Michigan Math. J. 65 (2016), 451–472

Clustering for Metric Graphs Using the p-Laplacian

Leandro M. Del Pezzo & Julio D. Rossi

Abstract. We deal with the clustering problem in a metric graph. We look for two clusters, and to this end, we study the first nonzero eigenvalue of the p-Laplacian on a quantum graph with Newmann or Kirchoff boundary conditions on the nodes. Then, an associated eigenfunction u_p provides two sets inside the graph, $\{u_p > 0\}$ and $\{u_p < 0\}$, which define the clusters. Moreover, we describe in detail the limit cases $p \to \infty$ and $p \to 1^+$.

1. Introduction

One of the major problems for networks is that of clustering. Clustering in a network means that we want to identify dense regions of it maximizing or minimizing some criterion. Here we deal with metric graphs Γ that are graphs in which we have a length for the edges and try to identify two clusters. Our approach to find two clusters in Γ is based on the following idea: given a sign-changing function u defined on the graph, just take $A = \{u > 0\}$ and $B = \{u < 0\}$ as clusters (note that the set $\{u = 0\}$ may be nontrivial, and therefore it may happen that $A \cup B \neq \Gamma$). In this work, we take u to be an eigenfunction for some differential operator; we take the p-Laplacian $-(|u'|^{p-2}u')'$ defined on the graph and study properties of this approach. We find two extreme cases: for $p = \infty$ (this is understood as the limit as $p \to \infty$), A and B are sets that have the diameter as large as possible (each one of them has the diameter equal to $\frac{\mathrm{diam}(\Gamma)}{2}$), whereas for $p = 1$ (understood as the limit as $p \to 1$), we find that A and B are sets with large total length and small number of "cuts" in the graph (small perimeter).

A quantum graph is a graph in which we associate a differential law with each edge and which models the interaction between the two nodes defining each edge. The use of quantum graphs (in contrast to more elementary graph models, such as simple unweighted or weighted graphs) opens up the possibility of modeling the interactions between agents identified by the graph vertices in a more detailed manner than with standard graphs. Quantum graphs are used to model thin tubular structures, so-called graph-like spaces, and they are their natural limits as the radius of a graph-like space tends to zero. On both, the graph-like spaces and the metric graphs, we can naturally define Laplace-like differential operators; see [2; 16; 26].

Received November 24, 2014. Revision received April 14, 2016.
Leandro M. Del Pezzo was partially supported by CONICET PIP 5478/1438 (Argentina), and Julio D. Rossi was partially supported by MTM2011-27998 (Spain).

Among properties that are relevant in the study of quantum graphs is the study of the spectrum of the associated differential operator. In particular, the so-called spectral gap (this concerns bounds for the first nonzero eigenvalue of the Laplacian with Neumann boundary conditions) has physical relevance and was extensively studied in recent years. We refer, for example, to [16; 17; 20; 18].

In this paper, we are interested in the eigenvalue problem that naturally arises when we consider the p-Laplacian $(|u'|^{p-2}u')'$ as the differential law on each side of the graph together with Newmann or Kirchoff boundary conditions (see [15]) at the nodes. To be concrete, we deal with the following problem: in a finite metric graph Γ, we consider the minimization problem

$$\lambda_{2,p}(\Gamma) = \inf\left\{ \frac{\int_\Gamma |u'(x)|^p\,dx}{\int_\Gamma |u(x)|^p\,dx} : u \in W^{1,p}(\Gamma), \right. \\ \left. \int_\Gamma |u|^{p-2}u(x)\,dx = 0, u \not\equiv 0 \right\}. \tag{1.1}$$

There is a minimizer (see Theorem 1.1) that is a nontrivial sign-changing weak solution to

$$\begin{cases} -(|u'|^{p-2}u')'(x) = \lambda_{2,p}(\Gamma)|u(x)|^{p-2}u(x) & \text{on the edges of } \Gamma, \\ \sum_{\mathrm{e}\in \mathrm{E_v}(\Gamma)} |\frac{\partial u}{\partial x_\mathrm{e}}|^{p-2}\frac{\partial u}{\partial x_\mathrm{e}}(\mathrm{v}) = 0 & \text{on the nodes.} \end{cases} \tag{1.2}$$

Our main results for this eigenvalue problem can be summarized as follows:

- For $1 < p < \infty$, we show that the infimum in (1.1) is attained at a sign-changing function. We provide examples that show that the set $\{u_p = 0\}$ may have nontrivial measure.
- We study the limit cases $p \to \infty$ and $p \to 1$. For $p = \infty$, we find a geometric characterization of the first nonzero eigenvalue, and for $p = 1$, we prove that there exist analogues of Cheeger sets in quantum graphs.

Now, let us present precise statements of our results. First, the following result follows by a standard compactness argument.

THEOREM 1.1. *Let Γ be a connected compact metric graph. Then, the infimum $\lambda_{2,p}(\Gamma)$ in (1.1) is attained and is the first nonzero Neumann eigenvalue for the p-Laplacian in Γ, that is, $\lambda_{2,p}(\Gamma)$ is the smallest positive number such that there exists a nontrivial $u_p \in W^{1,p}(\Gamma)$ such that*

$$\int_\Gamma |u_p'(x)|^{p-2}u_p'(x)v'(x)\,dx = \lambda \int_\Gamma |u_p(x)|^{p-2}u_p(x)v(x)\,dx \tag{1.3}$$

for all $v \in W^{1,p}(\Gamma)$.

Concerning the limit as $p \to \infty$, we have the following result.

THEOREM 1.2. *Let Γ be a connected compact metric graph, and u_p be a minimizer for (1.1) normalized by $\|u_p\|_{L^p(\Gamma)} = 1$. Let*

$$\Lambda_{2,\infty}(\Gamma) = \inf\left\{ \|v'\|_{L^\infty(\Gamma)} : \max_\Gamma v = \max_\Gamma -v = 1 \right\}. \tag{1.4}$$

Then,

$$\lim_{p\to\infty} \lambda_{2,p}(\Gamma)^{1/p} = \Lambda_{2,\infty}(\Gamma),$$

and there exists a subsequence $p_j \to \infty$ such that

$$u_{p_j} \to u_\infty$$

uniformly in Γ and weakly in $W^{1,q}(\Gamma)$ for every $q < \infty$. Moreover, any possible limit u_∞ is a minimizer for (1.4).

This value $\Lambda_{2,\infty}(\Gamma)$ can be characterized as

$$\Lambda_{2,\infty}(\Gamma) = \frac{2}{\operatorname{diam}(\Gamma)}.$$

For the limit as $p \to 1^+$, we have the following:

THEOREM 1.3. *Let Γ be a connected compact metric graph, and u_p be a minimizer for (1.1) normalized by $\|u_p\|_{L^1(\Gamma)} = 1$. Then, there exist a subsequence $p_j \to 1^+$ and $u_1 \in \mathrm{BV}(\Gamma)$ such that*

$$u_{p_j} \to u_1$$

in $L^1(\Gamma)$.

Moreover, any possible limit u_1 is a minimizer for

$$\Lambda_{2,1}(\Gamma) = \inf\left\{ \frac{\|v'\|_{\mathrm{BV}(\Gamma)}}{\|v\|_{L^1(\Gamma)}} : v \in \mathrm{BV}(\Gamma), \int_\Gamma \operatorname{sgn}(v)(x)\,dx = 0, v \not\equiv 0 \right\}. \tag{1.5}$$

This value $\Lambda_{2,1}(\Gamma)$ is the limit of $\lambda_{2,p}(\Gamma)$, that is,

$$\lim_{p\to 1} \lambda_{2,p}(\Gamma) = \Lambda_{2,1}(\Gamma).$$

We also have an analogue to Cheeger sets for metric graphs. Here and in what follows, we denote by $\ell(A)$ the measure (the total length) of a subset A of Γ.

THEOREM 1.4. *Let Γ be a connected compact metric graph, and A be a subset of Γ such that $\ell(A) = \frac{\ell(\Gamma)}{2}$ and $\mathrm{Per}(A) < \infty$. Then*

$$\frac{2\,\mathrm{Per}(A)}{\ell(\Gamma)} = \inf\left\{ \frac{\mathrm{Per}(E)}{\min\{\ell(E), |\Gamma \setminus E|\}} : E \subsetneq \Gamma, E \neq \emptyset \right\} \tag{1.6}$$

if only if

$$u = \chi_A - \chi_{\Gamma\setminus A}$$

is a minimizer for $\Lambda_{2,1}(\Gamma)$.

As we have mentioned at the beginning of this introduction, for a metric graph, one important problem is clustering. We want to identify two disjoint subsets A and B of the graph Γ that are similar in size (here we have to define in which sense we measure the size of a subset of a metric graph) and such that the resulting partition of Γ minimizes or maximizes some criterion (also to be specified). We remark again that, in general, we are not prescribing that $\Gamma = A \cup B$, and we can have $\Gamma \setminus (A \cup B) \neq \emptyset$.

For the case $p = \infty$, we let $A_\infty = \{u_\infty > 0\}$ and $B_\infty = \{u_\infty < 0\}$, and we have that A_∞ and B_∞ are two subsets of Γ with the same diameter that maximizes this common diameter, that is,

$$\operatorname{diam}(A_\infty) = \operatorname{diam}(B_\infty) = \frac{\operatorname{diam}(\Gamma)}{2}.$$

For $p = 1$, we let $A_1 = \{u_1 > 0\}$ and $B_1 = \{u_1 < 0\}$, and we obtain two subsets with total length $\ell(A_1)$ and $\ell(B_1)$ less than or equal to $\ell(\Gamma)/2$ that maximizes the sum $\ell(A_1) + \ell(B_1)$ and is such that the perimeter of them inside Γ is minimized.

