QUANDLE COLORING AND COCYCLE INVARIANTS OF COMPOSITE KNOTS AND ABELIAN EXTENSIONS

W. EDWIN CLARK, MASAHICO SAITO, LEANDRO VENDRAMIN

Abstract. Quandle colorings and cocycle invariants are studied for composite knots, and applied to chirality and abelian extensions. The square and granny knots, for example, can be distinguished by quandle colorings, so that a trefoil and its mirror can be distinguished by quandle coloring of composite knots. We investigate this and related phenomena. Quandle cocycle invariants are studied in relation to quandle coloring of the connected sum, and formulas are given for computing the cocycle invariant from the number of colorings of composite knots. Relations to corresponding abelian extensions of quandles are studied, and extensions are examined for the table of small connected quandles, called Rig quandles. Computer calculations are presented, and summaries of outputs are discussed.

Introduction

Sets with certain self-distributive operations called *quandles* have been studied since the 1940s in various areas with different names. The *fundamental quandle* of a knot was defined in a manner similar to the fundamental group [21, 24] of a knot, which made quandles an important tool in knot theory. The number of homomorphisms from the fundamental quandle to a fixed finite quandle has an interpretation as colorings of knot diagrams by quandle elements, and has been widely used as a knot invariant. Algebraic homology theories for quandles were defined [5, 19], and investigated in [22, 25, 26, 27]. Extensions of quandles by cocycles have been studied [1, 4, 16], and invariants derived thereof are applied to various properties of knots and knotted surfaces (see [8] and references therein).

Tables of small quandles have been made previously (e.g., [8, 15, 17]). Computations using GAP [34] significantly expanded the list for connected quandles. These quandles may be found in the GAP package Rig [33]. Rig includes all connected quandles of order less than 48. We refer to these quandles as *Rig* quandles, and use the notation $Q(n, i)$ for the i-th quandle of order n in the list of Rig quandles. As a matrix $Q(n, i)$ is the transpose

Key words and phrases. Quandles, knot colorings, cocycle invariants, connected sum, chirality.

of the quandle matrix $\mathsf{SmallQuandle}(n, i)$ in [33]. In this paper, however, we focus on Rig quandles of order less than 36. There are 431 such quandles.

In [11], it was investigated to what extent the number of quandle colorings of a knot by a finite quandle can distinguish the prime oriented knots with at most 12 crossings in the knot table at KnotInfo [14]. It is known that quandle colorings do not distinguish K from its reversed mirror, $rm(K)$. It is also known [10] that the quandle cocycle invariant can distinguish a trefoil 3_1 from its mirror image. Since 3_1 is reversible, it cannot be distinguished from its mirror by quandle colorings. However, we show here that quandle colorings can be used via connected sums to distinguish K from $rm(K)$ for many knots (we conjecture for all knots K such that $K \neq rm(K)$). In particular, for some reversible knots, we can distinguish K from $m(K)$ using this technique. For example, by distinguishing the square and granny knots by quandle colorings, we distinguish a trefoil from its mirror image. In this paper, we investigate this phenomenon, and other properties and applications of quandle invariants under connected sum. In particular, we relate quandle colorings of composite knots to quandle 2-cocycle invariants.

We also note that quandle colorings of the connected sum can be used to recover quandle cocycle invariants in many cases. It is well-known that quandle 2-cocycles give rise to abelian extensions of quandles, see for example [4]. We investigate the relations among abelian extensions that result from our computations, and their properties. As a result, several problems arise naturally.

An important part of this work depends on computer calculations. For that reason, we developed algorithms and techniques for computing quandle (co)homology groups and explicit quandle 2-cocycles, abelian extensions of quandles, dynamical cocycles and non-abelian extensions, colorings and quandle cocycle invariants of classical and virtual knots. The algorithms are freely available in the GAP package Rig. Several tables with all these calculations are available online at the Wiki page of Rig: `http://github.com/vendramin/rig/wiki`.

The paper is organized as follows. Preliminary material necessary for the paper follows this section, and it is shown that the number of quandle colorings by finite quandles can distinguish unknot in Section 2. Quandle colorings of composite knots are studied in Section 3. In Section 4, quandle colorings of composite knots are applied to distinguish knots from their reversed mirror images, relations to the quandle cocycle invariant are discussed, and computer calculations are presented. In Section 5, a method of computing quandle cocycle invariants from colorings of composite knots is studied. Relations to abelian extensions of quandles are examined in Section 6. Further considerations regarding extensions of Rig quandles are presented in Section 7. For convenience of the reader, we collect problems, questions and conjectures posed all over the text in Section 8.

1. Preliminaries

We briefly review some definitions and examples of quandles. More details can be found, for example, in [1, 8, 19].

A *quandle* X is a set with a binary operation $(a, b) \mapsto a * b$ satisfying the following conditions.

(1) For any $a \in X$, $a * a = a$.
(2) For any $b, c \in X$, there is a unique $a \in X$ such that $a * b = c$.
(3) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

A *quandle homomorphism* between two quandles X, Y is a map $f : X \to Y$ such that $f(a *_X b) = f(a) *_Y f(b)$, where $*_X$ and $*_Y$ denote the quandle operations of X and Y, respectively. A *quandle isomorphism* is a bijective quandle homomorphism, and two quandles are *isomorphic* if there is a quandle isomorphism between them.

Example 1.1. Any non-empty set X with the operation $a * b = a$ for any $a, b \in X$ is a quandle called a *trivial* quandle.

Example 1.2. A conjugacy class X of a group G is a quandle with the quandle operation $a * b = b^{-1}ab$. We call this a *conjugation quandle*.

Example 1.3. Let X and Y be quandles. Then $X \times Y$ is a quandle with $(x, y) * (x', y') = (x *_X x', y *_Y y')$ for all $x, x' \in X$ and $y, y' \in Y$.

Example 1.4 (Joyce [21]). A *generalized Alexander quandle* is defined by a pair (G, f) where G is a group, $f \in \mathrm{Aut}(G)$, and the quandle operation is defined by $x * y = f(xy^{-1})y$. If G is abelian, this is called an *Alexander* (or *affine*) quandle.

Example 1.5. A function $\phi : X \times X \to A$ for an abelian group A is called a *quandle 2-cocycle* [5] if it satisfies

$$\phi(x, y) - \phi(x, z) + \phi(x * y, z) - \phi(x * z, y * z) = 0$$

for any $x, y, z \in X$ and $\phi(x, x) = 0$ for any $x \in X$. For a quandle 2-cocycle ϕ, $E = X \times A$ becomes a quandle by

$$(x, a) * (y, b) = (x * y, a + \phi(x, y))$$

for $x, y \in X$, $a, b \in A$, denoted by $E(X, A, \phi)$ or simply $E(X, A)$, and it is called an *abelian extension* of X by A. The set of quandle 2-cocycles of X

with coefficients in A is denoted by $Z^2_Q(X,A)$. Two cocycles ϕ_1 and ϕ_2 are *cohomologous* if there is a function $\gamma\colon X \to A$ such that

$$\phi_2(x,y) = -\gamma(x) + \phi_1(x,y) + \gamma(x*y)$$

for any $x,y \in X$. The set of equivalence classes is a group and it is denoted by $H^2_Q(X,A)$. See [4] for more information on abelian extensions of quandles and [5, 6, 7] for more on quandle cohomology.

Example 1.6. In [1], *extensions by constant 2-cocycles* were defined as follows. For a quandle X and a set S, a *constant quandle cocycle* is a map

$$\beta : X \times X \to \mathrm{Sym}(S),$$

where $\mathrm{Sym}(S)$ is the symmetric group on S, such that $X \times S$ has a quandle structure by $(x,t)*(y,s) = (x*y, \beta_{x,y}(t))$ for $x,y \in X$ and $s,t \in S$ (see [1] for details). This quandle is denoted by $X \times_\beta S$. The map β satisfies the *constant cocycle condition* $\beta_{x*y,z}\beta_{x,y} = \beta_{x*z,y*z}\beta_{x,z}$ for any $x,y,z \in X$ and the quandle condition $\beta_{x,x} = \mathrm{id}$ for any $x \in X$. Following [1], we also call these extensions *non-abelian* extensions.

Let X be a quandle. The *right translation* $\mathcal{R}_a : X \to X$, by $a \in X$, is defined by $\mathcal{R}_a(x) = x*a$ for $x \in X$. Then $\mathcal{R}_a$ is a permutation of X by Axiom (2). The subgroup of $\mathrm{Aut}(X)$, the quandle automorphism group, generated by the permutations $\mathcal{R}_a$, $a \in X$, is called the *inner automorphism group* of X, and is denoted by $\mathrm{Inn}(X)$. A quandle is *connected* if $\mathrm{Inn}(X)$ acts transitively on X. A quandle is *homogeneous* if $\mathrm{Aut}(X)$ acts transitively on X. A quandle is *faithful* if the mapping $\varphi : X \to \mathrm{Inn}(X)$ defined by $\varphi(a) = \mathcal{R}_a$ is an injection from X to $\mathrm{Inn}(X)$. We note that abelian as well as non-abelian extensions are not faithful. The operation $\bar{*}$ on X defined by $a \mathbin{\bar{*}} b = \mathcal{R}_b^{-1}(a)$ is a quandle operation, and $(X, \bar{*})$ is called the *dual* quandle of $(X,*)$. A quandle X is called a *kei* [31], or *involutory*, if $(x*y)*y = x$ for all $x,y \in X$.

A *coloring* of an oriented knot diagram by a quandle X is a map $\mathcal{C}$ from the set of arcs $\mathcal{A}$ of the diagram to X such that the image of the map satisfies the relation depicted in Figure 1 at each crossing. More details can be found in [8, 16], for example. A coloring that assigns the same element of X to all the arcs is called trivial, otherwise non-trivial. The number of colorings of a knot diagram by a finite quandle is known to be independent of the choice of diagram, and hence is a knot invariant. We denote by $\mathrm{SCol}_X(K)$ and $\mathrm{Col}_X(K)$ the set and the number of colorings of K by X.

The fundamental quandle is defined in a manner similar to the fundamental group [21, 24]. A *presentation* of a quandle is defined in a manner similar to groups as well, and a presentation of the fundamental quandle is obtained from a knot diagram (see, for example, [18]), by assigning generators to arcs of a knot diagram, and relations corresponding to crossings. The set of colorings of a knot diagram K by a quandle X, then, is in one-to-one correspondence with the set of quandle homomorphisms from the fundamental quandle of K to X.

In this paper all knots are oriented. Let $m : \mathbb{S}^3 \to \mathbb{S}^3$ be an orientation reversing homeomorphism of the 3-sphere. For a knot K contained in $\mathbb{S}^3$, $m(K)$ is the mirror image of K, and $r(K)$ is the knot K with its orientation reversed. We regard m and r as maps on equivalence classes of knots. We consider the group $\mathcal{G} = \{1, r, m, rm\}$ acting on the set of all oriented knots. For each knot K let $\mathcal{G}(K) = \{K, r(K), m(K), rm(K)\}$ be the orbit of K under the action of $\mathcal{G}$.

For knots K and K', we write $K = K'$ to denote that there is an orientation preserving homeomorphism of $\mathbb{S}^3$ that takes K to K' preserving the orientations of K and K'. By a symmetry we mean that a knot (type) K remains unchanged under one of r, m, rm. As in the definition of *symmetry type* in [14] we say that a knot K is

- *reversible* if the only symmetry it has is $K = r(K)$,
- *negative amphicheiral* if the only symmetry it has is $K = rm(K)$,
- *positive amphicheiral* if the only symmetry it has is $K = m(K)$,
- *chiral* if it has none of these symmetries,
- *fully amphicheiral* if $K = r(K) = m(K) = rm(K)$, i.e. if K has all three symmetries.

The symmetry type of each knot on at most 12 crossings is given at [14]. Thus each of the 2977 knots K given there represents 1, 2 or 4 knots depending on the symmetry type. Among the 2977 knots, there are 1580 reversible, 47 negative amphicheiral, 1 positive amphicheiral, 1319 chiral, and 30 fully amphicheiral knots.

It is known [21, 24] that the fundamental quandles of K and K' are isomorphic if and only if $K = K'$ or $K = rm(K')$.

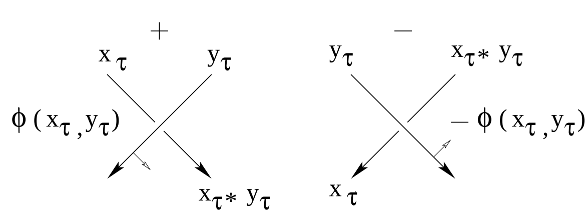


FIGURE 1. Colored crossings and cocycle weights

Let X be a quandle, and ϕ be a 2-cocycle with coefficient group A, a finite abelian group; we use multiplicative notation. We regard ϕ as a function

$\phi : X \times X \to A$. For a coloring of a knot diagram by a quandle X as depicted in Figure 1 at a positive (left) and negative (right) crossing, respectively, the pair (x_τ, y_τ) of colors assigned to a pair of nearby arcs is called the *source* colors. The third arc receives the color $x_\tau * y_\tau$.

The 2-cocycle (or cocycle, for short) invariant is an element of the group ring $\mathbb{Z}[A]$ defined by $\Phi_\phi(K) = \sum_{\mathcal{C}} \prod_\tau \phi(x_\tau, y_\tau)^{\epsilon(\tau)}$, where the product ranges over all crossings τ, the sum ranges over all colorings of a given knot diagram, (x_τ, y_τ) are source colors at the crossing τ, and $\epsilon(\tau)$ is the sign of τ as specified in Figure 1.

When $\mathbb{Z}_n$ is contained as a subgroup in $\mathbb{Z}_m$ and in $H^2_Q(X, \mathbb{Z}_m)$, and if a 2-cocycle $\phi : X \times X \to \mathbb{Z}_n$ is such that $[\phi]$ is a generator of the subgroup $\mathbb{Z}_n$ in $H^2_Q(X, \mathbb{Z}_m)$, then we say that ϕ is a generating 2-cocycle of the subgroup $\mathbb{Z}_n$.

