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Sharp Bounds for Fractional One-sided Operators

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Abstract In this paper, we characterize the sharp boundedness of the one-sided fractional maximal function for one-weight and two-weight inequalities. Also a new two-weight testing condition for the one-sided fractional maximal function is introduced extending the work of Martín-Reyes and de la Torre. Improving some extrapolation result for the one-sided case, we get weak sharp bounded estimates for one-sided fractional maximal function and weak and strong sharp bounded estimates for one-sided fractional integral.

Keywords One-sided fractional integrals, one-sided maximal fractional function, extrapolation, Sawyer weights

MR(2010) Subject Classification 42B20, 42B25

1 Introduction

Given an operator T defined on the set of medibles functions in $\mathbb{R}$, we say that T is a one-sided operator if $Tf(x)$ depends only on the values of f in the interval $[x,\infty)$, in other words $Tf(x) = T(f\chi_{[x,\infty)})(x)$. In a similar way, it can be considered $Tf(x) = T(f\chi_{(-\infty,x]})(x)$.

Examples of this kind of operators are the classical fractional integral Weyl operator I_α^+ and the fractional integral Riemann–Liouville operator I_α^-, that are defined in the following way: given $0<\alpha<1$ and $f \in L^1_{\rm loc}(\mathbb{R})$

$$I_\alpha^+ f(x) := \left(f(y) * \frac{1}{|y|^{1-\alpha}}\chi_{(-\infty,0)}(y)\right)(x) = \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}}dy$$

and

$$I_\alpha^- f(x) := \left(f(y) * \frac{1}{|y|^{1-\alpha}}\chi_{(0,\infty)}(y)\right)(x) = \int_{-\infty}^x \frac{f(y)}{(x-y)^{1-\alpha}}dy.$$

For $0<\alpha<n$, let us consider the convolution with the kernel $\frac{1}{|y|^{n-\alpha}}$, in this case, we obtain the Riezs potential operator I_α:

$$I_\alpha f(x) := \left(f(y) * \frac{1}{|y|^{n-\alpha}}\right)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

For each of this operators, there exists a maximal function that gives a Coifman type inequality. For the Weyl and Riemann–Liouville operators, these are the one-sided maximal fractional functions M_α^+ and M_α^- respectively and for the Riezs potential I_α, is the classical fractional maximal function M_α.

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This operators are defined in the following way:

Let $0 \le \alpha < n$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the fractional maximal operator is

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|dy.$$

For $0 \le \alpha < 1$ and $f \in L^1_{\text{loc}}(\mathbb{R})$, the one-sided fractional maximal operators are

$$M^+_\alpha f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_x^{x+h} |f(t)|dt, \quad M^-_\alpha f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_{x-h}^{x} |f(t)|dt.$$

If $\alpha = 0$, we write $M_\alpha = M$, $M^+_\alpha = M^+$ and $M^-_\alpha = M^-$ the classical Hardy–Littlewood maximal function.

The Coifman type inequality is the following: Let $p > 0$. Then

$$\int_{\mathbb{R}^n} |T_\alpha f(x)|^p w(x)dx \le C \int_{\mathbb{R}^n} |N_\alpha f(x)|^p w(x)dx,$$

where $T_\alpha = I_\alpha, I^+_\alpha, I^-_\alpha$, $N_\alpha = M_\alpha, M^+_\alpha, M^-_\alpha$ and $w \in A_\infty$, $w \in A^+_\infty$ or $w \in A^-_\infty$ as appropriate.

These operators were studied by different authors. Some well known results are:

• In 1928, Hardy and Littlewood gave the first norm inequality in Lebesgue norm for the Riezs potential for $n = 1$, see [7].

• In 1952, Hardy et al. proved the first norm inequalities for Lebesgue measure for the Weyl fractional integral operator, see [8].

• In 1974, Muckenhoupt and Wheeden introduced the $A_{p,q}$ classes of weights to study strong and weak inequalities with one weight for I_α and M_α, see [20].

• In 1984/88, Sawyer proved weighted inequalities for pair of weights for I_α and M_α, introducing new classes of weight, see [22] and [23].

• In 1988, Andersen and Sawyer introduced the $A^+_{p,q}$ to study weighted inequalities with one weight for I^+_α, see [1].

• In 1993, Martín-Reyes and de la Torre introduced the one-sided diadic maximal fractional function. They studied the relation between this maximal function and M^+_α and gave strong inequalities with different classes of weights, see [16].

• In 1997, Lorente and Martín-Reyes obtained the one-sided version of Sawyer's results for I^+_α introducing new classes of weights, see [12] and [13].

• In 1989, Gabidzashvili and Kokilashvili gave weighted weak (p, q) type inequalities, $1 \le p < \infty$, for I^+_α, see [10].

All these results did not consider how the inequality depends on the weight constant. In the last years, it has been studied how is this dependence, taking into account different classes of weights. Some of the results are the following:

• In 2009, Moen studied weighted inequalities for M_α for different classes of weights, obtaining sharp bounds respect to the weight constant, see [18].

• In 2010, Lacey et al. obtained weak and strong sharp bounds respect to the weight class $A_{p,q}$ constant for I_α, see [11].

• In 2013, Recchi studied for I_α, weak and strong sharp bound dependence of the constant in the extreme case, $w \in A_{1,q}$, see [21].

• In 2015, Martín-Reyes and de la Torre obtained strong sharp bounds with one weight in the class $A^+_{p,q}$ for M^+_α, see [17].

In this paper, we find the dependence of the weak and strong weighted-norm, for the one-sided operators, respect to the weight constant. Following Moen's ideas in [18], we prove sharp estimates for strong inequalities for the one-sided fractional function and different classes of weights. Finally improving extrapolation results for Sawyer classes, we are able to obtain sharp strong and weak bounds for the one-sided fractional operator respect to the weight constant, and weak two weighted sharp bounded estimates for one-sided fractional maximal function.

2 Description of Main Results

In order to state the main results, we will first define several classes of weights. A weight w will be a locally integrable function in $\mathbb{R}$ such that $w \geq 0$. First we start with Sawyer $A^+_{p,q}$ classes of weights, introduce by Andersen and Sawyer [1].

Definition 2.1 *Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, a pair of weights $(u, v) \in A^+_{p,q}$, if and only if*

$$\|(u,v)\|_{A^+_{p,q}} = \sup_{h>0} \operatorname*{ess\,sup}_{x\in\mathbb{R}} \left(\frac{1}{|h|}\int_{x-h}^{x} u(y)^q dy\right)\left(\frac{1}{h}\int_{x}^{x+h} v(y)^{-p'} dy\right)^{q/p'} \leq \infty,$$

where we understand for the case $q = \infty$ or $p = 1$, $\|\chi_{[x-h,x]}u\|_\infty$ and $\|\chi_{[x,x,+h]}v^{-1}\|_\infty$ respectively.

We say that $w \in A^+_p$ if and only if $(w^{1/p}, w^{1/p}) \in A^+_{p,p}$, we denote by $\|w\|_{A^+_p}$ the constant $\|(w^{1/p}, w^{1/p})\|_{A^+_{p,p}}$.

Lorente and Martín-Reyes proved in [12] and [13] that, for two positive locally integrable function u and v and $1 < p \leq q < \infty$, I^+_α is bounded from $L^p(v)$ into $L^q(u)$ if only if u and the function $\sigma = v^{1-p'}$ satisfy the testing conditions

$$[\sigma, u]_{S^+_{q',p'}} := \sup_I \left(\int_I u\right)^{-1/q'} \left(\int_I I^-_\alpha(\chi_I u)^{p'}\sigma\right)^{1/p'} < \infty,$$

and

$$[u, \sigma]_{S^-_{p,q}} := \sup_I \left(\int_I \sigma\right)^{-1/p} \left(\int_I I^+_\alpha(\chi_I \sigma)^{q} u\right)^{1/q} < \infty.$$

Moreover, their proof shows that actually

$$\|I^+_\alpha\|_{L^p(v)\to L^q(u)} \approx [\sigma, u]_{S^+_{q',p'}} + [u, \sigma]_{S^-_{p,q}}. \tag{2.1}$$

On the other hand, in their characterization of the weak two-weight type inequality, for I^+_α, they also showed that

$$\|I^+_\alpha\|_{L^p(v)\to L^{q,\infty}(u)} \approx [\sigma, u]_{S^+_{q',p'}}. \tag{2.2}$$

Combining (2.1) and (2.2), it follows that

$$\|I^+_\alpha\|_{L^p(v)\to L^q(u)} \approx \|I^+_\alpha\|_{L^p(v)\to L^{q,\infty}(u)} + \|I^-_\alpha\|_{L^{q'}(u^{1-q'})\to L^{p',\infty}(v^{1-p'})}.$$

If we set $u = w^q$ and $v = w^p$, we obtain the one-weight estimate

$$\|I^+_\alpha\|_{L^p(w^p)\to L^q(w^q)} \approx \|I^+_\alpha\|_{L^p(w^p)\to L^{q,\infty}(w^q)} + \|I^-_\alpha\|_{L^{q'}(w^{-q'})\to L^{p',\infty}(w^{-p'})}. \tag{2.3}$$

In [16], Martín-Reyes and de la Torre introduced the one-sided diadic maximal fractional function and studied the good weights for this diadic operator and the relation with the maximal fractional function.

From now on, each time we write I^- and I^+, we will mean contiguous intervals of equal length, not necessarily diadics. Given a pair of weight (u,v), we denote by σ the weight $v^{1-p'}$.

Let $x \in \mathbb{R}$ and consider the following family of diadic intervals

$$A_x^+ = \{I^- : I^- \text{ is a diadic interval such that } x \in I^-\}.$$

Definition 2.2 *Let $0 \le \alpha < 1$, $f \in L^1_{\mathrm{loc}(\mathbb{R})}$, the one-sided diadic maximal fractional function is defined by*

$$M^+_{\alpha,d} f(x) = \sup_{I^- \in A_x^+} \frac{1}{|I^+|^{1-\alpha}} \int_{I^+} |f(t)| dt.$$

Martín-Reyes and de la Torre showed that there exist constants C^1_α and C^2_α such that

$$M^+_\alpha f(x) \le C^1_\alpha M^+_{\alpha,d} f(x) \quad \text{and} \quad M^+_{\alpha,d} f(x) \le C^2_\alpha M^+_\alpha f(x). \tag{2.4}$$

They also introduced the following classes of weights.

Definition 2.3 *Given weights u, v, for $1 < p \le q$, we say that*

- $(u,v) \in S^+_{p,q,\alpha}$, *if, for all interval I, there exists a constant C such that,*

$$\int_I \sigma < \infty \quad \textit{and} \quad \left(\int_I (M^+_\alpha \sigma \chi_I)^q u \right)^{1/q} \le C \left(\int_I \sigma \right)^{1/p}.$$

- $(u,v) \in S^+_{p,q,\alpha,d}$ *if, for all intervals I^- and I^+, there exists a constant C_d such that*

$$\int_{I^- \cup I^+} \sigma < \infty \quad \textit{and} \quad \left(\int_{I^- \cup I^+} (M^+_{\alpha,d} \sigma \chi_{I^+})^q u \right)^{1/q} \le C_d \left(\int_{I^+} \sigma \right)^{1/p},$$

with $\int_{I^-} u > 0$.

