

# Verma and simple modules for quantum groups at non-abelian groups

Barbara Pogorelsky<sup>a</sup>, Cristian Vay<sup>b,\*</sup>

<sup>a</sup>Instituto de Matemática, Universidade Federal do Rio Grande do Sul, Av. Bento Gonçalves 9500, Porto Alegre, RS, 91509-900, Brazil

<sup>b</sup>Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, CIEM-CONICET, Medina Allende s/n, Ciudad Universitaria, 5000 Córdoba, República Argentina.

## Abstract

The Drinfeld double  $\mathfrak{D}$  of the bosonization of a finite-dimensional Nichols algebra  $\mathfrak{B}(V)$  over a finite non-abelian group  $G$  is called a *quantum group at a non-abelian group*. We introduce Verma modules over such a quantum group  $\mathfrak{D}$  and prove that a Verma module has simple head and simple socle. This provides two bijective correspondences between the set of simple modules over  $\mathfrak{D}$  and the set of simple modules over the Drinfeld double  $\mathfrak{D}(G)$ . As an example, we describe the lattice of submodules of the Verma modules over the quantum group at the symmetric group  $\mathbb{S}_3$  attached to the 12-dimensional Fomin-Kirillov algebra, computing all the simple modules and calculating their dimensions.

*Keywords:* Hopf algebras, Nichols algebras, Fomin-Kirillov algebras, Quantum groups, Verma modules, Representation Theory.

*2000 MSC:* 16W30

## 1. Introduction

The Drinfeld doubles of bosonizations of braided Hopf algebras over abelian groups, and their quotients by central group-likes, are known in the folklore as *quantum groups*. Such is the case of the *quantum enveloping algebra*  $U_q(\mathfrak{g})$  or the *small quantum group*  $u_q(\mathfrak{g})$  [4, 10]. These quantum groups have been intensely studied, both their intrinsic structures and their representation theories. However, to the best of our knowledge, there is no research which contemplates non-abelian groups. The purpose of our work is to give a first step in this direction.

More precisely, let  $G$  be a finite non-abelian group and  $V$  a Yetter-Drinfeld module over  $\mathbb{k}G$  with finite-dimensional Nichols algebra  $\mathfrak{B}(V)$  (we will work over an algebraically closed field  $\mathbb{k}$  of characteristic zero). We denote by  $\mathfrak{D}$  the Drinfeld double of the bosonization  $\mathfrak{B}(V)\#\mathbb{k}G$  and call it “*quantum group at a non-abelian group*” – recall that  $U_q(\mathfrak{g})$  is called *quantum group at a root of 1* if the indeterminate  $q$  is specialized to a root of 1 [9]. In this work, we deal with the category of representations of  $\mathfrak{D}$ . We use the methods coming from the theory of Lie algebras which were also used in the study of quantum groups over abelian groups. We find similarities as well as differences between our results and their analogues in the context of Lie algebras or the mentioned quantum groups. To explain these similarities and differences, we first recall briefly the situation in those frameworks.

Assume that  $U$  is either an enveloping algebra of a Lie algebra as in [5, Chapter 7] or a quantum group as in [8, Chapter 5], [10, Chapter 3]; the reader can find all the details of the following exposition in these chapters. Roughly speaking,  $U$  has a distinguished commutative and cocommutative Hopf subalgebra  $U^0$ . Hence

- (a) the maximal spectrum of  $U^0$  is an abelian group  $T$ . The elements of  $T$ , the algebra maps  $U^0 \rightarrow \mathbb{k}$ , are called *weights*. The module corresponding to the weight  $\alpha$  is denoted  $\mathbb{k}_\alpha$ .
- (b) The product of  $T$  is implemented by tensoring. That is, the tensor product of  $\mathbb{k}_\alpha$  and  $\mathbb{k}_\lambda$  is the module  $\mathbb{k}_{\alpha+\lambda}$ .

\*Corresponding author

*Email addresses:* barbara.pogorelsky@ufrgs.br (Barbara Pogorelsky), vay@famaf.unc.edu.ar (Cristian Vay)

Also,  $U$  admits a *triangular decomposition*, that means that there are subalgebras  $U^-$  and  $U^+$  such that the multiplication  $U^- \otimes U^0 \otimes U^+ \rightarrow U$  gives a linear isomorphism. Indeed,  $U$  is a  $\mathbb{Z}$ -graded algebra such that the degrees of  $U^0$ ,  $U^-$  and  $U^+$  are zero, negative and positive, respectively. Let  $U^{\geq 0}$  be the subalgebra generated by  $U^0$  and  $U^+$ . Given a weight  $\lambda$ , this can be seen as an  $U^{\geq 0}$ -module by letting  $U^+$  act trivially on it. We denote it again by  $\mathbb{k}_\lambda$ .

Let  $M$  be an  $U$ -module and  $M_\lambda = \{m \in M \mid h \cdot m = \lambda(h)m \text{ for all } h \in U^0\}$  its *weight space* of weight  $\lambda$ . We restrict our attention to the  $U$ -modules which decompose as the direct sum of their weight spaces. For instance,  $U$  regarded as a module with respect to the *adjoint action*. Then

(c)  $U_\alpha \cdot M_\lambda \subseteq M_{\alpha+\lambda}$  for all weights  $\alpha$  and  $\lambda$ .

An  $U$ -module  $M$  is called a *highest-weight module* (of weight  $\lambda$ ) if it is generated by an element  $v \in M_\lambda$  such that  $U^+v = 0$ . Notice that  $M = Uv = U^-v$  by the triangular decomposition of  $U$ . The basic examples of highest-weight modules are the *Verma modules*  $M(\lambda) = U \otimes_{U^{\geq 0}} \mathbb{k}_\lambda$ . These are essential in the study of the representation theory of  $U$  because

(d) Every Verma module has a unique simple quotient and every simple  $U$ -module is a quotient of a unique Verma module.

(e) Every Verma module is a free  $U^-$ -module of rank 1.

Other important features of the Verma modules can be found in the references above.

Let us consider now a quantum group  $\mathfrak{D}$  at a finite non-abelian group  $G$ . The role of  $U^0$  shall be played by the Drinfeld double  $\mathfrak{D}(G)$  of  $\mathbb{k}G$ . This is a semisimple but not commutative Hopf subalgebra of  $\mathfrak{D}$ . We will see that  $\mathfrak{D}$  admits a triangular decomposition

$$\mathfrak{D} = \mathfrak{B}(V) \otimes \mathfrak{D}(G) \otimes \mathfrak{B}(\bar{V})$$

where  $\bar{V}$  denotes the dual object of  $V$  in the category of  $\mathfrak{D}(G)$ -modules and  $\mathfrak{B}(\bar{V})$  is its Nichols algebra. In this setting the bosonization  $\mathfrak{D}^{\geq 0} = \mathfrak{B}(\bar{V}) \# \mathfrak{D}(G)$  shall play the role of  $U^{\geq 0}$ . We will calculate the commutation rules between the generators of  $\mathfrak{D}(G)$ ,  $V$  and  $\bar{V}$ , and deduce that  $\mathfrak{D}$  is a  $\mathbb{Z}$ -graded algebra with homogeneous spaces

$$\mathfrak{D}^n = \bigoplus_{n=j-i} \mathfrak{B}^i(V) \otimes \mathfrak{D}(G) \otimes \mathfrak{B}^j(\bar{V}).$$

The classification of the simple  $\mathfrak{D}(G)$ -modules is well-known, see for instance [1, Subsection 3.1]; unlike (a), there are simple modules of dimension greater than one. The simple  $\mathfrak{D}(G)$ -modules are parametrized by pairs  $(\mathcal{O}, \varrho)$ , where  $\mathcal{O}$  is a conjugacy class in  $G$  and  $\varrho$  is an irreducible representation of the centralizer of a fixed  $g \in \mathcal{O}$ . If  $M(g, \varrho)$  denotes the corresponding simple  $\mathfrak{D}(G)$ -module, cf. (4), then it becomes a  $\mathfrak{D}^{\geq 0}$ -module by letting  $\mathfrak{B}(\bar{V})$  act trivially on it. Therefore we can define the Verma modules for a quantum group at a non-abelian group as the induced modules

$$M(g, \varrho) = \mathfrak{D} \otimes_{\mathfrak{D}^{\geq 0}} M(g, \varrho).$$

Thus  $M(g, \varrho)$  is a free  $\mathfrak{B}(V)$ -module of rank  $\dim M(g, \varrho) = \#\mathcal{O}_g \cdot \dim(U, \varrho)$ , compare with (e).

Our main result asserts that (d) holds true in our context, *i. e.* every Verma module  $M(g, \varrho)$  has a unique simple quotient and every simple  $\mathfrak{D}$ -module is a quotient of a unique Verma module, Theorem 3. Therefore we obtain a bijective correspondence

$$\left\{ \begin{array}{c} \text{Simple } \mathfrak{D}(G)\text{-modules} \\ M(g, \varrho) \end{array} \right\} \begin{array}{c} \longleftrightarrow \\ \longleftrightarrow \end{array} \left\{ \begin{array}{c} \text{Simple } \mathfrak{D}\text{-modules} \\ L(g, \varrho) \end{array} \right\}$$

where  $L(g, \varrho)$  denotes the head of  $M(g, \varrho)$ . Moreover, we prove that the socle  $S(g, \varrho)$  of  $M(g, \varrho)$  is simple what provides another bijective correspondence between the set of simple  $\mathfrak{D}(G)$ -modules and the set of simple  $\mathfrak{D}$ -modules, Theorem 4. We also give a criterion to decide whether or not a Verma module is simple, Corollary 15, and show that the socle and the head are related by

$$(S(g, \varrho))^* \simeq L(\hat{g}^*, \hat{\varrho}^*)$$

where  $M(\hat{\mathfrak{g}}^*, \hat{\mathfrak{g}}^*) = (\mathfrak{B}^{top}(V) \otimes M(g, \varrho))^*$  and  $\mathfrak{B}^{top}(V)$  is the homogeneous component of maximum degree of  $\mathfrak{B}(V)$ , Theorem 5; recall that  $\mathfrak{B}^{top}(V)$  is one-dimensional.

In order to compute explicitly the simple  $\mathfrak{D}$ -modules, we have to study the submodules of the Verma modules. This is done in the abelian case using the properties (b) and (c) among others, which allow to obtain remarkable results under certain general assumptions. Although the  $\mathfrak{D}$ -modules decompose as the direct sum of simple  $\mathfrak{D}(G)$ -modules, our situation is more complex because (b) and (c) do not hold true. Here the tensor product between simple  $\mathfrak{D}(G)$ -modules is not necessarily simple and hence we have to know their fusion rules.

We give a general strategy to compute the highest-weight submodules of any  $\mathfrak{D}$ -module  $M$ . We use that  $\mathfrak{D}$  is a  $\mathbb{Z}$ -graded  $\mathfrak{D}(G)$ -module with respect to the adjoint action, which respects the triangular decomposition, and the fact that the action  $\mathfrak{D} \otimes M \rightarrow M$  is a morphism of  $\mathfrak{D}(G)$ -modules, §3.2. We carry out this strategy to compute the simple modules in a concrete example in Section 4 as we summarize below.

### 1.1. A quantum group at the symmetric group $\mathbb{S}_3$

The first genuine example of a finite-dimensional Nichols algebra over a non-abelian group is the Fomin-Kirillov algebra  $\mathcal{FK}_3$  [6]. It is isomorphic to the Nichols algebra  $\mathfrak{B}(V)$  of the Yetter-Drinfeld module  $V = \mathbb{k}\{x_{(12)}, x_{(23)}, x_{(13)}\}$  over  $\mathbb{k}\mathbb{S}_3$ . The action and coaction on  $V$  are

$$g \cdot x_{(ij)} = \text{sgn}(g) x_{g(ij)g^{-1}} \quad \text{and} \quad (x_{(ij)})_{(-1)} \otimes (x_{(ij)})_{(0)} = (ij) \otimes x_{(ij)}$$

for any transposition  $(ij)$  and  $g \in \mathbb{S}_3$  [13] where  $\text{sgn} : \mathbb{S}_3 \rightarrow \{\pm 1\}$  denotes the sign map.

Let now  $\mathfrak{D}$  be the Drinfeld double of  $\mathfrak{B}(V) \# \mathbb{k}\mathbb{S}_3$ . As an algebra,  $\mathfrak{D}$  is generated by

- the generators of  $\mathfrak{B}(V)$ :  $x_{(12)}, x_{(23)}, x_{(13)}$ ;
- the generators of  $\mathfrak{D}(\mathbb{S}_3)$ :  $g, \delta_g$  for all  $g \in \mathbb{S}_3$ ;
- the generators of  $\mathfrak{B}(\bar{V})$ :  $y_{(12)}, y_{(23)}, y_{(13)}$ ;

we shall see that  $V \simeq \bar{V}$  as  $\mathfrak{D}(\mathbb{S}_3)$ -modules. These elements are subjected to the next relations:

$$x_{(ij)}^2 = x_{(ij)}x_{(ik)} + x_{(jk)}x_{(ij)} + x_{(ik)}x_{(jk)} = x_{(ik)}x_{(ij)} + x_{(ij)}x_{(jk)} + x_{(jk)}x_{(ik)} = 0, \quad (\text{given by } \mathfrak{B}(V))$$

$$y_{(ij)}^2 = y_{(ij)}y_{(ik)} + y_{(jk)}y_{(ij)} + y_{(ik)}y_{(jk)} = y_{(ik)}y_{(ij)} + y_{(ij)}y_{(jk)} + y_{(jk)}y_{(ik)} = 0, \quad (\text{given by } \mathfrak{B}(\bar{V}))$$

$$\delta_h g = g \delta_{g^{-1}hg}, \quad (\text{given by } \mathfrak{D}(\mathbb{S}_3))$$

$$gx_{(ij)} = \text{sgn}(g) x_{g(ij)g^{-1}}, \quad \delta_g y_{(ij)} = y_{(ij)} \delta_{(ij)g}, \quad (\text{given by the bosonizations})$$

$$\delta_h x_{(ij)} = x_{(ij)} \delta_{(ij)h}, \quad y_{(ij)} g = \text{sgn}(g) g y_{g^{-1}(ij)g},$$

$$y_{(ij)} x_{(ij)} + x_{(ij)} y_{(ij)} = 1 + (ij)(\delta_{(ij)} - \delta_e), \quad (\text{given by the definition of } \mathfrak{D})$$

$$y_{(ik)} x_{(ij)} + x_{(ij)} y_{(jk)} = (ij)(\delta_{(ik)} - \delta_{(ik)(ij)}),$$

for all transpositions  $(ij)$ ,  $(ik)$  and  $g, h \in \mathbb{S}_3$ .

On the other hand, the simple  $\mathfrak{D}$ -modules are parametrized by the simple  $\mathfrak{D}(\mathbb{S}_3)$ -modules according to our main result. Let  $\sigma = (12)$  and  $\tau = (123)$  be permutations in  $\mathbb{S}_3$ . Then  $\mathcal{O}_e, \mathcal{O}_\sigma$  and  $\mathcal{O}_\tau$  are the conjugacy classes of  $\mathbb{S}_3$  and  $\mathfrak{D}(\mathbb{S}_3)$  has eight non-isomorphic simple modules. Namely,

$$M(e, +), \quad M(e, -), \quad M(e, \rho), \quad M(\sigma, +), \quad M(\sigma, -), \quad M(\tau, 0), \quad M(\tau, 1) \quad \text{and} \quad M(\tau, 2).$$

We recall the structures of them and their fusion rules in §2.5.1–§2.5.4.

We compute the lattice of submodules of the corresponding Verma modules  $M(g, \varrho)$  and classify the simple  $\mathfrak{D}$ -modules. In particular, we prove that

- $L(e, +) \simeq M(e, +)$  as  $\mathfrak{D}(\mathbb{S}_3)$ -modules and  $\dim L(e, +) = 1$ , Corollary 27.

- $L(e, \rho) \simeq M(e, \rho) \oplus M(\sigma, +) \oplus M(\tau, 0)$  as  $\mathcal{D}(\mathbb{S}_3)$ -modules and  $\dim L(e, \rho) = 7$ , Corollary 22.
- $L(\tau, 0) \simeq M(\tau, 0) \oplus M(\sigma, +) \oplus M(e, \rho)$  as  $\mathcal{D}(\mathbb{S}_3)$ -modules and  $\dim L(\tau, 0) = 7$ , Corollary 24.
- $L(\sigma, -) \simeq M(\sigma, -) \oplus M(\tau, 1) \oplus M(\tau, 2) \oplus M(\sigma, -)$  as  $\mathcal{D}(\mathbb{S}_3)$ -modules and  $\dim L(\tau, 0) = 10$ , Theorem 7.
- The Verma modules  $M(e, -)$ ,  $M(\tau, 1)$ ,  $M(\tau, 2)$  and  $M(\sigma, +)$  are simple, Theorem 6. Their dimensions are 12, 24, 24 and 36, respectively. As  $\mathcal{D}(\mathbb{S}_3)$ -modules they are the tensor product of  $\mathfrak{B}(V)$  with the associated simple  $\mathcal{D}(\mathbb{S}_3)$ -module.

We finish by pointing out other facts about these modules.

- The head  $L(g, \varrho)$  and the socle  $S(g, \varrho)$  are isomorphic, Theorem 6, Lemma 20 and Corollaries 27, except to

$$L(\tau, 0) \simeq S(e, \rho) \quad \text{and} \quad L(e, \rho) \simeq S(\tau, 0), \quad \text{Corollary 25.}$$

- The simple  $\mathcal{D}$ -modules are self-dual except to

$$(L(\tau, 0))^* \simeq L(e, \rho) \quad \text{and} \quad (L(e, \rho))^* \simeq L(\tau, 0) \quad \text{by Theorem 5.}$$

- $M(\sigma, -)$  has submodules which are not homogeneous, Lemma 26, and its maximal submodule is not generated by highest-weight submodules, Theorem 7.