Lemma 1.7. *If the second homology group $H^Q_2(X, \mathbb{Z})$ for X satisfies $H^Q_2(X, \mathbb{Z}) = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, $n_i > 0$ for all i, then we have*

$$H^2_Q(X, \mathbb{Z}_n) \cong \mathbb{Z}_{n'_1} \oplus \mathbb{Z}_{n'_2} \oplus \cdots \oplus \mathbb{Z}_{n'_k},$$

where $n'_i = \gcd(n_i, n)$.

Proof. It is known that $H^2_Q(X, A)$ is isomorphic to $\mathrm{Hom}(H^Q_2(X, \mathbb{Z}), A)$ by the universal coefficient theorem and from the fact that $H^Q_1(X, \mathbb{Z})$ is torsion free [7]. The result follows from the standard facts

$$\mathrm{Hom}(A_1 \oplus A_2 \oplus \cdots \oplus A_k, C) = \mathrm{Hom}(A_1, C) \oplus \mathrm{Hom}(A_2, C) \oplus \cdots \oplus \mathrm{Hom}(A_k, C)$$

and $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(n,m)}$, for positive integers n and m. $\square$

The groups $H^Q_2(X, \mathbb{Z})$ for some Rig quandles are found at [33]. Note that the groups given in [33] are rack homology $H^R_2(X, \mathbb{Z})$, and the relationship is given by $H^R_2(X, \mathbb{Z}) \cong H^Q_2(X, \mathbb{Z}) \oplus \mathbb{Z}$ [22].

The package Rig [33] includes cohomology groups, 2-cocycles, abelian extensions and cocycle invariants for some Rig quandles and some knots in the KnotInfo table [14]. Multiplication tables of Rig quandles, (co)homology groups, generating 2-cocycles, and abelian extensions of Rig quandles that we used for computations can be obtained online at the Wiki page of Rig: `http://github.com/vendramin/rig/wiki`.

2. Distinguishing the unknot by quandle colorings

We recall the following conjecture of [11].

Conjecture 2.1. *If K and K' are any two knots such that $K' \neq K$ and $K' \neq rm(K)$ then there is a finite quandle X such that $\mathrm{Col}_X(K) \neq \mathrm{Col}_X(K')$.*

In this section, we prove this conjecture when K' is the unknot. The idea is somewhat similar to that of Eisermann, see [16, Remark 59].

Proposition 2.2. *Let K be a non-trivial knot. Then there exists a finite quandle X such that K admits a non-trivial coloring with X.*

First we recall the facts we need for the proof, see for example [16].

(1) Papakyriakopoulos [28] proved that a knot is trivial if and only its longitude is trivial in the fundamental group of the complement of the knot, called the knot group, $\pi_1(\mathbb{S}^3 \setminus K)$.

(2) The *Wirtinger presentation* of the knot group of an oriented knot K is defined as follows. Label the arcs $x_1, x_2, \dots, x_n$. At the end of the arc x_{i-1} we undercross the arc $x_{k(i)}$ and continue on arc x_i. Let $\epsilon(i)$ be the sign of the crossing as in Figure 1. Then the knot group is

$$\pi_1(\mathbb{S}^3 \setminus K) \simeq \langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle,$$

where $r_i = {x_{k(i)}}^{-\epsilon(i)} x_{i-1} {x_{k(i)}}^{\epsilon(i)} x_i^{-1}$ for all i.

(3) The map $\partial\colon \pi_1(\mathbb{S}^3 \setminus K) \to \mathbb{Z}$ given by $\partial(x_i) = 1$ for all i is a group homomorphism. By [3], Remark 3.13, the longitude l_K can be written as a word w on all the generators $x_1, \dots, x_n$ with $\partial(w) = 0$.

(4) Recall that a group G is *residually finite* if every non-trivial $g \in G$ is mapped non-trivially into some finite quotient of G. As a consequence of [32] one obtains that every knot group is residually finite, see [20] for a proof.

Proof of Proposition 2.2. Since K is non-trivial, $l_K \neq 1$. Since knot groups are residually finite, there exists a finite group G and a surjective group homomorphism $f\colon \pi_1(\mathbb{S}^3 \setminus K) \to G$ such that $f(l_K) \neq 1$. Then f maps the conjugacy class of x_1 into a non-trivial conjugacy class X of G. From this it follows that the knot K admits a non-trivial coloring with the conjugation quandle X. □

3. Quandle colorings of composite knots

In this section we introduce the concept of *end monochromatic*, and show that if a knot K_1 or a knot K_2 is end monochromatic with a finite homogeneous quandle X, then $|X|\mathrm{Col}_X(K_1 \# K_2) = \mathrm{Col}_X(K_1)\mathrm{Col}_X(K_2)$.

A 1-tangle is a properly embedded arc in a 3-ball, and the equivalence of 1-tangles is defined by ambient isotopies of the 3-ball fixing the boundary (cf. [13]). A diagram of a 1-tangle is defined in a manner similar to a knot diagram, from a regular projection to a disk by specifying crossing information, see Figure 2(A). An orientation of a 1-tangle is specified by an arrow on a diagram as depicted. A knot diagram is obtained from a 1-tangle diagram by closing the end points by a trivial arc outside of a disk. This procedure is called the *closure* of a 1-tangle. If a 1-tangle is oriented, then the closure inherits the orientation.

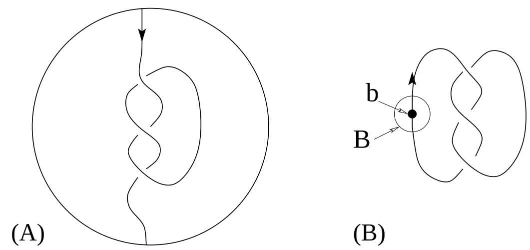


FIGURE 2. 1-tangles

A 1-tangle is obtained from a knot K as follows. Choose a base point $b \in K$ and a small open neighborhood B of b in the 3-sphere $\mathbb{S}^3$ such that $(B, K \cap B)$ is a trivial ball-arc pair (so that $K \cap B$ is unknotted in B, see Figure 2(B)). Then $(\mathbb{S}^3 \setminus \mathrm{Int}(B), K \cap (\mathbb{S}^3 \setminus \mathrm{Int}(B)))$ is a 1-tangle called the 1-tangle associated with K. The resulting 1-tangle does not depend on the choice of a base point. If a knot is oriented, then the corresponding 1-tangle inherits the orientation.

A quandle coloring of an oriented 1-tangle diagram is defined in a manner similar to those for knots. We do not require that the end points receive the same color for a quandle coloring of 1-tangle diagrams.

Definition 3.1. Let K be a 1-tangle diagram and X be a quandle. We say that (K, X) is *end monochromatic*, or K is *end monochromatic* with X, if any coloring of K by X assigns the same color on the two end points.

Two diagrams of the same 1-tangle are related by Reidemeister moves. The one-to-one correspondence of colorings under each Reidemeister move does not change the colors of the end points. Thus we have the following.

Lemma 3.2. *The property of being end monochromatic for a 1-tangle corresponding to a knot K and a base point b does not depend on the choice of the base point b.*

Thus, if a diagram of a 1-tangle corresponding to a knot K and some base point b is end monochromatic with X, then we say that a knot K is *end monochromatic* with X.

Lemma 3.3. *Let X be a finite homogeneous quandle, $x \in X$, and $\mathrm{Col}_{(X,x)}(K, b)$ be the number of colorings of a diagram K by X such that the arc that contains the base point b receives the color x. Then*

$$\mathrm{Col}_{(X,x)}(K, b) = \mathrm{Col}_X(K)/|X|$$

for any $x \in X$.

Proof. First we show that $\mathrm{Col}_{(X,x)}(K,b) = \mathrm{Col}_{(X,y)}(K,b)$ for any $x, y \in X$. Let $\mathrm{SCol}_{(X,x)}(K,b)$ be the set of colorings $\mathcal{C}$ such that $\mathcal{C}(\alpha) = x$, where α is the arc that contains b. Since X is homogeneous, there is an automorphism h of X such that $h(x) = y$. For any coloring $\mathcal{C} \in \mathrm{SCol}_{(X,x)}(K,b)$, $h_\#(\mathcal{C}) = h \circ \mathcal{C}$ satisfies $h_\#(\mathcal{C})(\alpha) = y$, hence h induces a bijective map $h_\# : \mathrm{SCol}_{(X,x)}(K,b) \to \mathrm{SCol}_{(X,y)}(K,b)$. Then we have

$$\mathrm{Col}_X(K) = \sum_{y \in X} \mathrm{Col}_{(X,y)}(K,b) = |X| \mathrm{Col}_{(X,x)}(K,b)$$

for any $x \in X$. $\square$

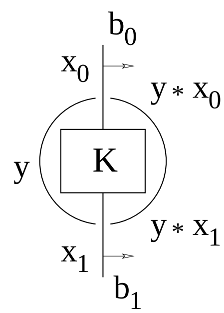


FIGURE 3. End monochromatic tangle

The following lemma was stated and proved in [29] for the 3 element dihedral quandle $Q(3,1)$ (and dihedral quandles in [30]) and generalized by Nosaka [27]. The idea of proof is illustrated by Figure 3, which was taken from [29].

Lemma 3.4 ([27])**.** *If a quandle* X *is faithful, then for any knot* K*,* (K, X) *is end monochromatic.*

Remark 3.5. There are many examples of knots K and quandles X where X is not faithful, but (K, X) is end monochromatic. For example, $Q(8,1)$, which is an abelian extension of $Q(4,1)$, is not faithful, but 5_1 and 8_5 are end monochromatic with $Q(8,1)$, where 5_1 has only trivial colorings, and 8_5 has non-trivial colorings with $Q(8,1)$. The smallest non-faithful quandle for which 3_1 is end monochromatic is $Q(12,1)$, which is an abelian extension of $Q(6,1)$.

In the following lemma, a formula is given for the number of colorings of composite knots. For a composite knot $K_1 \# K_2$, we assume that K_1 and K_2 are oriented, and the composite $K_1 \# K_2$ is defined in such a way that an orientation of the composite restricts to the orientation of each factor, and such an orientation is specified for the composite to make it an oriented knot, see Figure 4.

Lemma 3.6 (cf. [27, 29])**.** *If a knot K_1 or a knot K_2 is end monochromatic with a finite homogeneous quandle X, then*

$$|X|\mathrm{Col}_X(K_1\#K_2) = \mathrm{Col}_X(K_1)\mathrm{Col}_X(K_2).$$

Proof. Let b_1, b_2 be base points on diagrams of K_1 and K_2, respectively, with respect to which 1-tangles and connected sum are formed. Let $x \in X$. Let $\mathrm{SCol}_{(X,x)}(K_i, b_i)$, and $\mathrm{Col}_{(X,x)}(K_i, b_i)$, $i = 1, 2$, be the set and the number of colorings of K_i by X such that the arc that contains b_i receives the color x. Let c_1, c_2 be points on a diagram $K = K_1\#K_2$ that result from taking a connected sum with respect to b_1 and b_2 by connecting 1-tangles, see Figure 4.

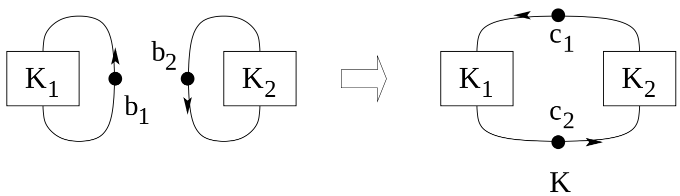


FIGURE 4. Taking connected sum

For colorings $\mathcal{C}_i \in \mathrm{Col}_{(X,x)}(K_i, b_i)$, $i = 1, 2$, a coloring $\mathcal{C} = \mathcal{C}_1\#\mathcal{C}_2$ of K is uniquely determined such that the colors of the arcs containing c_i, $i = 1, 2$, coincide and is x. Conversely, any coloring $\mathcal{C}$ of K has the property that the color of the arcs containing c_i, $i = 1, 2$, coincide, since $\mathcal{C}$ will also be a coloring of the tangles K_1 and K_2. If, say, K_1 is monochromatic with X then the colors of c_1 and c_2 must be the same. Hence there is a bijection

$$\bigcup_{x\in X} [\ \mathrm{SCol}_{(X,x)}(K_1, b_1) \times \mathrm{SCol}_{(X,x)}(K_2, b_2)\ ] \to \mathrm{SCol}_X(K).$$

By Lemma 3.3, we have $\mathrm{Col}_{(X,x)}(K_i, b_i) = \mathrm{Col}_X(K_i)/|X|$ for any $x \in X$, hence the left side above has the cardinality

$$|X|(\mathrm{Col}_X(K_1)/|X|)(\mathrm{Col}_X(K_2)/|X|),$$

as desired. □

Lemmas 3.4 and 3.6 imply the following.

Lemma 3.7 ([27])**.** *If X is a finite faithful quandle, then*

$$|X|\mathrm{Col}_X(K_1\#K_2) = \mathrm{Col}_X(K_1)\mathrm{Col}_X(K_2)$$

for knots K_1 and K_2.

Corollary 3.8. *If X is a finite faithful quandle and R, K are knots, then*

$$\mathrm{Col}_X(R\#K) = \mathrm{Col}_X(R\#rm(K)).$$

In particular, if X is a finite faithful quandle and K is reversible or positive-amphicheiral, respectively, then either $\mathrm{Col}_X(R\#K) = \mathrm{Col}_X(R\#m(K))$ *or* $\mathrm{Col}_X(R\#K) = \mathrm{Col}_X(R\#r(K))$.