We call the constant of the pair of weights $(u,v) \in S^+_{p,q,\alpha}$ to the smallest of the constants C and we denote it as $\|(u,v)\|_{S^+_{p,q,\alpha}}$ and we call the constant of the pair of weights $(u,v) \in S^+_{p,q,\alpha,d}$ to the smallest constant C_d and we denote it by $\|(u,v)\|_{S^+_{p,q,\alpha,d}}$.

In an analogous way for $1 < p \le q$, the classes $S^-_{p,q,\alpha}$ and $S^-_{p,q,\alpha,d}$ are defined.

Also we will need the following classes of pair of weights.

Definition 2.4 *Let $0 \le \alpha < 1$ and $1 < q < \infty$. Given u, v weights we say that*

- $(u,v) \in T^+_{q,\alpha}$ *if, for all interval I, there exists a constant $C > 0$ such that*

$$\int_I \sigma < \infty \quad \textit{and} \quad \left(\int_I (M^+ \sigma \chi_I)^{(1-\alpha)q} u dx \right)^{1/q} \le C \left(\int_I \sigma dx \right)^{1/q}.$$

- $(u,v) \in T^+_{q,\alpha,d}$ *if, for all intervals I^- and I^+, there exists a constant C_d*

$$\int_{I^- \cup I^+} \sigma < \infty \quad \textit{and} \quad \left(\int_{I^- \cup I^+} (M^+_d \sigma \chi_{I^+})^{(1-\alpha)q} u dx \right)^{1/q} \le C_d \left(\int_{I^+} \sigma dx \right)^{1/q},$$

with $\int_{I^-} u > 0$.

The smallest constant C will be called constant of the pair $(u,v) \in T^+_{q,\alpha}$ and will be denoted by $\|(u,v)\|_{T^+_{q,\alpha}}$. Also, the smallest constant C_d, will be called constant of the pair $(u,v) \in T^+_{q,\alpha,d}$ and will be denoted by $\|(u,v)\|_{T^+_{q,\alpha,d}}$.

Recently, in [17], Martín-Reyes and de la Torre proved the following result:

Theorem A ([17]) *Let $0 \le \alpha < 1$, $1 < p \le q < \infty$ with $1/p - 1/q = \alpha$, $w \in A^+_{p,q}$ and $f \in L^p(w)$. Then*

$$\|M^+_\alpha f\|_{L^q(w^q)} \le C\|w\|^{(1-\alpha)p'/q}_{A^+_{p,q}}\|f\|_{L^p(w^p)},$$

where the exponent $(1-\alpha)p'/q$ is sharp.

Now we are ready to state the results on sharp bounds depending on the constant of the weight for the one-sided maximal fractional operators. This results will be proved in Sections 5 and 6.

Theorem 2.5 *Let $0 \le \alpha < 1$, $1 < p \le q < \infty$, $(u,v) \in S^+_{p,q,\alpha,d}$ and $f \in L^p(v)$. There exists $C > 0$, which does not depend on the pair (u,v), such that*

$$\|M^+_{\alpha,d} f\|_{L^q(u)} \le C\|(u,v)\|_{S^+_{p,q,\alpha,d}}\|f\|_{L^p(v)}.$$

Corollary 2.6 *Let $0 \le \alpha < 1$, $1 < p \le q < \infty$, $(u,v) \in S^+_{p,q,\alpha}$ and $f \in L^p(v)$. Then*

$$\|M^+_\alpha f\|_{L^q(u)} \le C\|(u,v)\|_{S^+_{p,q,\alpha}}\|f\|_{L^p(v)},$$

where the constant C does not depend on the pair (u,v). Moreover, the dependence of the norm $\|M^+_\alpha\|_{L^p(v)\to L^q(u)}$ with respect to the constant $\|(u,v)\|_{S^+_{p,q,\alpha}}$ of the pair (u,v) is sharp.

Theorem 2.7 *Let $0 \le \alpha < 1$, $1 < p < \infty$, q such that $1/q = 1/p - \alpha$, $(u,v) \in T^+_{q,\alpha,d}$ and $f \in L^p(v)$. Then there exists $C > 0$ that does not depend on the pair (u,v), such that*

$$\|M^+_{\alpha,d} f\|_{L^q(u)} \le C\|(u,v)\|_{T^+_{q,\alpha,d}}\|f\|_{L^p(v)}.$$

Corollary 2.8 *Let $0 \le \alpha < 1$, $1 < p < \infty$, q such that $1/q = 1/p - \alpha$, $(u,v) \in T^+_{q,\alpha,d}$ and $f \in L^p(v)$.Then there exists $C > 0$*

$$\|M^+_\alpha f\|_{L^q(u)} \le C\|(u,v)\|_{T^+_{q,\alpha}}\|f\|_{L^p(v)},$$

where the constant C does not depend on the pair (u,v). Moreover, the dependence of the norm $\|M^+_\alpha\|_{L^p(v)\to L^q(u)}$ respect to the constant $\|(u,v)\|_{T^+_{q,\alpha}}$ of the pair of weights (u,v) is sharp.

Theorem 2.9 *Let $0 \le \alpha < 1$, $1 \le p \le q < \infty$, with $1/p - 1/q = \alpha$, $(u,v) \in A^+_{p,q}$ and $f \in L^p(v)$. Then there exists $C > 0$ such that*

$$\|M^+_\alpha f\|_{L^{q,\infty}(u^q)} \le C\|(u,v)\|^{1/q}_{A^+_{p,q}}\|f\|_{L^p(v^p)},$$

where the constant C does not depend on the pair (u,v). Moreover, the dependence of the norm $\|M^+_\alpha\|_{L^p(v^p)\to L^{q,\infty}(u^q)}$ with respect the constant $\|(u,v)\|_{A^+_{p,q}}$ of the pair of weights is sharp.

Now we give the results about sharp boundedness, depending on the constant of the weight for the Riemann–Liouville and the Weyl operators. To prove these results (see Sections 5 and 6) we will need extrapolation theorems that will be stated in Section 4 and proved in Section 5.

Theorem 2.10 *Let $0 < \alpha < 1$, $1 \le p < 1/\alpha$, q such that $1/q = 1/p - \alpha$. If $w \in A^+_{p,q}$ and $f \in L^p(w^p)$, then there exists $C > 0$ such that*

$$\|I^+_\alpha f\|_{L^{q,\infty}(w^q)} \le C\|w\|^{1-\alpha}_{A^+_{p,q}}\|f\|_{L^p(w^p)}, \tag{2.5}$$

where the constant C does not depend on the weight w. Moreover, the dependence of the norm $\|I^+_\alpha\|_{L^p(w^p)\to L^{q,\infty}(w^q)}$ respect to the constant $\|w\|_{A^+_{p,q}}$ of the weight w is sharp.

As a corollary we obtain the following strong estimate

Theorem 2.11 *Let $0 < \alpha < 1$, $1 < p < 1/\alpha$ and q such that $1/q = 1/p - \alpha$. If $w \in A^+_{p,q}$ and $f \in L^p(w^p)$, there exists $C > 0$ such that*

$$\|I^+_\alpha f\|_{L^q(w^q)} \le C\|w\|_{A^+_{p,q}}^{(1-\alpha)\max\{1,p'/q\}}\|f\|_{L^p(w^p)},$$

where the constant C does not depend on the weight w. Moreover, the dependence of the norm $\|I^+_\alpha\|_{L^p(w^p)\to L^q(w^q)}$ respect to the constant $\|w\|_{A^+_{p,q}}$ of the weight w is sharp.

The paper is organized as follows: In Section 3, we state and prove comparison of the different constants of the weights; in Section 4, we state some extrapolation results; in Section 5, we prove the main results and the extrapolation theorems and finally in Section 6, we show that the bounds of theorems in Section 2 are sharp.

3 Comparison of Different Weights Constants

In this section, we establish the relation between the different classes of weights defined previously. The following lemma gives the relation between $A^+_{p,q}$ and A^+_r. The proof follows immediately from the definitions.

Lemma 3.1 *Let $1 < p \le \infty$ and $1 \le q \le \infty$.*

(i) $(u,v) \in A^+_{p,q}$ *if, and only if* $(u^q, v^q) \in A^+_r$ *with* $r = 1 + q/p'$. *Moreover,*

$$\|(u,v)\|_{A^+_{p,q}} = \|(u^q, v^q)\|_{A^+_r}.$$

(ii) $(u,v) \in A^+_{p,q}$ *if, and only if* $(v^{-p'}, u^{-p'}) \in A^-_r$ *with* $r = 1 + p'/q$. *Moreover,*

$$\|(u,v)\|_{A^+_{p,q}}^{p'/q} = \|(v^{-p'}, u^{-p'})\|_{A^-_r}.$$

(iii) $(u,v) \in A^+_{p,\infty}$ *if, and only if* $(u^{-p'}, v^{-p'}) \in A^-_1$. *Moreover,*

$$\|(u,v)\|_{A^+_{p,\infty}} \approx \|(v^{-p'}, u^{-p'})\|_{A^-_1}^{1/p'}.$$

Martín-Reyes and de la Torre proved the following relation between $S^+_{p,q,\alpha,d}$ and $S^+_{p,q,\alpha,}$, and $A^+_{p,q}$ and $S^+_{p,q,\alpha}$:

Theorem B ([16]) *The pair $(u,v) \in S^+_{p,q,\alpha,d}$ if, and only if $(u,v) \in S^+_{p,q,\alpha}$. Moreover, there exist constants k_1 and k_2, only depending on p, q and α, such that*

$$\|(u,v)\|_{S^+_{p,q,\alpha,d}} \le k_1\|(u,v)\|_{S^+_{p,q,\alpha}} \quad \text{and} \quad \|(u,v)\|_{S^+_{p,q,\alpha}} \le k_2\|(u,v)\|_{S^+_{p,q,\alpha,d}}. \tag{3.1}$$

Theorem C ([16]) *Let $0 \le \alpha < 1$, $1 < p < 1/\alpha$, $1/q = 1/p - \alpha$ and w a weight. Then $w \in A^+_{p,q}$ if, and only if $(w^q, w^p) \in S^+_{p,q,\alpha}$. Moreover,*

$$\|w\|_{A^+_{p,q}} \le K_{p,q}\|(w^q, w^p)\|^q_{S^+_{p,q,\alpha}} \le C_{p,q}\|w\|_{A^+_{p,q}}^{(1-\alpha)p'}.$$

In the following results, we show the relation between $T^+_{q,\alpha}$ and $T^+_{q,\alpha,d}$.