In general, for intermediate $1 < p < \infty$, letting $A = \{u_p > 0\}$ and $B = \{u_p < 0\}$, we obtain something that interpolates between the two previous situations.

Let us end this introductions with a brief description of the ideas and techniques used in the proofs and of the previous bibliography.

The existence of eigenfunctions can be easily obtained by a compactness argument as for the usual p-Laplacian in a bounded domain of $\mathbb{R}^N$; see [12]. However, here we show examples that show that the set $\{u_p = 0\}$ may have nontrivial measure (it may contain some edges).

Eigenvalues on quantum graphs are by now a classical subject with an increasing number of recent references; we refer to [5; 11; 17; 18]. The literature on eigenfunctions of the p-Laplacian in a one-dimensional interval, also called p-trigonometric functions, is now quite extensive: we refer, in particular, to [22; 23; 24] and references therein.

Concerning the limit as $p \to \infty$ for the eigenvalue problem for the p-Laplacian in the usual PDE case; we refer to [3; 4; 13; 14; 27]. To study this limit, the main point is to use adequate test functions to obtain bounds that are uniform in p in order to gain compactness on a sequence of eigenfunctions.

Finally, for $p = 1$, we refer to [7; 10; 25], which deal with Cheeger sets in the Euclidean space. In this limit problem, the natural space that appears is that of bounded variation functions (that are not necessarily continuous; see [1]).

The paper is organized as follows. In Section 2, we collect some preliminaries; in Section 3, we deal with the first eigenvalue on a quantum graph and prove its upper and lower bounds; in Section 4, we study the limit as $p \to \infty$ of the first eigenvalue, whereas in the final section, Section 5, we look for the limit as $p \to 1^+$.

2. Preliminaries

We start with a brief review of the basic results that will be needed in subsequent sections. The known results are generally stated without proofs, but we provide references where the proofs can be found. Also, we introduce our notational conventions.

2.1. Neumann Eigenvalues for the p-Laplacian in One Dimension

First, we introduce a review about the one-dimensional Neumann eigenvalue problem for the p-Laplacian. For more details, see [21]. Let $p \in (1, \infty)$ and

$L > 0$. We consider the following eigenvalue problem for the p-Laplacian in an interval:

$$\begin{cases} -(|u'|^{p-2}u')'(x) = \lambda|u(x)|^{p-2}u(x) & \text{in } (0, L), \\ u'(0) = u'(L) = 0. \end{cases}$$

The eigenvalues λ are of the form

$$\lambda_{n+1,p} = \left(\frac{n\pi_p}{L}\right)^p \frac{p}{p'} \quad \forall n \in \mathbb{N}_0,$$

where $\pi_p = \frac{2\pi}{p\sin(\pi/p)}$, and $1/p + 1/p' = 1$. The eigenfunctions corresponding to the zero eigenvalue are the nonzero constants; those corresponding to $\lambda_{n,p}$ with $n > 0$ are

$$u_{n+1}(x) = \frac{\alpha L}{n\pi_p} \sin_p\left(\frac{n\pi_p}{L}\left(x - \frac{L}{2n}\right)\right), \quad \alpha \in \mathbb{R} \setminus \{0\},$$

where $\sin_p$ is the p-sine function.

Note that $\{\lambda_{n,p}\}$ coincides with the usual Neumann eigenvalues of the Laplacian when $p = 2$.

Finally, we want to remark that the first nonzero Neumann eigenvalue is

$$\lambda_{2,p} = \left(\frac{\pi_p}{L}\right)^p \frac{p}{p'}, \tag{2.1}$$

and the eigenfunctions u_2 corresponding to $\lambda_{2,p}$ have the property

$$\int_0^L |u_2(x)|^{p-2}u_2(x)\,dx = 0.$$

2.2. *Quantum Graphs*

We now recall some basic knowledge about quantum graphs; see, for instance, [2] and references therein.

A graph Γ consists of a finite or countable infinite set of vertices $\mathrm{V}(\Gamma) = \{\mathrm{v}_i\}$ and a set of edges $\mathrm{E}(\Gamma) = \{\mathrm{e}_j\}$ connecting the vertices. A graph Γ is said to be a finite graph if the numbers of edges and vertices are finite.

Two vertices u and v are called adjacent (denoted $\mathrm{u} \sim \mathrm{v}$) if there is an edge connecting them. An edge and a vertex on that edge are called incident. We will denote $\mathrm{v} \in \mathrm{e}$ when e and v are incident. We define $\mathrm{E_v}(\Gamma)$ as the set of all edges incident to v. The degree $d_{\mathrm{v}}(\Gamma)$ of a vertex $\mathrm{V}(\Gamma)$ is the number of edges that are incident to it, where a loop (an edge that connects a vertex to itself) is counted twice.

A graph Γ is called a directed graph if each of its edges is assigned a direction. Each edge e can be identified with an ordered pair $(\mathrm{v_e}, \mathrm{u_e})$ of vertices. The vertices $\mathrm{v_e}$ and $\mathrm{u_e}$ are the initial and terminal vertices of e. The edge $\hat{\mathrm{e}}$ is called the reversal of an edge e if $\mathrm{v}_{\hat{\mathrm{e}}} = \mathrm{u_e}$ and $\mathrm{u}_{\hat{\mathrm{e}}} = \mathrm{v_e}$.

DEFINITION 2.1 (see Definition 1.2.3 in [2]). Γ is called a metric graph if

(1) each directed edge e is assigned a positive length $\ell_{\mathrm{e}} \in (0, \infty]$;

(2) the lengths of the edges that are reversals of each other are assumed to be equal, that is, $\ell_{\mathrm{e}} = \ell_{\hat{\mathrm{e}}}$;
(3) a coordinate $x_{\mathrm{e}} \in I_{\mathrm{e}} = [0, \ell_{\mathrm{e}}]$ increasing in the direction of the edge is assigned on each edge;
(4) the relation $x_{\hat{\mathrm{e}}} = \ell_{\mathrm{e}} - x_{\mathrm{e}}$ holds between the coordinates on mutually reserved edges.

If a sequence of edges $\{\mathrm{e}_j\}_{j=1}^n$ forms a path, its length is defined as $\sum_{j=1}^n \ell_{\mathrm{e}_j}$. For two vertices v and u, the distance $d(\mathrm{v}, \mathrm{u})$ is defined as the length of the shortest path between them. A metric graph Γ becomes a metric measure space by defining the distance $d(x, y)$ of two points x and y of the graph (that are not necessarily vertices) to be the shortest path on Γ connecting these points, that is,

$$d(x,y) := \inf\left\{ \int_0^1 |\gamma'(t)|\, dt : \gamma : [0,1] \to \Gamma \text{ Lipschitz}, \gamma(0) = x, \gamma(1) = y \right\},$$

and the measure as the one obtained from the usual Lebesgue measure on each edge (we denote the measure of $A \subset \Gamma$ by $\ell(A)$). The total length of a metric graph (denoted $\ell(\Gamma)$) is the sum of the lengths of all edges, and its diameter (denoted by $\mathrm{diam}(\Gamma)$) is the maximum distance between two points in Γ.

A metric graph Γ is said to be connected or compact when it is connected or compact in the sense of a topological space.

A function u on a metric graph Γ is a collection of functions u_{e} defined on $(0, \ell_{\mathrm{e}})$ for all $\mathrm{e} \in \mathrm{E}(\Gamma)$, not just at the vertices as in discrete models.

Let $1 \le p \le \infty$. We say that u belongs to $L^p(\Gamma)$ if u_{e} belongs to $L^p(0, \ell_{\mathrm{e}})$ for all $\mathrm{e} \in \mathrm{E}(\Gamma)$ and

$$\|u\|_{L^p(\Gamma)}^p := \sum_{\mathrm{e} \in \mathrm{E}(\Gamma)} \|u_{\mathrm{e}}\|_{L^p(0,\ell_{\mathrm{e}})}^p < \infty.$$

The Sobolev space $W^{1,p}(\Gamma)$ is defined as the space of continuous functions u on Γ such that $u_{\mathrm{e}} \in W^{1,p}(I_{\mathrm{e}})$ for all $\mathrm{e} \in \mathrm{E}(\Gamma)$ and

$$\|u\|_{W^{1,p}(\Gamma)}^p := \sum_{\mathrm{e} \in \mathrm{E}(\Gamma)} \|u_{\mathrm{e}}\|_{L^p(0,\ell_{\mathrm{e}})}^p + \|u_{\mathrm{e}}'\|_{L^p(0,\ell_{\mathrm{e}})}^p < \infty.$$

Observe that the continuity condition in the definition of $W^{1,p}(\Gamma)$ implies that for each $\mathrm{v} \in \mathrm{V}(\Gamma)$, the function on all edges $\mathrm{e} \in \mathrm{E}_{\mathrm{v}}(\Gamma)$ takes the same value at v.

The space $W^{1,p}(\Gamma)$ is a Banach space for $1 \le p \le \infty$. It is reflexive for $1 < p < \infty$ and separable for $1 \le p < \infty$.