## Acknowledgments

The first author was financially supported by Capes - Brazil. The second author was partially supported by ANPCyT-Foncyt, CONICET and Secyt (UNC).

This work was carried out in part during the visit of the first author to the University of Córdoba (Argentina). She would like to thank the Faculty of Mathematics, Astronomy and Physics for its warm hospitality and support. Both authors are grateful to Nicolás Andruskiewitsch for drawing their attention to this problem and also for so many suggestions. The second author thanks Vyacheslav Futorny for very useful conversations during his visit to the University of São Paulo under the framework of the MATH-AmSud program. We also thank the referee for the careful reading of our article and for providing constructive comments to improve the exposition of this paper.

## 2. Preliminaries

Through this work  $\mathbb{k}$  denotes an algebraically closed field of characteristic zero. The dual of a vector space  $V$  will be denoted by  $V^*$ . If  $v \in V$  and  $f \in V^*$ , then  $\langle f, v \rangle$  denotes the evaluation of  $f$  in  $v$ . Let  $S$  be a set. We write  $\mathbb{k}S$  for the free vector space on  $S$ . Let  $A$  be an algebra. By an  $A$ -module, we mean a left  $A$ -module. If  $S$  is a subset of an  $A$ -module  $M$  and  $B \subseteq A$ , then  $BS$  denotes the set of all  $bs$  with  $b \in B$  and  $s \in S$ .

Let  $H$  be a finite-dimensional Hopf algebra. We denote by  $\Delta$ ,  $S$  and  $\varepsilon$  the comultiplication, the antipode and the counit of  $H$ . We will use the Sweedler notation  $\Delta(h) = h_{(1)} \otimes h_{(2)}$  for the comultiplication of any  $h \in H$ , and for the coaction  $\delta(m) = m_{(-1)} \otimes m_{(0)}$  of an element  $m$  belonging to an  $H$ -comodule.

Recall that  ${}^H_H\mathcal{YD}$  denotes the category of Yetter-Drinfeld modules over  $H$ , whose objects are the  $H$ -modules and  $H$ -comodules  $M$  such that for every  $h \in H$  and  $m \in M$  it holds that

$$(hm)_{(-1)} \otimes (hm)_{(0)} = h_{(1)} m_{(-1)} S(h_{(3)}) \otimes h_{(2)} m_{(0)}.$$

2.1.

We consider the Drinfeld double  $\mathfrak{D}(H)$  of  $H$  according to [12, Theorem 7.1.1]. Namely,  $\mathfrak{D}(H)$  is  $H \otimes H^*$  as coalgebra. Meanwhile, the multiplication and the antipode are given by

$$\begin{aligned} (h \otimes f)(h' \otimes f') &= \langle f_{(1)}, h'_{(1)} \rangle \langle f_{(3)}, \mathcal{S}_H(h'_{(3)}) \rangle (hh'_{(2)} \otimes f' f_{(2)}), \\ \mathcal{S}(h \otimes f) &= (1 \otimes \mathcal{S}_H^{-1}(f))(\mathcal{S}_H(h) \otimes \varepsilon), \quad \text{for every } h, h' \in H \text{ and } f, f' \in H^*. \end{aligned} \quad (1)$$

In consequence, we have that  $H$  and  $H^{*op}$  are Hopf subalgebras of  $\mathfrak{D}(H)$ .

Recall that the category  ${}^H_H\mathcal{YD}$  is braided equivalent to the category  ${}_{\mathfrak{D}(H)}\mathcal{M}$  of  $\mathfrak{D}(H)$ -modules. Namely, if  $M \in {}^H_H\mathcal{YD}$ , then  $M$  is a  $\mathfrak{D}(H)$ -module by setting

$$(hf) \cdot m = \langle f, m_{(-1)} \rangle hm_{(0)} \quad (2)$$

for every  $h \in H$ ,  $f \in H^*$  and  $m \in M$ .

2.2.

The Nichols algebra of  $V \in {}^H_H\mathcal{YD}$  is constructed as follows, see for instance [2, §2.1]. First, we consider the tensor algebra  $T(V)$  as a graded braided Hopf algebra in  ${}^H_H\mathcal{YD}$  by defining

$$\Delta(v) = v \otimes 1 + 1 \otimes v, \quad \mathcal{S}(v) = -v \quad \text{and} \quad \varepsilon(v) = 0$$

for all  $v \in V$ . Let  $\mathcal{J}(V)$  be the maximal ideal and coideal of  $T(V)$  generated by homogeneous elements of degree  $\geq 2$ . Then the Nichols algebra of  $V$  is the quotient

$$\mathfrak{B}(V) = T(V)/\mathcal{J}(V)$$

which is a graded braided Hopf algebra in  ${}^H_H\mathcal{YD}$ . Its homogeneous component of degree  $n \in \mathbb{N}$  will be denoted  $\mathfrak{B}^n(V)$ . Note that  $\mathfrak{B}^1(V) = V$  and  $\mathfrak{B}^0(V) = \mathbb{k}$ . Moreover, if  $\mathfrak{B}(V)$  is finite-dimensional, then its homogeneous component of maximum degree is one-dimensional and it is the space of left and right integrals, see for instance [1, §2.3 and §3.2].

2.3.

The bosonization  $\mathfrak{B}(V)\#H$  [16, 11] is the Hopf algebra structure defined on  $\mathfrak{B}(V) \otimes H$  in such a way that  $H$  is a Hopf subalgebra,  $\mathfrak{B}(V)$  is a subalgebra,

$$hv = (h_{(1)} \cdot v)\#h_{(2)} \quad \text{and} \quad \Delta(v) = v \otimes 1 + v_{(-1)} \otimes v_{(0)} \quad \text{for all } h \in H \text{ and } v \in V. \quad (3)$$

It is a graded Hopf algebra where its homogeneous component of degree  $n \in \mathbb{N}$  is  $\mathfrak{B}^n(V)\#H$ .

2.4.

Let  $G$  be a finite group. The unity element of  $G$  is denoted by  $e$ . We set  $\mathbb{k}^G = (\mathbb{k}G)^*$ , the dual Hopf algebra of the group algebra  $\mathbb{k}G$ . Let  $\{\delta_g\}_{g \in G}$  be the dual basis of the canonical basis  $\{g\}_{g \in G}$  of  $\mathbb{k}G$ . The comultiplication of an element  $\delta_g$  is

$$\Delta(\delta_g) = \sum_{t \in G} \delta_t \otimes \delta_{t^{-1}g}.$$

Let  $M$  be a  $\mathbb{k}^G$ -module and  $g \in G$ . Then  $M$  is  $G$ -graded with homogeneous component of degree  $g$ :

$$M[g] = \delta_g M = \{m \in M \mid f \cdot m = f(g)m \quad \forall f \in \mathbb{k}^G\}.$$

If  $S \subseteq M$ , we set  $S[g] = S \cap M[g]$ . We denote by  $\text{Supp } M$  the subset of  $G$  formed by those elements whose homogeneous component in  $M$  is non-zero. The one-dimensional  $\mathbb{k}^G$ -module of degree  $g$  will be denoted  $\mathbb{k}_g$ . If  $\mathbb{k}^G$  is a subalgebra of  $A$ , then we will consider  $A$  as a  $\mathbb{k}^G$ -algebra with the adjoint action, that is  $fa = \text{ad}(f)a = f_{(1)}a\mathcal{S}(f_{(2)})$  for any  $a \in A$  and  $f \in \mathbb{k}^G$ .

### 2.5. The Drinfeld double of a group algebra

We denote by  $\mathcal{D}(G)$  the Drinfeld Double of  $\mathbb{k}G$ . Since  $\mathbb{k}^G$  is a commutative algebra,  $\mathbb{k}^G$  and  $\mathbb{k}G$  are Hopf subalgebras of  $\mathcal{D}(G)$ . Then the algebra structure of  $\mathcal{D}(G)$  is completely determined by

$$\delta_h g = g \delta_{g^{-1}hg} \quad \forall g, h \in G, \quad \text{cf. (1).}$$

We will define Verma modules in §3.1 by inducing from the simple  $\mathcal{D}(G)$ -modules. These are well-known because they are equivalent to the simple objects in  ${}_{\mathbb{k}^G}^{\mathbb{k}G}\mathcal{YD}$  and a description of these last can be found for instance in [1, Subsection 3.1]. We recall this description but in the context of modules over  $\mathcal{D}(G)$ .

Let  $\mathcal{O}_g$  be the conjugacy class of  $g \in G$ ,  $\mathcal{C}_g$  the centralizer of  $g$  and  $(U, \varrho)$  an irreducible representation of  $\mathcal{C}_g$ . The  $\mathbb{k}G$ -module induced by  $(U, \varrho)$ ,

$$M(g, \varrho) = \text{Ind}_{\mathcal{C}_g}^G U = \mathbb{k}G \otimes_{\mathbb{k}\mathcal{C}_g} U, \quad (4)$$

is also a  $\mathbb{k}^G$ -module if we define the action by

$$f \cdot (x \otimes_{\mathbb{k}\mathcal{C}_g} u) = \langle f, xgx^{-1} \rangle x \otimes_{\mathbb{k}\mathcal{C}_g} u, \quad \text{for all } f \in \mathbb{k}^G, x \in G \text{ and } u \in U.$$

Then  $x \otimes_{\mathbb{k}\mathcal{C}_g} u$  is of  $G$ -degree  $xgx^{-1}$  and  $\text{Supp } M(g, \varrho) = \mathcal{O}_g$ . Note that  $\dim M(g, \varrho) = \#\mathcal{O}_g \cdot \dim U$ .

Therefore  $M(g, \varrho)$  is a  $\mathcal{D}(G)$ -module. Moreover,  $M(g, \varrho)$  is simple and every simple  $\mathcal{D}(G)$ -module is of this form by [1, Proposition 3.1.2].

**Definition 1.** A  $\mathcal{D}(G)$ -module is of weight  $(g, \varrho)$  if it is isomorphic to  $M(g, \varrho)$ .

Let  $\mathbb{S}_3$  be the group of bijections on  $\{1, 2, 3\}$ . We set  $\sigma = (12)$  and  $\tau = (123)$ . These two cycles generate  $\mathbb{S}_3$  and satisfy the relations  $\sigma^2 = e = \tau^3$  and  $\sigma\tau\sigma = \tau^{-1}$ . The conjugacy classes of  $\mathbb{S}_3$  are

$$\mathcal{O}_e = \{e\}, \quad \mathcal{O}_\sigma = \{(12), (13), (23)\} \quad \text{and} \quad \mathcal{O}_\tau = \{(123), (132)\}.$$

Next, we describe the simple  $\mathcal{D}(\mathbb{S}_3)$ -modules which we will consider in §4.

#### 2.5.1. Simple modules attached to $\sigma$

The centralizer  $\mathcal{C}_\sigma$  is just the cyclic subgroup generated by  $\sigma$ . Then  $\mathcal{C}_\sigma$  has only two irreducible representations: the trivial one and the induced by the sign map  $\text{sgn} : \mathbb{S}_3 \rightarrow \{\pm 1\}$ . Therefore the simple  $\mathcal{D}(\mathbb{S}_3)$ -modules attached to  $\sigma$  are

$$M(\sigma, +) := M(\sigma, \varepsilon) \quad \text{and} \quad M(\sigma, -) := M(\sigma, \text{sgn}).$$

Let us consider the set of symbols  $\{|\mathbf{12}\rangle_\pm, |\mathbf{23}\rangle_\pm, |\mathbf{13}\rangle_\pm\}$  as a basis of  $M(\sigma, \pm)$ . Sometimes we write  $|\sigma\tau^t\rangle_\pm$  instead of  $|\mathbf{ij}\rangle_\pm$ , if  $\sigma\tau^t = (ij)$ , and omit the subscript if there is no place for confusion. Hence the action of  $\mathcal{D}(\mathbb{S}_3)$  on  $M(\sigma, \pm)$  is defined in such a way that  $|\sigma\tau^t\rangle_\pm$  has  $\mathbb{S}_3$ -degree  $\sigma\tau^t$  and

$$\sigma \cdot |\sigma\tau^t\rangle_\pm = \pm |\sigma\tau^{-t}\rangle_\pm \quad \text{and} \quad \tau \cdot |\sigma\tau^t\rangle_\pm = |\sigma\tau^{t+1}\rangle_\pm.$$

#### 2.5.2. Simple modules attached to $\tau$

From now on, we fix a root of the unit  $\zeta$  of order 3. The centralizer  $\mathcal{C}_\tau$  is the cyclic subgroup generated by  $\tau$ . Then  $\mathcal{C}_\tau$  has (up to isomorphisms) three irreducible representations. These are given by the group maps  $\rho_\ell : \mathcal{C}_\tau = \langle \tau \rangle \mapsto \mathbb{k}^*$ ,  $\tau \mapsto \zeta^\ell$  for  $\ell = 0, 1, 2$ . Therefore the simple  $\mathcal{D}(\mathbb{S}_3)$ -modules attached to  $\tau$  are

$$M(\tau, \ell) := M(\tau, \rho_\ell) \quad \text{for } \ell = 0, 1, 2.$$

Let us consider the set of symbols  $\{|\mathbf{123}\rangle_\ell, |\mathbf{132}\rangle_\ell\}$  as a basis of  $M(\tau, \ell)$ . Sometimes we write  $|\tau^t\rangle_\ell$  instead of  $|\mathbf{ijk}\rangle_\ell$ , if  $\tau^t = (ijk)$ , and omit the subscript if there is no place for confusion. Hence the action of  $\mathcal{D}(\mathbb{S}_3)$  on  $M(\tau, \ell)$  is defined in such a way that  $|\tau^{\pm 1}\rangle_\ell$  is of  $\mathbb{S}_3$ -degree  $\tau^{\pm 1}$  and

$$\sigma\tau^t \cdot |\tau^{\pm 1}\rangle_\ell = \zeta^{\pm t\ell} |\tau^{\mp 1}\rangle_\ell \quad \text{and} \quad \tau^t \cdot |\tau^{\pm 1}\rangle_\ell = \zeta^{\pm t\ell} |\tau^{\pm 1}\rangle_\ell$$

for  $t = 0, 1, 2$ . It is not difficult to check that

$$M(\tau, 1) \longrightarrow M(\tau, 2), \quad |\tau^{\pm 1}\rangle_1 \longmapsto |\tau^{\mp 1}\rangle_2 \quad (5)$$

is an isomorphism of  $\mathbb{k}\mathbb{S}_3$ -modules.

### 2.5.3. Simple modules attached to $e$

Let  $\rho : \mathbb{S}_3 \rightarrow \mathrm{GL}_2(\mathbb{k})$  be the map defining the two-dimensional Specht  $\mathbb{S}_3$ -module. Then  $(\mathbb{k}, \varepsilon)$ ,  $(\mathbb{k}, \mathrm{sgn})$  and  $(\mathbb{k}^2, \rho)$  is a complete list of non-isomorphic irreducible  $\mathbb{S}_3$ -modules. Therefore the simple  $\mathcal{D}(\mathbb{S}_3)$ -modules attached to  $e$  are

$$M(e, +) = M(e, \varepsilon), \quad M(e, -) = M(e, \mathrm{sgn}) \quad \text{and} \quad M(e, \rho).$$

These are concentrated in  $\mathbb{S}_3$ -degree  $e$ . The modules  $M(e, \pm)$  are one-dimensional, we denote by  $|e\rangle_{\pm}$  its generators and omit the subscript if there is no place for confusion.

We can describe the  $\mathbb{k}\mathbb{S}_3$ -action on  $M(e, \rho)$  using the canonical representation of  $\mathbb{k}\mathbb{S}_3$  on the vector space spanned by  $\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ . In fact,  $\mathbb{k}\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$  decomposes into the direct sum  $(\mathbb{k}, \varepsilon) \oplus (\mathbb{k}^2, \rho)$  where the submodules of weight  $(\mathbb{k}, \varepsilon)$  and  $(\mathbb{k}^2, \rho)$  are spanned by  $\{\mathbf{1} + \mathbf{2} + \mathbf{3}\}$  and  $\{(\mathbf{1} - \mathbf{2}), (\mathbf{2} - \mathbf{3})\}$ , respectively.

Another special basis of  $M(e, \rho)$  is the set of symbol  $\{|\tau\rangle_{\rho}, |\tau^{-1}\rangle_{\rho}\}$  where

$$|\tau^{\pm 1}\rangle_{\rho} = \zeta^{\mp 1}\mathbf{1} + \zeta^{\pm 1}\mathbf{2} + \mathbf{3}$$

This basis is special because it gives the following isomorphisms of  $\mathbb{k}\mathbb{S}_3$ -modules

$$M(e, \rho) \longrightarrow M(\tau, 1), \quad |\tau^{\pm 1}\rangle_{\rho} \longmapsto |\tau^{\pm 1}\rangle_1. \quad (6)$$

We omit the subscript in  $|\tau^{\pm 1}\rangle_{\rho}$  if there is no place for confusion.

### 2.5.4. Fusion rules

Let  $W$  and  $N$  be simple  $\mathcal{D}(\mathbb{S}_3)$ -modules. We want to decompose the tensor products  $W \otimes N$  into a direct sum of simple  $\mathcal{D}(\mathbb{S}_3)$ -modules. First, we have a decomposition into the direct sum of two submodules which are not necessarily simple:

$$W \otimes N = \left( \bigoplus_{g \in \mathbb{S}_3} W[g] \otimes N[g] \right) \oplus \left( \bigoplus_{g, h \in \mathbb{S}_3, g \neq h} W[g] \otimes N[h] \right). \quad (7)$$

Note that the first submodule is zero if  $\mathrm{Supp} W \neq \mathrm{Supp} N$ .