Theorem 3.2 *Let $1 < q$, $0 \le \alpha < 1$ and $(1-\alpha)q > 1$. Then $(u,v) \in T^+_{q,\alpha}$ if, and only if $(u,v) \in T^+_{q,\alpha,d}$. Moreover, there exists C_1 and C_2 not depending on u and v such that*

$$\|(u,v)\|_{T^+_{q,\alpha,d}} \le C_1\|(u,v)\|_{T^+_{q,\alpha}} \le C_2\|(u,v)\|_{T^+_{q,\alpha,d}}. \tag{3.2}$$

Remark 3.3 Observe that if $\frac{1}{p}-\frac{1}{q}=\alpha$, then $(1-\alpha)q=(1-\frac{1}{p}+\frac{1}{q})q=\frac{q}{p'}+1>1$ and the hypothesis of the previous theorem is valid, so (3.2) is true.

The relation between the classes $A^+_{p,q}$ and $T^+_{q,\alpha}$, is the following:

Theorem 3.4 *Let $0\le\alpha<1$, $1<p<1/\alpha$, $1/q=1/p-\alpha$ and w a weight. Then, $w\in A^+_{p,q}$ if, and only if $(w^q,w^p)\in T^+_{p,\alpha}$. Moreover,*

$$\|w\|_{A^+_{p,q}}\le K_{p,q}\|(w^q,w^p)\|^q_{T^+_{p,\alpha}}\le C_{p,q}\|w\|^{(1-\alpha)p'}_{A^+_{p,q}}.$$

To prove the comparison results, we need some estimation in norms of different maximal functions.

Definition 3.5 *Let $0\le\alpha<1$ and μ be a positive regular Borel measure in $\mathbb{R}$. For $f\in L^1_{loc(\mathbb{R},d\mu)}$, the following maximal functions are defined as:*

$$M_{\alpha,\mu}f(x)=\sup_{I\ni x}\frac{1}{\mu(I)^{1-\alpha}}\int_I|f(t)|d\mu(t),$$
$$M^+_{\alpha,\mu}f(x)=\sup_{h>0}\frac{1}{\mu(x,x+h)^{1-\alpha}}\int_x^{x+h}|f(t)|d\mu(t),$$
$$M^-_{\alpha,\mu}f(x)=\sup_{h>0}\frac{1}{\mu(x,x+h)^{1-\alpha}}\int_{x-h}^{x}|f(t)|d\mu(t).$$

In [2], Bernal shows

Theorem D ([2]) *Let $0\le\alpha<1$, $1<p\le q<\infty$ with $1/p-1/q=\alpha$. Let μ be a positive regular Borel measure in $\mathbb{R}$ and $f\in L^p(\mu)$. Then there exists $C_{p,q}>0$ such that*

$$\|Nf\|_{L^q(\mu)}\le C_{p,q}\|f\|_{L^p(\mu)},$$

where N denotes any of the maximal functions $M_{\alpha,\mu}f(x)$, $M^+_{\alpha,\mu}f(x)$ or $M^-_{\alpha,\mu}f(x)$ from the previous definition. The constant $C_{p,q}$ changes according to the maximal function but does not depend on the measure μ.

Let us prove Theorems 3.2 and 3.4.

Proof of Theorem 3.2 By (2.4), we know that M^+_d and M^+ are equivalent. Therefore, if $(u,v)\in T^+_{q,\alpha,d}$, then $(u,v)\in T^+_{q,\alpha}$.

Reciprocally, let I^- be an interval. Then

$$\begin{aligned}&\Big(\int_{I^-\cup I^+}(M^+_d\sigma\chi_{I^+})^{(1-\alpha)q}udx\Big)^{1/q}\\&\le K\Big(\int_{I^-}(M^+\sigma\chi_{I^+})^{(1-\alpha)q}udx\Big)^{1/q}+K\Big(\int_{I^+}(M^+\sigma\chi_{I^+})^{(1-\alpha)q}udx\Big)^{1/q}\\&\le K\Big(\int_{I^-}(M^+\sigma\chi_{I^+})^{(1-\alpha)q}udx\Big)^{1/q}+K\|(u,v)\|_{T^+_{q,\alpha}}\Big(\int_{I^+}\sigma(x)dx\Big)^{1/q}.\end{aligned}$$

We only have to prove that, for all interval I^-, there exits a constant C not depending on u and v such that

$$\Big(\int_{I^-}(M^+\sigma\chi_{I^+})^{(1-\alpha)q}udx\Big)^{1/q}\le C\|(u,v)\|_{T^+_{q,\alpha}}\Big(\int_{I^+}\sigma(x)dx\Big)^{1/q}.$$

Let $I^- = [a,b)$ and $I^+ = [b,c)$. Let us suppose first that $\int_{I^-}\sigma \le \int_{I^+}\sigma$. Then

$$\begin{aligned}\left(\int_{I^-}(M^+\sigma\chi_{I^+})^{(1-\alpha)q}udx\right)^{1/q} &\le \left(\int_{I^-\cup I^+}(M^+\sigma\chi_{I^+})^{(1-\alpha)q}udx\right)^{1/q}\\ &\le \|(u,v)\|_{T^+_{q,\alpha}}\left(\int_{I^-\cup I^+}\sigma(x)dx\right)^{1/q}\\ &\le 2^{1/q}\|(u,v)\|_{T^+_{q,\alpha}}\left(\int_{I^+}\sigma(x)dx\right)^{1/q}.\end{aligned}$$

Now suppose that $\int_{I^-}\sigma \ge \int_{I^+}\sigma$, then we choose a sequence $x_0 = b > x_1 > x_2 > \cdots > x_k > \cdots > x_{N-1} > x_N = a$ such that for $k = 0,1,\ldots,N-1$, $\int_{x_k}^c \sigma = 2^k\int_b^c\sigma$ and $\int_a^c\sigma = r\int_b^c\sigma$, $2^{N-1} < r < 2^N$. It follows that $\int_{x_k}^{x_{k-1}}\sigma = 2^{k-1}\int_b^c\sigma$, $0 < k < N$, and $\int_a^{x_{N-1}}\sigma \le 2^{N-1}\int_b^c\sigma$. If, $x_k < x < x_{k-1}$, $1 < k \le N$, and $y \in I^+$, then

$$\int_x^y \sigma\chi_{(b,c)} = \int_b^y\sigma \le \int_b^c\sigma = 2^{-(k-2)}\int_{x_{k-1}}^{x_{k-2}}\sigma \le 2^{-(k-2)}\int_x^y\sigma.$$

Multiplying both sides of the inequality by $(y-x)^{-1}$ and taking suprema, we get that for each x such that $x_k < x < x_{k-1}$

$$M^+(\sigma\chi_{(b,c)})(x) \le 2^{-(k-2)}M^+(\sigma\chi_{(x,c)})(x), \quad k = 2,\ldots N,$$

and for $k=1$, we obtain trivially

$$M^+(\sigma\chi_{(b,c)})(x) \le M^+(\sigma\chi_{(x,c)})(x), \quad x_1 < x < b.$$

So finally,

$$\begin{aligned}&\int_a^b(M^+\sigma\chi_{(b,c)})^{(1-\alpha)q}udx\\ &= \sum_{k=1}^N\int_{x_k}^{x_{k-1}}(M^+\sigma\chi_{(b,c)})^{(1-\alpha)q}udx\\ &\le \sum_{k=2}^N 2^{-(k-2)(1-\alpha)q}\int_{x_k}^{x_{k-1}}(M^+\sigma\chi_{(x_k,c)})^{(1-\alpha)q}udx + \int_{x_1}^b(M^+\sigma\chi_{(x_1,c)})^{(1-\alpha)q}udx\\ &\le \sum_{k=2}^N 2^{-(k-2)(1-\alpha)q}\int_{x_k}^{c}(M^+\sigma\chi_{(x_k,c)})^{(1-\alpha)q}udx + \int_{x_1}^c(M^+\sigma\chi_{(x_1,c)})^{(1-\alpha)q}udx\\ &\le \|(u,v)\|_{T^+_{q,\alpha}}\left[\sum_{k=2}^N 2^{-(k-2)(1-\alpha)q}\int_{x_k}^c\sigma + \int_{x_1}^c\sigma\right]\\ &\le \|(u,v)\|_{T^+_{q,\alpha}}\left[\sum_{k=2}^N 2^{-(k-2)(1-\alpha)q}2^k + 2\right]\int_b^c\sigma,\end{aligned}$$

and as $(1-\alpha)q > 1$ the series converges and we obtain the desired result.

Proof of Theorem 3.4 Let $u = w^q$, $v = w^p$ and $\sigma = w^{-p'}$. Observe that if $(w^q, w^p) \in T^+_{q,\alpha}$, then $w \in A^+_{p,q}$. To prove this assumption, we will use the equivalence between the classes $T^+_{q,\alpha,d}$ and $T^+_{q,\alpha}$ (see Remark 3.3) and (3.2). By definition, $(w^q, w^p) \in T^+_{q,\alpha}$ means

$$\left(\int_{I^-\cup I^+}(M^+w^{-p'}\chi_{I^+})^{(1-\alpha)q}w^qdx\right)^{1/q} \le \|(u,v)\|_{T^+_{q,\alpha,d}}\left(\int_{I^+}w^{-p'}dx\right)^{1/q}.$$

Then

$$\left(\int_{I^-}(M^+w^{-p'}\chi_{I^+})^{(1-\alpha)q}w^q dx\right)^{1/q} \le \|(u,v)\|_{T^+_{q,\alpha,d}}\left(\int_{I^+}w^{-p'}dx\right)^{1/q}.$$

So, using that $(1-\alpha)q = 1+q/p'$, we get

$$\begin{aligned}\int_{I^-}w^q(x)dx\left(\frac{1}{|I^+|}\int_{I^+}w^{-p'}\right)^{1+q/p'} &\le \int_{I^-}(M_d^+(w^{-p'}\chi_{I^+})(x))^{1+q/p'}w(x)^q dx\\ &\le \|(u,v)\|^q_{T^+_{q,\alpha,d}}\left(\int_{I^+}w^{-p'}dx\right).\end{aligned}$$

Therefore,

$$\|w\|_{A^+_{p,q}} \le \|(w^q,w^p)\|^q_{T^+_{q,\alpha,d}} \le C\|(w^q,w^p)\|^q_{T^+_{q,\alpha}}.$$

Reciprocally, let I be an interval. For $x\in I$, there exists h_x such that $(x,x+h_x)\subset I$ and

$$M^+(w^{-p'}\chi_I)(x) \le \frac{3}{2h_x}\int_x^{x+h_x}w^{-p'}\chi_I dt.$$

Observe that

$$\begin{aligned}\frac{1}{h_x}\int_x^{x+h_x}w^{-p'}\chi_I dt &= \frac{2}{2h_x}\int_x^{x+\frac{h_x}{2}}w^{-p'}\chi_I dt + \frac{1}{h_x}\int_{x+\frac{h_x}{2}}^{x+h_x}w^{-p'}\chi_I dt\\ &\le \frac{1}{2}M^+(w^{-p'}\chi_I)(x) + \frac{1}{h_x}\int_{x+\frac{h_x}{2}}^{x+h_x}w^{-p'}\chi_I dt,\end{aligned}$$

then

$$M^+(w^{-p'}\chi_I)(x) \le 6\frac{1}{h_x}\int_{x+\frac{h_x}{2}}^{x+h_x}w^{-p'}\chi_I dt.$$