THEOREM 2.2. *Let Γ be a compact graph, and $1 < p < \infty$. The injection $W^{1,p}(\Gamma) \subset L^q(\Gamma)$ is compact for all $1 \le q \le \infty$.*

A quantum graph is a metric graph Γ equipped with a differential operator $\mathcal{H}$, accompanied by vertex conditions. In this work, we consider

$$\mathcal{H}(u)(x) := -\Delta_p u(x) = -(|u'(x)|^{p-2} u'(x))'.$$

Our vertex conditions are the following:

$$\sum_{\mathrm{e}\in \mathrm{E_v}(\Gamma)} \left|\frac{\partial u}{\partial x_{\mathrm{e}}}\right|^{p-2} \frac{\partial u}{\partial x_{\mathrm{e}}}(\mathrm{v}) = 0, \qquad \forall \mathrm{v} \in \mathrm{V}(\Gamma), \tag{2.2}$$

where the derivatives are assumed to be taken in the direction away from the vertex.

Throughout this work, $\int_\Gamma u(x)\,dx$ denotes $\sum_{\mathrm{e}\in\mathrm{E}(\Gamma)} \int_0^{\ell_{\mathrm{e}}} u_{\mathrm{e}}(x)\,dx$.

3. The First Nonzero Eigenvalue in Γ

Let Γ be a compact connected quantum graph, and $p \in (1,\infty)$. We say that the value $\lambda \in \mathbb{R}$ is a Neumann eigenvalue of the p-Laplacian in Γ if there exists a nontrivial function $u \in W^{1,p}(\Gamma)$ such that

$$\int_\Gamma |u'(x)|^{p-2} u'(x) v'(x)\,dx = \lambda \int_\Gamma |u(x)|^{p-2} u(x) v(x)\,dx \tag{3.1}$$

for all $v \in W^{1,p}(\Gamma)$, in which case, u is called an eigenfunction associated with λ.

Of course, the first eigenvalue is $\lambda = 0$ with eigenfunction $u \equiv 1$. Moreover, if $\lambda > 0$ is an eigenvalue and u is an associated eigenfunction, then, taking $v \equiv 1$ as a test function in (3.1), we have $\int_\Gamma |u(x)|^{p-2} u(x)\,dx = 0$.

Thus, the existence of the first nonzero eigenvalue $\lambda_{2,p}(\Gamma)$ is related to the problem of minimizing the quotient $\int_\Gamma |v'(x)|^p\,dx / \int_\Gamma |v(x)|^p\,dx$ among all functions $v \in W^{1,p}(\Gamma)$ such that $v \neq 0$ and $\int_\Gamma |v(x)|^{p-2} v(x)\,dx = 0$. This is exactly the contents of Theorem 1.1, which we prove next.

Proof of Theorem 1.1. Take a minimizing sequence u_n for $\lambda_{2,p}(\Gamma)$ and normalize it according to $\|u_n\|_{L^p(\Gamma)} = 1$. This sequence satisfies $\int_\Gamma |u_n(x)|^{p-2} u_n(x)\,dx = 0$, and its $W^{1,p}$-norm is bounded. Hence, by a standard compactness argument, using the compactness result of Theorem 2.2, it follows that there exists a subsequence u_{n_j} that converges strongly in $L^p(\Gamma)$ and weakly in $W^{1,p}(\Gamma)$. The limit of this subsequence satisfies $\|u\|_{L^p(\Gamma)} = 1$, $\int_\Gamma |u(x)|^{p-2} u(x)\,dx = 0$ and $\|u\|^p_{W^{1,p}(\Gamma)} = \lambda_{2,p}(\Gamma)$. Therefore, $\lambda_{2,p}(\Gamma)$ is attained, and it is the first nonzero Neumann eigenvalue of the p-Laplacian in Γ.

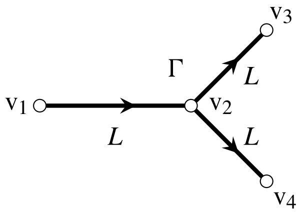
The fact that a minimizer satisfies (1.3) is standard, but we include a short proof here for completeness. Let u_p be a nontrivial minimizer. Then using Lagrange multipliers, we get the existence of $\lambda \in \mathbb{R}$ such that

$$\int_\Gamma |u_p'(x)|^{p-2} u_p'(x) v'(x)\,dx = \lambda \int_\Gamma |u_p(x)|^{p-2} u_p(x) v(x)\,dx \tag{3.2}$$

for all $v \in W^{1,p}(\Gamma)$ with $\int_\Gamma |v(x)|^{p-2} v(x)\,dx = 0$. Since $\int_\Gamma |u(x)|^{p-2} u(x)\,dx = 0$, we conclude that (3.2) also holds for $v = 1$ and, therefore, for every $v \in W^{1,p}(\Gamma)$. Finally, taking $v = u_p$, we get that $\lambda = \lambda_{2,p}(\Gamma)$. $\square$

Remark 3.1. In general, the second eigenvalue $\lambda_{2,p}(\Gamma)$ is not simple. For instance, let Γ be a simple graph with four vertices and three edges, that is,

$\mathrm{V}(\Gamma) = \{\mathrm{v}_1, \mathrm{v}_2, v_3, v_4\}$ and $\mathrm{E}(\Gamma) = \{[\mathrm{v}_1, \mathrm{v}_2], [\mathrm{v}_2, \mathrm{v}_3], [\mathrm{v}_2, \mathrm{v}_4]\}$,



$$\lambda_{2,p}(\Gamma) = (\tfrac{\pi_p}{2L})^p \tfrac{p}{p'}.$$

Observe that

$$u(x) = \begin{cases} \frac{2L}{\pi_p} \sin_p(\frac{\pi_p}{2L}(x - L)) & \text{if } x \in I_{[\mathrm{v}_1,\mathrm{v}_2]} = [0, L], \\ \frac{2L}{\pi_p} \sin_p(\frac{\pi_p}{2L} x) & \text{if } x \in I_{[\mathrm{v}_2,\mathrm{v}_3]} = [0, L], \\ 0 & \text{otherwise}, \end{cases}$$

$$v(x) = \begin{cases} \frac{2L}{\pi_p} \sin_p(\frac{\pi_p}{2L}(x - L)) & \text{if } x \in I_{[\mathrm{v}_1,\mathrm{v}_2]} = [0, L], \\ \frac{2L}{\pi_p} \sin_p(\frac{\pi_p}{2L} x) & \text{if } x \in I_{[\mathrm{v}_2,\mathrm{v}_4]} = [0, L], \\ 0 & \text{otherwise}, \end{cases}$$

are two linearly independent eigenfunctions associated with $\lambda_{2,p}(\Gamma)$.

Also, remark that in this example, the described eigenfunctions associated with $\lambda_{2,p}(\Gamma)$ vanish on an entire edge. Therefore, here we have that the set $\{u = 0\}$ is nontrivial.

These features correspond to a highly symmetric case. If we change the graph just by taking the same configuration but with three different lengths L_1, L_2, L_3 for the three different edges, then we have an eigenvalue whose associated eigenfunction vanishes only at one point (hence, its zero set has zero length). In fact, vanishing of an eigenfunction associated with the first nontrivial eigenvalue at the vertex v_2 is impossible since, for different lengths, we have different values of the first eigenvalue of the p-Laplacian with mixed boundary conditions ($u = 0$ at one endpoint and $u' = 0$ at the other). By the same reason an eigenfunction must vanish only inside the longest edge, and there is only one possibility for this point x_p (it must be the only one such that the first eigenvalue with mixed boundary conditions in the interval between the vertex v_i and the point x_p in the longest edge equals $\lambda_{2,p}(\Gamma)$).

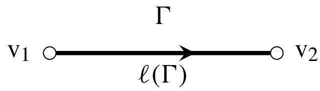
Our next result shows an upper bound and a lower bound for $\lambda_{2,p}(\Gamma)$, which depend on p, the length of a metric graph, and the number of elements in $\mathrm{E}(\Gamma)$. The proof is similar to that of [8, Thms. 3.5 and 3.8]. See also [19, Thm. 1].

THEOREM 3.2. *Let* Γ *be a connected compact metric graph, and* $p \in (1, \infty)$. *Then*

$$\left(\frac{\pi_p}{\ell(\Gamma)}\right)^p \frac{p}{p'} \leq \lambda_{2,p}(\Gamma) \leq \left(\frac{\mathrm{card}(E(\Gamma))\pi_p}{\ell(\Gamma)}\right)^p \frac{p}{p'},$$

where $\mathrm{card}(E(\Gamma))$ *is the number of elements in* $\mathrm{E}(\Gamma)$.

Note that the bounds given in the theorem are optimal. For instance, let Γ be a graph with only one edge, that is, $\mathrm{V}(\Gamma) = \{\mathrm{v}_1, \mathrm{v}_2\}$ and $\mathrm{E}(\Gamma) = \{[\mathrm{v}_1, \mathrm{v}_2]\}$,



Then, by (2.1) we have that

$$\lambda_{2,p}(\Gamma) = \left(\frac{\pi_p}{\ell(\Gamma)}\right)^p \frac{p}{p'},$$

and then the upper and lower bounds in Theorem 3.2 are attained and coincide.

4. The Limit as $p \to \infty$

In this section, we deal with the limit as $p \to \infty$ of the eigenvalue problem (1.1). We split the proof of Theorem 1.2 into several steps.