Proof. By Lemma 3.7,

$$\begin{aligned}\mathrm{Col}_X(R\#K) &= \mathrm{Col}_X(R)\mathrm{Col}_X(K)/|X| \\ &= \mathrm{Col}_X(R)\mathrm{Col}_X(rm(K))/|X| = \mathrm{Col}_X(R\#rm(K)).\end{aligned}$$

This completes the proof. □

According to this lemma, the situation of quandle colorings of composite knots may differ for non-faithful quandles, and indeed, the computer calculations reveal this. In the following sections we investigate these cases. We used the closed braid form for computer calculations of the number of quandle colorings as in [11]. In computing the number of colorings for composite knots, we formed the closed braid form as depicted in Figure 5. In the braid notation of [14], an m-braid is represented by $[a_1, ..., a_s]$, $a_i \in \mathbb{Z}$, where a_i represents the braid generator σ_k if $a_i = k > 0$, and σ_k^{-1} if $k < 0$. The sign of a_i, $\mathrm{sign}(a_i)$, is defined to be 1 (-1, respectively), if $k > 0$ (resp. $k < 0$). If $[a_1, ..., a_s]$ ($[b_1, ..., b_t]$, respectively) is an m-braid (resp. n-braid) representative for a knot K (resp. K'), then

$$[a_1, \ldots, a_s, b_1 + \mathrm{sign}(b_1)(m-1), \ldots, b_t + \mathrm{sign}(b_t)(m-1)]$$

is an $(m+n-1)$-braid representative for $K\#K'$. For example, for a trefoil 3_1, $s = 3$, $m = 2$, $t = 3$, $n = 2$, and $[1, 1, 1, 2, 2, 2]$ is a $(2 + 2 - 1)$-braid representative of $3_1\#3_1$. The orientations of each factor and the composite are defined by downward orientation of the braid form. It is known [2] that for the braid index Br, the formula $\mathrm{Br}(K_1\#K_2) = \mathrm{Br}(K_1) + \mathrm{Br}(K_2) - 1$ holds.

4. Distinguishing K from $rm(K)$ via colorings of composite knots

Since quandle colorings do not distinguish K from $rm(K)$, they do not distinguish $m(K)$ from $r(K)$. Consequently, in [11], distinguishing K from $m(K)$ by quandle colorings was examined only for chiral and negative-amphicheiral knots.

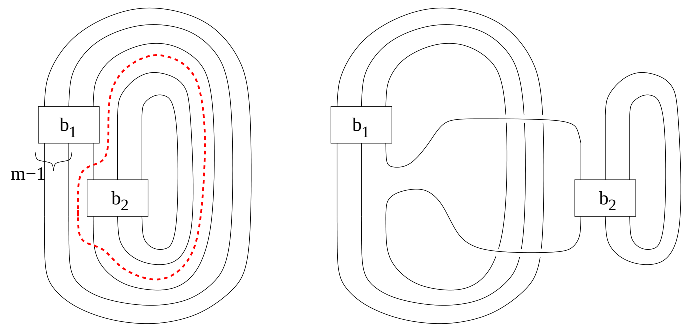


FIGURE 5. The connected sum of two closed braids.

In this section, we exhibit computational results on distinguishing reversible and chiral knots K from $rm(K)$ using quandle colorings of composite knots $R\#K$ and $R\#rm(K)$ for knots R and K.

Proposition 4.1. *Conjecture 2.1 implies that for any knot K such that $K \neq f(K)$ for some $f \in \mathcal{G}$, there is a finite quandle X and a prime knot P (with braid index 2) such that $\mathrm{Col}_X(P\#K) \neq \mathrm{Col}_X(P\#f(K))$.*

Proof. First we observe that for any knots K_1 and K_2 and $f \in \mathcal{G}$,

$$f(K_1\#K_2) = f(K_1)\#f(K_2),$$

and for any prime knot P and $f \in \mathcal{G}$, $f(P)$ is prime. Let $K = P_1\# \cdots \#P_n$ be the prime factorization of K. Then

$$f(K) = f(P_1)\# \cdots \#f(P_n)$$

is the prime factorization of $f(K)$. Let P be a prime knot such that P is not in $\mathcal{G}(P_i)$ for $i = 1, \dots, n$ and $P \neq rm(P)$ (take, for example, a $(2, n)$-torus knot, that is, the closure of a 2-braid, of a large crossing number for P). Clearly $P\#K \neq P\#f(K)$. The prime factorization of $rm(P\#K)$ is

$$rm(P)\#rm(P_1)\# \cdots \#rm(P_n)$$

and by the definition of P we again have by uniqueness of prime factorization that $rm(P\#K)$ is not equal to $P\#f(K)$. By the conjecture it follows that there is a finite quandle X such that $\mathrm{Col}_X(P\#K) \neq \mathrm{Col}_X(P\#f(K))$ □

As a corollary to the proof of Proposition 4.1, we obtain the following.

Corollary 4.2. *For any knot K such that $K \neq f(K)$ for some $f \in \mathcal{G}$, there exists a prime knot P such that the fundamental quandles of $P\#K$ and $P\#f(K)$ are not isomorphic.*

Recall from Corollary 3.8 that if X is a finite faithful quandle, then we cannot distinguish $R\#K$ from $R\#rm(K)$. Thus to apply this technique, we must use non-faithful quandles.

Remark 4.3. For reversible or chiral prime knots K up to 12 crossings and up to braid index 4, among the Rig quandles of order less than 36, only the quandles $Q(24,2)$ and $Q(27,14)$ distinguished $R\#K$ and $R\#m(K)$ for some closed 2-braids R by the condition

$$\mathrm{Col}_E(R\#K) \neq \mathrm{Col}_E(R\#rm(K)).$$

We noticed that these are abelian extensions of $Q(6,2)$ and $Q(9,6)$ with coefficient groups $\mathbb{Z}_4$ and $\mathbb{Z}_3$, respectively. In the remainder of the section, we give an interpretation of this method in terms of the quandle cocycle invariant, and extend this method to quandles of order larger than 36. Corollary 3.8 and Proposition 6.1 partly explain why only abelian extensions worked for this purpose among Rig quandles. Remark 4.6 suggests why many abelian extensions do not work.

Let X be a quandle, A be a finite abelian group, and $\phi \in Z^2_Q(X,A)$ be a 2-cocycle with coefficient group A. Let $\Phi_\phi(K) = \sum_{g\in A} a_g g \in \mathbb{Z}[A]$ be the cocycle invariant of a knot K. We write $C_g(\Phi_\phi(K)) = a_g$. In particular, $C_e(\Phi_\phi(K)) \in \mathbb{Z}$ denotes the coefficient of the identity element $e \in A$.

An examination of the proof of Theorem 4.1 in [4] reveals the following two lemmas. For convenience of the reader, we include a proof of Lemma 4.5.

Lemma 4.4 ([4])**.** *Let E be an abelian extension of X with respect to a 2-cocycle ϕ with coefficient group A. Let K be a knot that is end monochromatic with X. Then* $\mathrm{Col}_E(K) = C_e(\Phi_\phi(K))|A|$.

Lemma 4.5. *Suppose (K,X) is end monochromatic, and $E = E(X,A,\phi)$ is an abelian extension of X. Then (K,E) is end monochromatic if and only if* $\Phi_\phi(K) = \mathrm{Col}_X(K)\, e$.

Proof. In [4] an interpretation of the cocycle invariant as an obstruction to extending a coloring of a knot diagram K by X to a coloring by the abelian extension E of X with respect to a 2-cocycle ϕ was given as follows. Let $\mathcal{C}$ be a coloring of a 1-tangle S of K with initial and terminal end points b_0, b_1, respectively. Suppose (K,X) is end monochromatic, so that $\mathcal{C}(b_0) = \mathcal{C}(b_1) = x_0 \in X$. Let $a_0 \in A$ and assign a color $(x_0, a_0) \in E = X \times A$ to the arc at b_0. By traveling along the diagram from b_0 to b_1, a color of S by E is defined inductively using colors by X; if an under-arc colored by (x,a) goes under an over-arc colored by (y,b) at a positive crossing, then the other under-arc receives a color $(x*y, a\,\phi(x,y))$. The color extends at negative crossing as well. Then the coloring thus extended to S has the color

$(x_0, a_0\, d)$ at the arc at b_1, where $d \in A$ is the contribution of the cocycle invariant $d = \prod_\tau \phi(x_\tau, y_\tau)^{\epsilon(\tau)} \in A$. Thus the coloring by X extends to that by E if and only if d is the identity element. □

Remark 4.6. The examples mentioned in Remark 3.5 are explained by Lemma 4.5. Among Rig quandles of order less than 36, the following are abelian extensions and end monochromatic for all knots up to 9 crossings:

$$\begin{array}{lllll} Q(12,1), & Q(20,3), & Q(24,3), & Q(24,4), & Q(24,5), \\ Q(24,6), & Q(24,14), & Q(24,16), & Q(24,17), & Q(30,1), \\ Q(30,16), & Q(32,5), & Q(32,6), & Q(32,7), & Q(32,8). \end{array}$$

Thus we conjecture that this is the case for all knots. The corresponding quandle X for these abelian extensions E are found in [12], and they are, respectively:

$$\begin{array}{lllll} Q(6,1), & Q(10,1), & Q(12,6), & Q(12,5), & Q(12,8), \\ Q(12,9), & Q(12,7), & Q(12,8), & Q(12,8), & Q(15,2), \\ Q(15,7), & Q(16,4), & Q(16,4), & Q(16,5), & Q(16,6). \end{array}$$

Duplicates in the list of X are due to non-cohomologous 2-cocycles of the same quandle.

There are non-faithful quandles that are not abelian extensions, see Proposition 6.1, and we do not know any characterization of knots that are end monochromatic with such quandles. All prime knots up to 9 crossings are end monochromatic with $Q(30,4)$.

Definition 4.7 (e.g. [6])**.** For an element $a = \sum_h a_h h \in \mathbb{Z}[A]$, the element $\overline{a} = \sum_h a_h h^{-1} \in \mathbb{Z}[A]$ is called the *conjugate* of a.

Lemma 4.8 ([6])**.** $\Phi_\phi(K) = \overline{\Phi_\phi(rm(K))}$.

Definition 4.9. *The value of the quandle cocycle invariant $\Phi_\phi(K)$ of a knot K with respect to a 2-cocycle ϕ of a quandle X is called* asymmetric *if $\Phi_\phi(K) \neq \overline{\Phi_\phi(K)}$.*

Corollary 4.10. *If $\Phi_\phi(K)$ is asymmetric, then $K \neq rm(K)$.*

From the above corollary we can sometimes distinguish K from $rm(K)$ using the cocycle invariant for some quandles.

Proposition 4.11 ([27])**.** *Let ϕ be a 2-cocycle of a finite homogeneous quandle X with coefficient group A. Suppose that K_1 or K_2 is end monochromatic with X. Then*

$$|X|\Phi_\phi(K_1 \# K_2) = \Phi_\phi(K_1)\Phi_\phi(K_2).$$

The following corollary relates the condition

$$\mathrm{Col}_E(R\#K) \neq \mathrm{Col}_E(R\#rm(K))$$

to Corollary 4.10 via asymmetry of the cocycle invariant.

Corollary 4.12. *Let ϕ be a 2-cocycle of a finite connected faithful quandle X with coefficient group A. Assume that $\Phi_\phi(R) = r_e e + r_u u$ for $r_e, r_u \in \mathbb{N}$, the identity element e, and a non-identity element $u \in A$, and that $r_e = |X|$, that is, any non-trivial coloring contribute u to the cocycle invariant. Suppose a knot K satisfies*

$$\Phi_\phi(K) = k_e e + k_u u + k_{u^{-1}} u^{-1} + V,$$

where V does not contain terms in e, u or u^{-1}. Then $k_u \neq k_{u^{-1}}$ if and only if

$$\mathrm{Col}_E(R\#K) \neq \mathrm{Col}_E(R\#rm(K)),$$

where E is the abelian extension of X by ϕ.

Proof. By Proposition 4.11,

$$\begin{aligned} C_e(\Phi_\phi(R\#K)) &= (r_e k_e + r_u k_{u^{-1}})/|X|, \\ C_e(\Phi_\phi(R\#rm(K))) &= (r_e k_e + r_u k_u)/|X|. \end{aligned}$$

By Lemma 4.4, $k_u \neq k_{u^{-1}}$ if and only if

$$\begin{aligned} &\mathrm{Col}_E(R\#K) = |A|(r_e k_e + r_u k_{u^{-1}})/|X| \\ &\neq \ |A|(r_e k_e + r_u k_u)/|X| = \mathrm{Col}_E(R\#rm(K)). \end{aligned}$$

This completes the proof. □

We note that often computing the number of colorings has computational advantage over applying Corollary 4.10 by computing the cocycle invariant, even though Corollary 4.12 theoretically derives the condition

$$\mathrm{Col}_E(R\#K) \neq \mathrm{Col}_E(R\#rm(K))$$

from asymmetry of the cocycle invariant in many cases.

Example 4.13. Let $X = Q(6,2)$ and ϕ be a generating 2-cocycle over $\mathbb{Z}_4$ such that the abelian extension of X with respect to ϕ is $E = Q(24,2)$. Let us take an example of $R\#K$ and $R\#rm(K)$ for a trefoil $R = 3_1$ and $K = 6_1$. It was found in [10] that there is a multiplicative generator u of $\mathbb{Z}_4$ such that the trefoil has the cocycle invariant $\Phi_\phi(3_1) = 6 + 24u$ for $Q(6,2)$. With the same 2-cocycle, it is computed that $\Phi_\phi(K) = 6 + 24u^{-1}$. By Corollary 4.12, $\mathrm{Col}_E(R\#K) \neq \mathrm{Col}_E(R\#rm(K))$, where $E = Q(24,2)$. For a more complex knot K, however, it becomes difficult to compute the cocycle invariant, and easier to confirm the condition $\mathrm{Col}_E(R\#K) \neq \mathrm{Col}_E(R\#rm(K))$, which then implies that $K \neq rm(K)$ and K has an asymmetric invariant value.

We summarize outcomes of the methods described in this section, i.e. using Corollary 4.10 and cocycle invariants, or by directly computing

$$\mathrm{Col}_E(R\#K) \neq \mathrm{Col}_E(R\#rm(K)).$$

First we summarize our results for prime knots with 9 crossings or less using the cocycle invariant. Among 84 knots in the table up to 9 crossings, they are all reversible except:

- Fully amphicheiral knots: $4_1, 6_3, 8_3, 8_9, 8_{12}, 8_{18}$.
- Negative amphicheiral knot: 8_{17}.
- Chiral knots: $9_{32}, 9_{33}$.

The rest are 75 reversible knots. The colorings of $3_1\#K$ and $3_1\#rm(K)$ or the method described in Corollary 4.12 distinguished the following reversible knots from their mirrors.