Using the $A^+_{p,q}$ condition and the relation between α, q and p, we have

$$\begin{aligned}M^+(w^{-p'}\chi_I)(x)^{1+q/p'} &\le \left(\frac{6}{h_x}\int_{x+\frac{h_x}{2}}^{x+h_x}w^{-p'}\chi_I dt\right)^{q/p'(1+p'/q)}\\ &\le C_{p,q}\|w\|^{1+p'/q}_{A^+_{p,q}}\left(\frac{h_x}{2}\left(\int_x^{x+\frac{h_x}{2}}w^q\right)^{-1}\right)^{1+p'/q}\\ &= C_{p,q}\|w\|^{1+p'/q}_{A^+_{p,q}}\left(\frac{1}{w^q(x,x+\frac{h_x}{2})}\int_x^{x+\frac{h_x}{2}}w^{-q}w^q dx\right)^{1+p'/q}\\ &\le C_{p,q}\|w\|^{(1-\alpha)p'}_{A^+_{p,q}}\left(M^+_{w^q}(w^{-q}\chi_I)(x)\right)^{1+p'/q}.\end{aligned}$$

Recall that $M^+_{w^q}$ is bounded in $L^{1+p'/q}(w^q)$ and its norm does not depend on w^q (see Theorem D). Then

$$\begin{aligned}\int_I M^+(w^{-p'}\chi_I)(x)^{1+q/p'}w^q(x)dx &\le C_{p,q}\|w\|^{(1-\alpha)p'}_{A^+_{p,q}}\int_I\left(M^+_{w^q}(w^{-q}\chi_I)(x)\right)^{1+p'/q}w^q(x)dx\\ &\le C_{p,q}\|w\|^{(1-\alpha)p'}_{A^+_{p,q}}\int_I w^{-q(1+p'/q)}(x)w^q(x)dx\\ &= C_{p,q}\|w\|^{(1-\alpha)p'}_{A^+_{p,q}}\int_I w^{-p'}(x)dx.\end{aligned}$$

Therefore

$$\|(w^q, w^p)\|^q_{T^+_{q,\alpha}} \leq C_{p,q}\|w\|^{(1-\alpha)p'}_{A^+_{p,q}}.$$

4 Extrapolation Results

In this section, we will state some extrapolation results that are necessary to prove Theorems 2.10 and 2.9.

The first results of extrapolation for the classes $A_{p,q}$ were proved by Harboure et al. [5, 6]. Macías and Riveros in [14] obtained analogous results for the one-sided case. In these mentioned papers, the authors do not take into account the weight constant dependence. Dragicevic et al. in [3] and Lacey et al. in [11] improved the results in [6] taking into account the weight constant dependence.

Theorem 4.1 *Let T be a sublinear operator defined on $C_c^\infty(\mathbb{R})$. If the inequality*

$$\|Tf\|_{L^{q_0}(w^{q_0})} \leq c\|w\|^\gamma_{A^+_{p_0,q_0}}\|f\|_{L^{p_0}(w^{p_0})}$$

holds for some pair (p_0, q_0), $1 < p_0 \leq q_0 < \infty$ and for all weights w belonging to the class $A^+_{p_0,q_0}$, then for any pair (p, q) $1 < p \leq q < \infty$, satisfying $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}$, and for any weight $w \in A^+_{p,q}$, the inequality

$$\|Tf\|_{L^q(w^q)} \leq c\|w\|^{\gamma\max\{1,\frac{q_0}{p_0'}\frac{p'}{q}\}}_{A^+_{p,q}}\|f\|_{L^p(w^p)}$$

holds provided the left-hand side is finite.

Remark 4.2 Observe that if $p_0 = q_0$, the previous theorem can be written in the following way:

Let T be an operator defined in $C_c^\infty(\mathbb{R})$ and let $1 \leq p_0 < \infty$. If

$$\|Tf\|_{L^{p_0}(w^{p_0})} \leq c\|w^{p_0}\|^\gamma_{A^+_{p_0}}\|f\|_{L^{p_0}(w^{p_0})}$$

for all weight $w^{p_0} \in A^+_{p_0}$ and some $\gamma > 0$, then

$$\|Tf\|_{L^p(w^p)} \leq c\|w^p\|^{\gamma\max\{1,\frac{p_0-1}{p-1}\}}_{A^+_p}\|f\|_{L^p(w^p)}$$

for all $1 < p < \infty$ and $w^p \in A^+_p$.

Corollary 4.3 *Suppose that for some $1 \leq p_0 \leq q_0 < \infty$, an operator T satisfies the weak-type (p_0, q_0) inequality*

$$\|Tf\|_{L^{q_0,\infty}(w^{q_0})} \leq c\|w\|^\gamma_{A^+_{p_0,q_0}}\|f\|_{L^{p_0}(w^{p_0})}$$

for every $w \in A^+_{p_0,q_0}$ and some $\gamma > 0$. Then, for any pair (p, q), $1 < p \leq q < \infty$, satisfying $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}$, and for any weight $w \in A^+_{p,q}$ the weak inequality

$$\|Tf\|_{L^{q,\infty}(w^q)} \leq c\|w\|^{\gamma\max\{1,\frac{q_0}{p_0'}\frac{p'}{q}\}}_{A^+_{p,q}}\|f\|_{L^p(w^p)},$$

holds, provided the left hand side is finite.

Theorem 4.4 *Let T be a sublinear operator defined on $C_c^\infty(\mathbb{R})$, with values on the space of measurable functions. Let us assume that T verifies*

$$\|aT(f)\|_\infty \leq C(T, \|(a, b)\|_{A^+_{\beta,\infty}})\|fb\|_\beta$$

for every pair (a,b) of functions such that $(a,b)\in A^+_{\beta,\infty}$, $1<\beta\le\infty$.

If $1<p<\beta$, $\frac{1}{q}=\frac{1}{p}-\frac{1}{\beta}$ and $(u,v)\in A^+_{p,q}$, then there exists C, depending only on p,q, and of the operator T, such that

$$\lambda u^q\left(\{x:|Tf(x)|>\lambda\}\right)^{1/q}\le C(T,p,q)\|(u,v)\|^{1/q}_{A^+_{p,q}}\left(\int|f|^pv^pdx\right)^{\frac{1}{p}}$$

for all $\lambda>0$, provided the left-hand side is finite.

Remark 4.5 Observe that Theorems 4.1, 4.4 and Corollary 4.3 are obtained for $f\in C_c^\infty(\mathbb{R})$. This class of functions are dense in $L^p(w)$ for $w\in A_r$, $r\ge1$ and $1\le p<\infty$. If the operator T, for which the extrapolation result is applied, is only defined in $C_c^\infty(\mathbb{R})$, we can extend its definition, using a density argument, to all the space $L^p(w)$. Then the theorems or corollaries are still true with the same constants for every $f\in L^p(w)$.

5 Proof of the Results

Proof of Theorem 2.5 To proof this theorem, we will follow Jawerth's ideas, see [9]. Let $f\ge0$ and as usual, let $\sigma=v^{1-p'}$. Consider the maximal function $M^{N,+}_{\alpha,d}$, where we only consider the dyadic intervals of length at most 2^N. Let

$$\Omega_k=\{x\in\mathbb{R}:2^k<M^{N,+}_{\alpha,d}(f)(x)\le2^{k+1}\}$$

for $k\in\mathbb{Z}$. In order to study this sets, let us consider $O_k=\{x\in\mathbb{R}:2^k<M^{N,+}_{\alpha,d}(f)(x)\}$. As O_k is open, there exist dyadic maximal intervals $I^-_{k,j}$ of length less equal 2^N such that $O_k=\bigcup_j I^-_{k,j}$ and

$$2^k<\frac{1}{(I^+_{k,j})^{1-\alpha}}\int_{I^+_{k,j}}f.$$

Observe that, by the definition of $M^{N,+}_{\alpha,d}$, the intervals $I^-_{k,j}$ and $I^+_{k,j}$ are contiguous and of equal length. Let

$$E_{k,j}=I^-_{k,j}\cap\{x\in\mathbb{R}:2^k<M^{N,+}_{\alpha,d}(f)\le2^{k+1}\}.$$

The sets $E_{k,j}$ are pairwise disjoints and $\Omega_k=\bigcup_j E_{k,j}$. Then,

$$\begin{aligned}
&\int_{\mathbb{R}}\left(M^{N,+}_{\alpha,d}(f)\right)^qu(x)dx\\
&=\sum_{k\in\mathbb{Z}}\int_{\Omega_k}\left(M^{N,+}_{\alpha,d}(f)\right)^qu(x)dx\\
&\le\sum_{k,j}\int_{E_{k,j}}(2^{k+1})^qu(x)dx\\
&\le2^q\sum_{k,j}\int_{E_{k,j}}\left(\frac{1}{|I^+_{k,j}|^{1-\alpha}}\int_{I^+_{k,j}}f(y)dy\right)^qu(x)dx\\
&\le2^q\sum_{k,j}\left(\frac{1}{\sigma(I^+_{k,j})}\int_{I^+_{k,j}}f(y)\sigma^{-1}(y)\sigma(y)dy\right)^qu(E_{k,j})\left(\frac{\sigma(I^+_{k,j})}{|I^+_{k,j}|^{1-\alpha}}\right)^q\\
&\le2^q\int_X g\,d\mu,
\end{aligned}$$

where $X=\mathbb{Z}\times\mathbb{N}$ with the measure $\mu(k,j)=u(E_{k,j})\big(\frac{\sigma(I^+_{k,j})}{|I^+_{k,j}|^{1-\alpha}}\big)^q$, and

$$g(k,j)=\left(\frac{1}{\sigma(I^+_{k,j})}\int_{I^+_{k,j}} f(y)\sigma^{-1}(y)\sigma(y)dy\right)^p.$$

Let $\Gamma_\lambda=\{(k,j)\in X: g(k,j)>\lambda\}$ be the level set at the height $\lambda>0$ and let $B_\lambda=\cup\{I^+_{k,j}:(k,j)\in\Gamma_\lambda\}$.

Observe that if $\big(\frac{1}{\sigma(I^+_{k,j})}\int_{I^+_{k,j}} f(y)\sigma^{-1}(y)\sigma(y)dy\big)^q>\lambda$, then given $x\in I^+_{k,j}$, we have that $M_\sigma(f\sigma^{-1})^q(x)>\lambda$, therefore $B_\lambda\subseteq\{x:M_\sigma(f\sigma^{-1})^q(x)>\lambda\}$. Taking into account that the dyadic intervals $I^-_{k,j}$, (with $(k,j)\in\Gamma_\lambda$), have length at most 2^N, we can consider a subfamily $\{I^-_r\}$ of $\{I^-_{k,j}\}$ with $(k,j)\in\Gamma_\lambda$, such as I^+_r are maximal disjoints intervals. Then $B_\lambda=\bigcup I^+_r$, where this union is disjoint. Also if $I^+_{k,j}\subset I^+_r$, then $I^-_{k,j}\subset I^-_r\cup I^+_r$.