LEMMA 4.1. *We have*

$$\limsup_{p\to\infty} \lambda_{2,p}(\Gamma)^{1/p} \le \Lambda_{2,\infty}(\Gamma). \tag{4.1}$$

Proof. Let $w \in W^{1,\infty}(\Gamma)$ be admissible for $\Lambda_{2,\infty}(\Gamma)$, that is, $\max_\Gamma w = -\min_\Gamma w = 1$. Now, multiply the positive part w^+ of w by $a_p \in \mathbb{R}$ and the negative part w^- of w by $b_p \in \mathbb{R}$ to obtain

$$\int_\Gamma |z(x)|^{p-2} z(x)\, dx = 0$$

with

$$z(x) = a_p w^+(x) - b_p w^-(x).$$

Note that z is continuous in Γ and we can always assume that

$$\max_\Gamma |z| = 1,$$

hence, $a_p = 1$ or $b_p = 1$. Also note that

$$a_p \left(\int_\Gamma (w^+(x))^{p-1}\, dx\right)^{1/(p-1)} = b_p \left(\int_\Gamma (w^-(x))^{p-1}\, dx\right)^{1/(p-1)}$$

and hence

$$\lim_{p\to\infty} a_p = \lim_{p\to\infty} b_p = 1$$

since

$$\lim_{p\to\infty} \left(\int_\Gamma (w^+(x))^{p-1}\, dx\right)^{1/(p-1)} = \lim_{p\to\infty} \left(\int_\Gamma (w^-(x))^{p-1}\, dx\right)^{1/(p-1)} = 1.$$

Then, z is an admissible function for the minimization problem defining $\lambda_{2,p}(\Gamma)$. Hence, we get

$$\lambda_{2,p}^{1/p}(\Gamma) \le \frac{\|z'\|_{L^p(\Gamma)}}{\|z\|_{L^p(\Gamma)}}.$$

Now, we just observe that

$$\lim_{p\to\infty} \|z\|_{L^p(\Gamma)} = \|w\|_{L^\infty(\Gamma)} = 1$$

and

$$\lim_{p\to\infty} \|z'\|_{L^p(\Gamma)} = \|w'\|_{L^\infty(\Gamma)} = \max\{\|(w^+)'\|_{L^\infty(\Gamma)}; \|(w^-)'\|_{L^\infty(\Gamma)}\}.$$

Hence, it follows that

$$\limsup_{p\to\infty} \lambda_{2,p}(\Gamma)^{1/p} \le \|w'\|_{L^\infty(\Gamma)},$$

and we conclude

$$\limsup_{p\to\infty} \lambda_{2,p}(\Gamma)^{1/p} \le \Lambda_{2,\infty}(\Gamma).$$

□

As a second step, we prove that, up to a subsequence, u_p converges uniformly to a minimizer of $\Lambda_{2,\infty}(\Gamma)$.

LEMMA 4.2. *Let u_p be an eigenfunction associated to $\lambda_{2,p}(\Gamma)$ normalized with $\|u_p\|_{L^p(\Gamma)} = 1$. Then, up to taking a subsequence, u_p converge uniformly in Γ and weakly in $W^{1,r}(\Gamma)$ for any $1 < r < \infty$ to some $u_\infty \in W^{1,\infty}(\Gamma)$, which is a minimizer of $\Lambda_{2,\infty}(\Gamma)$.*

Moreover, we have

$$\lim_{p\to\infty} \lambda_{2,p}(\Gamma)^{1/p} = \Lambda_{2,\infty}(\Gamma).$$

Proof. We first notice that $\{u_p\}_{p\ge r}$ is bounded in $W^{1,r}(\Gamma)$ for any r. Indeed, by Hölder's inequality,

$$\int_\Gamma |u_p'(x)|^r\,dx \le \|u_p'\|^p_{L^p(\Gamma)}(\ell(\Gamma))^{1-r/p},$$

so that, by (4.1),

$$\|u_p'\|_{L^r(\Gamma)} \le \lambda_{2,p}(\Gamma)^{1/p}(\ell(\Gamma))^{1/r-1/p} \le C. \tag{4.2}$$

By Morrey's inequality $\{u_p\}_{p>r}$ is bounded in some Hölder space $C^{0,\alpha}(\Gamma)$, and then, up to a subsequence, $u_p \to u_\infty$ in $C(\Gamma)$. We can also assume that this convergence holds weakly in $W^{1,r}(\Gamma)$ for any r.

Let us prove that $\|u_\infty\|_{L^\infty(\Gamma)} = 1$. We have

$$\int_\Gamma |u_p(x)|^r\,dx \le \|u_p\|^p_{L^p(\Gamma)}(\ell(\Gamma))^{1-r/p},$$

so that by the normalization $\|u_p\|_{L^p(\Gamma)} = 1$ we get

$$\|u_p\|_{L^r(\Gamma)} \le (\ell(\Gamma))^{1/r-1/p}. \tag{4.3}$$

Letting $p, r \to \infty$ in (4.3), we see that $\|u_\infty\|_{L^\infty(\Gamma)} \le 1$. Now, suppose that $\|u_\infty\|_{L^\infty(\Gamma)} \le 1 - 2\varepsilon < 1$ for some $\varepsilon > 0$. Since $\|u_p\|_{L^\infty(\Gamma)} \to \|u_\infty\|_{L^\infty(\Gamma)}$ as $p \to \infty$, we have $\|u_p\|_{L^\infty(\Gamma)} \le 1 - \varepsilon$ for p large. Then

$$1 = \int_\Gamma |u_p(x)|^p\,dx \le (1-\varepsilon)^p\ell(\Gamma) \to 0$$

as $p \to \infty$, which is a contradiction with the normalization $\|u_p\|_{L^p(\Gamma)} = 1$.

We now verify that $\max_\Gamma u_\infty + \min_\Gamma u_\infty = 0$. From $\int_\Gamma |u_p(x)|^{p-2} u_p(x)\, dx = 0$ we obtain that

$$\int_{\{u_p \geq 0\}} |u_p(x)|^{p-1}\, dx = \int_{\{u_p \leq 0\}} |u_p(x)|^{p-1}\, dx.$$

We already know that $\|u_\infty\|_{L^\infty(\Gamma)} = 1$. Let us show that $\max_\Gamma u_\infty = 1$ and $\min_\Gamma u_\infty = -1$. We argue by contradiction. Assume, for example, that $\max_\Gamma u_\infty = 1$ but $\min_\Gamma u_\infty \geq -1 + 2\varepsilon$ for some $\varepsilon > 0$. Since $u_p \to u_\infty$ in $C(\Gamma)$, we also have $\min_\Gamma u_p \geq -1 + \varepsilon$ for p large. Then

$$\int_{\{u_p \geq 0\}} |u_p(x)|^{p-1}\, dx = \int_{\{u_p \leq 0\}} |u_p(x)|^{p-1}\, dx \leq (1-\varepsilon)^{p-1} \ell(\Gamma) \to 0$$

as $p \to \infty$. Since $\{u_p\}$ is bounded in $C(\Gamma)$ (because it converges), we obtain

$$1 = \int_\Gamma |u_p(x)|^p\, dx \leq C \int_\Gamma |u_p(x)|^{p-1}\, dx \to 0,$$

which is a contradiction.

Since $\|u_\infty\|_{L^\infty(\Gamma)} = 1$ and $\max_\Gamma u_\infty + \min_\Gamma u_\infty = 0$, we have that u_∞ is an admissible test-function for $\Lambda_{2,\infty}(\Gamma)$. It follows that $\Lambda_{2,\infty}(\Gamma) \leq \|u'_\infty\|_{L^\infty(\Gamma)}$. Since $u_p \to u_\infty$ weakly in $W^{1,r}(\Gamma)$ for any $\infty > r > 1$, we also have from (4.2) that

$$\|u'_\infty\|_{L^r(\Gamma)} \leq \liminf_{p\to\infty} \|u'_p\|_{L^r(\Gamma)} \leq |\Gamma|^{1/r} \liminf_{p\to\infty} \lambda_{2,p}(\Gamma)^{1/p}.$$

Letting $r \to \infty$, we obtain, using (4.1), that

$$\Lambda_{2,\infty}(\Gamma) \leq \|u'_\infty\|_{L^\infty(\Gamma)} \leq \liminf_{p\to\infty} \lambda_{2,p}(\Gamma)^{1/p} \leq \limsup_{p\to\infty} \lambda_{2,p}(\Gamma)^{1/p} \leq \Lambda_{2,\infty}(\Gamma),$$

from which the claim follows. □

Now, our goal is to show that $\Lambda_{2,\infty}(\Gamma) = \frac{2}{\operatorname{diam}(\Gamma)}$. As a first step, we prove an inequality.

LEMMA 4.3. *We have* $\Lambda_{2,\infty}(\Gamma) \geq \frac{2}{\operatorname{diam}(\Gamma)}$.

Proof. Given some admissible test-function u for the minimum defining $\Lambda_{2,\infty}(\Gamma)$, let $x \in \Gamma$ be a point where u attains its maximum, and $y \in \Gamma$ a point where u attains a minimum, so that $u(x) = 1$ and $u(y) = -1$. Consider also some curve $\gamma : [0, T] \to \Gamma$ joining y and x. Then

$$\begin{aligned} 2 &= u(x) - u(y) = u(\gamma(T)) - u(\gamma(0)) \\ &= \int_0^T u'(\gamma(s))\gamma'(s)\, ds = \|u'\|_{L^\infty(\Gamma)} \operatorname{Long}(\gamma). \end{aligned}$$

Taking the infimum over all such curves γ and all admissible u, we obtain

$$2 \leq \Lambda_{2,\infty}(\Gamma) d(x, y),$$

from which the claim follows. □

We now prove the reverse inequality.