- Using $Q(24, 2)$, the following knots are distinguished from mirrors: $3_1, 6_1, 7_4, 7_7, 8_{11},\ 9_1, 9_2, 9_4, 9_6, 9_{10}, 9_{11}, 9_{15}, 9_{17}, 9_{23}, 9_{29}$, $9_{34}, 9_{35}, 9_{37},\ 9_{38}, 9_{46}, 9_{47}, 9_{48}$.
- Using $Q(27, 14)$, the following knots are distinguished from mirrors: $3_1, 6_1, 7_4, 8_5, 8_{15}, 8_{19}, 8_{21},\ 9_2, 9_4, 9_{16}, 9_{17}, 9_{28}, 9_{29}, 9_{34}, 9_{38}, 9_{40}$.

Furthermore, computer calculations show that the following knots K in the KnotInfo table up to 12 crossings with braid index less that 4 have the property $\mathrm{Col}_E(3_1\#K) \neq \mathrm{Col}_E(3_1\#m(K))$.

- Both $E = Q(24, 2)$ and $Q(27, 14)$ have this property for: $10_5, 10_9, 10_{112}, 10_{159}, 12a_{0805}, 12a_{0878}, 12a_{1210}, 12a_{1248}$, $12a_{1283}, 12n_{0571}, 12n_{0666}, 12n_{0750}, 12n_{0751}$.
- Only $E = Q(24, 2)$ but not $Q(27, 14)$ has this property for: $11a_{355}, 12a_{1214}, 12n_{0574}, 12n_{0882}$.
- Only $E = Q(27, 14)$ but not $Q(24, 2)$ has this property for: $10_{64}, 10_{139}, 10_{141}, 11a_{338}, 12a_{1212}, 12n_{0604}, 12n_{0850}$.

Remark 4.14. To distinguish more knots from their mirrors using the property $\mathrm{Col}_E(R\#K) \neq \mathrm{Col}_E(R\#m(K))$ for some abelian extensions E and for some R, we further computed abelian extensions of some Rig quandles. We computed cohomology groups for some coefficient groups and found some 2-cocycles for Rig quandles up to order 23, and obtained 40 abelian extensions. This information is available online at `http://github.com/vendramin/rig/wiki`.

Let $\mathcal{E}$ be this set of quandles. It is likely that there are other abelian extensions that are not in this list.

There are 168 chiral, reversible or positive amphicheiral knots with braid index less than 4 and crossing number at most 12. Of these, we computed that 144 of knots have the property $\mathrm{Col}_E(R\#K) \neq \mathrm{Col}_E(R\#m(K))$ with $E \in \mathcal{E}$ and for $R = 3_1, 5_1$, or 9_1.

Remark 4.15. Reversible prime knots K, up to 12 crossings with braid index less than 4, distinguished from their mirror images by a quandle knot pair (X, R) are listed in Table 1. The table shows a quandle X, a knot R and knots K such that $\mathrm{Col}_X(R\#K) \neq \mathrm{Col}_X(R\#m(K))$. We recall that $Q(24, 2)$ and $Q(27, 14)$ are also abelian extensions.

TABLE 1. Some reversible prime knots K distinguished from their mirror images by a quandle knot pair (X, R).

X	R	K
$E(Q(12,3),\mathbb{Z}_5)$	3_1	$12n_{0472}$
$E(Q(12,3),\mathbb{Z}_5)$	9_1	$10_{46}, 10_{127}, 10_{155}, 12n_{0466}$
$E(Q(12,3),\mathbb{Z}_{10})$	3_1	$8_7, 8_{10}, 10_{116}, 10_{143}, 12a_{0576}$ $12a_{1220}, 12n_{0233}, 12n_{0234}, 12n_{0235}$ $12n_{0570}, 12n_{0722}, 12n_{0830}, 12n_{0887}$
$E(Q(12,3),\mathbb{Z}_{10})$	9_1	$5_2, 10_2, 10_{100}, 10_{125}, 10_{152}, 11a_{240}$ $12a_{0835}, 12a_{1203}, 12a_{1222}, 12n_{0242}, 12n_{0467}$
$E(Q(15,5),\mathbb{Z}_5)$	3_1	$12n_{0888}$
$E(Q(18,11),\mathbb{Z}_6)$	3_1	$8_{20},\ 10_{62},\ 12a_{0999},\ 12n_{0831}$
$E(Q(20,1),\mathbb{Z}_3)$	5_1	$12a_{1027}, 12a_{1233}, 12n_{0468}, 12n_{0721}$
$E(Q(20,1),\mathbb{Z}_6)$	5_1	$5_1, 10_{47}, 10_{48}, 10_{157}, 11a_{234}$ $12a_{0869}, 12a_{1114}, 12a_{1176}, 12a_{1199}$
$E(Q(20,2),\mathbb{Z}_3)$	5_1	$7_3, 12a_{0146}, 12a_{0369}, 12a_{0722}, 12n_{0822}$
$E(Q(20,2),\mathbb{Z}_6)$	5_1	$6_2, 8_{16}, 9_3, 10_{126}, 10_{161}, 12a_{0838}, 12a_{1246}, 12a_{1250}$ $12n_{0417}, 12n_{0725}, 12n_{0749}, 12n_{0820}, 12n_{0829}$
$Q(24,2)$	3_1	$3_1, 9_1, 9_6, 10_5, 10_9, 10_{112}, 10_{159}, 11a_{355}$ $12a_{0805}, 12a_{0878}, 12a_{1210}, 12a_{1214}, 12a_{1248}, 12a_{1283}$ $12n_{0571}, 12n_{0574}, 12n_{0666}, 12n_{0750}, 12n_{0751}, 12n_{0882}$
$Q(27,14)$	3_1	$8_5, 8_{19}, 8_{21}, 9_{16}, 10_{64}, 10_{139}, 10_{141}, 11a_{338}$ $12a_{1212}, 12n_{0604}, 12n_{0850}$
$E(Q(30,3),\mathbb{Z}_4)$	3_1	$12n_{0821}$

Remark 4.16. Chiral prime knots K, up to 12 crossings with braid index less than 4, distinguished from $rm(K)$ by a quandle knot pair (X, R) are listed in Table 2. The table shows a quandle X, a knot R and knots K such that such that $\mathrm{Col}_X(R\#K) \neq \mathrm{Col}_X(R\#rm(K))$.

Table 2. Some chiral prime knots K distinguished from $rm(K)$ by a quandle knot pair (X, R).

X	R	K
$E(Q(12,3),\mathbb{Z}_5)$	3_1	$10_{149}, 12n_{0344}, 12n_{0679}, 12n_{0688}$
$E(Q(12,3),\mathbb{Z}_{10})$	3_1	$12a_{0815}, 12a_{0898}, 12a_{0981}, 12n_{0708}$
$E(Q(12,3),\mathbb{Z}_{10})$	9_1	$12a_{1223}, 12n_{0748}$
$E(Q(18,11),\mathbb{Z}_6)$	3_1	$10_{82}, 12a_{1191}, 12a_{1215}, 12a_{1253}, 12n_{0675}$
$E(Q(20,1),\mathbb{Z}_3)$	5_1	$10_{148}, 12a_{1047}, 12a_{1227}$
$E(Q(20,1),\mathbb{Z}_6)$	5_1	$12a_{0824}, 12a_{0850}, 12a_{0859}, 12n_{0113}, 12n_{0114}, 12n_{0345}$
$E(Q(20,2),\mathbb{Z}_3)$	5_1	$12a_{1227}, 12a_{1235}, 12a_{1258}$
$E(Q(20,2),\mathbb{Z}_6)$	5_1	$12a_{0920}, 12n_{0709}$
$Q(24,2)$	3_1	$10_{106}, 12a_{0909}, 12a_{0916}, 12a_{1002}, 12a_{1120}, 12a_{1226}$ $12a_{1255}, 12n_{0640}, 12n_{0767}$
$Q(27,14)$	3_1	$10_{85}, 12a_{0864}, 12a_{1219}, 12a_{1221}, 12n_{0674}$
$E(Q(30,3),\mathbb{Z}_4)$	3_1	$12a_{1011}, 12a_{1051}, 12n_{0191}, 12n_{0684}$

5. Recovering cocycle invariants from colorings

In this section, we obtain formulas for computing the cocycle invariant from the number of colorings for some cases. The formulas give computational advantage in many cases. To obtain formulas, however, one needs information on concrete non-trivial invariant values for a few knots.

Proposition 5.1. *Let X, A, ϕ be as above. Suppose that X is end monochromatic with K. Suppose further that for an element $v \in A$ that is not the identity element e, there exists a knot R_v such that $\Phi_\phi(R_v) = r_e e + r_v v \in \mathbb{Z}[A]$. Then*

$$C_{v^{-1}}(\Phi_\phi(K)) = \frac{1}{r_v|A|}\big(\ |X|\mathrm{Col}_E(R_v\#K) - r_e\,\mathrm{Col}_E(K)\ \big).$$

Proof. By Proposition 4.11, we have $|X|\Phi_\phi(R_v\#K) = \Phi_\phi(R_v)\Phi_\phi(K)$. By assumption $\Phi_\phi(R_v)\Phi_\phi(K) = (r_e e + r_v v)(\sum_{u\in A} a_u u)$. The coefficient of the identity element in the left-hand side is $r_e a_e + r_v a_{v^{-1}}$. Hence we obtain $|X|C_e(\Phi_\phi(R_v\#K)) = r_e a_e + r_v a_{v^{-1}}$. Let E be the abelian extension of X with respect to ϕ. Then by Lemma 4.4, we have

$$\mathrm{Col}_E(R_v\#K)) = C_e(\Phi_\phi(R_v\#K))|A|$$

and $\mathrm{Col}_E(K) = a_e|A|$. By substitution and solving for $a_{v^{-1}}$, we obtain the lemma. $\square$

In the following examples, we focus on the Rig quandles of order up to 12 where the second cohomology group is non-trivial when the coefficient group is other than $\mathbb{Z}_2$. When the coefficient group A is cyclic of order n, even though we write $A = \mathbb{Z}_n$ (a notation usually used for the additive group of integers modulo n), we specify a multiplicative generator u, so that $A = \langle u \rangle$ where u has order n, and write A multiplicatively.

Example 5.2. Let $X = Q(6, 2)$ and ϕ be a generating 2-cocycle over $A = \mathbb{Z}_4$ such that the abelian extension of X with respect to ϕ is $E = Q(24, 2)$. Since X is faithful, any knot is end monochromatic with X.

The cocycle invariants of $X = Q(6, 2)$ using this cocycle are given in the wiki page of Rig at `http://github.com/vendramin/rig/wiki`, for knots up to 10 crossings. Some of the results are shown in Table 3. Knots that are not listed have the trivial invariant value 6. We abbreviate the identity element in the remaining of the paper. For example, $6+24u$ means $6e+24u$ for the identity element e. In particular, in order to use Proposition 5.1, we obtain the following invariant values:

$$\Phi_\phi(3_1) = 6 + 24u, \qquad \Phi_\phi(8_5) = 30 + 24u^2, \qquad \Phi_\phi(9_1) = 6 + 24u^3.$$

Proposition 5.1 implies that

$$\begin{aligned}
C_u(\Phi_\phi(K)) &= (\ 1/(24 \cdot 4)\ )\ (\ 6 \cdot \mathrm{Col}_E(9_1 \# K) - 6 \cdot \mathrm{Col}_E(K)\ ),\\
C_{u^2}(\Phi_\phi(K)) &= (\ 1/(24 \cdot 4)\ )\ (\ 6 \cdot \mathrm{Col}_E(8_5 \# K) - 30 \cdot \mathrm{Col}_E(K)\ ),\\
C_{u^3}(\Phi_\phi(K)) &= (\ 1/(24 \cdot 4)\ )\ (\ 6 \cdot \mathrm{Col}_E(3_1 \# K) - 6 \cdot \mathrm{Col}_E(K)\ ).
\end{aligned}$$

We also have

$$C_e(\Phi_\phi(K)) = (1/|A|)\mathrm{Col}_E(K) = (1/4)\mathrm{Col}_E(K)$$

from Lemma 4.4. Therefore we obtain

$$\begin{aligned}
\Phi_\phi(K) = \frac{1}{16} [\ & 4\,\mathrm{Col}_E(K) + (\mathrm{Col}_E(9_1 \# K) - \mathrm{Col}_E(K))u \\
& + (\mathrm{Col}_E(8_5 \# K) - 5\,\mathrm{Col}_E(K))u^2 + (\mathrm{Col}_E(3_1 \# K) - \mathrm{Col}_E(K))u^3\ \ ].
\end{aligned}$$

See the appendix for examples of cocycles invariants computed using this formula.

Remark 5.3. In computing the coloring numbers of knots by quandles, some computational techniques have been developed in [11], such as fixing a color of the first braid strand to reduce the computation time. On the other hand, to compute the cocycle invariant, every coloring must be computed, and the cocycle value must be evaluated for each coloring. The latter increases the computational time significantly. Thus the formula of Proposition 5.1 is useful in determining invariant values for higher crossing knots with lower braid indices.

TABLE 3. Some cocycle invariants for the quandle $Q(6,2)$.

cocycle invariant	knot
$6+24u$	$3_1, 7_7, 8_{11}, 9_2, 9_4, 9_{10}, 9_{11}, 9_{15}$
$54+72u$	9_{35}
$6+48u^2$	9_{40}
$30+24u^2$	$8_5, 8_{10}, 8_{15}, 8_{19}, 8_{20}, 8_{21}, 9_{16}, 9_{24}, 9_{28}$
$6+24u^3$	$6_1, 7_4, 9_1, 9_6, 9_{17}, 9_{23}, 9_{29}, 9_{34}, 9_{38}$
$6+72u+48u^2$	9_{48}
$6+48u+48u^3$	8_{18}
$6+24u+72u^3$	9_{47}
$54+24u+48u^3$	9_{46}
$6+48u+48u^2+24u^3$	9_{37}

Remark 5.4. There are discrepancies of representatives of knots and their mirrors in different notations in [14] for the following knots up to 9 crossings: 7_7, 9_{11}, 9_{17}, 9_{34}, 9_{46}, 9_{47}, 9_{48}. Specifically, the diagram of 7_7 listed agrees with the braid notation, but its PD notation seems to represent its mirror. In our first computation up to 9 crossings, we used the PD notation in [14], and the second computations for those with braid index less than 4 are performed using the braid notation. For up to 9 crossings, these calculations showed discrepancies for the above listed knots. The discrepancies are all related by conjugate values of the invariant. We note that in the following computations, these knots are not used for R in $\mathrm{Col}_E(R\#K)$ in the formulas.