Let us estimate $\mu(\Gamma_\lambda)$ using the weight condition:

$$\begin{aligned}\mu(\Gamma_\lambda)&=\sum_{(k,j)\in\Gamma_\lambda} u(E_{k,j})\left(\frac{\sigma(I^+_{k,j})}{|I^+_{k,j}|^{1-\alpha}}\right)^q\le\sum_{(k,j)\in\Gamma_\lambda}\int_{E_{k,j}} M^+_{\alpha,d}(\sigma\chi_{I^+_{k,j}})^q u(x)dx\\ &\le\sum_r\sum_{(k,j)\in\Gamma_\lambda,I^+_{k,j}\subset I^+_r}\int_{E_{k,j}} M^+_{\alpha,d}(\sigma\chi_{I^+_{k,j}})^q u(x)dx\\ &\le\sum_r\int_{I^-_r\cup I^+_r} M^+_{\alpha,d}(\sigma\chi_{I^+_r})^q u(x)dx\le\|(u,v)\|^q_{S^+_{p,q,\alpha,d}}\sum_r\sigma(I^+_r)^{q/p}\\ &\le\|(u,v)\|^q_{S^+_{p,q,\alpha,d}}\sigma(B_\lambda)^{q/p}\le\|(u,v)\|^q_{S^+_{p,q,\alpha,d}}\sigma\{x:M_\sigma(f\sigma^{-1})^q>\lambda\}^{q/p}.\end{aligned}$$

Making the substitution $\lambda=t^{q/p}$, $d\lambda=\frac{q}{p}t^{\frac{q}{p}}\frac{dt}{t}$, we get

$$\begin{aligned}\int_X g d\mu&=\int_0^\infty\mu(\Gamma_\lambda)d\lambda\le\|(u,v)\|^q_{S^+_{p,q,\alpha,d}}\int_0^\infty\sigma\{x:M_\sigma(f\sigma^{-1})^q>\lambda\}^{q/p}d\lambda\\ &=\frac{q}{p}\|(u,v)\|^q_{S^+_{p,q,\alpha,d}}\int_0^\infty(t\sigma\{x:M_\sigma(f\sigma^{-1})^p>t\})^{q/p}\frac{dt}{t},\end{aligned}$$

and using that $p\le q$,

$$\begin{aligned}&\int_0^\infty(t\sigma\{x:M_\sigma(f\sigma^{-1})^p>t\})^{q/p}\frac{dt}{t}\\ &=\sum_{l\in\mathbb{Z}}\int_{2^l}^{2^{l+1}}(t\sigma\{x:M_\sigma(f\sigma^{-1})^p>t\})^{q/p}\frac{dt}{t}\le 2^{q/p}\log 2\sum_{l\in\mathbb{Z}}(2^l\sigma\{x:M_\sigma(f\sigma^{-1})^p>2^l\})^{q/p}\\ &\le C\left(\sum_{l\in\mathbb{Z}}2^l\sigma\{x:M_\sigma(f\sigma^{-1})^p>2^l\}\right)^{q/p}\le C\left(\sum_{l\in\mathbb{Z}}\int_{2^{l-1}}^{2^l}\sigma\{x:M_\sigma(f\sigma^{-1})^p>t\}dt\right)^{q/p}\\ &=C\left(\int_0^\infty\sigma\{x:M_\sigma(f\sigma^{-1})^p>t\}dt\right)^{q/p}\le C\left(\int_{\mathbb{R}}M_\sigma(f\sigma^{-1})^p\sigma dx\right)^{q/p}\\ &\le C\left(\int_{\mathbb{R}}f^p\sigma^{1-p}dx\right)^{q/p}=C\left(\int_{\mathbb{R}}f^p v dx\right)^{q/p}, \qquad (5.1)\end{aligned}$$

where the last inequality holds using that the maximal function M_σ is bounded in $L^p(\sigma)$ (see Theorem D).

Finally, we get

$$\int_{\mathbb{R}} (M^{N,+}_{\alpha,d}(f))^q u(x)dx \le C\|(u,v)\|^q_{S^+_{p,q,\alpha,d}} \left(\int_{\mathbb{R}} f^p v dx\right)^{q/p}$$

and using the monotone convergence theorem, we obtain

$$\int_{\mathbb{R}} (M^{+}_{\alpha,d}(f))^q u(x)dx \le C\|(u,v)\|^q_{S^+_{p,q,\alpha,d}} \left(\int_{\mathbb{R}} f^p v dx\right)^{q/p}.$$

Observe that in the case $p = q$ the equation (5.1) is easier because $q/p = 1$, then we obtain that

$$\int_0^\infty \sigma(\{x : M_\sigma(f\sigma^{-1})^p > t\})dt = \int_{\mathbb{R}} M_\sigma(f\sigma^{-1})^p \sigma dx \le C\int_{\mathbb{R}} f^p\sigma^{1-p}dx = C\int_{\mathbb{R}} f^p v dx.$$

The proof of Corollary 2.6 is a consequence of Theorem 2.5 and (2.4) and (3.1). In Section 6, we will prove that the exponent of the constant $\|(u,v)\|^+_{S_{p,q,\alpha,d}}$ is sharp.

Proof of Theorem 2.7 To prove this result we considered, as in Theorem 2.5, the dyadic fractional maximal operator $M^{N,+}_{\alpha,d}$, and the sets $O_k = \{x \in \mathbb{R} : 2^k < M^{N,+}_{\alpha,d}(f)(x)\} = \bigcup_j I^-_{k,j}$, where the dyadic intervals $I^-_{k,j}$ and $I^+_{k,j}$ are contiguous and have equal length and satisfy

$$2^k < \frac{1}{(I^+_{k,j})^{1-\alpha}} \int_{I^+_{k,j}} f.$$

Also we consider

$$E_{k,j} = I^-_{k,j} \cap \{x \in \mathbb{R} : 2^k < M^{N,+}_{\alpha,d}(f) \le 2^{k+1}\} = I^-_{k,j} \cap \Omega_k,$$

which are pairwise disjoint and observe that $\Omega_k = \bigcup_j E_{k,j}$.

Then

$$\begin{aligned}
&\int_{\mathbb{R}} (M^{N,+}_{\alpha,d}(f))^q u(x)dx \\
&\quad\le 2^q \sum_{k,j} \int_{E_{k,j}} \left(\frac{1}{|I^+_{k,j}|^{1-\alpha}} \int_{I^+_{k,j}} f(y)dy\right)^q u(x)dx \\
&\quad\le 2^q \sum_{k,j} \left(\frac{1}{\sigma(I^+_{k,j})^{1-\alpha}} \int_{I^+_{k,j}} f(y)\sigma^{-1}(y)\sigma(y)dy\right)^q u(E_{k,j})\left(\frac{\sigma(I^+_{k,j})}{|I^+_{k,j}|}\right)^{(1-\alpha)q} \\
&\quad\le 2^q \int_X g d\mu,
\end{aligned}$$

where (X, μ) is de space $X = \mathbb{Z}\times\mathbb{N}$ with measure $\mu(k,j) = u(E_{k,j})\big(\frac{\sigma(I^+_{k,j})}{|I^+_{k,j}|}\big)^{(1-\alpha)q}$, and

$$g(k,j) = \left(\frac{1}{\sigma(I^+_{k,j})^{1-\alpha}} \int_{I^+_{k,j}} f(y)\sigma^{-1}(y)\sigma(y)dy\right)^q.$$

Define the level set, for $\lambda > 0$, as $\Gamma_\lambda = \{(k,j) \in X : g(k,j) > \lambda\}$. Let $B_\lambda = \bigcup\{I^+_{k,j} : (k,j) \in \Gamma_\lambda\}$. Observe that if $\big(\frac{1}{\sigma(I^+_{k,j})^{1-\alpha}} \int_{I^+_{k,j}} f(y)\sigma^{-1}(y)\sigma(y)dy\big)^q > \lambda$, then for all $x \in I^+_{k,j}$, we have $M_{\alpha,\sigma}(f\sigma^{-1})^q > \lambda$, so $B_\lambda \subseteq \{x : M_{\alpha,\sigma}(f\sigma^{-1})^q > \lambda\}$.

As dyadic intervals $I^-_{k,j}$ (with $(k,j) \in \Gamma_\lambda$) have length less equal 2^N, we can consider the subfamily $\{I^-_r\}$ of $\{I_{k,j}\}$ with $(k,j) \in \Gamma_\lambda$, such as I^+_r are maximal and pairwise disjoint. Observe that $B_\lambda = \bigcup I^+_r$, where the union is disjoint. Also if $I^+_{k,j} \subset I^+_r$, then $I^-_{k,j} \subset I^-_r \cup I^+_r$.

Let us estimate $\mu(\Gamma_\lambda)$ using the weight condition

$$\begin{aligned}\mu(\Gamma_\lambda) &= \sum_{(k,j)\in\Gamma_\lambda} u(E_{k,j})\Big(\frac{\sigma(I_{k,j}^+)}{|I_{k,j}^+|}\Big)^{(1-\alpha)q} \le \sum_{(k,j)\in\Gamma_\lambda}\int_{E_{k,j}} M_d^+(\sigma\chi_{I_{k,j}^+})^{(1-\alpha)q}u(x)dx\\ &\le \sum_r \sum_{(k,j)\in\Gamma_\lambda, I_{k,j}^+\subset I_r^+}\int_{E_{k,j}} M_d^+(\sigma\chi_{I_{k,j}^+})^{(1-\alpha)q}u(x)dx\\ &\le \sum_r \int_{I_r^-\cup I_r^+} M_d^+(\sigma\chi_{I_r^+})^{(1-\alpha)q}u(x)dx \le \|(u,v)\|^q_{T^+_{q,\alpha,d}}\sum_r \sigma(I_r^+)\\ &= \|(u,v)\|^q_{T^+_{q,\alpha,d}}\sigma(B_\lambda) \le \|(u,v)\|^q_{T^+_{q,\alpha,d}}\sigma\{x : M_{\alpha,\sigma}(f\sigma^{-1})^q > \lambda\}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_X g d\mu &= \int_0^\infty \mu(\Gamma_\lambda)d\lambda\\ &\le \|(u,v)\|^q_{T^+_{q,\alpha,d}}\int_0^\infty \sigma\{x : M_{\alpha,\sigma}(f\sigma^{-1})^q > \lambda\}d\lambda\\ &= \|(u,v)\|^q_{T^+_{q,\alpha,d}}\Big(\int_{\mathbb{R}} M_{\alpha,\sigma}(f\sigma^{-1})^q\sigma dx\Big)\\ &\le C\|(u,v)\|^q_{T^+_{q,\alpha,d}}\Big(\int_{\mathbb{R}} f^p\sigma^{1-p}dx\Big)^{q/p}\\ &= C\|(u,v)\|^q_{T^+_{q,\alpha,d}}\Big(\int_{\mathbb{R}} f^p v dx\Big)^{q/p},\end{aligned}$$

where the last inequality holds using that the maximal function $M_{\alpha,\sigma}$ is bounded from $L^p(\sigma)$ to $L^q(\sigma)$ with $1/q = 1/p - \alpha$ (see Theorem D).