LEMMA 4.4. *We have* $\Lambda_{2,\infty}(\Gamma) \le \frac{2}{\operatorname{diam}(\Gamma)}$.

Proof. Take two points $x_0, y_0 \in \Gamma$ such that $\operatorname{diam}(\Gamma) = d(x_0, y_0)$. Consider the function

$$u(z) = \frac{2}{\operatorname{diam}(\Gamma)}\left(d(z, x_0) - \frac{\operatorname{diam}(\Gamma)}{2}\right), \quad z \in \Gamma.$$

This function is admissible for the minimization problem for $\Lambda_{2,\infty}$ and has

$$\|u'\|_{L^\infty(\Gamma)} = \frac{2}{\operatorname{diam}(\Gamma)}.$$

This gives the desired upper bound.

Another possible choice of a test-function is

$$u(z) = C_y(z)_+ - C_x(z)_+,$$

where

$$C_y(z) = 1 - \frac{2}{\operatorname{diam}(\Gamma)} d(z, y) \quad \text{and} \quad C_x(z) = 1 - \frac{2}{\operatorname{diam}(\Gamma)} d(z, x)$$

are the cones centered at x and y of height 1 and slope $\frac{2}{\operatorname{diam}(\Gamma)}$. □

REMARK 4.5. In the example described in Remark 3.1 with three edges of the same length L, we have that this limit selects (extracting a subsequence u_{p_j} with $p_j \to \infty$) two edges as A_∞ and B_∞, and the third edge is just $\{u = 0\}$. Here the diameter of Γ is $2L$, and we obtain two sets of maximum diameter as A_∞ and B_∞.

When we consider the same configuration of the graph, but with three different lengths L_1, L_2, L_3 (assume that $L_1 > L_2 > L_3$) for the three different edges, we get that the diameter of Γ is $L_1 + L_2$ and our limit as $p \to \infty$ gives A_∞ as the segment of the longest edge of length $(L_1 + L_2)/2$ starting at v_1 and B_∞ as the rest of the graph.

5. The Limit as $p \to 1^+$

In this section, we study the other limit case, $p = 1$. We will use functions of bounded variation on the graph (denoted by $\mathrm{BV}(\Gamma)$) and the perimeter of a subset of the graph (denoted by $\operatorname{Per}(D)$). More precisely, BV functions on a compact interval are exactly those u that can be written as a difference $g - h$ where both g and h are bounded and monotone. In terms of derivatives, a function is in BV if its distributional derivative is a finite Radon measure whose total variation gives the BV-norm of the function. Given a set D, we say that it has finite perimeter if its characteristic function χ_D is in BV (and we define the perimeter of D as the total variation of the distributional derivative of χ_D). These concepts extend to a metric graph Γ by applying them on each edge. We refer to [1] for more precise definitions and properties of functions and sets in this context.

We start by showing two technical lemmas required in the proof of Theorem 1.3.

LEMMA 5.1. *Let Γ be a connected compact metric graph, and $v \in \mathrm{BV}(\Gamma)$ be such that*

$$\int_\Gamma \mathrm{sgn}(v)(x)\,dx = 0. \tag{5.1}$$

If there exists a constant $c \neq 0$ such that

$$\int_\Gamma \mathrm{sgn}(v-c)(x)\,dx = 0, \tag{5.2}$$

then $\|v-c\|_{L^1(\Gamma)} = \|v\|_{L^1(\Gamma)}$, and

$$\begin{aligned}
&\ell(\{x\colon v(x) \geq c\}) = \ell(\{x\colon v(x) \leq 0\}) \quad \textit{and} \quad \ell(\{x\colon 0 < v(x) < c\}) = 0\\
&\quad \textit{if } c > 0;\\
&\ell(\{x\colon v(x) \leq c\}) = \ell(\{x\colon v(x) \geq 0\}) \quad \textit{and} \quad \ell(\{x\colon c < v(x) < 0\}) = 0\\
&\quad \textit{if } c < 0.
\end{aligned}$$

Proof. We consider the case $c > 0$. The other case is similar. We begin by introducing the following notation: $E_0^+ = \{x\colon v(x) > 0\}$, $E_0^- = \{x\colon v(x) < 0\}$, $E_0 = \{x\colon v(x) = 0\}$, $E_c^+ = \{x\colon v(x) > c\}$, $E_c^- = \{x\colon v(x) < c\}$, $E_c = \{x\colon v(x) = c\}$, and $E_{0,c} = \{x\colon 0 < v(x) < c\}$. By (5.1) and (5.2) there exist $w_1 \in \mathrm{sgn}(v)$ and $w_2 \in \mathrm{sgn}(v-c)$ such that

$$0 = \int_\Gamma w_1(x)\,dx = \ell(E_0^+) + \int_{E_0} w_1(x)\,dx - \ell(E_0^-) \tag{5.3}$$

and

$$\begin{aligned}
0 &= \int_\Gamma w_2(x)\,dx = \ell(E_c^+) - \ell(E_c^-) + \int_{E_c} w_2(x)\,dx\\
&= \ell(E_c^+) - \ell(E_{0,c}) - \ell(E_0) - \ell(E_0^-) + \int_{E_c} w_2(x)\,dx\\
&\leq \ell(E_c^+) - \ell(E_{0,c}) + \int_{E_0} w_1(x)\,dx - \ell(E_0^-) + \int_{E_c} w_2(x)\,dx\\
&\quad (\|w_1\|_{L^\infty(\Gamma)} = 1)\\
&= \ell(E_c^+) - \ell(E_{0,c}) - \ell(E_0^+) + \int_{E_c} w_2(x)\,dx \quad \text{(by (5.3))}\\
&= -2\ell(E_{0,c}) + \int_{E_c} (w_2 - 1)(x)\,dx\\
&\leq -2\ell(E_{0,c}) \quad \text{(note that } \|w_2\|_{L^\infty(\Gamma)} = 1).
\end{aligned}$$

Observe that if we assume that $\ell(E_{0,c}) > 0$, then we arrive at a contradiction in the last inequality. Then $\ell(E_{0,c}) = 0$. Therefore,

$$\begin{aligned}
0 &= \int_\Gamma w_1(x)\,dx = \ell(E_0^+) + \int_{E_0} w_1(x)\,dx - \ell(E_0^-)\\
&= \ell(\{x\colon v(x) \geq c\}) + \int_{E_0} w_1(x)\,dx - \ell(E_0^-),
\end{aligned}$$

and

$$
\begin{aligned}
0 &= \int_\Gamma w_2(x)\,dx = \ell(E_c^+) + \ell(E_c^-) + \int_{E_c} w_2(x)\,dx \\
&= \ell(E_c^+) - \ell(\{x\colon v(x) \le 0\}) + \int_{E_c} w_2(x)\,dx.
\end{aligned}
$$

Subtracting these equations, we get

$$
\begin{aligned}
0 &= \ell(E_c) + \int_{E_0} w_1(x)\,dx + \ell(E_0) - \int_{E_c} w_2(x)\,dx \\
&= \int_{E_0} (w_1+1)(x)\,dx + \int_{E_c} (1-w_2)(x)\,dx.
\end{aligned}
$$

Therefore, $w_1 = -1$ in E_0 and $w_2 = 1$ in E_c due to $\|w_i\|_{L^\infty(\Gamma)} \le 1$ for $i = 1, 2$. Thus,

$$
\begin{aligned}
0 &= \int_\Gamma w_1(x)\,dx = \ell(\{x\colon v(x) \ge c\}) + \int_{E_0} w_1(x)\,dx - \ell(E_0^-) \\
&= \ell(\{x\colon v(x) \ge c\}) - \ell(\{x\colon v(x) \le 0\}),
\end{aligned}
$$

that is, $\ell(\{x\colon v(x) \ge c\}) = \ell(\{x\colon v(x) \le 0\})$.