Below we give a summary of the formula in Proposition 5.1 for Rig quandles of order up to 12, as examples to indicate how to use the formula, and to illustrate varieties of actual formulas obtained.

Example 5.5. Let $X = Q(9,6) = \mathbb{Z}_3[t]/(t^2+2t+1)$ and ϕ be a generating 2-cocycle over $A = \mathbb{Z}_3$ such that the abelian extension of X with respect to ϕ is $E = Q(27,14)$. Since X is faithful, any knot is end monochromatic with X. Computer calculation shows that $\Phi_\phi(3_1) = 27+54u$, where u is a multiplicative generator of A and it also implies that $\Phi_\phi(m(3_1)) = 27+54u^2$. Proposition 5.1 implies that

$$\begin{aligned} C_u(\Phi_\phi(K)) &= (\ 1/(54\cdot 3)\ )\ (\ 9\cdot \mathrm{Col}_E(m(3_1)\#K) - 27\cdot \mathrm{Col}_E(K)\ ), \\ &= (\ 1/18\ )\ (\ \mathrm{Col}_E(m(3_1)\#K) - 3\cdot \mathrm{Col}_E(K)\ ), \\ C_{u^2}(\Phi_\phi(K)) &= (\ 1/18\ )\ (\ \mathrm{Col}_E(3_1\#K) - 9\cdot \mathrm{Col}_E(K)\ ). \end{aligned}$$

Example 5.6. Let $X = Q(12,3)$. This quandle is not Alexander, not kei, not Latin, faithful, and $H^2_Q(X,A) = \mathbb{Z}_{10}$ for $A = \mathbb{Z}_{10}$. Let E be the abelian

TABLE 4. Some cocycle invariants for $Q(12, 6)$.

cocycle invariant	knot
108	$3_1, 6_1, 7_4, 7_7, 8_{11}$ $9_1, 9_2, 9_4, 9_6, 9_{10}, 9_{11}, 9_{15}, 9_{17}, 9_{23}, 9_{29}, 9_{34}, 9_{38}$
204	$8_5, 8_{10}, 8_{15}, 8_{19}, 8_{20}, 8_{21}, 9_{16}, 9_{24}, 9_{28}$
396	$8_{18}, 9_{40}, 9_{47}$
$492 + 192u^2$	$9_{35}, 9_{37}, 9_{46}, 9_{48}$
12	otherwise

extension corresponding to a cocycle that represents a generator of $\mathbb{Z}_{10}$. We obtain the following invariant values:

$\Phi_\phi(3_1) = 12 + 60u$, $\Phi_\phi(8_{19}) = 12 + 60u^2$, $\Phi_\phi(5_2) = 12 + 60u^3$,
$\Phi_\phi(m(9_{29})) = 12 + 60u^6$, $\Phi_\phi(5_1) = 12 + 60u^5$, $\Phi_\phi(9_{29}) = 12 + 60u^6$,
$\Phi_\phi(5_2) = 12 + 60u^7$, $\Phi_\phi(m(8_{19})) = 12 + 60u^8$, $\Phi_\phi(8_6) = 12 + 60u^9$.

One computes

$$\begin{aligned} C_u(\Phi_\phi(K)) &= (\ 1/(60 \cdot 12)\ )\ (\ 12 \cdot \mathrm{Col}_E(8_6\#K) - 12 \cdot \mathrm{Col}_E(K)\ ) \\ &= (1/60)\ (\ \mathrm{Col}_E(8_6\#K)) - \mathrm{Col}_E(K)\ ) \end{aligned}$$

and the other terms are similar with the corresponding knots listed above. We note that the coefficient of every term is computed by these formulas, but we needed to compute the invariant for up to 9 crossings for this conclusion, as u^4 and u^6 are missing up to 8 crossing knots.

Example 5.7. Let $X = Q(12, 5)$. This quandle is not Alexander, not kei, not Latin, faithful, and $H^2_Q(X, \mathbb{Z}_4) = \mathbb{Z}_4$. With a choice of a generating cocycle ϕ, up to 8 crossings, all knots have the cocycle invariant of the form $\Phi_\phi(K) = a + bu^2$, $a, b \in \mathbb{Z}$. Thus we conjecture that this is the case for all knots. The trefoil has the invariant value $\Phi_\phi(3_1) = 12 + 96u^2$. Hence we obtain

$$\begin{aligned} C_{u^2}(\Phi_\phi(K)) &= (\ 1/(96 \cdot 4)\ )\ (\ 12 \cdot \mathrm{Col}_E(3_1\#K) - 12 \cdot \mathrm{Col}_E(K)\ ) \\ &= (1/32)\ (\ \mathrm{Col}_E(3_1\#K) - \mathrm{Col}_E(K)\ ). \end{aligned}$$

If the conjecture does not hold and a knot with the term u or u^3 is found, then it can be used to evaluate other terms.

Example 5.8. Let $X = Q(12, 6)$. This quandle is not Alexander, not kei, not Latin, faithful, and $H^2_Q(X, \mathbb{Z}_4) = \mathbb{Z}_4$. With a generating 2-cocycle ϕ of $\mathbb{Z}_4$ the invariant values $\Phi_\phi(K)$ for K up to 9 crossing knots are listed in Table 4. Thus we conjecture that the invariant values are of the form

$$\Phi_\phi(K) = a + bu^2,$$

for $a, b \in \mathbb{Z}$ and for all knots K. One computes

$$\begin{aligned} C_{u^2}(\Phi_\phi(K)) &= (\ 1/(492 \cdot 4) \ ) \ (\ 12 \cdot \mathrm{Col}_E(9_{35} \# K) - 12 \cdot \mathrm{Col}_E(K) \ ) \\ &= (1/164) \ (\ \mathrm{Col}_E(9_{35} \# K) - 41 \cdot \mathrm{Col}_E(K) \ ). \end{aligned}$$

We note that we needed to compute the invariant for knots up to 9 crossing to obtain this formula.

Remark 5.9. The second cohomology groups for $Q(12,7)$, $Q(12,9)$ with coefficient group $\mathbb{Z}_4$ are $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$, respectively, and for choices of generating cocycles, the cocycle invariants are non-trivial. Situations and computations are similar to those for $Q(6,2)$ and $Q(12,5)$ for each factor, for up to 7 crossings.

Example 5.10. Let $X = Q(12,10)$. This quandle is not Alexander, not kei, not Latin, faithful, and $H^2_Q(X, \mathbb{Z}_6) = \mathbb{Z}_6$. With a generating cocycle ϕ of $\mathbb{Z}_6$, we obtain

$$\Phi_\phi(3_1) = 12 + 108u^3, \quad \Phi_\phi(8_5) = 120 + 216u^2, \quad \Phi_\phi(8_{15}) = 120 + 216u^4.$$

Since we observed, up to 8 crossings, one or more of the terms with u^2, u^3 and u^4 (and no terms of u or u^5), we conjecture that it is the case for all knots. One computes

$$\begin{aligned} C_{u^2}(\Phi_\phi(K)) &= (\ 1/(108 \cdot 6) \ ) \ (\ 12 \cdot \mathrm{Col}_E(3_1 \# K)) - 12 \cdot \mathrm{Col}_E(K) \ ), \\ &= (\ 1/54 \ ) \ (\ \mathrm{Col}_E(3_1 \# K)) - \mathrm{Col}_E(K) \ ), \\ C_{u^3}(\Phi_\phi(K)) &= (\ 1/(216 \cdot 6) \ ) \ (\ 12 \cdot \mathrm{Col}_E(8_{15} \# K)) - 120 \cdot \mathrm{Col}_E(K) \ ), \\ &= (\ 1/108 \ ) \ (\ \mathrm{Col}_E(8_{15} \# K)) - 12 \cdot \mathrm{Col}_E(K) \ ), \\ C_{u^4}(\Phi_\phi(K)) &= (\ 1/108 \ ) \ (\ \mathrm{Col}_E(8_5 \# K)) - 12 \cdot \mathrm{Col}_E(K) \ ). \end{aligned}$$

Remark 5.11. The 2-cocycle invariants discussed in this section are derived from the following invariant: Let $R_1, \ldots, R_n$ be knots and $X_1, \ldots, X_m$ be finite quandles. Then an invariant is defined for a knot K by

$$\mathrm{CL}_{X_1,\ldots,X_m,R_1,\ldots,R_n}(K) = [\ \mathrm{Col}_{X_i}(R_j \# K) \ ]_{i=1,\ldots,m,\, j=1,\ldots,n} \, .$$

It is, then, a natural question whether for any quandle 2-cocycle invariant $\Phi_\phi(K)$, there is a sequence of knots $R_1, \ldots, R_n$ and quandles $X_1, \ldots, X_m$ such that $\Phi_\phi(K)$ is derived from $\mathrm{CL}_{X_1,\ldots,X_m,R_1,\ldots,R_n}(K)$.

6. Properties of abelian extensions

Finding abelian extensions have, for example, the following applications: (1) non-triviality of the second cohomology group can be confirmed, (2) knots and their mirrors may be distinguished by colorings of composite knots as in Section 4, (3) they are useful in computing cocycle knot invariants via colorings as in Section 5.

We summarize our findings on extensions of Rig quandles in this section. Among the 790 Rig quandles of order < 48 there are 66 non-faithful quandles. All but 8 are extensions by $\mathbb{Z}_2$.

Proposition 6.1. *Among the non-faithful Rig quandles (of order less than 48), $Q(30,4)$, $Q(36,58)$, and $Q(45,29)$ are the only quandles that are not abelian extensions.*

Proof. Computations show that the only non-trivial quotient of $Q(30,4)$ is $X = Q(10,1)$. So it suffices to show that there is no abelian extension of X of order 30. We have $H_2^Q(X,\mathbb{Z}) \cong \mathbb{Z}_2$ [33]. To get an abelian extension of X of order 30 we would have to have a non-trivial 2-cocycle $X \times X \to \mathbb{Z}_3$ which would give an element of $H^2_Q(X,\mathbb{Z}_3) = \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}_3) = 0$, a contradiction.

The only non-trivial quotients of $Q(36,58)$ are $Q(4,1)$ and $Q(12,10)$. Since $H_2(Q(4,1)) \cong \mathbb{Z}_2$, a similar argument implies that $Q(36,58)$ is not an abelian extension of $Q(4,1)$. We have $H_2(Q(12,10)) \cong \mathbb{Z}_6$, and let f be a 2-cocycle that generates $H^2(Q(12,10),\mathbb{Z}_6) \cong \mathbb{Z}_6$. Then $2f$ and $4f$ take values in $\mathbb{Z}_3$, and computations show that the corresponding abelian extensions are both isomorphic to $Q(36,57)$. Since cohomologous cocycles give rise to isomorphic quandles, this implies that $Q(36,58)$ is not an abelian extension of $Q(12,10)$.

The only non-trivial quotient of $Q(45,29)$ is $Q(15,7)$. Since $H_2(Q(15,7)) \cong \mathbb{Z}_2$, a similar argument implies that $Q(45,29)$ is not an abelian extension of $Q(15,7)$.

Then one checks by computer that all the other non-faithful Rig quandles are abelian extensions. We note that many cases satisfy the condition in Lemma 7.1 below. □

In [1] Proposition 2.11, it was proved that if Y is a connected quandle and $X = \varphi(Y) \subset \mathrm{Inn}(Y)$, then each fiber has the same cardinality, and if S is a set with the same cardinality as a fiber, then there is a constant cocycle $\beta : X \times X \to \mathrm{Sym}(S)$ such that Y is isomorphic to $X \times_\beta S$.

Proposition 6.2. *The quandles $Q(30,4)$, $Q(36,58)$, and $Q(45,29)$ are non-abelian extensions of the quandles $Q(10,1)$, $Q(12,10)$ and $Q(15,7)$, respectively, by constant 2-cocycles.*

Proof. By calculation we see that the image of the mapping φ from $Q(30,4)$ (resp. $Q(36,58)$, $Q(45,29)$) to its inner-automorphism group is isomorphic to $Q(10,1)$ (resp. $Q(12,10)$, $Q(15,7)$). The claim follows from [1], Proposition 2.11. □

We noticed that some non-cohomologous cocycles give isomorphic extensions, such as $Q(36,57)$ over $Q(12,10)$ as in the proof of Proposition 6.1. We also had the following observation from computer calculations.

Remark 6.3. Let $X = Q(15,2)$, which has cohomology group $H^2_Q(X,\mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence there are three 2-cocycles that are non-trivial and pairwise non-cohomologous. There are, however, only two non-isomorphic abelian extensions of X by $\mathbb{Z}_2$, $Q(30,1)$ and $Q(30,5)$. Then calculations show that two non-cohomologous cocycles define the extension $Q(30,5)$. Similar examples are found for some 12 element quandles, see below.

Lemma 6.4. *For abelian groups B and C and a quandle X, let*

$$\phi_B : X \times X \to B \text{ and } \phi_C : X \times X \to C$$

be 2-cocycles with abelian extensions $E(X,B,\phi_B)$ and $E(X,C,\phi_C)$, respectively. Then for $A = B \times C$, $\phi = (\phi_B,\phi_C) : X \times X \to A$ is a 2-cocycle with abelian extension $E(X,A,\phi)$, and $E(X,A,\phi)$ is an abelian extension of $E(X,B,\phi_B)$ and $E(X,C,\phi_C)$.

Proof. Define $\phi'_C : E(X,B,\phi_B) \times E(X,B,\phi_B) \to C$ by

$$\phi'_C(\ (x_1,b_1),(x_2,b_2)\ ) = \phi_C(x_1,x_2).$$

Then ϕ'_C is a 2-cocycle of $E(X,B,\phi_B)$ with coefficient C.