This decomposition is useful for us because each submodule has a basis which is a transitive  $\mathbb{S}_3$ -set in the sense of the next lemma. Let  $\mathbf{B}_W$  and  $\mathbf{B}_N$  be the bases of  $W$  and  $N$  given in §2.5.1, 2.5.2 and 2.5.3. Then the sets

$$\mathbf{B}_1 = \bigcup_{g \in \mathbb{S}_3} \mathbf{B}_W[g] \otimes \mathbf{B}_N[g] \quad \text{and} \quad \mathbf{B}_2 = \bigcup_{g, h \in \mathbb{S}_3, g \neq h} \mathbf{B}_W[g] \otimes \mathbf{B}_N[h] \quad (8)$$

are bases of the first submodule and the second one in (7), respectively.

**Lemma 2.** *If  $\alpha, \beta \in \mathbf{B}_\ell$ ,  $\ell = 1, 2$ , then there is  $\pi \in \mathbb{S}_3$  such that  $\pi \cdot \alpha = \lambda \beta$  for some non-zero scalar  $\lambda$ .*

*Proof.* The sets  $\mathbf{B}_W[g]$  and  $\mathbf{B}_N[g]$ ,  $g \in \mathbb{S}_3$ , are either empty or have only one element  $|g\rangle$  except to  $M(e, \rho)$ , but in this case the basis is  $\{|\tau\rangle_{\rho}, |\tau^{-1}\rangle_{\rho}\}$ . In these bases, we see from the definition that  $\pi|g\rangle = \lambda|\pi g \pi^{-1}\rangle$  for some non-zero scalar  $\lambda$ . We conclude by remarking that  $\mathbb{S}_3$  acts transitively by conjugation on the sets  $\{g \times g \mid g \in \mathrm{Supp} W \cap \mathrm{Supp} N\}$  and  $\{g \times h \mid g \in \mathrm{Supp} W, h \in \mathrm{Supp} N, g \neq h\}$ .  $\square$

As a consequence of the above lemma we have the next remark which will be useful in §4 where the action of  $V$ , or  $\bar{V}$ , on  $N$  will play the role of  $\mu$ .

**Remark 3.** *Let  $\mu : W \otimes N \rightarrow N'$  be a map of  $\mathcal{D}(\mathbb{S}_3)$ -modules. Assume there is  $\alpha \in \mathbf{B}_1$ , respectively  $\alpha \in \mathbf{B}_2$ , such that  $\mu(\alpha) = 0$ . Hence  $\mu$  restricted to  $\bigoplus_{g \in \mathbb{S}_3} W[g] \otimes N[g]$ , respectively  $\bigoplus_{g, h \in \mathbb{S}_3, g \neq h} W[g] \otimes N[h]$ , is zero since  $\mathbb{S}_3$  acts transitively on the basis  $\mathbf{B}_1$ , respectively  $\mathbf{B}_2$ .*

We next list the precise fusion rules only for those tensor products which will appear in Section 4. We give the assignments (or describe the submodules) which realize the listed isomorphisms but we leave to the reader the verification that these really are maps of  $\mathfrak{D}(\mathbb{S}_3)$ -modules (or  $\mathfrak{D}(\mathbb{S}_3)$ -submodules).

- $M(e, -) \otimes M(e, -) \simeq M(e, +)$  and  $M(\sigma, \pm) \otimes M(e, -) \simeq M(\sigma, \mp)$ .

The isomorphisms are given by  $m \otimes |e\rangle \mapsto m$ .

- $M(e, \rho) \otimes M(e, -) \simeq M(e, \rho)$  and
- $M(\tau, \ell) \otimes M(e, -) \simeq M(\tau, \ell)$  for all  $\ell = 0, 1, 2$ .

The assignments  $|\tau^{\pm 1}\rangle \otimes |e\rangle \mapsto \pm |\tau^{\pm 1}\rangle$  give these isomorphisms.

In the sequel, by abuse of notation,  $i\ell$  and  $\ell + i$  denote the multiplication and sum module 3.

- $M(\tau, \ell) \otimes M(\tau, \ell) \simeq M(e, +) \oplus M(e, -) \oplus M(\tau, 2\ell)$  for all  $\ell = 0, 1, 2$ .

We obtain this isomorphism keeping in mind that

$$M(e, \pm) \simeq \mathbb{k} \left\{ |\tau\rangle_\ell \otimes |\tau^{-1}\rangle_\ell \pm |\tau^{-1}\rangle_\ell \otimes |\tau\rangle_\ell \right\} \text{ and } M(\tau, 2\ell) \simeq \mathbb{k} \left\{ |\tau\rangle_\ell \otimes |\tau\rangle_\ell, |\tau^{-1}\rangle_\ell \otimes |\tau^{-1}\rangle_\ell \right\}.$$

- $M(\tau, \ell) \otimes M(e, \rho) \simeq M(\tau, \ell + 1) \oplus M(\tau, \ell + 2)$  for all  $\ell = 0, 1, 2$ .

The isomorphism follows by considering the submodules

$$\left\{ |\tau^{\pm 1}\rangle_\ell \otimes |\tau^{\pm 1}\rangle_\rho \right\} \text{ and } \left\{ |\tau^{\pm 1}\rangle_\ell \otimes |\tau^{\mp 1}\rangle_\rho \right\}. \quad (9)$$

- $M(\tau, i) \otimes M(\tau, j) \simeq M(e, \rho) \oplus M(\tau, k)$  with  $\{i, j, k\} = \{0, 1, 2\}$ .

Here we use that

$$\begin{aligned} M(e, \rho) &\simeq \mathbb{k} \left\{ |\tau\rangle_i \otimes |\tau^{-1}\rangle_j, |\tau^{-1}\rangle_i \otimes |\tau\rangle_j \right\} \text{ and} \\ M(\tau, k) &\simeq \mathbb{k} \left\{ |\tau\rangle_i \otimes |\tau\rangle_j, |\tau^{-1}\rangle_i \otimes |\tau^{-1}\rangle_j \right\}. \end{aligned} \quad (10)$$

- $M(\tau, \ell) \otimes M(\sigma, -) \simeq M(\sigma, +) \oplus M(\sigma, -) \simeq M(\sigma, -) \otimes M(\tau, \ell)$  for all  $\ell = 0, 1, 2$ .

In the first isomorphism  $|\sigma\tau^i\rangle_\pm \in M(\sigma, \pm)$  identifies with the element

$$\zeta^{i\ell} |\tau\rangle_\ell \otimes |\sigma\tau^{i+1}\rangle_- \mp \zeta^{-i\ell} |\tau^{-1}\rangle_\ell \otimes |\sigma\tau^{i+2}\rangle_- \text{ for } i = 0, 1, 2, \quad (11)$$

meanwhile in the second isomorphism,  $|\sigma\tau^i\rangle_\pm$ , for  $i = 0, 1, 2$ , identifies with

$$\zeta^{i\ell} |\sigma\tau^{i+2}\rangle_- \otimes |\tau\rangle_\ell \mp \zeta^{-i\ell} |\sigma\tau^{i+1}\rangle_- \otimes |\tau^{-1}\rangle_\ell = \zeta^{i\ell} (1 \pm \sigma\tau^i) |\sigma\tau^{i+2}\rangle_- \otimes |\tau\rangle_\ell \quad (12)$$

- $M(\tau, \ell) \otimes M(\sigma, +) \simeq M(\sigma, +) \oplus M(\sigma, -)$  for all  $\ell = 0, 1, 2$ .

Here we take the assignments  $\zeta^{i\ell} |\tau\rangle_\ell \otimes |\sigma\tau^{i+1}\rangle_+ \pm \zeta^{i\ell} |\tau^{-1}\rangle_\ell \otimes |\sigma\tau^{i+2}\rangle_+ \mapsto |\sigma\tau^i\rangle_\pm$  for  $i = 0, 1, 2$ .

- $M(\sigma, -) \otimes M(\sigma, \pm) \simeq M(e, \mp) \oplus M(e, \rho) \oplus \bigoplus_{\ell=0,1,2} M(\tau, \ell)$ .

For this isomorphism we use that  $\mathbb{k} \left\{ \sum_{i=0}^2 |\sigma\tau^i\rangle_- \otimes |\sigma\tau^i\rangle_\pm \right\}$  is a one-dimensional submodule; the maps

$$\begin{aligned} M(e, \rho) &\longrightarrow M(\sigma, -) \otimes M(\sigma, -), \quad |\tau^j\rangle_\rho \mapsto \sum_{i=0}^2 \zeta^{-ij} |\sigma\tau^i\rangle_- \otimes |\sigma\tau^i\rangle_- \text{ and} \\ M(e, \rho) &\longrightarrow M(\sigma, -) \otimes M(\sigma, +), \quad |\tau^j\rangle_\rho \mapsto \sum_{i=0}^2 j \zeta^{-ij} |\sigma\tau^i\rangle_- \otimes |\sigma\tau^i\rangle_+, \end{aligned} \quad (13)$$



with  $j = \pm 1$ , define inclusions of  $\mathfrak{D}(\mathbb{S}_3)$ -modules and  $M(\tau, \ell)$  is included as  $\mathfrak{D}(\mathbb{S}_3)$ -module by identifying the element  $|\tau^i\rangle_\ell$  of  $M(\tau, \ell)$  with the element

$$\begin{aligned} (\zeta^\ell + \zeta^{-\ell}\tau^{-i} + \tau^i)|\sigma\rangle_- \otimes |\sigma\tau^i\rangle_- &\in M(\sigma, -) \otimes M(\sigma, -), \quad \text{respectively} \\ i(\zeta^\ell + \zeta^{-\ell}\tau^{-i} + \tau^i)|\sigma\rangle_- \otimes |\sigma\tau^i\rangle_+ &\in M(\sigma, -) \otimes M(\sigma, +), \end{aligned} \quad (14)$$

for  $i = \pm 1$  and  $\ell = 0, 1, 2$ .

- $M(\sigma, -) \otimes M(e, \rho) \simeq M(\sigma, +) \oplus M(\sigma, -)$ .

Here we have to identify  $|\sigma\tau^i\rangle_\pm \in M(\sigma, \pm)$ , for  $i = 1, 2, 3$ , with the element

$$\zeta^i|\sigma\tau^i\rangle_- \otimes |\tau\rangle_\rho \mp \zeta^{-i}|\sigma\tau^i\rangle_- \otimes |\tau^{-1}\rangle_\rho = \zeta^i(1 \pm \sigma\tau^i)|\sigma\tau^i\rangle_- \otimes |\tau\rangle_\rho. \quad (15)$$

### 3. A quantum group at a non-abelian group

Through this section, we fix a finite non-abelian group  $G$  and a Yetter-Drinfeld module  $V \in {}_{\mathbb{k}G}^{\mathbb{k}G}\mathcal{YD}$  such that its Nichols algebra  $\mathfrak{B}(V)$  is finite-dimensional. We denote by  $\mathfrak{D}$  the Drinfeld double of the bosonization  $\mathfrak{B}(V)\#\mathbb{k}G$ . For shortness we say that  $\mathfrak{D}$  is a *quantum group at a non-abelian group*.

In the first part of the section we describe the algebra structure of  $\mathfrak{D}$ . Then we introduce and study the Verma modules for  $\mathfrak{D}$ .

**Definition 4.** We set  $\bar{V}$  to be  $V^*$  endowed with the Yetter-Drinfeld module structure over  $\mathbb{k}G$  defined by the following properties:

$$\langle f \cdot y, x \rangle = \langle f, \mathfrak{S}(x_{(-1)}) \rangle \langle y, x_{(0)} \rangle \quad \text{and} \quad \langle y, g \cdot x \rangle = \langle y_{(-1)}, g \rangle \langle y_{(0)}, x \rangle \quad (16)$$

for every  $y \in \bar{V}$ ,  $x \in V$ ,  $g \in G$  and  $f \in \mathbb{k}G$ .

It is a straightforward computation to check that these structures satisfy the compatibility for Yetter-Drinfeld modules. Also, this is a consequence of the next lemma. Recall the Hopf algebra structure of a bosonization in §2.3.

**Lemma 5.** The algebra map  $\varphi : \mathfrak{B}(\bar{V})\#\mathbb{k}G \longrightarrow (\mathfrak{B}(V)\#\mathbb{k}G)^{*op}$  defined by

$$\begin{aligned} \langle \varphi(f), g \rangle &= \langle f, g \rangle \quad \text{and} \quad \langle \varphi(f), \mathfrak{B}^n(V)\#\mathbb{k}G \rangle = 0, \\ \langle \varphi(y), x\#g \rangle &= \langle y, x \rangle \quad \text{and} \quad \langle \varphi(y), \mathfrak{B}^m(V)\#\mathbb{k}G \rangle = 0 \end{aligned}$$

for all  $g \in G$ ,  $f \in \mathbb{k}G$ ,  $x \in V$ ,  $y \in \bar{V}$ ,  $n > 0$  and  $m \neq 1$ , is an isomorphism of graded Hopf algebras.

In particular, the Hilbert series of the Nichols algebras  $\mathfrak{B}(V)$  and  $\mathfrak{B}(\bar{V})$  are equals.

*Proof.* We can deduce that  $(\mathfrak{B}(V)\#\mathbb{k}G)^{*op} \simeq R\#\mathbb{k}G$  where  $R$  is the Nichols algebra of its homogeneous space of degree 1 following for instance [3, Section 2]. Also, we see from the definition in the statement that  $\varphi : \bar{V}\#\mathbb{k}G \longrightarrow (\mathfrak{B}(V)\#\mathbb{k}G)^{*op}$ , with  $\varphi(y\#f) = \varphi(f)\varphi(y)$  for all  $y \in \bar{V}$  and  $f \in \mathbb{k}G$ , is a linear map which is bijective in degree 0 and 1. Therefore the lemma follows if we show that

$$\Delta\varphi(\delta_g) = \sum_{t \in G} \varphi(\delta_t) \otimes \varphi(\delta_{t^{-1}g}), \quad \varphi(\delta_g\delta_h) = \varphi(\delta_h)\varphi(\delta_g), \quad (17)$$

$$\Delta\varphi(y) = \varphi(y) \otimes 1 + \varphi(y_{(-1)}) \otimes \varphi(y_{(0)}) \quad \text{and} \quad \varphi(y)\varphi(\delta_g) = \sum_{t \in G} \varphi(\delta_{t^{-1}g})\varphi(\delta_t \cdot y) \quad (18)$$

for all  $g, h \in G$  and  $y \in \bar{V}$ . In fact, (17) ensures that  $\varphi_{\mathbb{k}G}$  is a Hopf algebra map. By (18), we determine that  $\bar{V}$  is the space of coinvariants in degree 1 of  $(\mathfrak{B}(V)\#\mathbb{k}G)^{*op}$  with respect to the projection over  $\mathbb{k}G$  and the corresponding Yetter-Drinfeld structure is given by Definition 4. Hence  $R$  is the Nichols algebra of  $\bar{V}$ .

We check the first equality of (18), the remainder ones can be checked in a similar way. As  $\mathfrak{B}(V)\#kG$  is a graded Hopf algebra it is enough to see that

$$\begin{aligned}\langle \Delta\varphi(y), a\otimes(x\#b) \rangle &= \langle \varphi(y), a(x\#b) \rangle = \langle \varphi(y), (a \cdot x)\#ab \rangle = \langle y, a \cdot x \rangle = \langle y_{(-1)}, a \rangle \langle y_{(0)}, x \rangle = \langle \varphi(y_{(-1)}), a \rangle \langle \varphi(y_{(0)}), x \rangle \\ &= \langle \varphi(y), a \rangle \langle 1, x\#b \rangle + \langle \varphi(y_{(-1)}), a \rangle \langle \varphi(y_{(0)}), x\#b \rangle = \langle \varphi(y) \otimes 1 + \varphi(y_{(-1)}) \otimes \varphi(y_{(0)}), a \otimes (x\#b) \rangle\end{aligned}$$

for all  $a, b \in G$  and  $x \in V$ .  $\square$

**Convention 6.** We identify the Hopf subalgebra  $(\mathfrak{B}(V)\#kG)^{*op}$  of  $\mathfrak{D}$  with  $\mathfrak{B}(\bar{V})\#kG$  by invoking the above lemma.

**Lemma 7.** The quantum group  $\mathfrak{D}$  at a non-abelian group can be presented as an algebra generated by the elements belonging to  $V, \bar{V}, kG$  and  $k^G$  subject to their relations in  $\mathfrak{B}(V), \mathfrak{B}(\bar{V}), kG$  and  $k^G$ , plus the commutation rules

$$g x = (g \cdot x) g, \quad \delta_g y = \sum_{t \in G} (\delta_t \cdot y) \delta_{t^{-1}g}, \quad (19)$$

$$\delta_g x = \sum_{t \in G} \langle \delta_t, x_{(-1)} \rangle x_{(0)} \delta_{t^{-1}g}, \quad y g = \langle y_{(-1)}, g \rangle g y_{(0)}, \quad (20)$$

$$y x - \langle y_{(-1)}, x_{(-1)} \rangle x_{(0)} y_{(0)} = \langle y, x \rangle 1 + \langle y_{(-2)}, x_{(-2)} \rangle \langle y_{(0)}, \mathcal{S}(x_{(0)}) \rangle x_{(-1)} y_{(-1)}, \quad (21)$$

$$\delta_h g = g \delta_{g^{-1}hg}. \quad (22)$$

for all  $g, h \in G, x \in V$  and  $y \in \bar{V}$ .

*Proof.* The equations (19) correspond to the bosonization, see (3). Meanwhile (20), (21) and (22) follow from (1).  $\square$

**Lemma 8.** The subalgebra of  $\mathfrak{D}$  generated by  $kG$  and  $k^G$  is a Hopf subalgebra isomorphic to  $\mathfrak{D}(G)$  and it is the coradical of  $\mathfrak{D}$ . In particular,  $\mathfrak{D}$  is non-pointed.

*Proof.* It follows from Lemma 5 and (22).  $\square$

Due to the above lemmata, a quantum group at a non-abelian group has a triangular decomposition, that is

$$\mathfrak{D} = \mathfrak{B}(V) \otimes \mathfrak{D}(G) \otimes \mathfrak{B}(\bar{V}), \quad (23)$$

and it is a  $\mathbb{Z}$ -graded algebra by setting

$$\deg V = -1, \quad \deg \mathfrak{D}(G) = 0, \quad \deg \bar{V} = 1. \quad (24)$$

**Convention 9.** We consider  $V$  and  $\bar{V}$  as Yetter-Drinfeld modules over  $\mathfrak{D}(G)$  with the adjoint action and the same coaction as  $kG$ -comodule and  $k^G$ -comodule, respectively.