Finally,

$$
\begin{aligned}
\int_\Gamma |v-c|(x)\,dx &= \int_{\{x\colon v(x)\ge c\}} (v-c)(x)\,dx + \int_{\{x\colon v(x)\le 0\}} (c-v)(x)\,dx \\
&= \int_\Gamma |v(x)|\,dx + c|\{x\colon v(x) \ge c\}| - c|\{x\colon v(x) \le c\}| \\
&= \int_\Gamma |v(x)|\,dx,
\end{aligned}
$$

and the proof is complete. □

LEMMA 5.2. *Let Γ be a connected compact metric graph, and $v \in L^1(\Gamma)$ and $\{v_n\}_{n\in\mathbb{N}}$ be such that*

$$
\int_\Gamma \operatorname{sgn}(v_n)(x)\,dx = 0 \quad \forall n \in \mathbb{N}, \quad \textit{and} \quad v_n \to v \quad \textit{strongly in } L^1(\Gamma). \tag{5.4}
$$

Then,

$$
\int_\Gamma \operatorname{sgn}(v)(x)\,dx = 0.
$$

Proof. For any $n \in \mathbb{N}$, by (5.4) there exists $w_n \in \operatorname{sgn}(v_n)$ such that

$$
\int_\Gamma w_n(x)\,dx = 0. \tag{5.5}
$$

Moreover, $\|w_n\|_{L^\infty(\Gamma)} \le 1$ for all $n \in \mathbb{N}$. Therefore, there exist a function w and a subsequence, still denoted $\{w_n\}_{n\in\mathbb{N}}$, such that

$$
w_n \rightharpoonup w \quad \text{weakly in } L^q(\Omega) \text{ for any } 1 < q < \infty.
$$

Thus, using (5.5), we have

$$\int_\Gamma w(x)\,dx = \lim_{n\to\infty}\int_\Gamma w_n(x)\,dx = 0,$$

and for any $\varphi \in C^\infty(\Gamma)$, we have

$$\left|\int_\Gamma w(x)\varphi(x)\,dx\right| = \left|\lim_{n\to\infty}\int_\Gamma w_n\varphi(x)\,dx\right| \le \int_\Gamma |\varphi(x)|\,dx.$$

Then $w \in L^\infty(\Omega)$, and $\|w\|_{L^\infty(\Omega)} \le 1$. In addition, by (5.4),

$$w_n \to \operatorname{sgn}(v) \quad \text{a.e. in } \{x : v(x) \ne 0\}$$

as $n \to \infty$. Thus, $w \in \operatorname{sgn}(u_1)$ and $\int_\Gamma w(x)\,dx = 0$, that is, $\int_\Gamma \operatorname{sgn}(v)(x)\,dx = 0$. □

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. We split the proof into three steps.

Setp 1. First, we show that $\{u_p\}_{1<p\le 2}$ is bounded in $W^{1,1}(\Gamma)$.

Let $\varphi \in C^\infty(\Gamma)$ be such that φ_e is odd with respect to the center of I_e for any $e \in \mathrm{E}(\Gamma)$. Then

$$\int_\Gamma |\varphi(x)|^{p-2}\varphi(x)\,dx = 0 \quad \forall p \in (1,\infty).$$

Then, using that u_p is a minimizer for $\lambda_{2,p}(\Gamma)$ with $\|u_p\|_{L^p(\Omega)=1}$ and Hölder's inequality, we get

$$\|u_p'\|^p_{L^1(\Gamma)} \le \|u_p'\|^p_{L^p(\Gamma)}(\ell(\Gamma))^{p-1} \le \frac{\|\varphi'\|^p_{L^p(\Gamma)}}{\|\varphi\|^p_{L^p(\Gamma)}}(\ell(\Gamma))^{p-1}.$$

Therefore, $\{u_p\}_{1<p\le 2}$ is bounded in $W^{1,1}(\Gamma)$.

Setp 2. Next, we show that

$$\liminf_{p\to 1^+}\lambda_{2,p}(\Gamma)^{1/p} \ge \Lambda_{2,1}(\Gamma).$$

Let $\{u_{p_n}\}_{n\in\mathbb{N}}$ be a subsequence of $\{u_p\}_{p\in(1,2)}$ such that $p_n \to 1^+$ as $n \to \infty$ and

$$\lim_{n\to\infty}\|u_{p_n}'\|_{L^{p_n}(\Gamma)} = \liminf_{p\to 1^+}\lambda_{2,p}(\Gamma)^{1/p}. \tag{5.6}$$

By step 1, $\{u_{p_n}\}_{n\in\mathbb{N}}$ is bounded in $W^{1,1}(\Gamma)$. Then, by Theorem 8.8 in [6] and Theorem 1 in [9, Sect. 5.2.1] there exist a constant $C > 0$, u_1, and a subsequence, still denoted $\{u_{p_n}\}_{n\in\mathbb{N}}$, such that

$$\|u_{p_n}\|_{L^\infty(\Omega)} \le C \quad \forall n \in \mathbb{N}, \tag{5.7}$$

$$u_{p_n} \to u_1 \quad \text{strongly in } L^q(\Gamma) \text{ for any } q \in [1,\infty), \tag{5.8}$$

$$u_{p_n} \to u_1 \quad \text{a.e. in } \Gamma, \tag{5.9}$$

and

$$\begin{aligned}\|u_1'\|(\Gamma) &\le \liminf_{n\to\infty}\|u_{p_n}'\|_{L^1(\Gamma)} \le \liminf_{n\to\infty}\|u_{p_n}'\|_{L^p(\Gamma)}(\ell(\Gamma))^{(p_n-1)/p_n} \\ &= \liminf_{p\to 1^+}\lambda_{2,p}(\Gamma)^{1/p}. \end{aligned}\tag{5.10}$$

Moreover, by (5.7), (5.8), and Holder's inequality we have that

$$\begin{aligned}\int_\Gamma |u_1(x)|\,dx &= \lim_{n\to\infty}\int_\Gamma |u_{p_n}(x)|\,dx \le \lim_{n\to\infty} \|u_{p_n}\|_{L^{p_n}(\Omega)}(\ell(\Gamma))^{(p_n-1)/p_n} = 1\\ &\le \lim_{n\to\infty} C^{p_n-1}\|u_{p_n}\|_{L^1(\Omega)}(\ell(\Gamma))^{(p_n-1)/p_n} = \int_\Gamma |u_1(x)|\,dx.\end{aligned}$$

Then $\|u_1\|_{L^1(\Gamma)} = 1$.

On the other hand, by (5.7) we have that $\{|u_{p_n}|^{p_n-2}u_{p_n}\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Gamma)$. Therefore, there exist a function w and a subsequence, still denoted $\{u_{p_n}\}_{n\in\mathbb{N}}$, such that

$$|u_{p_n}|^{p_n-2}u_{p_n} \rightharpoonup w \quad \text{weakly in } L^q(\Omega) \text{ for any } 1<q<\infty.$$

Thus,

$$\int_\Gamma w(x)\,dx = \lim_{n\to\infty}\int_\Gamma |u_{p_n}(x)|^{p_n-2}u_{p_n}(x)\,dx = 0,$$

and for any $\varphi\in C^\infty(\Gamma)$, we have

$$\begin{aligned}\left|\int_\Gamma w(x)\varphi(x)\,dx\right| &= \left|\lim_{n\to\infty}\int_\Gamma |u_{p_n}(x)|^{p_n-2}u_{p_n}(x)\varphi(x)\,dx\right|\\ &\le \lim_{n\to\infty} C^{p_n-1}\int_\Gamma |\varphi(x)|\,dx \quad \text{(by (5.7))}\\ &= \int_\Gamma |\varphi(x)|\,dx.\end{aligned}$$

Then $w\in L^\infty(\Omega)$, and $\|w\|_{L^\infty(\Omega)}\le 1$. In addition, by (5.9),

$$|u_{p_n}|^{p_n-2}u_{p_n} \to \operatorname{sgn}(u_1)$$

a.e. in $\{x : u_1(x)\neq 0\}$ as $n\to\infty$. Thus, $w\in\operatorname{sgn}(u_1)$ and $\int_\Gamma w(x)\,dx = 0$, that is, $\int_\Gamma \operatorname{sgn}(u_1)(x)\,dx = 0$.

Then $u\in \mathrm{BV}(\Gamma)$ and $\int_\Gamma \operatorname{sgn}(u_1)(x)\,dx = 0$, and therefore, by (5.10),

$$\Lambda_{2,p}(\Gamma) \le \|u_1'\|(\Gamma) \le \liminf_{p\to 1^+}\lambda_{2,p}(\Gamma)^{1/p}$$

since $\|u_1\|_{L^1(\Gamma)} = 1$.

Setp 3. Finally, we show that

$$\limsup_{p\to 1^+}\lambda_{2,p}(\Gamma)^{1/p} \le \Lambda_{2,1}(\Gamma).$$

Let $\{p_n\}_{n\in\mathbb{N}}\subset(1,2)$ be such that $p_n\to 1^+$ and

$$\limsup_{p\to 1^+}\lambda_{2,p}(\Gamma)^{1/p} = \lim_{n\to\infty}\lambda_{2,p_n}(\Gamma)^{1/p_n}. \tag{5.11}$$

Given $v\in \mathrm{BV}(\Gamma)\setminus\{0\}$ such that $\int_\Gamma \operatorname{sgn}(v)(x)\,dx = 0$, there exists $\{\varphi_j\}_{j\in\mathbb{N}}\subset C^\infty(\Gamma)$ such that

$$\varphi_j \to v \quad \text{strongly in } L^1(\Gamma), \tag{5.12}$$

$$\varphi_j \to v \quad \text{a.e. in } \Gamma, \tag{5.13}$$

$$\|\varphi_j'\|_{L^1(\Gamma)} \to \|v'\|_{\mathrm{BV}(\Gamma)}. \tag{5.14}$$

Moreover, there exists a constant $C > 0$ such that

$$\|\varphi_j\|_{L^\infty(\Gamma)} \le C \quad \forall j \in \mathbb{N}. \tag{5.15}$$

Fix $j \in \mathbb{N}$. For any $n \in \mathbb{N}$, there exists $c_{j,n} \in [\min_{x\in\Gamma} \varphi_j(x), \max_{x\in\Gamma} \varphi_j(x)]$ such that

$$\int_\Gamma |\varphi_j(x) - c_{j,n}|^{p_n-2}(\varphi_j(x) - c_{j,n})\, dx = 0. \tag{5.16}$$

By (5.15) there exist $c_j \in [\min_{x\in\Gamma} \varphi_j(x), \max_{x\in\Gamma} \varphi_j(x)]$ and a subsequence, still denoted $\{c_{j,n}\}_{n\in\mathbb{N}}$, such that $c_{j,n} \to c_j$ as $n \to \infty$. Moreover, proceeding as in step 2, we can check that there exists $w_j \in \operatorname{sgn}(\varphi_j - c_j)$ such that $\int_\Gamma w_j(x)\, dx = 0$, that is, $\int_\Gamma \operatorname{sgn}(\varphi_j(x) - c_j)\, dx = 0$.