Finally, we get

$$\int_{\mathbb{R}}(M_{\alpha,d}^{N,+}(f))^q u(x)dx \le C\|(u,v)\|^q_{T^+_{p,\alpha,d}}\Big(\int_{\mathbb{R}} f^p v dx\Big)^{q/p}.$$

By using the monotone convergence theorem, we obtain

$$\int_{\mathbb{R}}(M_{\alpha,d}^{+}(f))^q u(x)dx \le C\|(u,v)\|^q_{T^+_{p,\alpha,d}}\Big(\int_{\mathbb{R}} f^p v dx\Big)^{q/p}.$$

The proof of Corollary 2.8 is a consequence of Theorem 2.7 and inequalities (2.4) and (3.2). Also in Section 6, we will see that the exponent of the constant $\|(u,v)\|_{T^+_{q,\alpha}}$ is sharp.

For the proof of Theorem 2.9 we will use the extrapolation Theorem 4.4. In Section 6, we will prove that the exponent $1/q$ of the constant $\|(u,v)\|_{A^+_{p,q}}$ is sharp.

Proof of Theorem 2.9 We will see M_α^+ satisfying the hypotheses of Theorem 4.4. Let $\beta = 1/\alpha$ and $(a,b) \in A^+_{\beta,\infty}$. Let $h > 0$ and fix $x \in \mathbb{R}$,

$$\frac{1}{h^{1-\alpha}}\int_x^{x+h}|f|dx = \frac{1}{h^{1-\alpha}}\int_x^{x+h}|f|bb^{-1}dx \le \frac{1}{h^{1-\alpha}}\Big(\int_x^{x+h}|f|^{\frac{1}{\alpha}}b^{\frac{1}{\alpha}}dx\Big)^{\alpha}\Big(\int_x^{x+h}b^{\frac{-1}{1-\alpha}}dx\Big)^{1-\alpha}.$$

As a is finite for almost everywhere, let x such that $a(x) \le \|a\chi_{[x-h,x]}\|_\infty$. Then

$$a(x)\Big(\frac{1}{h^{1-\alpha}}\int_x^{x+h}|f|dx\Big)$$

$$\leq \|a\chi_{[x-h,x]}\|_\infty \frac{1}{h^{1-\alpha}}\left(\int_x^{x+h}|f|^{1/\alpha}b^{1/\alpha}dx\right)^\alpha\left(\int_x^{x+h}b^{\frac{-1}{1-\alpha}}dx\right)^{1-\alpha}$$
$$\leq \|(a,b)\|_{A^+_{\beta,\infty}}\|fb\|_{L^{1/\alpha}},$$

taking supreme over h and next over x, we will obtain

$$\|aM^+_\alpha f\|_\infty \leq \|(a,b)\|_{A^+_{\beta,\infty}}\|fb\|_{1/\alpha}. \tag{5.2}$$

Now we can apply Theorem 4.4. Next for all weights (u,v) in $A^+_{p,q}$, with $1/p-1/q=\alpha$, we get

$$\|M^+_\alpha\|_{L^p(v^p)\to L^{q,\infty}(u^q)} \leq C(T,q,p)\|(u,v)\|^{1/q}_{A^+_{p,q}}.$$

Proof of Theorem 2.10 We apply weak extrapolation Corollary 4.3, with $q_0=\frac{1}{1-\alpha}$, $p_0=1$ and w^q. For this, we only have to see that

$$\|I^+_\alpha f\|_{L^{q_0,\infty}(u)} \leq C\|f\|_{L^1((M^-u)^{\frac{1}{q_0}})} \tag{5.3}$$

for all weight u. Since by $w\in A^+_{1,q_0}$ is equivalent to $M^-(w^{q_0})(x)\leq \|w\|_{A^+_{1,q_0}}w^{q_0}(x)$, for almost everything x, and from the estimate (5.3) we conclude

$$\|I^+_\alpha f\|_{L^{q_0,\infty}(w^{q_0})} \leq C\|w\|^{1-\alpha}_{A^+_{1,q_0}}\|f\|_{L^1(w)}.$$

From the weak extrapolation Corollary 4.3, with $\gamma=1-\alpha$, we obtain

$$\|I^+_\alpha f\|_{L^{q,\infty}(w^q)} \leq C\|w\|^{(1-\alpha)\max\{1,\frac{q_0}{p_0'}\frac{p'}{q}\}}_{A^+_{p,q}}\|f\|_{L^1(w^p)}$$

for all $1\leq p<\frac{1}{\alpha}$ and q with $1/q=1/p-\alpha$. Also as $p_0=1$,

$$\|I^+_\alpha f\|_{L^{q,\infty}(w^q)} \leq C\|w\|^{(1-\alpha)}_{A^+_{p,q}}\|f\|_{L^1(w^p)},$$

we give the estimate.

In order to prove (5.3), we note that $\|\cdot\|_{L^{q_0,\infty}}$ is equivalent to a norm since $q_0>1$. Hence, we may use Minkowski's integral inequality as follows

$$\|I^+_\alpha f\|_{L^{q_0,\infty}(u)} = \left\|\int_\cdot^\infty \frac{|f(y)|}{(y-\cdot)^{1-\alpha}}dy\right\|_{L^{q_0,\infty}(u)}$$
$$\leq C_{q_0}\int_{\mathbb{R}}|f(y)|\sup_{\lambda>0}\lambda u\left(\left\{x\in(-\infty,y):(y-x)^{\alpha-1}>\lambda\right\}\right)^{\frac{1}{q_0}}dy.$$

We can finally calculate the inner norm by

$$\sup_{\lambda>0}\lambda u\left(\left\{x\in(-\infty,y):(y-x)^{\alpha-1}>\lambda\right\}\right)^{\frac{1}{q_0}}$$
$$=\left(\sup_{t>0}\frac{1}{t}u\left\{x\in(-\infty,y):(y-x)<t\right\}\right)^{\frac{1}{q_0}}$$
$$=\left(\sup_{t>0}\frac{1}{t}\int_{y-t}^t u(t)dt\right)^{\frac{1}{q_0}} = (M^-(u)(y))^{\frac{1}{q_0}}.$$

Proof of Theorem 2.11 This result is an immediate consequence of Theorem 2.10 and the equation (2.3). For a weight $w\in A^+_{p,q}$, we have

$$\|I^+_\alpha\|_{L^p(w^p)\to L^q(w^q)} \approx \|I^+_\alpha\|_{L^p(w^p)\to L^{q,\infty}(w^q)} + \|I^-_\alpha\|_{L^{q'}(w^{-q'})\to L^{p',\infty}(w^{-p'})}$$

$$\approx \|w\|_{A^+_{p,q}}^{1-\alpha} + \|w^{-1}\|_{A^-_{q',p'}}^{1-\alpha} \approx \|w\|_{A^+_{p,q}}^{(1-\alpha)\max\{1,p'/q\}},$$

where the last inequality is obtain using Lemma 3.1 ($\|w^{-1}\|_{A^-_{q',p'}} = \|w^q\|_{A_{1+q/p'}}^{p'/q} = \|w\|_{A^+_{p,q}}^{p'/q}$).

Now we will prove the extrapolation results. To prove the extrapolation results, we need some previous lemmas. Here we follow the ideas of Garcia Cuerva and Rubio de Francia in [4] and also we take in consideration Theorem A, to obtain special control of the constant.

Lemma 5.1 *Let $v \in A^+_p$ and $1 \le p_0 < p < \infty$. Then, for all $h \ge 0$ in $L^{(p/p_0)'}(v)$, there exist $S(h) \ge h$ such that*

- $\|S(h)\|_{L^{(p/p_0)'}(v)} \le C_{p,p_0} \|v\|_{A^+_p}^{\frac{p-p_0}{p-1}} \|h\|_{L^{(p/p_0)'}(v)}$,
- $(hv, S(h)v) \in A^+_{p_0}$ *with* $\|(hv, S(h)v)\|_{A^+_{p_0}} \le C\|v\|_{A^+_p}^{\frac{p_0-1}{p-1}}$ *if* $p_0 > 1$ *and* $\|(hv, S(h)v)\|_{A^+_1} \le 1$ *if* $p_0 = 1$.

Proof We defined $S(h) = \left(v^{-1} M^-(h^{\frac{p-1}{p-p_0}} v)\right)^{\frac{p-p_0}{p-1}}$. It is easy to check that $S(h) \ge h$ for almost all point x. Estimating directly the norm, using Theorem A, we obtain

$$\begin{aligned}\|S(h)\|_{L^{(p/p_0)'}(v)} &= \left(\int v^{1-p'} \left(M^-(h^{\frac{p-1}{p-p_0}} v)\right)^{p'}\right)^{\frac{p-p_0}{p}} \\ &\le C_{p,p_0} \|v^{1-p'}\|_{A^-_{p'}}^{p-p_0} \left(\int h^{(p/p_0)'} v^{p'} v^{1-p'}\right)^{\frac{1}{(p/p_0)'}} \\ &\le C_{p,p_0} \|v\|_{A^+_p}^{\frac{p-p_0}{p-1}} \|h\|_{L^{(p/p_0)'}(v)}.\end{aligned}$$

To see the second part, we observe that if $p_0 = 1$, then $vS = M^-(hv)$. Therefore $(hv, S(h)v) \in A^+_1$ with constant equal 1.