That is possible because the rules (19) and (20) guarantee that  $V$  and  $\bar{V}$  are stable by the adjoint action of  $\mathfrak{D}(G)$ , i. e.  $\text{ad}(h)x = h_{(1)}x\mathcal{S}(h_{(2)}) \in V$  and  $\text{ad}(h)y = h_{(1)}y\mathcal{S}(h_{(2)}) \in \bar{V}$  for all  $h \in \mathfrak{D}(G), x \in V$  and  $y \in \bar{V}$ . Also,  $\mathfrak{D}(G) = kG \otimes k^G$  as coalgebra.

We extend these structures to  $\mathfrak{B}(V)$  and  $\mathfrak{B}(\bar{V})$ . Hence the bosonization

$$\mathfrak{D}^{\leq 0} = \mathfrak{B}(V)\#\mathfrak{D}(G), \quad \text{respectively} \quad \mathfrak{D}^{\geq 0} = \mathfrak{B}(\bar{V})\#\mathfrak{D}(G), \quad (25)$$

identifies with the subalgebra of  $\mathfrak{D}$  generated by  $\mathfrak{D}(G)$  and  $V$ , respectively  $\bar{V}$ .

**Remark 10.** The adjoint action of  $\mathfrak{D}(G)$  on  $V$  coincides with the action defined by the equivalence of categories between  ${}_{kG}^k\mathcal{Y}\mathfrak{D}$  and  ${}_{\mathfrak{D}(G)}\mathcal{M}$ , see (2).

We would like to remark other facts about  $V$  and  $\bar{V}$ . We refer to [1] for details about the item (iv) below.

**Lemma 11.** (i)  $\bar{V}$  is the dual object of  $V$  in the tensor category  ${}_{\mathfrak{D}(G)}\mathcal{M}$ .

- (ii)  $\mathfrak{B}(V)$  and  $\mathfrak{B}(\bar{V})$  are the Nichols algebras of  $V$  and  $\bar{V}$  in  $\mathfrak{D}(G)\mathcal{YD}$ , respectively.
- (iii)  $\mathfrak{B}(V)$  and  $\mathfrak{B}(\bar{V})$  are the Nichols algebras of  $V$  and  $\bar{V}$  in  $\mathfrak{D}(G)\mathcal{M}$ , respectively.
- (iv)  $\mathfrak{B}(\bar{V})$  is isomorphic to the opposite and copposite Hopf algebra  $\mathfrak{B}(V)^{*bop}$  in  $\mathfrak{D}(G)\mathcal{M}$ .

*Proof.* (i) Let  $y \in \bar{V}$ ,  $x \in V$  and  $g \in G$ . By (20), we have that  $\langle \text{ad}(g)y, x \rangle = \langle y_{(-1)}, g^{-1} \rangle \langle y_{(0)}, x \rangle$ . On the other hand, (16) and (19) imply that  $\langle y, \text{ad}(g^{-1})x \rangle = \langle y, g^{-1} \cdot x \rangle = \langle y_{(-1)}, g^{-1} \rangle \langle y_{(0)}, x \rangle$ . Then  $\langle \text{ad}(g)y, x \rangle = \langle y, \text{ad } S(g)x \rangle$ . In a similar way, we see that  $\langle \text{ad}(\delta_g)y, x \rangle = \langle y, \text{ad } S(\delta_g)x \rangle$ .

(ii)  $\mathfrak{B}(V)$  and  $\mathfrak{B}(\bar{V})$  are braided Hopf algebras in  $\mathfrak{D}(G)\mathcal{YD}$ , because  $\mathfrak{D}^{\leq 0}$  and  $\mathfrak{D}^{\geq 0}$  are Hopf algebras, which satisfy the defining properties of a Nichols algebra

(iii) follows from (ii) because the braiding of  $\mathfrak{D}(G)\mathcal{YD}$  on  $V$  coincides with that of  $\mathfrak{D}(G)\mathcal{M}$  and the same holds for  $\bar{V}$ .

(iv) Let  $\tilde{V}$  be the dual object of  $V$  in  $\mathbb{k}^G\mathcal{YD}$ . By [1, Theorem 3.2.30],  $\mathfrak{B}(\tilde{V}) \simeq \mathfrak{B}(V)^{*bop}$  in  $\mathbb{k}^G\mathcal{YD}$ . We said before that the adjoint action of  $\mathfrak{D}(G)$  on  $V$  coincides with the action defined by the functor given the equivalence of categories between  $\mathbb{k}^G\mathcal{YD}$  and  $\mathfrak{D}(G)\mathcal{M}$ . Then, by (i),  $\bar{V}$  is the image of  $\tilde{V}$  by this functor and (iv) follows because the braidings of these categories are equal via this functor.  $\square$

### 3.1. Verma modules

A classical technique in Representation Theory is to study modules induced by simple modules of a subalgebra. Such is the case of the Verma modules for quantum groups, see for instance [5, Chapter 7], [8, Chapter 5] and [10, Chapter 3]. Following this idea, we shall induce from the subalgebra  $\mathfrak{D}^{\geq 0}$ .

Every simple  $\mathfrak{D}^{\geq 0}$ -module is isomorphic to a simple  $\mathfrak{D}(G)$ -module where  $\mathfrak{B}(\bar{V})$  acts via the counit. This holds because  $\mathfrak{B}(\bar{V})$  is local and hence  $\ker(\varepsilon)\#\mathfrak{D}(G)$  is the Jacobson radical of  $\mathfrak{D}^{\geq 0}$ .

**Definition 12.** Let  $M(g, \varrho)$  be a simple  $\mathfrak{D}(G)$ -module. The Verma module  $\mathbb{M}(g, \varrho)$  is the  $\mathfrak{D}$ -module induced by  $M(g, \varrho)$  seen as a module over  $\mathfrak{D}^{\geq 0}$ . Explicitly,

$$\mathbb{M}(g, \varrho) = \mathfrak{D} \otimes_{\mathfrak{D}^{\geq 0}} M(g, \varrho).$$

We fix a simple  $\mathfrak{D}(G)$ -module  $M(g, \varrho)$  and set  $\mathbb{M} = \mathbb{M}(g, \varrho)$ . Immediately from the definition, we get that  $\mathbb{M}$  is free as  $\mathfrak{B}(V)$ -module of rank  $\dim M(g, \varrho) = \#0_g \cdot \dim \varrho$ . Moreover,

$$\mathbb{M} = \mathfrak{B}(V) \otimes M(g, \varrho) \quad \text{in } \mathfrak{D}(G)\mathcal{M} \tag{26}$$

since  $h(x \otimes m) = (hx) \otimes m = \text{ad}(h_{(1)})xh_{(2)} \otimes m = \text{ad}(h_{(1)})x \otimes h_{(2)}m$ , for  $h \in \mathfrak{D}(G)$ ,  $x \in \mathfrak{B}(V)$  and  $m \in M(g, \varrho)$ , and the last term is the definition of the action in the tensor product of two  $\mathfrak{D}(G)$ -modules.

Also  $\mathbb{M}$  inherits the  $\mathbb{Z}$ -grading of  $\mathfrak{D}$  and its homogeneous spaces are  $\mathfrak{D}(G)$ -submodules. Namely, its homogeneous space of degree  $n \leq 0$  is

$$\mathbb{M}^n = \mathfrak{B}^{-n}(V) \otimes M(g, \varrho).$$

Thus  $\mathbb{M}$  turns out to be an  $\mathbb{Z}$ -graded  $\mathfrak{D}$ -module since

$$V\mathbb{M}^n = \mathbb{M}^{n-1} \quad \text{and} \quad \bar{V}\mathbb{M}^n \subseteq \mathbb{M}^{n+1}. \tag{27}$$

In fact, the first equality holds as the action by an element of  $V$  is just the multiplication in the Nichols algebra and this is generated by  $V$ . We proceed by induction to prove the second inclusion. If  $n = 0$ , then  $\mathbb{M}^0 = \mathbb{k}1 \otimes M(g, \varrho)$  on which  $\bar{V}$  acts by zero. If  $n < 0$ , then

$$\bar{V}\mathbb{M}^{n-1} = \bar{V}V\mathbb{M}^n \subseteq V\bar{V}\mathbb{M}^n + \mathfrak{D}(G)\mathbb{M}^n \subseteq \mathbb{M}^n,$$

where the first inclusion follows from (21) and the second one by inductive hypothesis.

As  $\mathfrak{D}^{\leq 0}$ -module,  $\mathbb{M}$  is generated by any element of degree 0, that is

$$\mathbb{M} = \mathfrak{D}^{\leq 0}(1 \otimes m) \quad \forall m \in M. \tag{28}$$

In fact,  $\mathfrak{D}^{\leq 0}(1 \otimes m) = \mathfrak{B}(V) \otimes \mathfrak{D}(G)m = \mathfrak{B}(V) \otimes M(g, \varrho)$  since  $M(g, \varrho)$  is  $\mathfrak{D}(G)$ -simple. However we have more generators for a Verma module as  $\mathfrak{D}^{\leq 0}$ -module.

**Lemma 13.** *If  $m \in M^0$  is non-zero, then  $M = \mathcal{D}^{\leq 0}(m + n)$  for any  $n \in \oplus_{n < 0} M^n$ .*

*Proof.* By (28) the lemma follows if  $n = 0$ . Otherwise, we write  $n = n_1 + n_2$  with  $0 \neq n_1 \in M^{n_1}$  and  $n_2 \in \oplus_{n < n_1} M^n$ . By (28) there is  $z \in \mathcal{D}^{\leq 0}$  such that  $zm = n_1$ , moreover  $z \in \mathfrak{B}^{n_1}(V) \# \mathcal{D}(G)$ . Then

$$(m + n) - z(m + n) = m + (n_2 - zn_1 - zn_2) \in \mathcal{D}^{\leq 0}(m + n)$$

and the maximum degree of  $(n_2 - zn_1 - zn_2)$  is smaller than  $n_1$ . Hence the lemma follows by induction in the maximum degree of  $n$  since  $\mathfrak{B}(V)$  is finite-dimensional and  $VM^{-n_{top}} = 0$  if  $n_{top}$  is the maximum degree of  $\mathfrak{B}(V)$ .  $\square$

Using the above lemma we prove one of the main properties of a Verma module.

**Theorem 1.** *A Verma module has a unique maximal  $\mathcal{D}$ -submodule and it is homogeneous.*

*Proof.* If  $N$  is a strict  $\mathcal{D}$ -submodule of  $M$ , then there exists a non-zero negative integer  $n_N$  such that  $N \subseteq \oplus_{n < n_N} M^n$  by Lemma 13. Hence the sum  $X$  of all strict  $\mathcal{D}$ -submodules is the unique maximal  $\mathcal{D}$ -submodule of  $M$ .

Let  $\sum_n n_n \in X$  with  $n_n \in M^n$ . If we see that  $n_n \in X$  for all  $n$ , then  $X$  is homogeneous.

Otherwise, without loss of generality, we can assume  $n_n \notin X$  with  $n$  maximal, then  $\mathcal{D}n_n = M$ . Thus there is  $z \in \mathfrak{B}^n(\bar{V}) \# \mathcal{D}(G)$  such that  $0 \neq zn_n = 1 \otimes m \in M^0$  and hence  $z \sum_n n_n = 1 \otimes m + \tilde{n}$  with  $\tilde{n} \in \oplus_{n < 0} M^n$ . Then  $\mathcal{D} \sum_n n_n = M$  by Lemma 13 but this is not possible because  $\sum_n n_n \in X \subsetneq M$ .  $\square$

As it is common, we introduce the highest-weight modules in such a way that a Verma module is a highest-weight module. The weights in our case are the simple  $\mathcal{D}(G)$ -modules which can have dimension greater than one.

**Definition 14.** *Let  $N$  be a  $\mathcal{D}$ -module and  $M \subset N$  a simple  $\mathcal{D}(G)$ -submodule of weight  $(g, \varrho)$ . Assume that  $N$  is generated as  $\mathcal{D}$ -module by  $M$ .*

*We say that  $N$  is a highest-weight module of weight  $(g, \varrho)$  if  $\bar{V}M = 0$ .*

*We say that  $N$  is a lowest-weight module of weight  $(g, \varrho)$  if  $VM = 0$ .*

Hence we have that

$$N = \mathcal{D}M = \mathfrak{B}(V)M = \mathcal{D}^{\leq 0}m \quad \forall m \in M \quad (29)$$

if  $N$  is a highest-weight module, and

$$N = \mathcal{D}M = \mathfrak{B}(\bar{V})M = \mathcal{D}^{\geq 0}m \quad \forall m \in M \quad (30)$$

in case that  $N$  is a lowest-weight module. These follow from the decomposition (23) of  $\mathcal{D}$  and since  $M$  is  $\mathcal{D}(G)$ -simple.

We set  $M_{\text{soc}} = \mathfrak{B}^{n_{top}}(V) \otimes M(g, \varrho)$  where  $n_{top}$  is the maximum degree of the Nichols algebra. Note that  $M_{\text{soc}}$  is simple as a  $\mathcal{D}(G)$ -module since  $\mathfrak{B}^{n_{top}}(V)$  is one-dimensional.

**Theorem 2.** *The socle of the Verma module  $M$  is simple as a  $\mathcal{D}$ -module and equals  $\mathfrak{B}(\bar{V})M_{\text{soc}}$ .*

*Proof.* The socle is simple if we show that  $M_{\text{soc}} \subset \mathcal{D}m$  for any homogeneous element  $m \neq 0$  of degree  $-n$  in  $M$  with  $n < n_{top}$ . To show that, we write  $m = \sum_i z_i \otimes m_i$  with  $z_i \in \mathfrak{B}^n(V)$  and  $\{m_i\} \subset M(g, \varrho)$  linearly independent. We pick  $z_i$ . Since  $\mathfrak{B}^{n_{top}}(V)$  is the space of integrals of the Nichols algebra, there is  $x_1 \in V$  such that  $0 \neq x_1 z_i \in \mathfrak{B}^{n+1}(V)$ . Then  $0 \neq x_1 m \in \mathcal{D}m$  is an homogeneous element of degree  $-n - 1$ . Hence  $x_{n_{top}-n} \cdots x_1 m \neq 0$  for appropriated  $x_{n_{top}-n}, \dots, x_1 \in V$  and therefore  $M_{\text{soc}} \subset \mathcal{D}m$  because  $M_{\text{soc}}$  is  $\mathcal{D}(G)$ -simple.

Finally, the socle is a lowest-weight module because it is generated by  $M_{\text{soc}}$ . Therefore the socle is equal to  $\mathfrak{B}(\bar{V})M_{\text{soc}}$  by (30).  $\square$

As a direct consequence we obtain a criterion for the simplicity of a Verma module. Recall that the Hilbert series of  $\mathfrak{B}(V)$  and  $\mathfrak{B}(\bar{V})$  are equal by Lemma 5.

**Corollary 15.** *Let  $x_{top} \in \mathfrak{B}^{n_{top}}(V)$ ,  $y_{top} \in \mathfrak{B}^{n_{top}}(\bar{V})$  and  $m \in M$ . The Verma module  $M$  is simple as  $\mathcal{D}$ -module if and only if  $y_{top}(x_{top} \otimes m) \neq 0$ .*

*Proof.* By (27),  $y_{top}(x_{top} \otimes m) \in M^0$  and hence it generates  $M$  by (28). Then the socle, which is simple by the above theorem, is exactly  $M$ .

Assume now that  $M$  is a simple  $\mathcal{D}$ -module. In particular,  $M$  is generated by  $x_{top} \otimes m$  and thus there is an element  $z \in \mathcal{D}$  such that  $0 \neq z(x_{top} \otimes m) \in M^0$ . By (27),  $z \in \mathcal{D}^{n_{top}} = \sum_{j=-n_{top}} \mathfrak{B}^i(V) \mathcal{D}(G) \mathfrak{B}^j(\bar{V})$ . Since  $n_{top}$  is the maximum degree of  $\mathfrak{B}(V)$  and  $\mathfrak{B}(\bar{V})$ , we have that  $z \in \mathcal{D}(G) \mathfrak{B}^{n_{top}}(\bar{V})$ . Finally, we can take  $z = y_{top}$  because  $\mathfrak{B}^{n_{top}}(\bar{V})$  is one-dimensional.  $\square$

Due to the above theorems we can introduce the following  $\mathcal{D}$ -modules.

**Definition 16.** Let  $M(g, \varrho)$  be a simple  $\mathcal{D}(G)$ -module. Then

- $X(g, \varrho)$  denotes the maximal  $\mathcal{D}$ -submodule of  $M(g, \varrho)$ .
- $L(g, \varrho)$  denotes the head of  $M(g, \varrho)$ .
- $S(g, \varrho)$  denotes the socle of  $M(g, \varrho)$  as  $\mathcal{D}$ -module.

The following theorem states that the correspondence  $M(g, \varrho) \leftrightarrow L(g, \varrho)$ , between the sets of simple  $\mathcal{D}(G)$ -modules and simple  $\mathcal{D}$ -modules, is bijective.

**Theorem 3.** (i) Let  $M(g, \varrho)$  be a simple  $\mathcal{D}(G)$ -module. Then  $L(g, \varrho)$  is the unique simple highest-weight module of weight  $(g, \varrho)$ .

(ii) Every simple  $\mathcal{D}$ -module is isomorphic to  $L(g, \varrho)$  for a unique simple  $\mathcal{D}(G)$ -module  $M(g, \varrho)$ .