Then,

$$\begin{aligned}
\limsup_{p\to 1^+} \lambda_{2,p}(\Gamma)^{1/p} &= \lim_{n\to\infty} \lambda_{2,p_n}(\Gamma)^{1/p_n} \le \liminf_{n\to\infty} \frac{\|(\varphi_j - c_{j,n})'\|_{L^{p_n}(\Gamma)}}{\|(\varphi_n - c_{j,n})\|_{L^{p_n}(\Gamma)}} \\
&\le \liminf_{n\to\infty} \frac{\|(\varphi_j - c_{j,n})'\|_{L^\infty(\Gamma)}^{(p_n-1)/p_n} \|(\varphi_{n_i} - c_{n_i,p_i})'\|_{L^1(\Gamma)}^{1/p_n}}{(\ell(\Gamma))^{(1-p_n)/p_n} \|(\varphi_j - c_j)\|_{L^1(\Gamma)}} \\
&= \frac{\|(\varphi_j - c_j)'\|_{L^1(\Gamma)}}{\|\varphi_j - c_j\|_{L^1(\Gamma)}}.
\end{aligned} \tag{5.17}$$

On the other hand, since $c_j \in [\min_{x\in\Gamma} \varphi_j(x), \max_{x\in\Gamma} \varphi_j(x)]$ for all $j \in \mathbb{N}$, by (5.15) there exist $c \in \mathbb{R}$ and a subsequence, still denoted $\{c_j\}_{j\in\mathbb{N}}$, such that $c_j \to c$ as $j \to \infty$. Then, by (5.12) we have that $\varphi_j - c_j \to v - c$ strongly in $L^1(\Gamma)$. Therefore, by Lemma 5.2, $\int_\Gamma \operatorname{sgn}(v(x) - c)\, dx = 0$. Hence, by (5.17), (5.14), and Lemma 5.2 we obtain

$$\begin{aligned}
\limsup_{p\to 1^+} \lambda_{2,p}(\Gamma)^{1/p} &\le \lim_{j\to\infty} \frac{\|(\varphi_j - c_j)'\|_{L^1(\Gamma)}}{\|\varphi_j - c_j\|_{L^1(\Gamma)}} = \lim_{j\to\infty} \frac{\|\varphi_j'\|_{L^1(\Gamma)}}{\|\varphi_j - c_j\|_{L^1(\Gamma)}} \\
&= \frac{\|v'\|(\Gamma)}{\|v - c\|_{L^1(\Gamma)}} = \frac{\|v'\|(\Gamma)}{\|v\|_{L^1(\Gamma)}}.
\end{aligned}$$

Since v is arbitrary, we have that

$$\limsup_{p\to 1^+} \lambda_{2,p}(\Gamma)^{1/p} \le \Lambda_{2,1}(\Gamma).$$

Therefore, from this inequality and step 2 we conclude that

$$\lim_{p\to 1^+} \lambda_{2,p}(\Gamma) = \Lambda_{2,1}(\Gamma)$$

and that u_1 is a minimizer for (1.5). □

The next result gives a curious property that we include here just for completeness, although it is not needed in the proofs of our main results.

LEMMA 5.3. *Let* Γ *be a connected compact metric graph,* $\varphi \in C^\infty(\Gamma)$ *be such that*

$$\int_\Gamma \operatorname{sgn}(\varphi)(x)\,dx = 0, \tag{5.18}$$

and $\{c_p\}_{p>1}$ *be a subset of* $(\min_{x\in\Gamma}\varphi(x), \max_{x\in\Gamma}\varphi(x))$ *such that*

$$\int_\Gamma |\varphi(x) - c_p|^{p-2}(\varphi(x) - c_p)\,dx = 0. \tag{5.19}$$

Then $c_p \to 0$ *as* $p \to 1^+$.

Proof. We show that all convergent subsequences of $\{c_p\}_{p>1}$ converge to 0. Let $\{c_{p_i}\}_{i\in\mathbb{N}}$ be a subsequence of $\{c_p\}_{p>1}$ such that

$$p_i \to 1^+ \quad \text{and} \quad c_{p_i} \to c \in \left[\min_{x\in\Gamma}\varphi(x), \max_{x\in\Gamma}\varphi(x)\right]$$

as $i \to \infty$. We will see that $c = 0$.

It is clear that there exists a constant $C > 0$ such that

$$\|\varphi - c_{p_i}\|_{L^\infty(\Omega)} \le C \quad \forall i \in \mathbb{N}. \tag{5.20}$$

Then $\{|\varphi - c_{p_i}|^{p_i-2}(\varphi - c_{p_i})\}_{i\in\mathbb{N}}$ is bounded in $L^q(\Gamma)$ for all $q \in [1,\infty]$. Therefore, there exist $v \in L^q(\Omega)$ and a subsequence that, still denoted $\{|\varphi - c_{p_i}|^{p_i-2}(\varphi - c_{p_i})\}_{i\in\mathbb{N}}$, such that

$$|\varphi - c_{p_i}|^{p_i-2}(\varphi - c_{p_i}) \rightharpoonup v \quad \text{weakly in } L^q(\Omega)$$

for any $1 < q < \infty$. Thus,

$$\int_\Gamma v(x)\,dx = \lim_{i\to\infty}\int_\Gamma |\varphi(x) - c_{p_i}|^{p_i-2}(\varphi(x) - c_{p_i})\,dx = 0, \quad \text{(by (5.19))}$$

and for any $\phi \in C^\infty(\Gamma)$, we have

$$\begin{aligned}\left|\int_\Gamma v(x)\phi(x)\,dx\right| &= \left|\lim_{i\to\infty}\int_\Gamma |\varphi(x) - c_{p_i}|^{p_i-2}(\varphi(x) - c_{p_i})\phi(x)\,dx\right| \\ &\le \lim_{i\to\infty} C^{p_i-1}\int_\Gamma |\phi(x)|\,dx \quad \text{(by (5.20))} \\ &= \int_\Gamma |\phi(x)|\,dx.\end{aligned}$$

Then $v \in L^\infty(\Omega)$, $\|v\|_{L^\infty(\Omega)} \le 1$, and

$$\int_\Gamma v(x)\,dx = 0. \tag{5.21}$$

In addition,

$$|\varphi - c_{p_i}|^{p_i-2}(\varphi - c_{p_i}) \to \operatorname{sgn}(\varphi - c)$$

a.e. in $\{x : \varphi(x) - c \ne 0\}$ as $i \to \infty$. Therefore, $v \in \operatorname{sgn}(\varphi - c)$.

On the other hand, by (5.18) there exists $w \in \operatorname{sgn}(\varphi)$ such that

$$0 = \int_\Gamma w(x)\,dx = \ell(E_0^+) + \int_{E_0} w(x)\,dx - \ell(E_0^-), \tag{5.22}$$

where $E_0^+ = \{x\colon \varphi(x) > 0\}$, $E_0^- = \{x\colon \varphi(x) < 0\}$, and $E_0 = \{x\colon \varphi(x) = 0\}$.

We now suppose by contraction that $c \neq 0$. We only consider the case $c > 0$. The case $c < 0$ is similar.

Taking $E_c^+ = \{x\colon \varphi(x) > c\}$, $E_c^- = \{x\colon \varphi(x) < c\}$, $E_c = \{x\colon \varphi(x) = c\}$, and $E_{0,c} = \{x\colon 0 < \varphi(x) < c\}$, we have that

$$\begin{aligned}
0 &= \int_\Gamma v(x)\,dx \quad \text{(by (5.21))}\\
&= \ell(E_c^+) - \ell(E_c^-) + \int_{E_c} v(x)\,dx \quad (v \in \operatorname{sgn}(\varphi - c))\\
&= \ell(E_c^+) - \ell(E_{0,c}) - \ell(E_0) - \ell(E_0^-) + \int_{E_c} v(x)\,dx\\
&\leq \ell(E_c^+) - \ell(E_{0,c}) + \int_{E_0} w(x)\,dx - \ell(E_0^-) + \int_{E_c} v(x)\,dx \quad (\|w\|_{L^\infty(\Gamma)} \leq 1)\\
&= \ell(E_c^+) - \ell(E_{0,c}) - \ell(E_0^+) + \int_{E_c} v(x)\,dx \quad \text{(by (5.22))}\\
&= -2\ell(E_{0,c}) + \int_{E_c} (v-1)(x)\,dx \leq -2\ell(E_{0,c}) \quad (\|v\|_{L^\infty(\Gamma)} \leq 1).
\end{aligned}$$

If $\ell(E_{0,c}) > 0$, then we arrive at a contradiction. If $\ell(E_{0,c}) = 0$, then we have two possibilities: either $\varphi \geq c$ or $\varphi \leq 0$. In the case $\varphi \geq c$, we get a contradiction with (5.22). Finally, if $\varphi \leq 0$, then we arrive at a contradiction with (5.21). Consequently, $c = 0$. □