Define $f : E(X,A,\phi) \to E(X,B,\phi_B) \times C$ by $f(\ (x,(b,c)\ ) = (\ (x,b),\ c\ )$, which is clearly bijective. Then one computes

$$\begin{aligned}
&f(\ (x_1,(b_1,c_1)) * (x_2,(b_2,c_2))\ ) \\
&= f(\ (x_1 * x_2,\ (b_1,c_1) + \phi(x_1,x_2))\ ) \\
&= f(\ (x_1 * x_2,\ (b_1 + \phi_B(x_1,x_2), c_1 + \phi_C(x_1,x_2))\ ) \\
&= (\ (x_1 * x_2,\ b_1 + \phi_B(x_1,x_2)),\ c_1 + \phi_C(x_1,x_2)\ ),
\end{aligned}$$

and

$$\begin{aligned}
&f(\ (x_1,(b_1,c_1))\ ) * f(\ (x_2,(b_2,c_2)\ ) \\
&= (\ (x_1,b_1),\ c_1\ ) * (\ (x_2,b_2),\ c_2)\ ) \\
&= (\ (x_1 * x_2, b_1 + \phi_B(x_1,x_2)),\ c_1 + \phi'_C(\ (x_1,c_1),(x_2,c_2)\ )\ ) \\
&= (\ (x_1 * x_2, b_1 + \phi_B(x_1,x_2)),\ c_1 + \phi_C(x_1,x_2)\ ),
\end{aligned}$$

as desired. □

Similarly, we obtain the following.

Lemma 6.5. *Let B and C be abelian groups and $A = B \times C$, X be a quandle, and $\phi : X \times X \to A$ be a 2-cocycle with abelian extension $E(X,A,\phi)$. Further, let p_B and p_C be the projections from A onto B and*

C respectively. Then $p_B\phi : X \times X \to B$ is a 2-cocyle giving abelian extension $E(X, B, p_B\phi)$, and $E(X, A, \phi)$ is isomorphic to $E(E(X, B, p_B\phi), C, \phi')$, where $\phi'((x_1, b_1), (x_2, b_2)) = p_C\phi(x_1, x_2)$ for $(x_1, b_1), (x_2, b_2) \in X \times B$.

Lemma 6.5 is generalized as follows.

Proposition 6.6. *Let X be a finite quandle, and $0 \to C \xrightarrow{\iota} A \xrightarrow{p_B} B \to 0$ be an exact sequence of finite abelian groups. Let $\phi : X \times X \to A$ be a quandle 2-cocycle. Then $E(X, A, \phi)$ is an abelian extension of $E(X, B, p_B\phi)$ with coefficient group C.*

Proof. Let $s : B \to A$ be a section of the map p_B, that is, $p_B s = \mathrm{id}_B$. Then

$$p_B(s(b_1 + b_2) - s(b_1) - s(b_2)) = 0.$$

Thus $s(b_1 + b_2) - s(b_1) - s(b_2)$ lies in the kernel of p_B so we can write

$$s(b_1 + b_2) - s(b_1) - s(b_2) = \iota(c)$$

for some $c \in C$. Let $\eta : B \times B \to C$ be given by $\eta(b_1, b_2) = c$. Then $p_B(a - sp_B(a)) = 0$ and hence we can write $a - sp_B(a) = \iota(p_C(a))$ where $p_C : A \to C$. This yields

$$\iota p_C(a) + sp_B(a) = a$$

for all $a \in A$.

Define $\phi' : E(X, B, p_B\phi) \times E(X, B, p_B\phi) \to C$ by

$$\phi'(\,(x_1, b_1), (x_2, b_2)\,) = p_C\phi(x_1, x_2) - \eta(b_1, p_B\phi(x_1, x_2))$$

for $(x_i, b_i) \in E(X, B, p_B\phi) = X \times B$, $i = 1, 2$. To show that ϕ' is a 2-cocycle it suffices to show that $E(E(X, B, p_B\phi), C, \phi')$ is a quandle. For this it suffices to show that the mapping

$$f : E(E(X, B, p_B\phi), C, \phi') \to E(X, A, \phi)$$

defined by $f(\,((x, b), c)\,) = (x, s(b) + \iota(c))$ is a bijection and preserves the product. To show that f is a bijection, since the domain and codomain of f have the same cardinality, it suffices to show that f is a surjection. Given $(x, a) \in X \times A$ we see that

$$f((x, p_B(a)), p_C(a)) = (x, sp_B(a) + \iota p_C(a)) = (x, a).$$

Finally to show that f preservers the product we compute:

$$\begin{aligned} & f(\ ((x_1,b_1),c_1) * ((x_2,b_2),c_2)\ ) \\ = \quad & f(\,(\,(x_1,b_1)*(x_2,b_2),\ c_1+\phi'(\,(x_1,b_1),(x_2,b_2)\,)\,)\,) \\ = \quad & f(\,(\,(x_1*x_2, b_1+p_B\phi(x_1,x_2)),\ c_1+p_C\phi(x_1,x_2)-\eta(b_1,\phi(x_1,x_2))\,)\,) \\ = \quad & (\,x_1*x_2,\ s(b_1+p_B\phi(x_1,x_2))+\iota(c_1+p_C\phi(x_1,x_2)-\eta(b_1,\phi(x_1,x_2))\,)\,) \\ = \quad & (\,x_1*x_2,\ s(b_1)+sp_B\phi(x_1,x_2)+\iota\eta(b_1,p_B\phi(x_1,x_2)) \\ & \qquad +\iota(c_1)+\iota p_C\phi(x_1,x_2)-\iota\eta(b_1,p_B\phi(x_1,x_2))\,) \\ = \quad & (\,x_1*x_2,\ s(b_1)+\iota(c_1)+\phi(x_1,x_2)\,), \end{aligned}$$

and

$$\begin{aligned} & f(\ ((x_1,b_1),c_1)\ ) * f(\ ((x_2,b_2),c_2)\ ) \\ = \quad & (x_1,\ s(b_1)+\iota(c_1)) * (x_2,\ s(b_2)+\iota(c_2)) \\ = \quad & (x_1*x_2,\ s(b_1)+\iota(c_1)+\phi(x_1,x_2)), \end{aligned}$$

as desired. □

If we suppress the 2-cocycle in the notation $E(X,A,\phi)$ and write merely $E(X,A)$ then the above Lemma 6.4 and Proposition 6.6 may be stated more simply.

Corollary 6.7. (i) *If $E(X,B)$ and $E(X,C)$ are abelian extensions, then so is $E(X,B\times C)$, and*

$$E(X,B\times C)=E(E(X,B),C).$$

(ii) *If $E(X,A)$ is a finite abelian extension of a quandle X and C is a subgroup of the finite abelian group A then*

$$E(X,A)=E(E(X,A/C),C).$$

We note that if $E(X,A)$ is connected, then $E(X,A/C)$ is connected since the epimorphic image of a connected quandle is connected.

We examine some connected abelian extensions of Rig quandles of order up to 12. In the following, we use the notation $E\xrightarrow{n}X$ if $E=E(X,\mathbb{Z}_n.\phi)$ for some 2-cocycle ϕ such that E is connected. $E_2\xRightarrow{m}E_1\xRightarrow{d}X$ if there is a short exact sequence $0\to\mathbb{Z}_m\to\mathbb{Z}_n\to\mathbb{Z}_d\to 0$ such that $\mathbb{Z}_n\subset H^2_Q(X,\mathbb{Z}_n)$ and E_1,E_2 are corresponding extensions as in Proposition 6.6. In this case $E_2\xrightarrow{n}X$ where $n=md$. The notation $\emptyset\xrightarrow{1}X$ indicates that $H^2_Q(X,A)=0$ for any coefficient group A, and hence there is no non-trivial abelian extension. It is noted to the left when all quandles in question are keis.

$$\begin{aligned}
&\emptyset \xrightarrow{1} Q(8,1) \xrightarrow{2} Q(4,1)\\
\text{(Kei)}\quad &\emptyset \xrightarrow{1} Q(24,1) \xrightarrow{2} Q(12,1) \xrightarrow{2} Q(6,1)\\
&\emptyset \xrightarrow{1} Q(24,2) \xRightarrow{2} Q(12,2) \xRightarrow{2} Q(6,2)\\
\text{(Kei)}\quad &\emptyset \xrightarrow{1} Q(27,1) \xrightarrow{3} Q(9,2) = Q(3,1) \times Q(3,1)\\
&\emptyset \xrightarrow{1} Q(27,6) \xrightarrow{3} Q(9,3) = \mathbb{Z}_3[t]/(t^2+1)\\
&\emptyset \xrightarrow{1} Q(27,14) \xrightarrow{3} Q(9,6) = \mathbb{Z}_3[t]/(t^2+2t+1)\\
&\emptyset \xrightarrow{1} Q(24,8) = Q(3,1) \times Q(8,1) \xrightarrow{2} Q(12,4) = Q(3,1) \times Q(4,1)
\end{aligned}$$

In the following, we list abelian extensions of Rig quandles that contain quandles of order higher than 35. The notation $Q(n,-)$ indicates that it is a quandle of order $n > 35$ and is not a Rig quandle. The notation $? \longrightarrow Q(n,-)$ indicates that we do not know if non-trivial abelian extension exists for the quandle $Q(n,-)$ in question. Except for the quandle $Q(120,-)$ in the third line, we have explicit quandle operation tables for the quandles appearing in the list and hence we can prove by computer that such quandles are connected.

$$\begin{aligned}
&? \longrightarrow Q(120,-) \xrightarrow{6} Q(20,3) \xrightarrow{2} Q(10,1)\\
&? \longrightarrow Q(120,-) \xRightarrow{5} Q(24,7) \xRightarrow{2} Q(12,3)\\
&? \longrightarrow Q(120,-) \xRightarrow{2} Q(60,-) \xRightarrow{5} Q(12,3)\\
&? \longrightarrow Q(48,-) \xRightarrow{2} Q(24,4) \xRightarrow{2} Q(12,5)\\
&? \longrightarrow Q(48,-) \xRightarrow{2} Q(24,3) \xRightarrow{2} Q(12,6)
\end{aligned}$$

It is interesting to remark that all quandles appearing in the first and the last lines are keis.

These observations raise the following questions.

• What is a condition on cocycles for abelian, or non-abelian extensions to be connected?

In [1], a condition for an extension to be connected was given in terms of elements of the inner automorphism group.

• Is there an infinite sequence of abelian extensions of connected quandles $\cdots \to Q_n \to \cdots \to Q_1$?

We note that sequences of abelian extensions of connected quandles terminate as much as we were able to compute.

• Is any abelian extension of a finite kei a kei?

In relation to this question, below we observe a condition of 2-cocycles that give extensions that are keis.

Lemma 6.8. *Let X be a kei, ϕ be a 2-cocycle with coefficient group A, and E be the abelian extension of X with respect to ϕ. Then E is a kei if and only if $\phi(x,y)+\phi(x*y,y)=0\in A$ for any $x,y\in X$, in additive notation.*

Proof. One computes, for any $x,y\in X$ and $a,b\in A$,

$$\begin{aligned}&[\ (x,a)*(y,b)\ ]*(y,b)\\ &= \quad (x*y,a+\phi(x,y))*(y,b)=(\ (x*y)*y,a+\phi(x,y)+\phi(x*y,y)).\end{aligned}$$

For any $x,y\in X$ and $a,b\in A$, the right-hand side is equal to (x,a) if and only if $\phi(x,y)+\phi(x*y,y)=0$ for any $x,y\in X$. $\square$

Remark 6.9. Let $X=Q(12,7)$. Then $H^2_Q(X,\mathbb{Z}_4)\cong\mathbb{Z}_2\times\mathbb{Z}_4$. By computer calculation, there is a particular generating 2-cocycle of the $\mathbb{Z}_2$-factor, $Q(24,15)\xrightarrow{2}Q(12,7)$. Notice that $H^2_Q(Q(24,15),\mathbb{Z}_4)\cong\mathbb{Z}_4$. We also note that there are epimorphisms $Q(24,14)\to Q(12,7)$ and $Q(24,18)\to Q(12,7)$, where $H^2_Q(Q(24,14),\mathbb{Z}_4)\cong\mathbb{Z}_4$ and $H^2_Q(Q(24,14),\mathbb{Z}_2)\cong\mathbb{Z}_2\times\mathbb{Z}_2$. Hence there is a quandle of order 48 corresponding to the $\mathbb{Z}_4$-factor of $H^2_Q(X,\mathbb{Z}_4)$, that has epimorphic image $Q(24,14)$ or $Q(24,18)$.

Remark 6.10. Let $X=Q(12,8)$. Then $H^2_Q(X,\mathbb{Z}_2)\cong(\mathbb{Z}_2)^3$. There are three epimorphisms from Rig quandles of order less than 36:

$$Q(24,5)\to X,\qquad Q(24,16)\to X,\qquad Q(24,17)\to X$$

and their cohomology groups with $A=\mathbb{Z}_2$ are $(\mathbb{Z}_2)^3$, $(\mathbb{Z}_2)^2$, and $(\mathbb{Z}_2)^2$, respectively. We note that there are 7 cocycles that are not cohomologous each other, yet there are only 3 extensions as in Remark 6.3.

Remark 6.11. Let $X=Q(12,9)$. Then $H^2_Q(X,\mathbb{Z}_4)\cong\mathbb{Z}_4\times\mathbb{Z}_4$. There are two extensions in Rig quandles of order less than 36:

$$Q(24,6)\to Q(12,9),\quad Q(24,19)\to Q(12,9)$$

and with $A=\mathbb{Z}_4$ their cohomology groups are $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$ and $\mathbb{Z}_2\times\mathbb{Z}_4$, respectively. There are 3 cocycles that give order 2 extensions, yet there are two extensions as in Remark 6.3.

Remark 6.12. Let $X=Q(12,10)$. Then $H^2_Q(X,\mathbb{Z}_6)\cong\mathbb{Z}_6$. There is one extension among Rig quandles of order less than 36, $Q(24,20)\to Q(12,10)$ and we have $H^2_Q(Q(24,20),\mathbb{Z}_3)\cong\mathbb{Z}_3\times\mathbb{Z}_3$. One of the order 3 cocycle corresponds to an extension of X of order 6.

7. Finding extensions of higher order

We further investigated extensions among non-faithful quandles over Rig quandles. Extensions of some of the Rig quandles of order greater than 12 and less than 28 can be found in `http://github.com/vendramin/rig/wiki`. The computations of cocycles become difficult for quandles of order 28. Thus we take an approach of constructing non-faithful connected quandles and identify extensions as follows.