*Proof.* (i)  $L(g, \varrho)$  is a simple  $\mathcal{D}$ -module by Theorem 1. As  $M(g, \varrho)$  is generated by  $M(g, \varrho)$ ,  $L(g, \varrho)$  is so. Moreover,  $\bar{V}M(g, \varrho) = 0$  then  $L(g, \varrho)$  is a highest-weight module. The uniqueness follows from the fact that a highest-module  $L$  of weight  $(g, \varrho)$  is a quotient of  $M(g, \varrho)$  since  $M(g, \varrho)$  turns out to be a  $\mathcal{D}^{\geq 0}$ -submodule of  $L$ . If also  $L$  is  $\mathcal{D}$ -simple, then  $L \simeq L(g, \varrho)$  since  $X(g, \varrho)$  is the unique maximal submodule of  $M(g, \varrho)$ .

(ii) Every  $\mathcal{D}$ -module  $L$  has a simple  $\mathcal{D}^{\geq 0}$ -module, say  $M(g, \varrho)$ . Then we have a morphism  $M(g, \varrho) \rightarrow L$  of  $\mathcal{D}$ -modules and this map is surjective if  $L$  is  $\mathcal{D}$ -simple. Hence  $L(g, \varrho) \simeq L$ . On the other hand, if there exists another  $L(g', \varrho')$  isomorphic to  $L$ , then  $M(g, \varrho) \simeq M(g', \varrho')$  because they are highest weights of  $L$ .  $\square$

The correspondence  $M(g, \varrho) \leftrightarrow S(g, \varrho)$  also is bijective by the next theorem. We set  $M(\hat{g}, \hat{\varrho})$  to be the simple  $\mathcal{D}(G)$ -module isomorphic to  $\mathfrak{B}^{n_{top}}(V) \otimes M(g, \varrho)$ .

**Theorem 4.** (i) Let  $M(g, \varrho)$  be a simple  $\mathcal{D}(G)$ -module. Then  $S(g, \varrho)$  is the unique simple lowest-weight module of weight  $M(\hat{g}, \hat{\varrho})$ .

(ii) Every simple  $\mathcal{D}$ -module is isomorphic to  $S(g, \varrho)$  for a unique simple  $\mathcal{D}(G)$ -module  $M(g, \varrho)$ .

*Proof.* The socles of the Verma modules are lowest-weight modules by Theorem 2. Also, the socles of non-isomorphic Verma modules are non-isomorphic because their lowest-weight components are not. Hence the uniqueness in (i) and (ii) follow from the fact that the sets of simple  $\mathcal{D}$ -modules and simple  $\mathcal{D}(G)$ -modules are in bijective correspondence by Theorem 3.  $\square$

We denote by  $M(g^*, \varrho^*)$  the dual  $\mathcal{D}(G)$ -module of  $M(g, \varrho)$ .

**Theorem 5.** The dual  $\mathcal{D}$ -module  $(S(g, \varrho))^*$  is a highest-weight module of weight  $M(\hat{g}^*, \hat{\varrho}^*)$ . Therefore  $(S(g, \varrho))^* \simeq L(\hat{g}^*, \hat{\varrho}^*)$  as  $\mathcal{D}$ -modules.

*Proof.* Let  $S_i$ ,  $i \geq -n_{top}$ , be the homogeneous component of degree  $i$  of  $S(g, \varrho)$ . Then  $(S(g, \varrho))^* = \oplus_i S_i^*$  and  $S_{-n_{top}}^* \simeq M(\hat{g}^*, \hat{\varrho}^*)$  as  $\mathcal{D}(G)$ -modules. Since  $(S(g, \varrho))^*$  is simple, it is generated by  $S_{-n_{top}}^*$ . Moreover,  $\bar{V}S_{-n_{top}}^* = 0$ . In fact,  $\langle \bar{V}S_{-n_{top}}^*, S_i \rangle = \langle S_{-n_{top}}^*, \mathfrak{S}(\bar{V})S_i \rangle = 0$  because  $\mathfrak{S}(\bar{V})S_i \subseteq S_{i+1}$  and  $-n_{top} < i + 1$  for all  $i \geq -n_{top}$ . Hence the theorem follow from Theorem 3.  $\square$

**Remarks 17.** (i) Even though the maximal submodule of  $M$  is homogeneous, a submodule is not necessarily homogeneous, cf. Lemma 26.

(ii) The maximal submodule is not necessarily generated by highest-weight submodules, cf. Theorem 7.

(iii) The head and the socle of a Verma module are not necessarily isomorphic, cf. Corollary 25.

(iv) There are examples with  $M(\hat{g}, \hat{\varrho}) \neq M(g, \varrho)$ . For instance, let  $R$  be the Nichols algebra considered in [15]. Then  $R^{n_{top}}$  is not necessarily trivial as Yetter-Drinfeld module.

### 3.2. Highest and lowest weight modules

We fix a  $\mathfrak{D}$ -module  $N$  and a  $\mathfrak{D}(G)$ -submodule  $M \subset N$ . We will explain how we can compute the  $\mathfrak{D}$ -submodule generated by  $M$  under the hypothesis that it is either a lowest-weight or highest-weight module.

We denote by  $\mu$ , and call it *action map*, the restriction to  $V \otimes M$  of the action of  $\mathfrak{D}$  over  $N$ . By abuse of notation, we also denote by  $\mu$  the restriction to  $\bar{V} \otimes M$ . The key of our idea is the simple observation that

$$\text{the action map } \mu \text{ is a morphism in the category } {}_{\mathfrak{D}(G)}\mathcal{M}. \quad (31)$$

Indeed, we want to see that  $\mu(h(z \otimes m)) = h\mu(z \otimes m)$  for any  $z \in V \cup \bar{V}$ ,  $h \in \mathfrak{D}(G)$  and  $m \in M$ . The action on the tensor product is  $h(z \otimes m) = h_{(1)} \cdot z \otimes h_{(2)}m$ . Then we apply the action map and obtain  $(h_{(1)} \cdot z)h_{(2)}m = \text{ad}(h_{(1)})zh_{(2)}m = h_{(1)}z\mathcal{S}(h_{(2)})h_{(3)}m = h(zm) = h\mu(z \otimes m)$ .

By (29) and (30), the  $\mathfrak{D}$ -submodule generated by  $M$  is either  $\mathfrak{B}(\bar{V})M$  or  $\mathfrak{B}(V)M$ . Hence we can compute  $\mathfrak{D}M$  following the algorithm described in the next remark.

**Remark 18.** *Keep the above hypothesis and notation.*

(I) *Decompose the tensor product  $\bar{V} \otimes M$ , or  $V \otimes M$  depending on the case, into the direct sum  $\oplus S_\ell$  of simple  $\mathfrak{D}(G)$ -modules.*

*This is possible because  $\mathfrak{D}(G)$  is semisimple. Moreover, its simple modules are well-know, recall §2.5.*

(II) *Apply the action map to each simple  $\mathfrak{D}(G)$ -module  $S_\ell$ .*

*The restriction of the action map to  $S_\ell$  is either zero or an isomorphism by Schur's Lemma. Therefore the image of the action map is isomorphic as  $\mathfrak{D}(G)$ -module to the direct sum of the simple modules  $S_\ell$  that are not annulled. Note that  $\mu(S_\ell) = 0$  if and only if  $\mu(w) = 0$  for some  $w \in S_\ell$ .*

(III) *Repeat the process with the  $\mathfrak{D}(G)$ -submodule  $\mu(\bar{V} \otimes M)$ , or  $\mu(V \otimes M)$  depending on the case, instead of  $M$ .*

*We have to repeat the process as many times as the maximum degree of  $\mathfrak{B}(\bar{V})$  or  $\mathfrak{B}(V)$ .*

(IV) *The  $\mathfrak{D}$ -submodule generated by  $M$  is the sum of all  $\mathfrak{D}(G)$ -submodules obtained in the step (II).*

## 4. The quantum group at the symmetric group $\mathbb{S}_3$ attached to the 12-dimensional Fomin-Kirillov algebra

Throughout this section  $V = \mathbb{k}\{x_{(12)}, x_{(23)}, x_{(13)}\}$  is the Yetter-Drinfeld module over  $\mathbb{k}\mathbb{S}_3$  given by

$$g \cdot x_{(ij)} = \text{sgn}(g) x_{g(ij)g^{-1}} \quad \text{and} \quad (x_{(ij)})_{(-1)} \otimes (x_{(ij)})_{(0)} = (ij) \otimes x_{(ij)}$$

for any transposition  $(ij)$  and  $g \in \mathbb{S}_3$ . Let  $\bar{V} \in {}_{\mathbb{k}\mathbb{S}_3}^{\mathbb{k}\mathbb{S}_3} \mathcal{YD}$  be the Yetter-Drinfeld module attached to  $V$  from Definition 4. From [13] we know that  $\mathfrak{B}(V)$  is isomorphic to the 12-dimensional Fomin-Kirillov algebra introduced in [6].

We denote by  $\mathfrak{D}$  the Drinfeld double of the bosonization  $\mathfrak{B}(V) \# \mathbb{k}\mathbb{S}_3$ . The aim of this section is to apply the results of the previous section in the specific example of this quantum group.

We have to consider  $V$  and  $\bar{V}$  as  $\mathfrak{D}(\mathbb{S}_3)$ -modules with the adjoint action in  $\mathfrak{D}$ . Using Remark 10, we see that  $V \simeq M(\sigma, -)$  via the assignment

$$V \longrightarrow M(\sigma, -), \quad x_{(ij)} \longmapsto |ij\rangle$$

for every transposition  $(ij)$ .

By Lemma 11,  $\bar{V}$  is the dual object of  $V$  in the category  ${}_{\mathfrak{D}(\mathbb{S}_3)}\mathcal{M}$ . We denote by  $\{y_{(12)}, y_{(23)}, y_{(13)}\}$  the basis of  $\bar{V}$  dual to  $\{x_{(12)}, x_{(23)}, x_{(13)}\}$ , that is  $\langle y_{(ij)}, x_{(lk)} \rangle = \delta_{(ij),(lk)}$ . Then it is not difficult to check that  $\bar{V} \simeq M(\sigma, -)$  via the assignment

$$\bar{V} \longrightarrow M(\sigma, -), \quad y_{(ij)} \longmapsto |ij\rangle$$

for every transposition  $(ij)$ .

The defining relations of the Nichols algebra  $\mathfrak{B}(V)$  are

$$\begin{aligned} & x_{(12)}^2, \quad x_{(13)}^2, \quad x_{(23)}^2, \\ & x_{(12)}x_{(13)}+x_{(23)}x_{(12)} + x_{(13)}x_{(23)} \quad \text{and} \\ & x_{(13)}x_{(12)}+x_{(12)}x_{(23)} + x_{(23)}x_{(13)}. \end{aligned} \tag{32}$$

We denote by  $\mathbb{B}$  the basis of  $\mathfrak{B}(V)$  which is obtained by choosing one element per row of the next list and multiply them from top to bottom, see e. g. [7]:

$$\begin{aligned} & 1, x_{(12)}, \\ & 1, x_{(13)}, x_{(13)}x_{(12)}, \\ & 1, x_{(23)}. \end{aligned}$$

We set  $\mathbb{B}^n = \mathbb{B} \cap \mathfrak{B}^n(V)$ ,  $n \geq 0$ . The element of maximum degree in  $\mathbb{B}$  is

$$x_{top} = x_{(12)}x_{(13)}x_{(12)}x_{(23)} \in \mathbb{B}^4.$$

**Lemma 19.** *We have the following isomorphisms of  $\mathcal{D}(\mathbb{S}_3)$ -modules.*

- (i)  $\mathfrak{B}^0(V) \simeq \mathfrak{B}^4(V) \simeq M(e, +)$ ,
- (ii)  $\mathfrak{B}^1(V) \simeq \mathfrak{B}^3(V) \simeq M(\sigma, -)$ ; the last isomorphism is given by the assignment

$$|\mathbf{12}\rangle \mapsto x_{(13)}x_{(12)}x_{(23)}, \quad |\mathbf{23}\rangle \mapsto -x_{(12)}x_{(13)}x_{(12)}, \quad |\mathbf{13}\rangle \mapsto x_{(12)}x_{(13)}x_{(23)}.$$

- (iii)  $\mathfrak{B}^2(V) \simeq M(\tau, 1) \oplus M(\tau, 2)$ , the isomorphism is given by

$$\begin{aligned} |\tau\rangle_\ell & \mapsto (\zeta^\ell - 1)x_{(12)}x_{(23)} + (\zeta^{-\ell} - 1)x_{(13)}x_{(12)}, \\ |\tau^{-1}\rangle_\ell & \mapsto (\zeta^\ell - \zeta^{-\ell})x_{(12)}x_{(13)} + (1 - \zeta^{-\ell})x_{(13)}x_{(23)} \quad \text{for } \ell = 1, 2. \end{aligned}$$

*Proof.* (i) For  $\mathfrak{B}^0(V)$  the isomorphism is clear. For  $\mathfrak{B}^4(V)$  it is enough to see that

$$\begin{aligned} \sigma \cdot x_{top} &= \text{sgn}^4(\sigma)x_{(12)(12)(12)x_{(12)(13)(12)}x_{(12)(12)(12)}x_{(12)(23)(12)}} = x_{(12)}x_{(23)}x_{(12)}x_{(13)} \\ &= -x_{(12)}x_{(13)}x_{(23)}x_{(13)} = x_{(12)}x_{(13)}x_{(12)}x_{(23)} = x_{top} \quad \text{by (32)}. \end{aligned}$$

To prove (ii) we note that  $\sigma \cdot x_{(13)}x_{(12)}x_{(23)} = -x_{(13)}x_{(12)}x_{(23)}$  and

$$\sigma \cdot x_{(12)}x_{(13)}x_{(23)} = \text{sgn}(\sigma)^3 x_{(12)(12)(12)x_{(12)(13)(12)}x_{(12)(23)(12)}} = -x_{(12)}x_{(23)}x_{(13)} = x_{(12)}x_{(13)}x_{(12)}.$$

(iii) follows from (14) and using (32). □

#### 4.1. Description of the action on a Verma module

We fix a simple  $\mathcal{D}$ -module  $M$  and take the basis  $\mathbf{B}_M$  of  $M$  which consists of elements of the form  $|\mathbf{g}\rangle$ ,  $\mathbf{g} \in \mathbb{S}_3$ , recall §2.5.1, §2.5.2 and §2.5.3.

Let  $\mathbf{M}$  be the Verma module of  $M$ . Since  $\mathcal{D} \simeq \mathfrak{B}(V) \otimes \mathcal{D}(\mathbb{S}_3) \otimes \mathfrak{B}(\bar{V})$ , a basis of  $\mathbf{M}$  is the set of elements

$$x|\mathbf{g}\rangle = x \otimes |\mathbf{g}\rangle \quad \forall x \in \mathbb{B}, |\mathbf{g}\rangle \in \mathbf{B}_M.$$

Then the action of  $\mathfrak{B}(V)$  on  $\mathbf{M}$  is given just by the multiplication:

$$z \cdot x|\mathbf{g}\rangle = (zx)|\mathbf{g}\rangle \quad \forall z \in \mathfrak{B}(V).$$

The action of  $\mathcal{D}(\mathbb{S}_3)$  is the diagonal action by (26):

$$h \cdot x|\mathbf{g}\rangle = \text{ad } h_{(1)}(x) \otimes h_{(2)} \cdot |\mathbf{g}\rangle \quad \forall h \in \mathcal{D}(\mathbb{S}_3).$$

Computing the action of  $\mathfrak{B}(\bar{V})$  is more laborious. We have to use the commutation rules between the generators of  $\mathfrak{B}(\bar{V})$  and  $\mathfrak{B}(V)$  given by (21). In our case (21) is rewritten as follow

$$y_{(ij)}x_{(ij)} = 1 + (ij)(\delta_{(ij)} - \delta_e) - x_{(ij)}y_{(ij)} \quad \text{and} \quad (33)$$

$$y_{(ik)}x_{(ij)} = (ij)(\delta_{(ik)} - \delta_{(ik)(ij)}) - x_{(ij)}y_{(jk)} \quad (34)$$

for all distinct transpositions  $(ij)$  and  $(ik)$ . However, if we know the action of  $y_{(12)}$  on  $M$ , then we can deduce the action of the remainder generators of  $\mathfrak{B}(\bar{V})$ . In fact, let  $(ij) \neq (12)$  and  $t \in \mathbb{S}_3$  such that  $t(ij)t^{-1} = (12)$ . Hence

$$y_{(ij)} \cdot x|\mathbf{g}\rangle = \text{sgn}(t)t y_{(12)} t^{-1} \cdot x|\mathbf{g}\rangle. \quad (35)$$

In the Appendix we give explicitly the action of  $y_{(12)}$  on each element  $x|\mathbf{g}\rangle$  in the basis of  $M$ . We leave this for the Appendix because it is a very long list and we don't want to bore the reader now. We shall use these computations in the next subsections without previous mention.

#### 4.2. The simple Verma modules

**Theorem 6.** *The Verma modules  $M(e, -)$ ,  $M(\sigma, +)$ ,  $M(\tau, 1)$  and  $M(\tau, 2)$  are  $\mathcal{D}$ -simple. Therefore  $L(g, \varrho) \simeq S(g, \varrho)$  holds for these weights.*

*Proof.* By Corollary 15, it is enough to check that  $y_{top}(x_{top}v) \neq 0$  for some  $v \in M(g, \varrho)$ . In the case  $M(e, -)$ , using the calculations done in the appendix we have that  $y_{top}(x_{top}|e\rangle) = -12|e\rangle \neq 0$ . For  $M(\sigma, +)$ , in the same way we obtain  $y_{top}(x_{top}|\sigma\tau\rangle) = 2|\sigma\tau\rangle \neq 0$ . Finally, in both cases  $M(\tau, 1)$  and  $M(\tau, 2)$  we have that  $y_{top}(x_{top}|\tau^{-1}\rangle) = -3|\tau^{-1}\rangle \neq 0$ .