If $p_0 > 1$, we take I^- and I^+ contiguous intervals with the same length, then for all $t \in I^+$ we get

$$\frac{1}{|I^-|} \int_{I^-} h^{\frac{p-1}{p-p_0}} v \le 2M^-(h^{\frac{p-1}{p-p_0}} v)(t).$$

Therefore

$$\begin{aligned}&\left(\frac{1}{|I^-|}\int_{I^-} hv\right)\left(\frac{1}{|I^+|}\int_{I^+} (S(h)v)^{\frac{-1}{p_0-1}}\right)^{p_0-1} \\ &= \left(\frac{1}{|I^-|}\int_{I^-} hv^{\frac{p-p_0}{p-1}} v^{1-\frac{p-p_0}{p-1}}\right)\left(\frac{1}{|I^+|}\int_{I^+} \left(\left(v^{-1}M^-(h^{\frac{p-1}{p-p_0}} v)\right)^{\frac{p-p_0}{p-1}} v\right)^{\frac{-1}{p_0-1}}\right)^{p_0-1} \\ &\le \left(\frac{1}{|I^-|}\int_{I^-} h^{\frac{p-1}{p-p_0}} v\right)^{\frac{p-p_0}{p-1}} \left(\frac{1}{|I^-|}\int_{I^-} v\right)^{\frac{p_0-1}{p-1}} \\ &\quad \cdot \sup_{t\in I^+} (M^-(h^{\frac{p-1}{p-p_0}} v)(t)^{-\frac{p-p_0}{p-1}} \left(\frac{1}{|I^+|}\int_{I^+} v^{\frac{-1}{p-1}}\right)^{p_0-1} \\ &\le C\left(\frac{1}{|I^-|}\int_{I^-} v\right)^{\frac{p_0-1}{p-1}} \left(\frac{1}{|I^+|}\int_{I^+} v^{\frac{-1}{p-1}}\right)^{(p-1)\frac{p_0-1}{p-1}} \\ &\le C\|w\|_{A^+_p}^{\frac{p_0-1}{p-1}}.\end{aligned}$$

Lemma 5.2 *Let $v \in A^+_p$ and $1 \le p_0 < p < \infty$. Then for all $h \ge 0$ in $L^{(p/p_0)'}(v)$, there exists $H \ge h$ such that*

- $\|H\|_{L^{(p/p_0)'}(v)} \le 2\|h\|_{L^{(p/p_0)'}(v)}$;
- $Hv \in A^+_{p_0}$ *with* $\|Hv\|_{A^+_{p_0}} \le C(p,p_0)\|v\|_{A^+_p}$.

Let $v \in A^-_p$ *and* $1 \le p_0 < p < \infty$. *Then for all* $h \ge 0$ *in* $L^{(p/p_0)'}(v)$, *there exists* $H \ge h$ *such that*

- $\|H\|_{L^{(p/p_0)'}(v)} \le 2\|h\|_{L^{(p/p_0)'}(v)}$;
- $Hv \in A^-_{p_0}$ *with* $\|Hv\|_{A^-_{p_0}} \le C(p,p_0)\|v\|_{A^-_p}$.

Proof We will only prove the first part of the lemma. Let $v \in A^+_p$ and $1 \le p_0 < p < \infty$. Define H via the following convergent Neumann series:

$$H = \sum_{n=0}^{\infty} \frac{S^n(h)}{2^n\|S\|^n},$$

where S is defined in Lemma 5.1 and $\|S\| = \|S\|_{L^{(p/p_0)'}(v)}$. It is clear that $\|H\|_{L^{(p/p_0)'}(v)} \le 2\|h\|_{L^{(p/p_0)'}(v)}$. To prove the second item, first we observe that

$$S(H) \le 2\|S\|(H-h) \le 2\|S\|H.$$

Using Lemma 5.1, for $p_0 = 1$, we get $\|S\|_{L^{p'}(v)} \le C_p\|v\|_{A^+_p}$. Furthermore,

$$M^-(Hv) = M^-(Hv)v^{-1}v = S(H)v \le 2\|S\|Hv \le 2C_p\|v\|_{A^+_p}Hv.$$

Now suppose $p_0 > 1$. By Lemma 5.1, the pair $(hv, S(h)v)$ lies in $A^+_{p_0}$ with constant bounded by $\|v\|_{A^+_p}^{\frac{p_0-1}{p-1}}$ and $\|S\|_{L^{(p/p_0)'}(v)} \le C_{p,p_0}\|v\|_{A^+_p}^{\frac{p-p_0}{p-1}}$. We can now estimate $\|Hv\|_{A^+_{p_0}}$:

$$\begin{aligned}\Big(\frac{1}{|I^-|}\int_{I^-} Hv\Big)\Big(\frac{1}{|I^+|}\int_{I^+}(Hv)^{\frac{-1}{p_0-1}}\Big)^{p_0-1} &\le \Big(\frac{1}{|I^-|}\int_{I^-} Hv\Big)\Big(\frac{1}{|I^+|}\int_{I^+}(S(H)v)^{\frac{-1}{p_0-1}}\Big)^{p_0-1} 2\|S\| \\ &\le C_{p,p_0}\|v\|_{A^+_p}^{\frac{p_0-1}{p-1}}\|v\|_{A^+_p}^{\frac{p-p_0}{p-1}} = C_{p,p_0}\|v\|_{A^+_p}.\end{aligned}$$

Proof of Theorem 4.1 Let $w \in A^+_{p,q}$. Assume first $p > p_0$, thus $q > q_0$ and

$$\Big(\int |Tf|^q w^q\Big)^{1/q} = \Big(\int |Tf|^{q_0} g w^q\Big)^{1/q_0}$$

holds with some $g \ge 0$, $\|g\|_{L^{(q/q_0)'}(w^q)} = 1$. We observe that $w \in A^+_{p,q}$ if and only if $w^q \in A^+_r$ with $r = 1 + q/p'$. Let us put $r_0 = 1 + q_0/p_0'$, $h = g$ and $v = w^q$. As $r/r_0 = q/q_0$, by Lemma 5.2, there exists $H \ge g$ such that $\|H\|_{L^{(q/q_0)'}(w^q)} \le 2$ and $Hw^q \in A^+_{r_0}$ with $\|Hw^q\|_{A^+_{r_0}} \le C\|w^q\|_{A^+_r}$. This implies $H^{1/q_0}w^{q/q_0} \in A^+_{p_0,q_0}$ with $\|H^{1/q_0}w^{q/q_0}\|_{A^+_{p_0,q_0}} \le C\|w\|_{A^+_{p,q}}$. Therefore, noting that $p_0q/q_0 = p_0 + q\,(1 - p_0/p)$,

$$\begin{aligned}\Big(\int |Tf|^q w^q\Big)^{1/q} &\le \Big(\int |Tf|^{q_0}(H^{1/q_0}w^{q/q_0})^{q_0}\Big)^{1/q_0} \le C\|w\|^{\gamma}_{A^+_{p,q}}\Big(\int |f|^{p_0}(H^{1/q_0}w^{q/q_0})^{p_0}\Big)^{1/p_0} \\ &= C\|w\|^{\gamma}_{A^+_{p,q}}\Big(\int |f|^{p_0}w^{p_0}H^{p_0/q_0}w^{q(\frac{1}{(p/p_0)'})}\Big)^{1/p_0}.\end{aligned}$$

Using Hölder's inequality, we have

$$\Big(\int |Tf|^q w^q\Big)^{1/q} \le C\|w\|^{\gamma}_{A^+_{p,q}}\Big[\Big(\int |f|^p w^p\Big)^{\frac{p_0}{p}}\Big(\int H^{\frac{p_0}{q_0}(\frac{p}{p_0})'}w^{q\frac{1}{(\frac{p}{p_0})'}(\frac{p}{p_0})'}\Big)^{\frac{1}{(p/p_0)'}}\Big]^{\frac{1}{p_0}}$$

$$= C\|w\|^{\gamma}_{A^+_{p,q}} \left(\int |f|^p w^p\right)^{\frac{1}{p}} \left(\int H^{(\frac{r}{r_0})'} w^q\right)^{\frac{1}{p_0}-\frac{1}{p}}$$
$$\leq C\|w\|^{\gamma}_{A^+_{p,q}} \|f\|_{L^p(w^p)}.$$

Now if $p_0 > p$, then $q_0 > q$ and we have

$$\left(\int |f|^p w^p\right)^{\frac{1}{p}} = \left(\int (|fw^{p'}|^{p_0})^{\frac{p}{p_0}} w^{-p'}\right)^{\frac{p_0}{p}\frac{1}{p_0}}.$$

Therefore, there exists (see [8], Theorem 210) $g \geq 0$ such that

$$\int g^{\frac{p}{p-p_0}} w^{-p'} dx = 1 \quad \text{and} \quad \left(\int |f|^p w^p\right)^{\frac{1}{p}} = \left(\int |fw^{p'}|^{p_0} g w^{-p'}\right)^{\frac{1}{p_0}}.$$

Let $h = g^{-p_0'/p_0}$, $v = w^{-p'}$, $r = 1 + p'/q$ and $r_0 = 1 + p_0'/q_0$. Since $(r/r_0)'(-p_0'/p_0) = p/(p-p_0)$, we have $\int h^{(\frac{r}{r_0})'} v dx = 1$. On the other hand, $w \in A^+_{p,q}$ if and only if $w^{-p'} \in A^-_r$ with $r = 1 + p'/q$ with $\|w^{-p'}\|_{A^-_r} = \|w\|^{p'/q}_{A^+_{p,q}}$. By Lemma 5.2, there exists $H \geq h$ such that $\int H^{(\frac{r}{r_0})'} w^{-p'} \leq 2$ and $Hw^{-p'} \in A^-_{r_0}$ with $\|Hw^{-p'}\|_{A^-_{r_0}} \leq C\|w^{-p'}\|_{A^-_r}$. Hence $[Hv^{-p'}]^{-1/p_0'} \in A^+_{p_0,q_0}$ with $\|[Hv^{-p'}]^{-1/p_0'}\|_{A^+_{p_0,q_0}} \leq C\|w\|^{\frac{q_0}{p_0'}\frac{p'}{q}}_{A^+_{p,q}}$. Thus

$$\left(\int |f|^p w^p\right)^{\frac{1}{p}} = \left(\int |fw^{p'}|^{p_0} g w^{-p'}\right)^{\frac{1}{p_0}} = \left(\int |f|^{p_0} h^{-\frac{p_0}{p_0'}} w^{-p'(1-p_0)}\right)^{\frac{1}{p_0}}$$
$$\geq \left(\int |f|^{p_0} \left[H^{-\frac{1}{p_0'}} w^{\frac{p'}{p_0'}}\right]^{p_0}\right)^{\frac{1}{p_0}} \geq C\|w\|^{-\frac{q_0}{p_0'}\frac{p'}{q}}_{A^+_{p,q}} \left(\int |Tf|^{q_0} \left[H^{-\frac{1}{p_0'}} w^{\frac{p'}{p_0'}}\right]^{q_0}\right)^{\frac{1}{q_0}}.$$

By Hölder's inequality, we have

$$\left(\int |f|^p w^p\right)^{\frac{1}{p}} \geq C\|w\|^{-\frac{q_0}{p_0'}\frac{p'}{q}}_{A^+_{p,q}} \left(\int |Tf|^q w^q\right)^{\frac{1}{q}} \left(\int H^{(\frac{r}{r_0})'} w^{-p'}\right)^{\frac{q-q_0}{q_0 q}}$$
$$\geq C\|w\|^{-\frac{q_0}{p_0'}\frac{p'}{q}}_{A^+_{p,q}} \left(\int |Tf|^q w^q\right)^{\frac{1}{q}}.$$

Proof of Corollary 4.3 Theorem 4.1 does not require T to be a linear operator. Then we can simply apply the result to the operator $T_\lambda f = \lambda \chi_{\{|Tf|>\lambda\}}$. Fix $\lambda > 0$, then

$$\|T_\lambda f\|_{L^{q_0}(w^{q_0})} = \lambda w^{q_0}(\{x : |Tf(x)| > \lambda\})^{1/q_0}$$
$$\leq \|Tf\|_{L^{q_0,\infty}(w^{q_0})} \leq C\|w\|^{\gamma}_{A^+_{p_0,q_0}} \|f\|_{L^{p_0}(w^{p_0})},$$

where the constant C is independent of λ. If $w \in A^+_{p,q}$, by Theorem 4.1, T_λ maps $L^p(w^p) \to L^q(w^q)$ for all $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}$, and we obtain

$$\|T_\lambda f\|_{L^q(w^q)} \leq C\|w\|^{\gamma \max\{1, \frac{q_0}{p_0'}\frac{p'}{q}\}}_{A^+_{p,q}} \|f\|_{L^p(w^p)},$$

where the constant c is independent of λ, therefore,

$$\|Tf\|_{L^{q,\infty}(w^q)} = \sup_{\lambda>0} \|T_\lambda f\|_{L^q(w^q)} \leq C\|w\|^{\gamma \max\{1, \frac{q_0}{p_0'}\frac{p'}{q}\}}_{A^+_{p,q}} \|f\|_{L^p(w^p)}.$$

Before proving Theorem 4.4, we need some previous results.