We considered Rig quandles of order less than 36. To find extensions of Rig quandles, we made a list $\mathcal{NF}$ of 315 non-faithful connected generalized Alexander quandles with respect to pairs (G, f) for non-abelian groups G and $f \in \mathrm{Aut}(G)$ (see Section 1). We considered all groups or order n, $36 \leq n < 128$, and for $n = 128$, only the first 172 groups in GAP Small Groups library (the library contains all the 2328 groups of size 128). All possible automorphisms $f \in \mathrm{Aut}(G)$ were considered up to conjugacy. For example, there are 39 non-abelian groups of order 108 which give 74 connected non-faithful quandles of order 108.

Lemma 2.3 and Proposition 2.11 in [1] were used to determine abelian extensions and non-abelian extensions by constant cocycles among quandles in $\mathcal{NF}$ over Rig quandles of order less than 36. Specifically, quotient quandles are computed, *dynamical* cocycles ([1], Lemma 2.3) are computed, whether the cocycles are constant is determined, and whether the extensions are abelian is determined.

We note that most examples computed for abelian extensions are 2-fold epimorphisms, and observe the following.

Lemma 7.1. *Let Y be a finite connected quandle of even order $2n$, and assume that $\varphi(Y) = X \subset \mathrm{Inn}(Y)$ with $|X| = n$. Then Y is isomorphic to an abelian extension of X by $\mathbb{Z}_2$.*

Proof. As in Proposition 6.2, it follows from Proposition 2.11 of [1] that Y is isomorphic to an extension $X \times_\beta S$ by a constant cocycle β, where a set S consists of two elements. Let $S = \{0, 1\}$, and we identify S with $\mathbb{Z}_2$. Then $\mathrm{Sym}(S)$ consists of two elements, the identity and the transposition of 0 and 1. We define $\phi : X \times X \to \mathbb{Z}_2$ by $\phi(x, y) = 0$ if $\beta_{x,y} = \mathrm{id}$ and $\phi(x, y) = 1$ if $\beta_{x,y}$ is the transposition. Then $\beta_{x,y}(t) = t + \phi(x, y)$ for $t \in \mathbb{Z}_2$ and ϕ is a 2-cocycle. $\square$

Remark 7.2. Among Rig quandles of order less than 36, the following have 2-fold extensions among quandles in $\mathcal{NF}$.

$$\begin{aligned} &Q(18,i), &&i = 1,3,4,5,6,7,8. \\ &Q(24,i), &&i = 6,10,11,13,18,22,23. \\ &Q(28,i), &&i = 1,2,3,4,5,6,7,8,9. \end{aligned}$$

Other than these, we found that $Q(12,3)$ has a 5-fold abelian extension, and $Q(15,2)$ has a 4-fold non-abelian extension in $\mathcal{NF}$. We remark that the 5-fold extension of $Q(12,3)$ was predicted by Lemma 6.5, see the list in Section 6 for $Q(12,3)$. Thus this specific extension is found in $\mathcal{NF}$.

We observe the following generalization of Lemma 7.1. Let Y be a finite connected quandle, and let $\varphi(Y) = X \subset \mathrm{Inn}(Y)$ with $|X| = n$. It follows again from Proposition 2.11 of [1] that Y is isomorphic to an extension $X \times_\beta S$ by a constant cocycle β.

Lemma 7.3. *Let X and Y be as above. If $|Y| = kn$ where k is a prime power, and the subgroup H_β of $\mathrm{Sym}(S)$ generated by $\{\, \beta_{x,y} \,|\, x, y \in X \,\}$ is cyclic of order k, then Y is isomorphic to an abelian extension of X.*

Proof. Since $|Y| = kn$, we have $|S| = k$. Since H_β is cyclic of order k and k is a prime power, it is generated by a k-cycle σ. We can identify S with $\mathbb{Z}_k$ in such a way that $\sigma = (1, 2, \dots, k)$. Then $\sigma(t) = 1+t$ for any $t \in \mathbb{Z}_k$. Hence for any $x, y \in X$, $\beta_{x,y} = \sigma^i$ for some $i \in \mathbb{Z}_k$, so that $\beta_{x,y}(t) = t + \phi(x, y)$ with $\phi(x, y) = i$. $\square$

Remark 7.4. Although homology groups of Rig quandles have been computed in [33], as mentioned earlier, explicit 2-cocycles have not been computed for Rig quandles of order greater than 23. The above computations of extensions give rise to explicit 2-cocycles, and also may be used for computations of cocycle invariants as in Section 5. Furthermore, the computations identify the pairs (G, f) of generalized Alexander quandles that are abelian extensions of Rig quandles.

8. Problems, questions and conjectures

For convenience of the reader, we collect here questions, problems and conjectures discussed all over the text.

Problem 8.1. *Compute explicit 2-cocycles and extensions of Rig quandles of order ≥ 24.*

In Remark 4.6 we made the following conjecture.

Conjecture 8.2. *Let X be one of the following quandles:*

$$\begin{array}{lllll} Q(12,1), & Q(20,3), & Q(24,3), & Q(24,4), & Q(24,5), \\ Q(24,6), & Q(24,14), & Q(24,16), & Q(24,17), & Q(30,1), \\ Q(30,16), & Q(32,5), & Q(32,6), & Q(32,7), & Q(32,8). \end{array}$$

Then every knot K is end monochromatic with X.

In Examples 5.7, 5.8 and 5.10 we made the following conjectures.

Conjecture 8.3. *Let $X = Q(12,5)$ and ϕ be the 2-cocycle choosen in Example 5.7. Then for each knot K the cocycle invariant Φ_ϕ is of the form $\Phi_\phi(K) = a + bu^2$, where $a, b \in \mathbb{Z}$.*

Conjecture 8.4. *Let $X = Q(12,6)$ and ϕ be the 2-cocycle choosen in Example 5.8. Then for each knot K the cocycle invariant Φ_ϕ is of the form $\Phi_\phi(K) = a + bu^2$, where $a, b \in \mathbb{Z}$.*

Conjecture 8.5. *Let $X = Q(12,10)$ and ϕ be the 2-cocycle choosen in Example 5.10. For each knot K write $\Phi_\phi(K) = a+bu+cu^2+du^3+eu^4+fu^5$, where $a,b,c,d,e,f \in \mathbb{Z}$. Then $b = f = 0$ for all K.*

In Section 6 we posed the following questions.

Question 8.6. *What is a condition on cocycles for abelian, or non-abelian extensions to be connected?*

Question 8.7. *Is there an infinite sequence of abelian extensions of connected quandles $\cdots \to Q_n \to \cdots \to Q_1$?*

Question 8.8. *Is any abelian extension of a finite kei a kei?*

Appendix: Cocycle invariants for $Q(6,2)$

In this appendix we list the cocycle invariant $\Phi_\phi(K)$ for the quandle $X = Q(6,2)$ and the 2-cocycle over $\mathbb{Z}_4$ discussed in Example 5.2. The list is for all knots in [14] that have braid index 4 or less, and 12 crossings or less. Knots with only trivial colorings (the invariant value 6) are not listed. These values are computed using the formula described in Example 5.2 and programs similar to those in [11]. Note that if the 2-cocycle invariant below has the form $a + bu + cu^2 + du^3$ where $b \neq d$ then by Lemma 4.3 each of the corresponding knots K satisfes $K \neq rm(K)$.

Table 5: Some cocycle invariants for the quandle $Q(6,2)$ of the form $a+bu$ for some $a,b\in\mathbb{Z}$.

cocycle invariant	knot
54	$10_{62}, 10_{65}, 10_{140}, 10_{143}, 10_{165}$ $11a_{108}, 11a_{109}, 11a_{139}, 11a_{157}$ $11n_{85}, 11n_{106}, 11n_{118}, 11n_{119}$ $12a_{0290}, 12a_{0375}, 12a_{0390}, 12a_{0571}$ $12a_{0668}, 12a_{0672}, 12a_{0941}, 12a_{0949}$ $12a_{1184}, 12a_{1191}, 12a_{1207}, 12a_{1215}$ $12n_{0425}, 12n_{0426}, 12n_{0533}, 12n_{0807}$ $12n_{0811}, 12n_{0812}, 12n_{0831}, 12n_{0868}$
198	10_{99}
$6+24u$	$3_1, 8_{11}, 9_4, 9_{10}, 9_{17}, 9_{34}$ $10_5, 10_9, 10_{40}, 10_{103}, 10_{106}$ $10_{136}, 10_{146}, 10_{158}, 10_{159}, 10_{163}$ $11a_{73}, 11a_{99}, 11a_{146}, 11a_{171}, 11a_{175}$ $11a_{176}, 11a_{184}, 11a_{196}, 11a_{216}, 11a_{239}$ $11a_{248}, 11a_{306}, 11a_{346}, 11a_{353}, 11n_{13}$ $11n_{14}, 11n_{86}, 11n_{98}, 11n_{109}$ $11n_{125}, 11n_{137}, 11n_{138}, 11n_{158}$ $12a_{0234}, 12a_{0346}, 12a_{0409}, 12a_{0411}$ $12a_{0422}, 12a_{0509}, 12a_{0519}, 12a_{0523}$ $12a_{0567}, 12a_{0588}, 12a_{0617}, 12a_{0626}, 12a_{0718}$ $12a_{0723}, 12a_{0878}, 12a_{0894}, 12a_{0904}, 12a_{0907}$ $12a_{0916}, 12a_{0923}, 12a_{0944}$ $12a_{0986}, 12a_{1002}, 12a_{1025}, 12a_{1029}$ $12a_{1060}, 12a_{1079}, 12a_{1115}, 12a_{1120}$ $12a_{1136}, 12a_{1170}, 12a_{1177}, 12a_{1180}$ $12a_{1197}, 12a_{1201}, 12a_{1214}, 12a_{1226}$ $12a_{1247}, 12a_{1248}, 12a_{1262}, 12a_{1270}$ $12a_{1272}, 12a_{1276}, 12n_{0147}, 12n_{0329}$ $12n_{0369}, 12n_{0377}, 12n_{0409}, 12n_{0413}$ $12n_{0419}, 12n_{0439}, 12n_{0493}, 12n_{0502}$ $12n_{0543}, 12n_{0597}, 12n_{0653}, 12n_{0655}$ $12n_{0657}, 12n_{0660}, 12n_{0667}, 12n_{0668}$ $12n_{0752}, 12n_{0767}, 12n_{0782}, 12n_{0803}$ $12n_{0825}, 12n_{0866}, 12n_{0284}$
$54+72u$	$12n_{0546}$
$150+24u$	$11n_{126}, 12n_{0440}$

Table 6: Some cocycle invariants for the quandle $Q(6,2)$ of the form $a+bu+cu^2$ for some $a,b,c \in \mathbb{Z}$ with $c \neq 0$.

cocycle invariant	knot
$6+48u^2$	$9_{40}, 10_{61}, 10_{64}, 10_{66}$ $10_{139}, 10_{141}, 10_{142}, 10_{144}, 10_{164}$ $11a_{106}, 11a_{194}, 11a_{223}, 11a_{232}$ $11a_{244}, 11a_{338}, 11a_{340}, 11n_{87}$ $11n_{104}, 11n_{105}, 11n_{107}, 11n_{145}$ $11n_{146}, 11n_{173}, 11n_{183}, 11n_{184}, 11n_{185}$ $12a_{0428}, 12a_{0670}, 12a_{0737}, 12a_{0739}, 12a_{0855}$ $12a_{0864}, 12a_{0970}, 12a_{1111}, 12a_{1147}, 12a_{1212}$ $12a_{1219}, 12a_{1221}, 12n_{0483}, 12n_{0484}, 12n_{0536}$ $12n_{0627}, 12n_{0779}$
$6+48u+96u^2$	$12a_{0701}, 12a_{0987}$
$30+24u^2$	$8_5, 8_{10}, 8_{15}, 8_{19}, 8_{20}, 8_{21}, 9_{16}, 9_{24}, 9_{28}$ $10_{76}, 10_{77}, 10_{82}, 10_{84}, 10_{85}, 10_{87}$ $11a_{71}, 11a_{72}, 11a_{245}, 11a_{261}$ $11a_{264}, 11a_{305}, 11a_{351}$ $11n_{38}, 11n_{121}$ $12a_{0577}, 12a_{0578}, 12a_{0852}$ $12a_{0861}, 12a_{0930}, 12a_{0979}$ $12a_{0981}, 12a_{0982}, 12a_{0999}$ $12a_{1000}, 12a_{1059}, 12a_{1061}$ $12a_{1100}, 12a_{1187}, 12a_{1252}$ $12a_{1253}, 12a_{1261}, 12a_{1284}$ $12a_{1285}, 12n_{0084}, 12n_{0106}$ $12n_{0107}, 12n_{0290}, 12n_{0291}$ $12n_{0572}, 12n_{0573}, 12n_{0575}$ $12n_{0576}, 12n_{0577}, 12n_{0578}$ $12n_{0638}, 12n_{0674}, 12n_{0675}$ $12n_{0700}, 12n_{0753}, 12n_{0833}$ $12n_{0845}, 12n_{0850}$
$30+168u^2$	$12n_{0604}$
$54+144u^2$	$12n_{0508}$
$54+48u+48u^2$	$12a_{0742}, 12n_{0380}$
$78+48u+24u^2$	$12a_{0574}, 12n_{0571}, 12n_{0574}$
$102+96u^2$	$12n_{0518}$
$126+72u^2$	$12a_{0647}, 12n_{0605}$
$150+48u^2$	$12a_{1288}, 12n_{0888}$

Table 7: Some cocycle invariants for the quandle $Q(6,2)$ of the form $a+bu+cu^2+du^3$ for some $a,b,c,d\in\mathbb{Z}$ with $d\neq 0$.