Therefore the isomorphism  $L(g, \varrho) \simeq S(g, \varrho)$  holds because  $\mathfrak{B}^4(V) \simeq M(e, +)$ .  $\square$

#### 4.3. The Verma module $M(\sigma, -)$

The aim of this subsection is to prove the next theorem which describes the submodules of  $M(\sigma, -)$ . As a byproduct, we find  $L(\sigma, -)$  and the remaining simple  $\mathcal{D}$ -modules as subquotients of  $M(\sigma, -)$ .

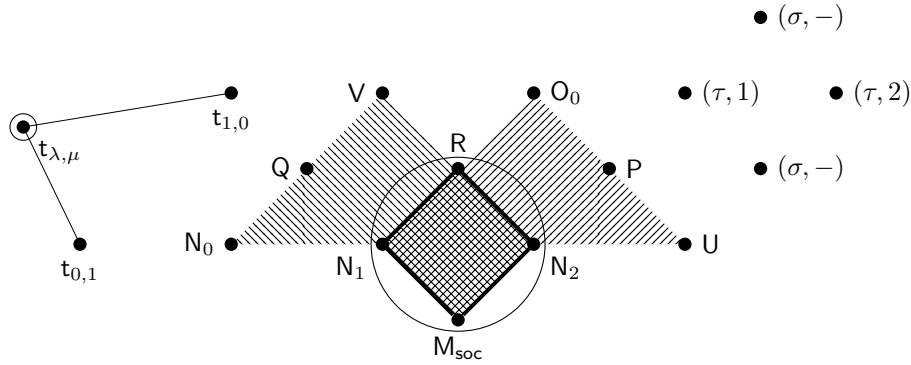


Figure 1: Submodules of  $M(\sigma, -)$

**Theorem 7.** *Every  $\mathcal{D}$ -submodule of  $M(\sigma, -)$  is obtained by adding the  $\mathcal{D}$ -submodules*

$$S(\sigma, -), \quad OPU, \quad VQN \quad \text{and} \quad T_{\lambda, \mu} \quad \text{with} \quad \lambda, \mu \in \mathbb{k}.$$

*In particular,  $X(\sigma, -) = OPU + VQN + T_{1,0} + T_{0,1}$  and*

$$L(\sigma, -) \simeq M(\sigma, -) \oplus M(\tau, 1) \oplus M(\tau, 2) \oplus M(\sigma, -).$$

*as  $\mathcal{D}(\mathbb{S}_3)$ -modules. Moreover,  $\{L(\sigma, -), L(\tau, 0), L(e, \rho), L(e, +), L(\sigma, -)\}$  are the composition factors of  $M(\sigma, -)$ .*



The submodules mentioned in the statement will be given in the successive lemmas but the Figure 1 helps to visualise them. Namely, each dot represents a weight of  $M(\sigma, -)$ . This follows from the fusion rules given in §2.5.4 since  $M(\sigma, -) = \mathfrak{B}(V) \otimes M(\sigma, -)$ :  $O_0$  is of weight  $(\tau, 0)$ ,  $U$  and  $V$  are of weight  $(e, \rho)$ ,  $P$  and  $Q$  are of weight  $(\sigma, +)$ ,  $R$  and  $M_{\text{soc}}$  are of weight  $(\sigma, -)$ ,  $t_{\lambda, \mu}$  is of weight  $(e, +)$  for any  $\lambda, \mu \in \mathbb{k}$ ,  $N_\ell$  is of weight  $(\tau, \ell)$  for  $\ell = 0, 1, 2$ . Of course, the weights in the same row are in the same homogeneous component and the degree decreases from the top to the bottom.

Then, the dots connected by a thick line represent the socle  $S(\sigma, -)$ , Lemma 20. The dots over the northwest lines form VQN, Lemma 21, and those over the northeast lines form OPU, Lemma 23. The dots enclosed by circles represent a  $\mathfrak{D}$ -submodule  $T_{\lambda, \mu}$ , Lemma 26. The isolated dots on the right hand side represent the weights of  $L(\sigma, -)$ .

We start by calculating the socle of the Verma module and then we describe the highest-weight submodules. For that, we shall use the algorithm described in Remark 18. In order to avoid extra computations, we approximate the kernel of the action map before applying this algorithm using Remark 3.

Let  $N_\ell$  and  $R$  be the  $\mathfrak{D}(\mathbb{S}_3)$ -submodules of  $M(\sigma, -)$  generated by

- $n_\ell = \zeta^\ell x_{(13)x_{(12)}x_{(23)}}|\mathbf{23}\rangle + \zeta^{-\ell} x_{(12)x_{(13)}x_{(23)}}|\mathbf{12}\rangle - x_{(12)x_{(13)}x_{(12)}}|\mathbf{13}\rangle$ , for  $\ell = 0, 1, 2$ , and
- $r = -(x_{(12)x_{(13)}} + x_{(13)x_{(23)}})|\mathbf{12}\rangle - (x_{(12)x_{(23)}} + x_{(13)x_{(12)}})|\mathbf{13}\rangle$ ,

respectively. By (14),  $n_\ell$  identifies with the element  $|\tau\rangle_\ell$  and belongs to the submodule of weight  $(\tau, \ell)$ . Hence  $N_\ell \simeq M(\tau, \ell)$  for  $\ell = 0, 1, 2$ .

**Lemma 20.** *The socle of  $M(\sigma, -)$  is*

$$S(\sigma, -) = M_{\text{soc}} \oplus N_1 \oplus N_2 \oplus R,$$

where  $M_{\text{soc}} \simeq M(\sigma, -) \simeq R$  and  $N_\ell \simeq M(\tau, \ell)$  as  $\mathfrak{D}(\mathbb{S}_3)$ -modules,  $\ell = 1, 2$ .

Moreover,  $S(\sigma, -)$  is a highest-weight module of weight  $(\sigma, -)$  and therefore

$$S(\sigma, -) \simeq L(\sigma, -).$$

*Proof.* We use the algorithm proposed in Remark 18 to compute the socle. Recall that  $S(\sigma, -) = \mathfrak{B}(\bar{V})M_{\text{soc}}$  by Theorem 2.

As  $\mathfrak{D}(\mathbb{S}_3)$ -module, we have that

$$\bar{V} \otimes M_{\text{soc}} = \mathbb{k}\{y_{(ij)} \otimes x_{\text{top}}|\mathbf{ij}\} \mid i \neq j\} \oplus \mathbb{k}\{y_{(ij)} \otimes x_{\text{top}}|\mathbf{jk}\} \mid i \neq j \neq k \neq i\}$$

by (7). Since  $y_{(12)}(x_{\text{top}}|\mathbf{12}\rangle) = 0$ , the action map is zero in the first submodule by Remark 3. The second submodule decomposes into the direct sum  $M(\tau, 0) \oplus M(\tau, 1) \oplus M(\tau, 2)$  where  $(\zeta^\ell + \zeta^{-\ell}\tau^{-1} + \tau)(y_{(12)} \otimes x_{\text{top}}|\mathbf{23}\rangle)$  belongs to the submodule of weight  $(\tau, \ell)$  by (14). The action map applied to these elements gives

$$\begin{aligned} (\zeta^\ell + \zeta^{-\ell}\tau^{-1} + \tau)y_{(12)}(x_{\text{top}}|\mathbf{23}\rangle) &= (\zeta^\ell + \zeta^{-\ell}\tau^{-1} + \tau)(-x_{(12)x_{(13)}x_{(23)}}|\mathbf{12}\rangle + x_{(13)x_{(12)}x_{(23)}}|\mathbf{23}\rangle) \\ &= (\zeta^{-\ell} - \zeta^\ell)x_{(12)x_{(13)}x_{(23)}}|\mathbf{12}\rangle + (\zeta^\ell - 1)x_{(13)x_{(12)}x_{(23)}}|\mathbf{23}\rangle + (\zeta^{-\ell} - 1)x_{(12)x_{(13)}x_{(12)}}|\mathbf{13}\rangle, \end{aligned}$$

which is zero iff  $\ell = 0$ . Otherwise, we obtain  $\frac{1}{1-\zeta^{-\ell}}n_\ell$  and hence

$$\bar{V}M_{\text{soc}} = N_1 \oplus N_2,$$

recall Remark 18.

Now, we calculate  $\bar{V}N_\ell$ . By (12),  $\bar{V} \otimes N_\ell \simeq M(\sigma, +) \oplus M(\sigma, -)$  as  $\mathfrak{D}(\mathbb{S}_3)$ -modules and the element  $\zeta^\ell(1 \pm (23))y_{(12)} \otimes n_\ell$  belongs to the submodule of weight  $(\sigma, \pm)$ . We apply the action map to these elements and obtain

$$r_\ell^\pm := \zeta^\ell(1 \pm (23))y_{(12)}n_\ell = \zeta^\ell(1 \pm (23))(x_{(23)x_{(12)}}|\mathbf{12}\rangle + x_{(23)x_{(13)}}|\mathbf{13}\rangle) = \zeta^\ell(1 \mp 1)(x_{(23)x_{(12)}}|\mathbf{12}\rangle + x_{(23)x_{(13)}}|\mathbf{13}\rangle).$$

Hence  $r_\ell^+ = 0$  and  $0 \neq \frac{1}{\zeta^{\ell 2}}r_\ell^- = r \in R[(23)]$ . Therefore

$$R = \bar{V}N_1 = \bar{V}N_2 \simeq M(\sigma, -) \quad \text{for } \ell = 1, 2.$$

Finally, we consider  $y_{(12)\otimes(13)}r \in \overline{V}[(12)]\otimes R[(12)]$  and  $y_{(12)\otimes r} \in \overline{V}[(12)]\otimes R[(23)]$ . Since  $y_{(12)(13)}r = 0 = y_{(12)}r$ , we see that  $\overline{V}R = 0$  by Remark 3. Therefore

$$S(\sigma, -) = \mathfrak{B}(\overline{V})M_{\text{soc}} = M_{\text{soc}} \oplus N_1 \oplus N_2 \oplus R.$$

Moreover,  $S(\sigma, -)$  is simple and generated by  $R \simeq M(\sigma, -)$  with  $\overline{V}R = 0$ . Therefore  $S(\sigma, -) = L(\sigma, -)$ .  $\square$

Let  $V$  and  $Q$  be the  $\mathcal{D}(\mathbb{S}_3)$ -submodules of  $M(\sigma, -)$  generated by

- $v = \zeta^{-1}x_{(23)}|\mathbf{23}\rangle + \zeta x_{(13)}|\mathbf{13}\rangle + x_{(12)}|\mathbf{12}\rangle$  and
- $q = x_{(12)x_{(23)}}|\mathbf{23}\rangle - x_{(12)x_{(13)}}|\mathbf{13}\rangle,$

respectively. Note that  $V$  is of weight  $(e, \rho)$  by (13).

**Lemma 21.** *Let  $VQN = S(\sigma, -) \oplus V \oplus Q \oplus N_0$ . Then*

- (i)  $VQN = \mathcal{D}V$  is a highest-weight submodule of weight  $(e, \rho)$ .
- (ii)  $VQN = \mathcal{D}Q = \mathcal{D}N_0$ .
- (iii)  $Q$  is of weight  $(\sigma, +)$ .

*Proof.* Since  $y_{(12)}v = 0$ ,  $\overline{V}V = 0$  by Remark 3 and thus  $\mathcal{D}V$  is a highest-weight module. Hence we will use Remark 18 to compute  $\mathcal{D}V = \mathfrak{B}(V)V$ .

By (15),  $V\otimes V \simeq M(\sigma, +) \oplus M(\sigma, -)$  where  $(1 \pm \sigma)x_{(12)}\otimes v$  belongs to the submodule of weight  $(\sigma, \pm)$ . The action map applied to these elements gives

$$(1 + \sigma)x_{(12)}v = (\zeta^{-1} - \zeta)q \quad \text{and} \quad (1 - \sigma)x_{(12)}v = (13)r$$

Hence  $VV = R \oplus Q$ . In particular,  $Q \simeq M(\sigma, +)$  which proves (iii).

Now, we just have to compute  $VQ$  since  $VR \subset S(\sigma, -)$ . As  $\mathcal{D}(\mathbb{S}_3)$ -modules,  $V\otimes Q \simeq M(e, -) \oplus M(e, \rho) \oplus \bigoplus_{\ell=0,1,2} M(\tau, \ell)$ , cf. §2.5.4. The action map on the components of weight  $(e, -)$  and  $(e, \rho)$  is zero since  $x_{(12)}q = 0$ , recall Remark 3. Meanwhile, the action map on the components of weight  $(\tau, \ell)$  is not zero. In fact,  $(\zeta^\ell + \zeta^{-\ell}\tau^{-1} + \tau)x_{(12)\otimes(13)}q$  belongs to the submodule of weight  $(\tau, \ell)$  by (14) and

$$\begin{aligned} (\zeta^\ell + \zeta^{-\ell}\tau^{-1} + \tau)x_{(12)(13)}q &= (\zeta^\ell + \zeta^{-\ell}\tau^{-1} + \tau)(1 - \zeta^2)(x_{(12)x_{(13)}x_{(23)}}|\mathbf{12}\rangle - x_{(12)x_{(13)}x_{(12)}}|\mathbf{13}\rangle) \\ &= (\zeta^2 - 1)(\zeta^{-\ell}x_{(12)x_{(13)}x_{(23)}}|\mathbf{12}\rangle - x_{(12)x_{(13)}x_{(12)}}|\mathbf{13}\rangle + \zeta^\ell x_{(13)x_{(12)}x_{(23)}}|\mathbf{23}\rangle). \end{aligned}$$

Hence  $VQ = N_0 \oplus N_1 \oplus N_2$ . Since  $VN_\ell \subseteq M_{\text{soc}}$ , we conclude that  $\mathcal{D}V = VQN$  and (i) follows.

Finally, we proof (ii) by noting that  $V \subset \mathcal{D}Q$  and  $V \subset \mathcal{D}N_0$  since

$$\frac{1}{2}y_{(12)}q = \frac{1}{3}y_{(12)y_{(13)}}n_0 = (\zeta^{-1} - \zeta)^{-1}(1 - \sigma)v \in V.$$

$\square$

Using the characterization of the simple modules given in Theorem 3, the next result follows directly from the above lemma.

**Corollary 22.** *The quotient  $VQN/S(\sigma, -)$  is a simple highest-weight module of weight  $(e, \rho)$ . Therefore*

$$L(e, \rho) \simeq M(e, \rho) \oplus M(\sigma, +) \oplus M(\tau, 0),$$

as  $\mathcal{D}(\mathbb{S}_3)$ -modules.  $\square$

Let  $U$ ,  $P$  and  $O_\ell$  be the  $\mathcal{D}(\mathbb{S}_3)$ -submodules generated by

- $u = -\zeta^{-1}x_{(12)x_{(13)}x_{(12)}}|\mathbf{23}\rangle + \zeta x_{(12)x_{(13)}x_{(23)}}|\mathbf{13}\rangle + x_{(13)x_{(12)}x_{(23)}}|\mathbf{12}\rangle,$

- $\mathfrak{p} = -(2x_{(13),x_{(12)}} + x_{(12),x_{(23)}})|\mathbf{23}\rangle - (2x_{(13),x_{(23)}} + x_{(12),x_{(13)}})|\mathbf{13}\rangle$  and
- $\mathfrak{o}_\ell = \zeta^{-\ell}x_{(13)}|\mathbf{12}\rangle + \zeta^\ell x_{(12)}|\mathbf{23}\rangle + x_{(23)}|\mathbf{13}\rangle$ , for  $\ell = 0, 1, 2$ ,

respectively. We see that  $\mathbf{U}$  is of weight  $(e, \rho)$  by (13) and Lemma 19, and  $\mathbf{O}_\ell$  is of weight  $(\tau, \ell)$  by (14).

**Lemma 23.** *Let  $\mathbf{OPU} = \mathbf{S}(\sigma, -) \oplus \mathbf{U} \oplus \mathbf{P} \oplus \mathbf{O}_0$ . Then*

- $\mathbf{OPU} = \mathfrak{D}\mathbf{O}_0$  is a highest-weight submodule of weight  $(\tau, 0)$ .
- $\mathbf{OPU} = \mathfrak{D}\mathbf{U} = \mathfrak{D}\mathbf{P}$ .
- $\mathbf{P}$  is of weight  $(\sigma, +)$ .

*Proof.* We have that  $\mathfrak{D}\mathbf{O}_0$  is a highest-weight module, since  $y_{(12)}\mathfrak{o}_0 = 0$  and Remark 3 provides  $\overline{V}\mathbf{O}_0 = 0$ . Then we compute  $\mathfrak{D}\mathbf{O}_0 = \mathfrak{B}(V)\mathbf{O}_0$ .

By (12),  $V\otimes\mathbf{O}_0 \simeq M(\sigma, +) \oplus M(\sigma, -)$  where  $(1 \pm \sigma)x_{(13)}\otimes\mathfrak{o}_0$  belongs to the submodule of weight  $(\sigma, \pm)$ . The action map gives

$$(1 + \sigma)x_{(13)}\mathfrak{o}_0 = -\mathfrak{p} \quad \text{and} \quad (1 - \sigma)x_{(13)}\mathfrak{o}_0 = (13)r$$

and we obtain  $V\mathbf{O}_0 = \mathbf{R} \oplus \mathbf{P}$ . In particular,  $\mathbf{P} \simeq M(\sigma, +)$  which proves (iii).