Proof of Theorem 1.4. We begin by observing that

$$\Lambda_{2,1}(\Gamma) \leq \inf\left\{ \frac{\operatorname{Per}(E)}{\min\{\ell(E), \ell(\Gamma \setminus E)\}} : D \subsetneq \Gamma,\, E \neq \emptyset \right\}.$$

Therefore, if $u = \chi_A - \chi_{\Gamma\setminus A}$ is a minimizer for $\Lambda_{2,1}(\Gamma)$, then

$$\Lambda_{2,1}(\Gamma) = \frac{\|u'\|(\Gamma)}{\|u\|_{L^1(\Gamma)}} = \frac{2\operatorname{Per}(A)}{\ell(\Gamma)} \geq \inf\left\{ \frac{\operatorname{Per}(E)}{\min\{\ell(E), \ell(\Gamma \setminus E)\}} : D \subsetneq \Gamma,\, E \neq \emptyset \right\},$$

that is,

$$\Lambda_{2,1}(\Gamma) = \frac{2\operatorname{Per}(A)}{\ell(\Gamma)} = \inf\left\{ \frac{\operatorname{Per}(E)}{\min\{\ell(E), \ell(\Gamma \setminus E)\}} : E \subsetneq \Gamma,\, E \neq \emptyset \right\}.$$

On the other hand, suppose that (1.6) is valid. For any $v \in \mathrm{BV}(\Gamma)$ such that $\int_\Gamma \operatorname{sgn}(v)(x)\,dx = 0$, $v \neq 0$, by the coarea formula (see [9, Thm. 1 in Sect. 5.5]) we have that

$$\|v'\|(\Gamma) = \int_{-\infty}^{\infty} \operatorname{Per}(E_t^+)\,dt, \tag{5.23}$$

where $E_t^+ = \{x\colon v(x) > t\}$.

Since $\int_\Gamma \operatorname{sgn}(v)(x)\,dx = 0$, we also have that $\ell(E_t^+) \le \ell(\{x : v(x) \le t\})$ for all $t \ge 0$ and $\ell(E_t^+) \ge \ell(\{x : v(x) \le t\})$ for all $t < 0$. Then, by (5.23) and (1.6) we get

$$\begin{aligned}
\|v'\|(\Gamma) &= \int_0^\infty \operatorname{Per}(E_t^+)\,dt + \int_{-\infty}^0 \operatorname{Per}(E_t^+)\,dt \\
&\ge \frac{2\operatorname{Per}(A)}{\ell(\Gamma)}\left(\int_0^\infty \ell(E_t^+)\,dt + \int_{-\infty}^0 \ell(\{x : v(x) \le t\})\,dt\right) \\
&\ge \frac{2\operatorname{Per}(A)}{\ell(\Gamma)}\|v\|_{L^1(\Gamma)}.
\end{aligned}$$

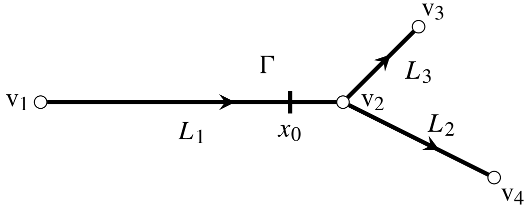
Then,

$$\Lambda_{2,1}(\Gamma) \ge \frac{2\operatorname{Per}(A)}{\ell(\Gamma)} = \frac{\|u'\|(\Gamma)}{\|u\|_{L^1(\Gamma)}}. \tag{5.24}$$

Finally, we observe that $\ell(x : u(x) > 0) = \ell(A) = \ell(\Gamma \setminus A) = \ell(\{x : u(x) \le 0\})$. Then $\int_\Gamma \operatorname{sgn}(u)(x)\,dx = 0$, and by (5.24), u is a minimizer for $\Lambda_{2,1}(\Gamma)$. □

REMARK 5.4. In the example described in Remark 3.1 with three edges of the same length L, we have that this limit selects (as for the case $p = \infty$) two edges as A_∞ and B_∞ and the third edge is just $\{u = 0\}$. Here we have only one "cut" in our graph Γ (the perimeter of A and B inside Γ is one).

Now, let us consider the same configuration of the graph, but with three different lengths L_1, L_2, L_3 for the three different edges, and let us assume that $L_1 > L_2 > L_3$ with $L_1 > L_2 + L_3$. In this case, we get that this limit finds a point $x_0 \in \Gamma$ that divides Γ in two sets A and B with the same total length. The position of x_0 is the point in L_1 whose distance to v_1 is $(L_1 + L_2 + L_3)/2$.



References

[1] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variations and free discontinuity problems,* Oxford University Press, London, 2000.

[2] G. Berkolaiko and P. Kuchment, *Introduction to quantum graphs,* Math. Surveys Monogr., 186, xiv + 270 pp., American Mathematical Society, Providence, RI, 2013.

[3] I. Birindelli and F. Demengel, *First eigenvalue and maximum principle for fully nonlinear singular operators,* Adv. Differential Equations 11 (2006), no. 1, 91–119.

[4] ———, *Eigenvalue, maximum principle and regularity for fully non linear homogeneous operators,* Commun. Pure Appl. Anal. 6 (2007), no. 2, 335–366.

[5] A. N. Bondarenko and V. A. Dedok, *Spectral surgery for the Schrödinger operator on graphs,* Dokl. Math. 85 (2012), no. 3, 367–368.

[6] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations,* Universitext, xiv + 599 pp., Springer, New York, 2011.

[7] V. Caselles, A. Chambolle, and M. Novaga, *Some remarks on uniqueness and regularity of Cheeger sets,* Rend. Semin. Mat. Univ. Padova 123 (2010), 191–201.

[8] L. M. Del Pezzo and J. D. Rossi, *The first eigenvalue of the p-Laplacian on quantum graphs,* preprint, ⟨http://cms.dm.uba.ar/Members/ldpezzo/QG.pdf⟩; Anal. Math. Phys. (to appear).

[9] L. Evans and R. Gariepy, *Measure theory and fine properties of functions,* Stud. Adv. Math., viii + 268 pp., CRC Press, Boca Raton, FL, 1992.

[10] A. Figalli, F. Maggi, and A. Pratelli, *A note on Cheeger sets,* Proc. Amer. Math. Soc. 137 (2009), 2057–2062.

[11] L. Friedlander, *Genericity of simple eigenvalues for a metric graph,* Israel J. Math. 146 (2005), 149–156.

[12] J. Garcia-Azorero and I. Peral, *Existence and non-uniqueness for the p-Laplacian: nonlinear eigenvalues,* Comm. Partial Differential Equations 12 (1987), 1389–1430.

[13] P. Juutinen, *Principal eigenvalue of a very badly degenerate operator and applications,* J. Differential Equations 236 (2007), 532–550.

[14] P. Juutinen, P. Lindqvist, and J. J. Manfredi, *The ∞-eigenvalue problem,* Arch. Ration. Mech. Anal. 148 (1999), 89–105.

[15] V. Kostrykin and R. Schrader, *Kirchhoff's rule for quantum wires,* J. Phys. A 32 (1999), no. 4, 595–630.

[16] P. Kuchment, *Quantum graphs: I. Some basic structures,* Waves Random Media 14 (2004), S107–S128.

[17] P. Kurasov, *On the spectral gap for Laplacians on metric graphs,* Acta Phys. Polon. A 124 (2013), 1060–1062.

[18] P. Kurasov, G. Malenova, and S. Naboko, *Spectral gap for quantum graphs and their edge connectivity,* J. Phys. A 46 (2013), 275309.

[19] P. Kurasov and S. Naboko, *On Rayleigh theorem for quantum graphs,* Institut Mittag-Leffler Report No. 4, 2012/2013.

[20] ______, *Rayleigh estimates for differential operators on graphs,* J. Spectr. Theory 4 (2014), no. 2, 211–219.

[21] J. Lang and D. Edmunds, *Eigenvalues, embeddings and generalised trigonometric functions,* Lecture Notes in Math., 2016, xii + 220 pp., Springer, Heidelberg, 2011.

[22] P. Lindqvist, *Note on a nonlinear eigenvalue problem,* Rocky Mountain J. Math. 23 (1993), 281–288.

[23] ______, *Some remarkable sine and cosine functions,* Ric. Mat. XLIV (1995), 269–290.

[24] P. Lindqvist and J. Peetre, *Two remarkable identities, called Twos, for inverses to some Abelian integrals,* Amer. Math. Monthly 108 (2001), 403–410.

[25] E. Parini, *An introduction to the Cheeger problem,* Surv. Math. Appl. 6 (2011), 9–22.

[26] O. Post, *Spectral analysis on graph-like spaces,* Lecture Notes in Math., 2012.

[27] J. D. Rossi and N. Saintier, *On the first nontrivial eigenvalue of the ∞-Laplacian with Neumann boundary conditions,* Houston J. Math. 42 (2016), 613–635.

L. M. Del Pezzo
CONICET and Departamento de Matemática
FCEyN
Universidad de Buenos Aires
Pabellon I
Ciudad Universitaria (1428)
Buenos Aires
Argentina

ldpezzo@dm.uba.ar

J. D. Rossi
CONICET and Departamento de Matemática
FCEyN
Universidad de Buenos Aires
Pabellon I
Ciudad Universitaria (1428)
Buenos Aires
Argentina

jrossi@dm.uba.ar