cocycle invariant	knot
$6+24u^3$	$6_1, 7_4, 7_7, 9_1, 9_6, 9_{11}, 9_{23}, 9_{29}, 9_{38}$ $10_{14}, 10_{19}, 10_{21}, 10_{32}$ $10_{108}, 10_{112}, 10_{113}, 10_{114}$ $10_{122}, 10_{145}, 10_{147}, 10_{160}$ $11a_{179}, 11a_{203}, 11a_{236}, 11a_{274}$ $11a_{286}, 11a_{300}, 11a_{318}, 11a_{335}$ $11a_{355}, 11a_{365}, 11n_{65}, 11n_{66}, 11n_{92}$ $11n_{94}, 11n_{95}, 11n_{99}, 11n_{122}, 11n_{136}$ $11n_{143}, 11n_{148}, 11n_{149}, 11n_{153}, 11n_{176}$ $11n_{182}, 12a_{0236}, 12a_{0321}, 12a_{0496}$ $12a_{0580}, 12a_{0762}, 12a_{0805}$ $12a_{0806}, 12a_{0807}, 12a_{0809}$ $12a_{0876}, 12a_{0909}, 12a_{0952}$ $12a_{0972}, 12a_{1036}, 12a_{1091}$ $12a_{1101}, 12a_{1129}, 12a_{1157}$ $12a_{1196}, 12a_{1200}, 12a_{1210}$ $12a_{1216}, 12a_{1224}, 12a_{1237}$ $12a_{1239}, 12a_{1255}, 12n_{0330}$ $12n_{0368}, 12n_{0375}, 12n_{0412}$ $12n_{0438}, 12n_{0441}, 12n_{0443}$ $12n_{0464}, 12n_{0500}, 12n_{0603}$ $12n_{0640}, 12n_{0641}, 12n_{0717}$ $12n_{0738}, 12n_{0740}, 12n_{0750}$ $12n_{0751}, 12n_{0754}, 12n_{0769}$ $12n_{0770}, 12n_{0781}, 12n_{0791}$ $12n_{0823}, 12n_{0832}, 12n_{0836}$ $12n_{0865}, 12n_{0874}, 12n_{0875}$ $12n_{0882}$
$6+48u^2+72u^3$	$9_{48}, 11a_{293}, 12a_{0895}$
$6+144u^2+24u^3$	10_{98}
$6+24u+72u^3$	$12n_{0666}$
$6+24u+48u^2+48u^3$	$11n_{164}, 12n_{0402}$
$6+24u+96u^2+24u^3$	$11n_{167}$
$6+48u+48u^3$	$8_{18}, 12a_{1260}, 12n_{0403}$
$6+48u+48u^2+24u^3$	$12n_{0565}$
$6+72u+24u^3$	$9_{47}, 12n_{0549}$

$30 + 120u^2 + 24u^3$	$12n_{0737}$
$30 + 24u + 72u^2 + 24u^3$	$12a_{0576}, 12n_{0570}$
$54 + 72u^3$	$11a_{314}$
$54 + 48u^2 + 48u^3$	$11a_{332}, 12n_{0386}$
$54 + 24u + 48u^2 + 24u^3$	$12a_{0297}, 12n_{0379}$
$54 + 48u + 24u^3$	$9_{46}, 11a_{291}, 12n_{0567}$
$78 + 24u^2 + 48u^3$	$12a_{1283}$
$78 + 24u + 24u^2 + 24u^3$	$12n_{0883}$
$78 + 48u + 72u^2 + 48u^3$	$11a_{44}, 11a_{47}, 11a_{57}$ $11a_{231}, 11a_{263}, 11n_{71}$ $11n_{72}, 11n_{73}, 11n_{74}$ $11n_{75}, 11n_{76}, 11n_{77}$ $11n_{78}, 11n_{81}$ $12a_{0167}, 12a_{0692}, 12a_{0801}$
$102 + 48u^3$	$12n_{0806}$
$102 + 24u + 24u^3$	$11a_{277}, 12a_{1225}$

Acknowledgements. MS was partially supported by the National Institutes of Health under Award Number R01GM109459. The content of this paper is solely the responsibility of the authors and does not necessarily represent the official views of NIH. LV was partially supported by Conicet, ICTP, UBACyT 20020110300037 and PICT-2014-1376. We thank Santiago Laplagne for the computer were some calculations were performed.

References

[1] Andruskiewitsch, N.; Graña, M., *From racks to pointed Hopf algebras*, Adv. in Math., 178 (2003) 177–243.

[2] Birman, J.S.; Menasco, W., *Studying links via closed braids IV: Composite links and split links,* Invent. Math. 102 (1990) 115–139.

[3] Burde, G.; Zieschang, H., *Knots,* Second edition, de Gruyter Studies in Mathematics 5, Walter de Gruyter & Co., Berlin, 2003.

[4] Carter, J.S.; Elhamdadi, M.; Nikiforou, M.A.; Saito, M., *Extensions of quandles and cocycle knot invariants,* J. Knot Theory Ramifications, 12 (2003) 725–738.

[5] Carter, J.S.; Jelsovsky, D.; Kamada, S.; Langford, L.; Saito, M., *Quandle cohomology and state-sum invariants of knotted curves and surfaces,* Trans. Amer. Math. Soc., 355 (2003) 3947–3989.

[6] Carter, J.S.; Jelsovsky, D.; Kamada, S.; Saito, M., *Computations of quandle cocycle invariants of knotted curves and surfaces,* Adv. in Math., 157 (2001) 36-94.

[7] Carter, J. S.; Jelsovsky, D.; Kamada, S.; Saito, M., *Quandle homology groups, their Betti numbers, and virtual knots,* J. Pure Appl. Algebra, 157 (2001), no. 2-3, 135–155.

[8] Carter, J.S.; Kamada, S.; Saito, M., Surfaces in 4-space, Encyclopaedia of Mathematical Sciences, 142, Springer Verlag, 2004.

[9] Carter, J.S.; Kamada, S.; Saito, M., *Geometric interpretations of quandle homology,* J. Knot Theory Ramifications, 10 (2001) 345-386.

[10] Carter, J.S.; Saito, M.; Satoh, S., *Ribbon concordance of surface-knots via quandle cocycle invariants,* J. Aust. Math. Soc., 80 (2006) 131–147.

[11] Clark, W.E.; Elhamdadi, M.; Saito, M.; Yeatman, T., *Quandle colorings of knots and applications*, J. of Knot Theory and Its Ramifications, 23 (2014) 1450035, 29 pp.

[12] Clark, W.E.; Yeatman, T., *Properties of Rig quandles,* `http://math.usf.edu/~saito/QuandleColor/characterization.pdf`.

[13] Conway, J.H. *An Enumeration of Knots and Links, and Some of Their Algebraic Properties,* In J. Leech (editor), Computational Problems in Abstract Algebra. Oxford, England. Pergamon Press (1970) 329–358.

[14] Cha, J.C.; Livingston, C., *KnotInfo: Table of Knot Invariants,* `http://www.indiana.edu/~knotinfo`, May 26, 2011.

[15] Clauwens, F.J.B.J., *Small connected quandles,* `http://front.math.ucdavis.edu/1011.2456`.

[16] Eisermann, M., *Homological characterization of the unknot,* J. Pure Appl. Algebra, 177 (2003), no. 2, 131–157.

[17] Elhamdadi, M.; MacQuarrie, J; Restrepo, R., *Automorphism groups of quandles*, J. Algebra Appl., 11 (2012) 1250008, 9 pp.

[18] Fenn, R.; Rourke, C., *Racks and links in codimension two*, J. Knot Theory Ramifications, 1 (1992) 343–406.

[19] Fenn, R.; Rourke, C.; Sanderson, B., *Trunks and classifying spaces,* Appl. Categ. Structures, 3 (1995) 321–356.

[20] Hempel, J., *Residual finiteness for 3-manifolds,* Combinatorial group theory and topology (Alta, Utah, 1984) 379–396, Ann. of Math. Stud., 111, Princeton Univ. Press, Princeton, NJ, 1987.

[21] Joyce, D., *A classifying invariant of knots, the knot quandle,* J. Pure Appl. Algebra, 23 (1982) 37–65.

[22] Litherland, L.N.; Nelson, S., *The Betti numbers of some finite racks,* J. Pure Appl. Algebra, 178 (2003) 187–202.

[23] Manturov, V., Knot theory, CRC press, 2004.

[24] Matveev, S., *Distributive groupoids in knot theory.* (Russian) Mat. Sb. (N.S.) 119(161) (1982) 78–88 (160).

[25] Mochizuki, T., *The third cohomology groups of dihedral quandles,* J. Knot Theory Ramifications, 20 (2011) 1041–1057.

[26] Niebrzydowski, M.; Przytycki, J. H., *Homology of dihedral quandles,* J. Pure Appl. Algebra, 213 (2009) 742–755.

[27] Nosaka, T., *On homotopy groups of quandle spaces and the quandle homotopy invariant of links,* Topology Appl., 158 (2011) 996–1011.

[28] Papakyriakopoulos, C. D., *On Dehn's lemma and the asphericity of knots,* Ann. of Math. (2) 66 (1957) 1–26.

[29] Przytycki, J.H., 3-*colorings and other elementary invariants of knots,* Knot theory (Warsaw, 1995), 275–295, Banach Center Publ., 42, Polish Acad. Sci., Warsaw, 1998.

[30] Przytycki, J.H., *From 3-moves to Lagrangian tangles and cubic skein modules,* Advances in Topological Quantum Field Theory, Proceedings of the NATO ARW on New Techniques in Topological Quantum Field Theory, John M. Bryden (ed), 2004, 71–125.

[31] Takasaki, M., *Abstraction of symmetric transformation,* (in Japanese), Tohoku Math. J., 49 (1942/3) 145–207.

[32] Thurston, W.P., *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry,* Bull. Amer. Math. Soc. (N.S.) 6 (1982) 357–381.

[33] Vendramin, L., *Rig – A GAP package for racks and quandles.* Available at `http://github.com/vendramin/rig/`.

[34] Vendramin, L., *On the classification of quandles of low order*, J. Knot Theory Ramifications, 21 (2012), no. 9, 1250088.

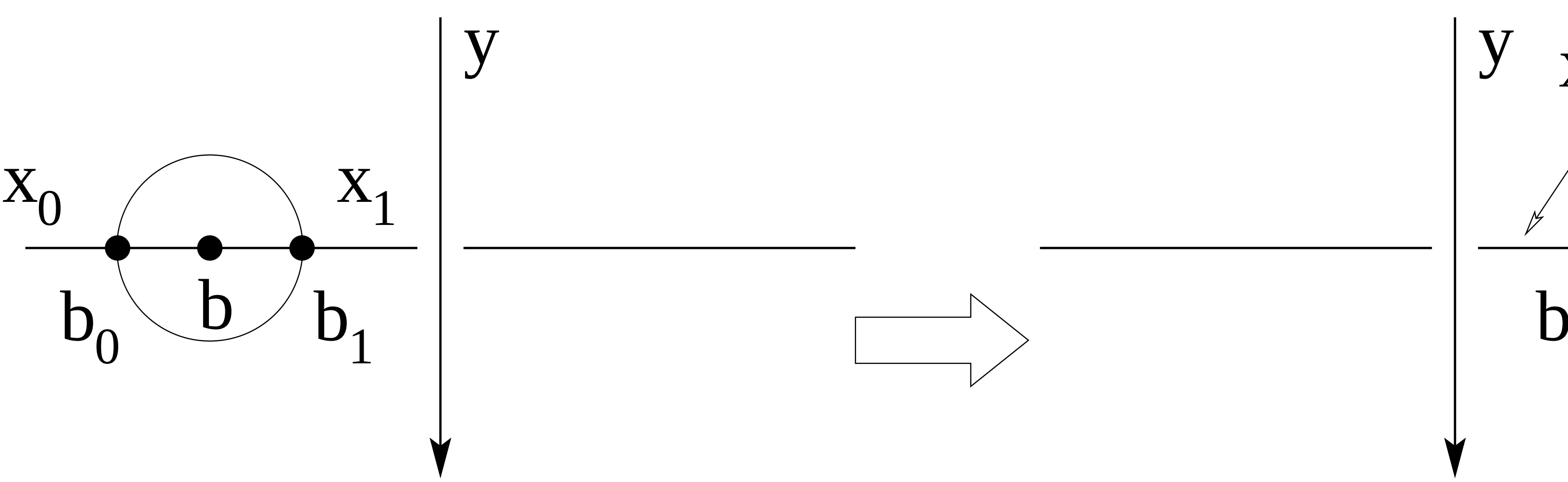
E. Clark and M. Saito: Department of Mathematics and Statistics, University of South Florida, Tampa, Florida, U.S.A.

E-mail address: `wclark@mail.usf.edu`

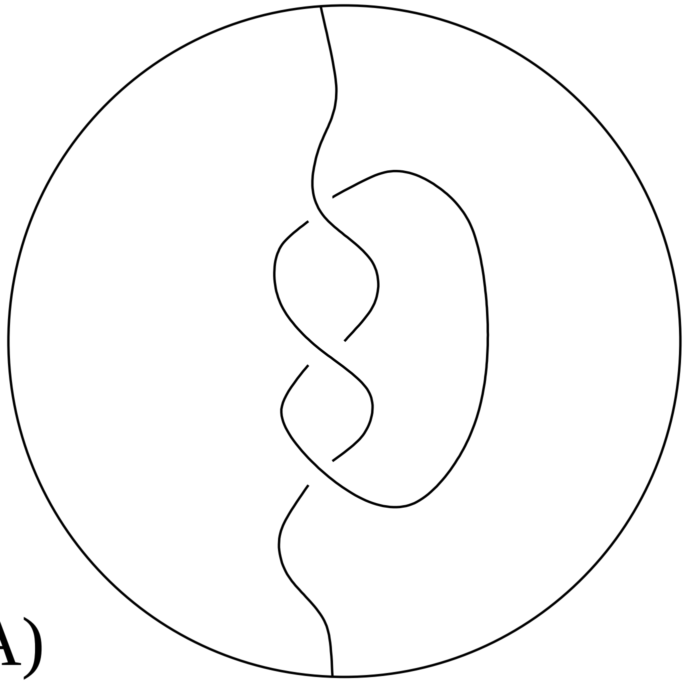
E-mail address: `saito@usf.edu`

L. Vendramin: Departamento de Mathemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Buenos Aires, Argentina

E-mail address: `lvendramin@dm.uba.ar`



y
x0
x1
b0
b
b1
y
b

(A) 

(B)