimate (6.4) follows as in the proof of Corollary 4.2. We conclude the result by using Lebesgue differentiation theorem and a covering argument (Besicovitch–Vitali type theorem, see [5]). □

Theorem 6.7. *Let $\Omega' \Subset \Omega$. There exists a constant $C > 0$, depending only on Ω' and universal parameters such that, for any $x_0 \in \mathfrak{F}(u_0, \Omega')$, there holds*

$$\mathcal{H}^{n-1}(\mathfrak{F}(u_0, \Omega') \cap B_\rho(x_0)) \leq C\rho^{n-1}.$$

Proof. From Theorem 6.4, for $j \gg 1$ one has

$$\left[\mathcal{N}_\delta(\mathfrak{F}(u_0, \Omega')) \cap B_\rho(x_0)\right] \subset \left[\mathcal{N}_{4\delta}(\partial\{u^{\varepsilon_j} > C_1\varepsilon_j\}) \cap B_{2\rho}(x_0)\right].$$

Assuming $\varepsilon_j \leq \delta \leq \rho \ll \operatorname{dist}(\Omega', \partial\Omega)$, the hypotheses of Theorem 5.1 are fulfilled, implying the following estimate for the δ-neighborhood,

$$\mathcal{L}^n(\mathcal{N}_\delta(\mathfrak{F}(u_0, \Omega')) \cap B_\rho(x_0)) \leq C\delta\rho^{n-1}.$$

Now, let $\{B_j\}_{j\in\mathbb{N}}$ be a covering of $\mathfrak{F}(u_0, \Omega') \cap B_\rho(x_0)$ by balls with radii $\delta > 0$ and centered at free boundary points on $\mathfrak{F}(u_0, \Omega') \cap B_\rho(x_0)$. Then

$$\bigcup_j B_j \subset \mathcal{N}_\delta(\mathfrak{F}(u_0, \Omega')) \cap B_{\rho+\delta}(x_0).$$

Therefore, there exists a constant $\overline{C} > 0$ with universal dependence such that

$$\begin{aligned}\mathcal{H}^{n-1}_{\delta}(\mathfrak{F}(u_0,\Omega')\cap B_\rho(x_0)) &\leq \overline{C}\sum_j \mathcal{L}^{n-1}(\partial B_j)\\ &= n\frac{\overline{C}}{\delta}\mathcal{L}^n(B_j)\\ &\leq n\frac{\overline{C}}{\delta}\mathcal{L}^n(\mathcal{N}_\delta(\mathfrak{F}(u_0,\Omega'))\cap B_{\rho+\delta}(x_0))\\ &\leq C(n)(\rho+\delta)^{n-1}\\ &= C(n)\rho^{n-1}+o(\delta).\end{aligned}$$

Letting $\delta \to 0^+$ we finish the proof. □

As an immediate consequence of Theorem 6.7 we conclude that $\mathfrak{F}(u_0)$ has locally finite perimeter. Moreover, the reduced free boundary $\mathfrak{F}^\star(u_0) := \partial_{\text{red}}\{u_0 > 0\}$ has a total $\mathcal{H}^{n-1}$ measure in the sense that $\mathcal{H}^{n-1}(\mathfrak{F}(u_0)\setminus\mathfrak{F}^\star(u_0)) = 0$ (Theorem 6.7 in [1]). In particular, the free boundary has an outward vector for $\mathcal{H}^{n-1}$ almost everywhere in $\mathfrak{F}^*(u_0)$.

7. Final comments

We finish the paper by analyzing the one-dimensional profile representing the corresponding free boundary condition. Let

$$u^\varepsilon_{xx}(u^\varepsilon_x)^2 = \zeta_\varepsilon(u^\varepsilon) \quad \text{in} \quad (-1,1), \tag{7.1}$$

where ζ_ε given by

$$\zeta_\varepsilon(s) = \frac{1}{\varepsilon}\zeta\left(\frac{s}{\varepsilon}\right)$$

is a high energy activation potential, i.e., a non-negative smooth function supported in $[0,\varepsilon]$. The limiting configuration satisfies (in the viscosity sense)

$$\Delta_\infty u_0 = 0 \quad \text{in} \quad \{u_0 > 0\}\cap(-1,1).$$

Multiplying (7.1) by u^ε_x we get

$$u^\varepsilon_{xx}(u^\varepsilon_x)^3 = \zeta_\varepsilon(u^\varepsilon).u^\varepsilon_x = \frac{d}{dx}\Xi_\varepsilon(u^\varepsilon), \tag{7.2}$$

where

$$\Xi_\varepsilon(t) = \int_0^{\frac{t}{\varepsilon}}\zeta(s)ds \to \left(\int\zeta(s)ds\right)\chi_{\{t>0\}}$$

as $\varepsilon \to 0^+$, i.e.,

$$\Xi_\varepsilon(u^\varepsilon) \to \int \zeta(s)ds, \quad \text{as} \quad \varepsilon \to 0^+$$

provided $u_0(x) > 0$. Using change of variable

$$u_x^\varepsilon(x) = w,$$

we re-write

$$\int \frac{d}{dx}\Xi_\varepsilon(u^\varepsilon) = \int (u^\varepsilon)_x^3 u_{xx}^\varepsilon dx = \int w^3 dw.$$

Hence, by computing the anti-derivatives at (7.2) and letting $\varepsilon \to 0^+$ we obtain the following characterization for limiting condition

$$|u_0'| = \sqrt[4]{4\int \zeta(s)ds} \quad \text{on} \quad \partial\{u_0 > 0\}.$$

Therefore, the corresponding one-dimensional limiting free boundary problem is given by

$$\begin{cases} \Delta_\infty u_0 = 0 & \text{in} \quad \{u_0 > 0\} \cap (-1, 1), \\ u_0 = 0 & \text{in} \quad \partial\{u_0 > 0\}, \\ |u_0'| = \sqrt[4]{4\int \zeta(s)ds} & \text{on} \quad \partial\{u_0 > 0\}. \end{cases}$$

Furthermore, if for some direction x_i we have

$$u_{x_i x_i}^\varepsilon (u_{x_i}^\varepsilon)^2 \leq \zeta_\varepsilon(u^\varepsilon) \quad \text{in} \quad \Omega,$$

then by repeating the previous argument (since u^ε is increasing in direction x_i), we conclude

$$\left|\frac{\partial u_0}{\partial x_i}\right| \leq \sqrt[4]{4\int \zeta(s)ds} \quad \text{on} \quad \partial\{u_0 > 0\}$$

in every regular point of the free boundary.

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