Now, we calculate  $V\mathbf{P}$ . As  $\mathfrak{D}(\mathbb{S}_3)$ -modules,  $V\otimes\mathbf{P} \simeq M(e, -) \oplus M(e, \rho) \oplus \bigoplus_{\ell=0,1,2} M(\tau, \ell)$ , cf. §2.5.4. Using (14) we see that the action map on the components of weight  $(e, -)$  and  $(\tau, 0)$  is zero since  $x_{(12)}\mathfrak{p} + x_{(13)}(23)\mathfrak{p} + x_{(23)}(13)\mathfrak{p} = 0$  and  $(1 + \tau + \tau^{-1})x_{(12)}(13)\mathfrak{p} = 0$ . Meanwhile, from (13) and (14), the action map on the components of weight  $(\tau, 1)$ ,  $(\tau, 2)$  and  $(e, \rho)$  is not zero. In fact, for  $\ell = 1, 2$ :

$$\begin{aligned} (\zeta^\ell + \zeta^{-\ell}\tau^{-1} + \tau)x_{(12)}(13)\mathfrak{p} &= (\zeta^\ell + \zeta^{-\ell}\tau^{-1} + \tau)(x_{(12),x_{(13)},x_{(23)}}|\mathbf{12}\rangle + x_{(12),x_{(13)},x_{(12)}}|\mathbf{13}\rangle) \\ &= (\zeta^\ell - 1)x_{(12),x_{(13)},x_{(23)}}|\mathbf{12}\rangle + (\zeta^\ell - \zeta^{-\ell})x_{(12),x_{(13)},x_{(12)}}|\mathbf{13}\rangle + (1 - \zeta^{-\ell})x_{(13),x_{(12)},x_{(23)}}|\mathbf{23}\rangle. \end{aligned}$$

For the component  $(e, \rho)$  we have that  $x_{(12)}\mathfrak{p} + \zeta x_{(13)}(23)\mathfrak{p} + \zeta^{-1}x_{(23)}(13)\mathfrak{p} = 2(\zeta - \zeta^{-1})\mathfrak{u}$ . Hence  $V\mathbf{P} = \mathbf{U} \oplus \mathbf{N}_1 \oplus \mathbf{N}_2$ . Since  $V\mathbf{N}_\ell \subseteq \mathbf{M}_{\text{soc}}$ ,  $V\mathbf{U} \subseteq \mathbf{M}_{\text{soc}}$  and  $\mathbf{R} \subset \mathbf{S}(\sigma, -)$ , we conclude that  $\mathfrak{D}\mathbf{O}_0 = \mathbf{OPU}$  and (i) follows.

For the proof of (ii) we note that  $\mathbf{O}_0 \subset \mathfrak{D}\mathbf{P}$  and  $\mathbf{O}_0 \subset \mathfrak{D}\mathbf{U}$  since

$$(1 + \tau + \tau^2)y_{(12)}(13)\mathfrak{p} = \zeta(1 + \tau + \tau^2)y_{(12)}(13)(1 + \sigma)y_{(12)}\mathfrak{u} = 2(1 - \zeta)\mathfrak{o}_0 \in \mathbf{O}_0.$$

□

The next result is a direct consequence of the above lemma.

**Corollary 24.** *The quotient  $\mathbf{OPU}/\mathbf{S}(\sigma, -)$  is a simple highest-weight module of weight  $(\tau, 0)$ . Therefore*

$$\mathbf{L}(\tau, 0) \simeq M(\tau, 0) \oplus M(\sigma, +) \oplus M(e, \rho)$$

as  $\mathfrak{D}(\mathbb{S}_3)$ -modules. □

**Corollary 25.** *As  $\mathfrak{D}$ -modules,  $\mathbf{L}(\tau, 0) \simeq \mathbf{S}(e, \rho)$  and  $\mathbf{L}(e, \rho) \simeq \mathbf{S}(\tau, 0)$ .*

*Proof.* Since  $\mathbf{U} \subset \mathbf{M}^{-3}(\sigma, -)$ ,  $V\mathbf{U} \subset \mathbf{M}_{\text{soc}}$ . Hence  $\mathbf{U}$  is a lowest-weight in the quotient  $\mathbf{OPU}/\mathbf{S}(\sigma, -)$ . Therefore this quotient is isomorphic to  $\mathbf{S}(e, \rho)$  by Theorem 4. The proof of the second isomorphism is similar. □

The  $\mathfrak{D}(\mathbb{S}_3)$ -submodules of  $M(\sigma, -)$  of weight  $(e, +)$  are

- $\mathfrak{t}_{\lambda,\mu} = \mathbb{k}(\lambda|\mathfrak{e} + \mu|\mathfrak{e}^3)$  where  $\lambda, \mu \in \mathbb{k}$ ,

$$|\mathfrak{e}\rangle^1 = x_{(12)}|\mathbf{12}\rangle + x_{(23)}|\mathbf{23}\rangle + x_{(13)}|\mathbf{13}\rangle \quad \text{and} \quad |\mathfrak{e}\rangle^3 = x_{(13),x_{(12)},x_{(23)}}|\mathbf{12}\rangle - x_{(12),x_{(13)},x_{(12)}}|\mathbf{23}\rangle + x_{(12),x_{(13)},x_{(23)}}|\mathbf{13}\rangle.$$

**Lemma 26.** *Let  $\mathbf{T}_{\lambda,\mu} = \mathfrak{t}_{\lambda,\mu} \oplus \mathbf{S}(\sigma, -)$  for any  $\lambda, \mu \in \mathbb{k}$ . Then  $\mathfrak{D}\mathfrak{t}_{\lambda,\mu} = \mathbf{T}_{\lambda,\mu}$ . In particular,  $\mathbf{T}_{1,0}$  is a highest-weight submodule of weight  $(e, +)$ .*

*Proof.* We see that  $y_{(12)}|e\rangle^1 = 0$  and  $y_{(12)}|e\rangle^3 = -(13)r \in \mathbb{R}$ . Meanwhile,  $x_{(12)}|e\rangle^1 = -(13)r$  and clearly,  $x_{(12)}|e\rangle^3 \in M_{\text{soc}}$ . Then  $\overline{V}T_{\lambda,\mu}, \overline{V}T_{\lambda,\mu} \subset \mathbb{S}(\sigma, -)$  and the lemma follows.  $\square$

**Corollary 27.** *The quotient  $T_{1,0}/\mathbb{S}(\sigma, -)$  is a simple highest-weight module of weight  $(e, +)$ . Therefore*

$$L(e, +) \simeq M(e, +)$$

as  $\mathcal{D}(\mathbb{S}_3)$ -modules. Moreover,  $L(e, +) \simeq \mathbb{S}(e, +)$  as  $\mathcal{D}$ -modules.

*Proof.* The first part follows as the above corollaries. In particular,  $L(e, +)$  is one-dimensional and then it is also a lowest-weight module. Hence  $L(e, +) \simeq \mathbb{S}(e, +)$  holds.  $\square$

*Proof of Theorem 7.* Let  $N$  be a  $\mathcal{D}$ -submodule of  $M(\sigma, -)$ . Then  $N = \oplus_t S_t$  where  $S_t$  is  $\mathcal{D}(\mathbb{S}_3)$ -simple. Since  $N = \sum_t \mathcal{D}S_t$ , it is enough to compute the  $\mathcal{D}$ -submodule generated by  $S_t$  case-by-case according to the weight of  $S_t$ . Recall the weights of  $M(\sigma, -)$  from Figure 1.

- (Case 1) If  $S_t$  is of weight  $(e, +)$ , then  $\mathcal{D}S_t = T_{\lambda,\mu}$  for some  $\lambda, \mu \in \mathbb{k}$  by Lemma 26.
- (Case 2) If  $S_t$  is of weight  $(e, \rho)$ , then there is an element  $\mathbf{a} + \mathbf{b} \in S_t$  with  $\mathbf{a} \in U$  and  $\mathbf{b} \in V$ . Assume  $\mathbf{a} \neq 0 \neq \mathbf{b}$ , otherwise  $\mathcal{D}S_t$  is either UPO or VQN by Lemmas 21 and 23. Then  $\overline{V}S_t = \overline{V}U = \mathbb{R} \oplus P$  because  $\overline{V}V = 0$ , and hence  $\mathcal{D}P = \text{OPU} \subset \mathcal{D}S_t$ . Thus  $(\mathbf{a} + \mathbf{b}) - \mathbf{a} = \mathbf{b} \in \mathcal{D}S_t$  and therefore  $\mathcal{D}S_t = \text{OPU} + \text{VQN}$ .
- (Case 3) If  $S_t$  is of weight  $(\tau, 0)$  or  $(\sigma, +)$ . Proceeding as above, we can see that  $\mathcal{D}S_t \subseteq \text{OPU} + \text{VQN}$ .
- (Case 4) If  $S_t$  is of weight  $(\tau, \ell)$  with  $\ell \neq 0$ , then there is an element  $\mathbf{c} + \mathbf{d} \in S_t$  with  $\mathbf{c} \in N_\ell$  and  $\mathbf{d} \in O_\ell$ . Moreover,  $\mathbf{d} \in \mathcal{D}S_t$  as  $N_\ell \subseteq \mathbb{S}(\sigma, -) \subseteq \mathcal{D}S_t$ . Then either  $\mathcal{D}S_t = \mathbb{S}(\sigma, -)$ , if  $\mathbf{d} = 0$ , or  $\mathcal{D}S_t = M(\sigma, -)$  because  $y_{(12)}\mathbf{d} = (\zeta^{-\ell} - \zeta^\ell)|\mathbf{23}\rangle \in \mathcal{D}S_t$ .
- (Case 5) If  $S_t$  is of weight  $(\sigma, -)$ , then either  $\mathcal{D}S_t = \mathbb{S}(\sigma, -)$  or there is  $0 \neq \mathbf{y} + \mathbf{t} \in S_t$  with  $\mathbf{y} \in M^0(\sigma, -)$  and  $\mathbf{t} \in M^{-2}(\sigma, -) \setminus \mathbb{S}(\sigma, -)$ . If  $\mathbf{y} + \mathbf{t} \neq 0$ , then  $\mathcal{D}S_t = M(\sigma, -)$  by Lemma 13 when  $\mathbf{y} \neq 0$  or by noting that  $\overline{V}\mathbf{t} = M^0(\sigma, -)$  when  $\mathbf{t} \neq 0$ .

$\square$

#### 4.4. The Verma module $M(e, +)$

**Theorem 8.** *The proper  $\mathcal{D}$ -submodules of  $M(e, +)$  are  $\mathbb{S}(e, +) \subset X(e, +)$  where*

- (i)  $X(e, +) = \oplus_{n < 0} M^n(e, +)$  is a highest-weight submodule of weight  $(\sigma, -)$ .
- (ii)  $\mathbb{S}(e, +) = M_{\text{soc}}$  is a highest-weight submodule of weight  $(e, +)$ .

Therefore  $\{L(e, +), L(\sigma, -), L(e, +)\}$  are the composition factors of  $M(e, +)$ .

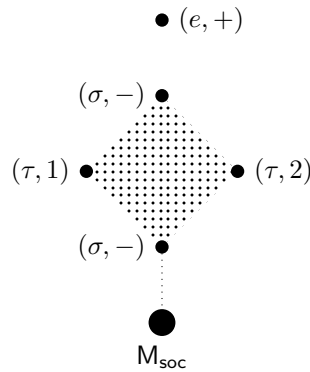


Figure 2: Submodules of  $M(e, +)$

In the Figure 2, we have schemed the submodules of  $M(e, +)$ . The weights are represented by dots, notice that  $M(e, +) \simeq \mathcal{B}(V)$  as  $\mathcal{D}(\mathbb{S}_3)$ -module. The big dot in the bottom represents the socle  $\mathbb{S}(e, +)$ . The dotted area represents the maximal  $\mathcal{D}$ -submodule  $X(e, +)$  and hence the dot in the top corresponds to  $L(e, +)$ .

*Proof.* By Corollary 27,  $L(e, +) \simeq M(e, +)$  as  $\mathfrak{D}(\mathbb{S}_3)$ -module and hence  $X(e, +) = \bigoplus_{n < 0} M^n(e, +)$ . Since  $X(e, +)$  is homogeneous by Theorem 1,  $\overline{V}M^{-1}(e, +) = 0$ . Moreover,  $M^{-1}(e, +)$  generates  $X(e, +)$  because  $M(e, +) \simeq \mathfrak{B}(V)$  as  $\mathfrak{B}(V)$ -modules. Therefore  $X(e, +)$  is a highest-weight submodule of weight  $(\sigma, -)$  and (i) follows.

By (i) and Theorem 3,  $X(e, +)$  has a quotient isomorphic to  $L(\sigma, -)$ . By Theorem 7,  $L(\sigma, -) \simeq M(\sigma, -) \oplus M(\tau, 1) \oplus M(\tau, 2) \oplus M(\sigma, -)$  as  $\mathfrak{D}(\mathbb{S}_3)$ -modules. Then, we deduce that the unique  $\mathfrak{D}$ -submodule of  $X(e, +)$  is  $M_{\text{soc}}$  by inspection in the weights of the Verma module, see Figure 2. Therefore  $S(e, +) = M_{\text{soc}}$  and (ii) follows.

The last sentence of the statement is immediate.  $\square$

#### 4.5. The Verma module $M(\tau, 0)$

Let  $J \subset M^{-1}(\tau, 0)$  and  $G \subset M^{-2}(\tau, 0)$  be the  $\mathfrak{D}(\mathbb{S}_3)$ -submodules of weight  $(\sigma, -)$  and  $(e, \rho)$  with basis

- $j_i = (1 - \sigma\tau^i)x_{\sigma\tau^{i+2}}|\tau\rangle$ ,  $i = 0, 1, 2$ , see (12);
- $g = (x_{(13)x(23)} - \zeta^2 x_{(12)x(13)})|\tau\rangle + (x_{(13)x(12)} - \zeta^2 x_{(12)x(23)})|\tau^{-1}\rangle$  and  $\sigma g$ ,

recall (10) and Lemma 19 (iii).

**Theorem 9.** *The proper  $\mathfrak{D}$ -submodules of  $M(\tau, 0)$  are  $S(\tau, 0) \subset X(\tau, 0)$  where*

- (i)  $X(\tau, 0) = \mathfrak{D}J$  is a highest-weight submodule of weight  $(\sigma, -)$ .
- (ii)  $S(\tau, 0) = \mathfrak{D}G$  is a highest-weight submodule of weight  $(e, \rho)$ .

Therefore  $\{L(\tau, 0), L(\sigma, -), L(e, \rho)\}$  are the composition factors of  $M(\tau, 0)$ .

The weights of  $M(\tau, 0)$  are represented by dots in the Figure 3 which can be computed using the fusion rules in §2.5.4. The weights connected by a line form the socle  $S(\tau, 0)$  and those over the dotted area form the maximal  $\mathfrak{D}$ -submodule  $X(\tau, 0)$ . The weights on the left hand side correspond to  $L(\tau, 0)$ .

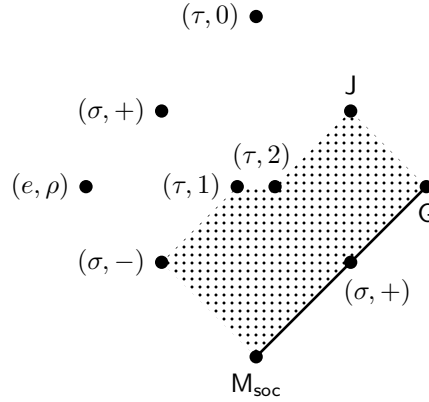


Figure 3: Submodules of  $M(\tau, 0)$

*Proof.* (ii) By Corollary 25,  $S(\tau, 0)$  is a highest-weight module of weight  $(e, \rho)$ . We see that  $\overline{V}G = 0$  using Remark 3. Therefore  $S(\tau, 0) = \mathfrak{D}G$ .

(i) By Corollary 24,  $L(\tau, 0) \simeq M(\tau, 0) \oplus M(\sigma, +) \oplus M(e, \rho)$  as  $\mathfrak{D}(\mathbb{S}_3)$ -module. Hence the sum of the simple  $\mathfrak{D}(\mathbb{S}_3)$ -submodules over the dotted area in Figure 3 have to form the maximal  $\mathfrak{D}$ -submodule  $X(\tau, 0)$ . Since  $X(\tau, 0)$  is homogeneous,  $\overline{V}J = 0$  and then  $\mathfrak{D}J$  is a highest-weight submodule of weight  $(\sigma, -)$ . On the other hand,  $\mathfrak{D}J$  has a quotient isomorphic to  $L(\sigma, -)$  and contains the socle  $S(\tau, 0)$ . Therefore  $X(\tau, 0) = \mathfrak{D}J$  and  $S(\tau, 0)$  is the unique  $\mathfrak{D}$ -submodule of  $X(\tau, 0)$ .  $\square$

#### 4.6. The Verma module $M(e, \rho)$

The proof of the next theorem and the description of Figure 4 are similar to the above subsection. Let  $E \subset M^{-1}(e, \rho)$  and  $C \subset M^{-2}(e, \rho)$  be the  $\mathfrak{D}(\mathbb{S}_3)$ -submodules of weight  $(\sigma, -)$  and  $(\tau, 0)$  with basis

- $e_i = \zeta^i(1 - \sigma\tau^i)x_{\sigma\tau^{i+2}}|\tau\rangle$ ,  $i = 0, 1, 2$ , see (15);
- $c = (\zeta x_{(13),x(12)} - x_{(12),x(23)})|\tau\rangle_\rho + (x_{(12),x(23)} - \zeta^{-1}x_{(13),x(12)})|\tau^{-1}\rangle_\rho$  and  $\sigma c$ ,

recall (9) and Lemma 19 (iii).

**Theorem 10.** *The proper  $\mathfrak{D}$ -submodules of  $M(e, \rho)$  are  $S(e, \rho) \subset X(e, \rho)$  where*

- $X(e, \rho) = \mathfrak{D}E$  is a highest-weight submodule of weight  $(\sigma, -)$ .
- $S(e, \rho) = \mathfrak{D}C$  is a highest-weight submodule of weight  $(\tau, 0)$ .

Therefore  $\{L(e, \rho), L(\sigma, -), L(\tau, 0)\}$  are the composition factors of  $M(e, \rho)$ . □

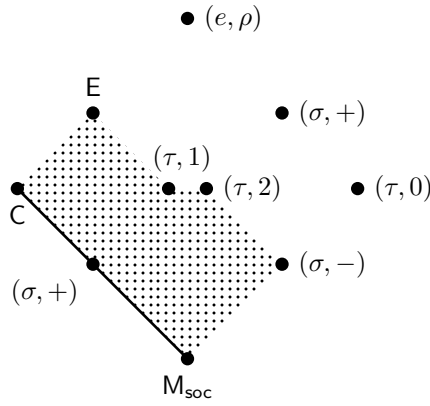


Figure 4: Submodules of  $M(e, \rho)$

## Appendix

Here we compute the action of  $y_{(12)} \in \mathfrak{D}$  on the Verma Modules. We noticed in (35) that it suffices to calculate the action of  $y_{(12)}$  to know the action of the generators  $y_{(23)}$  and  $y_{(13)}$ .

For the modules  $M(e, \pm)$  and  $M(e, \rho)$  we have only one list since all elements have weight  $e$ . For the module  $M(\sigma, \pm)$  we have three lists (as the elements may have weight (12), (13) or (23)) and for the module  $M(\tau, \ell)$  we have two lists (for the possible weights (123) and (132)).

### List 1: Action on $M(e, \pm)$

$$\begin{aligned}
 y_{(12)} \cdot (x_{(12)}|e\rangle_{\pm}) &= (1 \mp 1)|e\rangle_{\pm} \\
 y_{(12)} \cdot (x_{(13)}|e\rangle_{\pm}) &= 0 \\
 y_{(12)} \cdot (x_{(23)}|e\rangle_{\pm}) &= 0 \\
 y_{(12)} \cdot (x_{(12),x(13)}|e\rangle_{\pm}) &= x_{(13)}|e\rangle_{\pm} \\
 y_{(12)} \cdot (x_{(12),x(23)}|e\rangle_{\pm}) &= x_{(23)}|e\rangle_{\pm} \\
 y_{(12)} \cdot (x_{(13),x(12)}|e\rangle_{\pm}) &= \mp x_{(23)}|e\rangle_{\pm} \\
 y_{(12)} \cdot (x_{(13),x(23)}|e\rangle_{\pm}) &= -x_{(13)}(1 \mp 1)|e\rangle_{\pm} \\
 y_{(12)} \cdot (x_{(12),x(13),x(12)}|e\rangle_{\pm}) &= x_{(13),x(12)}|e\rangle_{\pm} \pm x_{(12),x(23)}|e\rangle_{\pm} \\
 y_{(12)} \cdot (x_{(12),x(13),x(23)}|e\rangle_{\pm}) &= x_{(12),x(13)}(1 \mp 1)|e\rangle_{\pm} + x_{(13),x(23)}|e\rangle_{\pm}
 \end{aligned}$$

$$y_{(12)} \cdot (x_{(13)}x_{(12)}x_{(23)}|e\rangle_{\pm}) = 0$$

$$y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(12)}x_{(23)}|e\rangle_{\pm}) = x_{(13)}x_{(12)}x_{(23)}(1 \mp 1)|e\rangle_{\pm}$$

List 2: Action on  $M(e, \rho)$

$$y_{(12)} \cdot (x_{(12)}|\tau^{\pm 1}\rangle_{\rho}) = 1|\tau^{\pm 1}\rangle_{\rho} - 1|\tau^{\mp 1}\rangle_{\rho}$$

$$y_{(12)} \cdot (x_{(13)}|\tau^{\pm 1}\rangle_{\rho}) = 0$$

$$y_{(12)} \cdot (x_{(23)}|\tau^{\pm 1}\rangle_{\rho}) = 0$$

$$y_{(12)} \cdot (x_{(12)}x_{(13)}|\tau^{\pm 1}\rangle_{\rho}) = x_{(13)}|\tau^{\pm 1}\rangle_{\rho}$$

$$y_{(12)} \cdot (x_{(12)}x_{(23)}|\tau^{\pm 1}\rangle_{\rho}) = x_{(23)}|\tau^{\pm 1}\rangle_{\rho}$$

$$y_{(12)} \cdot (x_{(13)}x_{(12)}|\tau^{\pm 1}\rangle_{\rho}) = -\zeta^{\pm 1}x_{(23)}|\tau^{\mp 1}\rangle_{\rho}$$

$$y_{(12)} \cdot (x_{(13)}x_{(23)}|\tau^{\pm 1}\rangle_{\rho}) = -x_{(13)}|\tau^{\pm 1}\rangle_{\rho} + \zeta^{\pm 1}x_{(13)}|\tau^{\mp 1}\rangle_{\rho}$$

$$y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(12)}|\tau^{\pm 1}\rangle_{\rho}) = x_{(13)}x_{(12)}|\tau^{\pm 1}\rangle_{\rho} + \zeta^{\mp 1}x_{(12)}x_{(23)}|\tau^{\mp 1}\rangle_{\rho}$$

$$y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(23)}|\tau^{\pm 1}\rangle_{\rho}) = x_{(12)}x_{(13)}|\tau^{\pm 1}\rangle_{\rho} - \zeta^{\pm 1}x_{(12)}x_{(13)}|\tau^{\mp 1}\rangle_{\rho} + x_{(13)}x_{(23)}|\tau^{\pm 1}\rangle_{\rho}$$

$$y_{(12)} \cdot (x_{(13)}x_{(12)}x_{(23)}|\tau^{\pm 1}\rangle_{\rho}) = 0$$

$$y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(12)}x_{(23)}|\tau^{\pm 1}\rangle_{\rho}) = x_{(13)}x_{(12)}x_{(23)}|\tau^{\pm 1}\rangle_{\rho} - x_{(13)}x_{(12)}x_{(23)}|\tau^{\mp 1}\rangle_{\rho}$$

List 3: Action on  $M(\sigma, \pm)$

$$y_{(12)} \cdot (x_{(12)}|\mathbf{12}\rangle_{\pm}) = (1 \pm 1)|\mathbf{12}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(13)}|\mathbf{12}\rangle_{\pm}) = \pm 1|\mathbf{23}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(23)}|\mathbf{12}\rangle_{\pm}) = \pm 1|\mathbf{13}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(12)}x_{(13)}|\mathbf{12}\rangle_{\pm}) = x_{(13)}|\mathbf{12}\rangle_{\pm} \mp x_{(12)}|\mathbf{23}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(12)}x_{(23)}|\mathbf{12}\rangle_{\pm}) = x_{(23)}|\mathbf{12}\rangle_{\pm} \mp x_{(12)}|\mathbf{13}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(13)}x_{(12)}|\mathbf{12}\rangle_{\pm}) = 0$$

$$y_{(12)} \cdot (x_{(13)}x_{(23)}|\mathbf{12}\rangle_{\pm}) = -x_{(13)}|\mathbf{12}\rangle_{\pm} \pm x_{(12)}|\mathbf{23}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(12)}|\mathbf{12}\rangle_{\pm}) = x_{(13)}x_{(12)}|\mathbf{12}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(23)}|\mathbf{12}\rangle_{\pm}) = x_{(13)}x_{(23)}|\mathbf{12}\rangle_{\pm} + x_{(12)}x_{(13)}|\mathbf{12}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(13)}x_{(12)}x_{(23)}|\mathbf{12}\rangle_{\pm}) = 0$$

$$y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(12)}x_{(23)}|\mathbf{12}\rangle_{\pm}) = x_{(13)}x_{(12)}x_{(23)}(1 \pm 1)|\mathbf{12}\rangle_{\pm}$$

List 4: Action on  $M(\sigma, \pm)$

$$y_{(12)} \cdot (x_{(12)}|\mathbf{13}\rangle_{\pm}) = 1|\mathbf{13}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(13)}|\mathbf{13}\rangle_{\pm}) = 0$$

$$y_{(12)} \cdot (x_{(23)}|\mathbf{13}\rangle_{\pm}) = 0$$

$$y_{(12)} \cdot (x_{(12)}x_{(13)}|\mathbf{13}\rangle_{\pm}) = x_{(13)}|\mathbf{13}\rangle_{\pm} \pm x_{(23)}|\mathbf{23}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(12)}x_{(23)}|\mathbf{13}\rangle_{\pm}) = x_{(23)}|\mathbf{13}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(13)}x_{(12)}|\mathbf{13}\rangle_{\pm}) = \pm x_{(23)}|\mathbf{13}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(13)}x_{(23)}|\mathbf{13}\rangle_{\pm}) = -x_{(13)}|\mathbf{13}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(12)}|\mathbf{13}\rangle_{\pm}) = x_{(13)}x_{(12)}|\mathbf{13}\rangle_{\pm} \mp x_{(12)}x_{(23)}|\mathbf{13}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(23)}|\mathbf{13}\rangle_{\pm}) = (x_{(13)}x_{(23)} + x_{(12)}x_{(13)})|\mathbf{13}\rangle_{\pm} \mp (x_{(13)}x_{(12)} + x_{(12)}x_{(23)})|\mathbf{23}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(13)}x_{(12)}x_{(23)}|\mathbf{13}\rangle_{\pm}) = \pm x_{(13)}x_{(12)}|\mathbf{12}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(12)}x_{(23)}|\mathbf{13}\rangle_{\pm}) = x_{(13)}x_{(12)}x_{(23)}|\mathbf{13}\rangle_{\pm} \mp x_{(12)}x_{(13)}x_{(12)}|\mathbf{12}\rangle_{\pm}$$

List 5: Action on  $M(\sigma, \pm)$

$$y_{(12)} \cdot (x_{(12)}|\mathbf{23}\rangle_{\pm}) = 1|\mathbf{23}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(13)}|\mathbf{23}\rangle_{\pm}) = 0$$

$$y_{(12)} \cdot (x_{(23)}|\mathbf{23}\rangle_{\pm}) = 0$$

$$y_{(12)} \cdot (x_{(12)}x_{(13)}|\mathbf{23}\rangle_{\pm}) = x_{(13)}|\mathbf{23}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(12)}x_{(23)}|\mathbf{23}\rangle_{\pm}) = x_{(23)}|\mathbf{23}\rangle_{\pm} \pm x_{(13)}|\mathbf{13}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(13)}x_{(12)}|\mathbf{23}\rangle_{\pm}) = \mp x_{(13)}|\mathbf{13}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(13)}x_{(23)}|\mathbf{23}\rangle_{\pm}) = -x_{(13)}(1 \pm 1)|\mathbf{23}\rangle_{\pm}$$

$$y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(12)}|\mathbf{23}\rangle_{\pm}) = x_{(13)}x_{(12)}|\mathbf{23}\rangle_{\pm} \mp x_{(13)}x_{(23)}|\mathbf{13}\rangle_{\pm}$$

$$\begin{aligned}
y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(23)}|\mathbf{23}\rangle_{\pm}) &= x_{(13)}x_{(23)}|\mathbf{23}\rangle_{\pm} - x_{(12)}x_{(13)}(1 \pm 1)|\mathbf{23}\rangle_{\pm} \\
y_{(12)} \cdot (x_{(13)}x_{(12)}x_{(23)}|\mathbf{23}\rangle_{\pm}) &= \mp x_{(12)}x_{(13)}|\mathbf{12}\rangle_{\pm} \mp x_{(13)}x_{(23)}|\mathbf{12}\rangle_{\pm} \\
y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(12)}x_{(23)}|\mathbf{23}\rangle_{\pm}) &= \pm x_{(12)}x_{(13)}x_{(23)}|\mathbf{12}\rangle_{\pm} + x_{(13)}x_{(12)}x_{(23)}|\mathbf{23}\rangle_{\pm}
\end{aligned}$$

List 6: Action on  $M(\tau, \ell)$

$$\begin{aligned}
y_{(12)} \cdot (x_{(12)}|\mathbf{123}\rangle_{\ell}) &= 1|\mathbf{123}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(13)}|\mathbf{123}\rangle_{\ell}) &= 0 \\
y_{(12)} \cdot (x_{(23)}|\mathbf{123}\rangle_{\ell}) &= -\zeta^{\ell}|\mathbf{132}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(12)}x_{(13)}|\mathbf{123}\rangle_{\ell}) &= x_{(13)}|\mathbf{123}\rangle_{\ell} - x_{(23)}|\mathbf{132}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(12)}x_{(23)}|\mathbf{123}\rangle_{\ell}) &= x_{(23)}|\mathbf{123}\rangle_{\ell} + \zeta^{\ell}x_{(12)}|\mathbf{132}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(13)}x_{(12)}|\mathbf{123}\rangle_{\ell}) &= 0 \\
y_{(12)} \cdot (x_{(13)}x_{(23)}|\mathbf{123}\rangle_{\ell}) &= -x_{(13)}|\mathbf{123}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(12)}|\mathbf{123}\rangle_{\ell}) &= x_{(13)}x_{(12)}|\mathbf{123}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(23)}|\mathbf{123}\rangle_{\ell}) &= (x_{(13)}x_{(23)} + x_{(12)}x_{(13)})|\mathbf{123}\rangle_{\ell} + (x_{(13)}x_{(12)} + x_{(12)}x_{(23)})|\mathbf{132}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(13)}x_{(12)}x_{(23)}|\mathbf{123}\rangle_{\ell}) &= \zeta^{-\ell}x_{(12)}x_{(13)}|\mathbf{132}\rangle_{\ell} + \zeta^{-\ell}x_{(13)}x_{(23)}|\mathbf{132}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(12)}x_{(23)}|\mathbf{123}\rangle_{\ell}) &= x_{(13)}x_{(12)}x_{(23)}|\mathbf{123}\rangle_{\ell} - \zeta^{-\ell}x_{(12)}x_{(13)}x_{(23)}|\mathbf{132}\rangle_{\ell}
\end{aligned}$$

List 7: Action on  $M(\tau, \ell)$

$$\begin{aligned}
y_{(12)} \cdot (x_{(12)}|\mathbf{132}\rangle_{\ell}) &= 1|\mathbf{132}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(13)}|\mathbf{132}\rangle_{\ell}) &= -\zeta^{\ell}|\mathbf{123}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(23)}|\mathbf{132}\rangle_{\ell}) &= 0 \\
y_{(12)} \cdot (x_{(12)}x_{(13)}|\mathbf{132}\rangle_{\ell}) &= x_{(13)}|\mathbf{132}\rangle_{\ell} + \zeta^{\ell}x_{(12)}|\mathbf{123}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(12)}x_{(23)}|\mathbf{132}\rangle_{\ell}) &= x_{(23)}|\mathbf{132}\rangle_{\ell} - x_{(13)}|\mathbf{123}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(13)}x_{(12)}|\mathbf{132}\rangle_{\ell}) &= x_{(13)}|\mathbf{123}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(13)}x_{(23)}|\mathbf{132}\rangle_{\ell}) &= \zeta^{\ell}x_{(12)}|\mathbf{123}\rangle_{\ell} - x_{(13)}|\mathbf{132}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(12)}|\mathbf{132}\rangle_{\ell}) &= x_{(13)}x_{(12)}|\mathbf{132}\rangle_{\ell} + x_{(13)}x_{(23)}|\mathbf{123}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(23)}|\mathbf{132}\rangle_{\ell}) &= x_{(13)}x_{(23)}|\mathbf{132}\rangle_{\ell} + x_{(12)}x_{(13)}|\mathbf{132}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(13)}x_{(12)}x_{(23)}|\mathbf{132}\rangle_{\ell}) &= -\zeta^{-\ell}x_{(13)}x_{(12)}|\mathbf{123}\rangle_{\ell} \\
y_{(12)} \cdot (x_{(12)}x_{(13)}x_{(12)}x_{(23)}|\mathbf{132}\rangle_{\ell}) &= x_{(13)}x_{(12)}x_{(23)}|\mathbf{132}\rangle_{\ell} + \zeta^{-\ell}x_{(12)}x_{(13)}x_{(12)}|\mathbf{123}\rangle_{\ell}
\end{aligned}$$

## References

## References

- [1] N. Andruskiewitsch, M. Graña, Braided Hopf algebras over non abelian finite groups, *Bol. Acad. Ciencias (Córdoba)* 63 (1999) 45–78.
- [2] N. Andruskiewitsch, H. J. Schneider, Pointed Hopf algebras, “New directions in Hopf algebras”, MSRI series Cambridge Univ. Press (2002), 1–68.
- [3] M. Beattie, Duals of pointed Hopf algebras, *J. Algebra* 262 (2003) 54–76.
- [4] V. G. Drinfeld, Quantum groups, *Proceedings of the International Congress of Mathematicians*, Vol. 1, 2 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.
- [5] J. Dixmier, Enveloping algebras, North-Holland Mathematical Library, Vol. 14. Translated from the French. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. xvi+375 pp.
- [6] S. Fomin, A. N. Kirillov, Quadratic algebras, Dunkl elements and Schubert calculus, *Progr. Math.* 172 (1999) 146–182.
- [7] M. Graña, Zoo of finite-dimensional Nichols algebras of non-abelian group type, <http://mate.dm.uba.ar/~matiasg/zoo.html>.
- [8] J. C. Jantzen, Lectures on quantum groups, Graduate Studies in Mathematics 6. AMS, Providence, RI, 1996. viii+266 pp.
- [9] G. Lusztig, Quantum groups at roots of 1, *Geom. Dedicata* 35 (1990) 89–113.
- [10] G. Lusztig, Introduction to quantum groups, *Progress in Mathematics* 110. Birkhäuser Boston, Inc., Boston, MA, 1993.
- [11] S. Majid, Crossed products by braided groups and bosonization, *J. Algebra* 163 (1994) 165–190.
- [12] S. Majid, Foundations of quantum group theory, Cambridge University Press, Cambridge (1995).
- [13] A. Milinski, H. J. Schneider, Pointed indecomposable Hopf algebras over Coxeter groups, *Contemp. Math.* 267 (2000) 215–236.
- [14] W. Nichols, Bialgebras of type one, *Comm. Algebra* 6 (1978) 1521–1552.
- [15] B. Pogorelsky, C. Vay, Representations of copointed Hopf algebras arising from the tetrahedron rack, *Ann. Univ. Ferrara Sez. VII Sci. Mat.* 60 (2014) 407–427.
- [16] D. Radford, Hopf algebras with a projection, *J. Algebra* 92 (1985) 322–347.