Lemma 5.3 *If $\Phi \in L^1$ and $\int |\Phi| = 1$, then there exists h such that $\int |h| \leq 2$ and $\|\Phi h^{-1}\|_\infty = 1$.*

Proof Let

$$h(x) = \begin{cases} \Phi(x), & \text{if } \Phi(x) \neq 0, \\ \mathrm{e}^{-\pi|x|^2}, & \text{if } \Phi(x) = 0. \end{cases}$$

Clearly, $\|\Phi h^{-1}\|_\infty = 1$ and

$$\int |h| = \int_{\{x:\Phi(x)\neq 0\}} |\Phi| + \int_{\{x:\Phi(x)=0\}} \mathrm{e}^{-\pi|x|^2} \leq 2.$$

Corollary 5.4 *Given $f \in L^p(v)$, then there exists $g \in L^p(v^{-\frac{1}{p-1}})$, $g \geq 0$, such that*

$$\int g^p v^{-\frac{1}{p-1}} \leq 2 \quad \textit{and} \quad \Big(\int |f|^p v\Big)^{\frac{1}{p}} = \|f v^{\frac{1}{p-1}} g^{-1}\|_\infty.$$

Proof If $\int |f|^p v \neq 0$ take

$$\Phi = \frac{|f|^p v}{\int |f|^p v},$$

then $\int |\Phi| = 1$. Let h be the function given by Lemma 4.6. If we put

$$g^p = \begin{cases} v^{\frac{1}{p-1}} h, & v \neq 0, \\ 0, & v = 0, \end{cases}$$

it follows immediately that $\int g^p v^{-\frac{1}{p-1}} = \int h \leq 2$ and

$$1 = \|\Phi h^{-1}\|_\infty = \frac{1}{\int |f|^p v} \||f|^p v^{\frac{p}{p-1}} g^{-p}\|_\infty.$$

The case $\int |f|^p v = 0$ is clear.

Finally, we recall some definitions concerning the Lorentz $L(p,q,\mu)$ spaces. Let f be a measurable function on a measure space $(M, \mathcal{M}, \mu)$. The non-increasing rearrangement f^* of f is defined as

$$f^*(t) = \inf\{s : \mu(\{x : |f(x)| > s\}) \leq t\}$$

for $t > 0$. The function f is said to belong to the Lorentz space $L(p,q,\mu)$ if

$$\|f\|_{p,q,\mu} = \Big(\frac{p}{q}\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t}\Big)^{\frac{1}{q}} < \infty,$$

whenever $1 < p < \infty$ and $1 < q < \infty$, and

$$\|f\|_{p,\infty,\mu} = \sup_{t>0} t^{\frac{1}{p}} f^*(t),$$

when $1 < p \leq \infty$ and $q = \infty$. For more details see [24].

Proof of Theorem 4.4 Let $f \in C_c^\infty(\mathbb{R})$, $0 < m = \int |f|^p v^p$, and $(u,v) \in A^+_{p,q}$. We define

$$b(x) = \begin{cases} |f(x)|^{p/\beta-1} v(x)^{p/\beta} m^{\frac{1}{q}}, & \text{if } |f(x)| > 0, \\ \mathrm{e}^{\pi\frac{|x|^2}{q}} v(x), & \text{if } |f(x)| = 0. \end{cases}$$

Thus $\|fv\|_p = \|fb\|_\beta$ and $\int b^{-q}v^q dx \le 2$. Set

$$a(x) = (M^+ b^{-\beta'}(x))^{-\frac{1}{\beta'}}.$$

It follows immediately that $(a,b) \in A^+_{\beta,\infty}$ with $\|(a,b)\|_{A^+_{\beta,\infty}} \le 4$. Let

$$E_\lambda = \{x : |Tf(x)| > \lambda\}.$$

Hence, by the Hölder's inequality for Lorentz spaces, we have

$$u^q(E_\lambda) = \int_{E_\lambda} u^q = \int \chi_{E_\lambda}(x) a^{-1}(x) a(x) u^q(x) dx$$
$$\le \|\chi_{E_\lambda}\|_{(1+1/q,1,au^q)} \|a^{-1}\|_{(q+1,\infty,au^q)}.$$

In order to estimate the second factor above, we observe that

$$\lambda^{q+1} \int_{\{x: a(x)^{-1} > \lambda\}} au^q \le \lambda^q \int_{\{x: M^+ b^{-\beta'}(x) > \lambda^{\beta'}\}} u^q. \tag{5.4}$$

Recalling $(u,v) \in A^+_{p,q}$ implies $(u^q, v^q) \in A^+_s$ for $s = 1 + q/p'$ with $\|(u^q,v^q)\|_{A^+_s} = \|(u,v)\|_{A^+_{p,q}}$, it follows that M^+ is weakly bounded from $L^s(v^q)$ into $L^s(u^q)$ with norm $\|M^+\|_{L^s(v^q,\infty)\to L^s(u^q)} \approx \|(u^q,v^q)\|^{1/s}_{A^+_s}$ (see [15]). Therefore

$$(5.4) \le C \frac{\lambda^q}{(\lambda^{\beta'})^s} \|(u,v)\|_{A^+_{p,q}} \int (b^{-\beta'})^s v^q = C\|(u,v)\|_{A^+_{p,q}} \int b^{-q} v^q \le 2C \|(u,v)\|_{A^+_{p,q}}.$$

We consider the non-increasing rearrangement of a^{-1} respect to the measure au^q,

$$(a^{-1})^*(t) = \inf\{y : au^q(\{x : |a^{-1}| > y\}) \le t\}$$
$$\le \inf\left\{y : \frac{C\|(u,v)\|_{A^+_{p,q}}}{y^{q+1}} \le t\right\} = \left(\frac{C\|(u,v)\|_{A^+_{p,q}}}{t}\right)^{\frac{1}{q+1}}.$$

Then, we have

$$\|a^{-1}\|_{(q+1,\infty,au^q)} = \sup_{t>0} t^{\frac{1}{q+1}} (a^{-1})^*(t) \le (C\|(u,v)\|_{A^+_{p,q}})^{\frac{1}{q+1}}.$$

A non-increasing rearrangement of χ_{E_λ} with respect to the measure au^q is $\chi_{[0,R)}$ with $R = \int_{E_\lambda} au^q$, then

$$\|\chi_{E_\lambda}\|_{(1+\frac{1}{q},1,au^q)} = \frac{q}{q+1} \int_0^R t^{\frac{q}{q+1}} \frac{dt}{t} = \frac{q}{q+1} \int_0^R t^{-\frac{1}{q+1}} dt = R^{\frac{q}{q+1}}.$$

On the other hand, using $\|(a,b)\|_{A^+_{\beta,\infty}} \le 4$, we get

$$R = \int_{E_\lambda} au^q \le \lambda^{-1} \int_{E_\lambda} |Tf| au^q \le \lambda^{-1} \|aTf\|_\infty \int_{E_\lambda} u^q \le C\lambda^{-1} \|fb\|_\beta \int_{E_\lambda} u^q.$$

Then

$$\|\chi_{E_\lambda}\|_{(1+\frac{1}{q},1,au^q)} \le C\lambda^{-\frac{q}{q+1}} \|fb\|_\beta^{\frac{q}{q+1}} \left(\int_{E_\lambda} u^q\right)^{\frac{q}{q+1}}.$$

Since $f \in C_0^\infty(\mathbb{R}))$, it follows that

$$u^q(E_\lambda) \le C\lambda^{-\frac{q}{q+1}} \|fv\|_p^{\frac{q}{q+1}} (C\|(u,v)\|_{A^+_{p,q}})^{\frac{1}{q+1}} (u^q(E_\lambda))^{\frac{q}{q+1}},$$

and as $u^q(E_\lambda)$ is finite, we get

$$u^q(\{x : |Tf(x)| > \lambda\}) \le C\|(u,v)\|_{A^+_{p,q}} \left(\frac{1}{\lambda^p} \int f^p v^p\right)^{\frac{q}{p}}.$$

6 Proof of Sharp Estimates

In this section, we will show that the constants obtained in the previous theorems are the best possible, in other words the dependence of the different constants of the weights cannot be improved.

Let us prove now that the constant in Corollary 2.8 is sharp. Consider the pair (w_δ^q, w_δ^p), by Theorem 3.4, we have

$$\|(w_\delta^q, w_\delta^p)\|_{T^+_{q,\alpha}} \leq C_{p,q}\|w_\delta\|_{A^+_{p,q}}^{(1-\alpha)p'/q},$$

therefore the dependence here is sharp, if not it won't be the one in Theorem A.

In a similar way, it can be proved that the constant in Corollary 2.6 is sharp.

Now we will show that the dependency of the norm related to the one-sided fractionary maximal respect to the constant $\|(u,v)\|_{A^+_{p,q}}$ in Theorem 2.9 is sharp. We follow Muckenhoupt's ideas in [19]. We will prove that

$$\|(u,v)\|_{A^+_{p,q}}^{1/q} \leq 2^{(1-\alpha)+\frac{1}{q}}\|M\|_{L^p(v^p)\to L^{q,\infty}(u^q)}.$$

Suppose first that $p>1$. Given a pair of fix intervals (a,b) and (b,c) with $a<b<c$ and $b-a=c-b$, we define $A=\left(\int_b^c v^{-p'}(y)dy\right)^{q/p'}$. If $A=\infty$, $(u,v)\in A^+_{p,q}$ implies $\int_a^b u^q(y)dy=0$. Either the case $A=\infty$ or $A=0$, trivially we get

$$0=\left(\frac{1}{b-a}\int_a^b u^q(y)dy\right)\left(\frac{1}{b-c}\int_b^c v^{-p'}(y)dy\right)^{q/p'} \leq \|M^+_\alpha\|^q_{L^p(v^p)\to L^{q,\infty}(u^q)}.$$

Let $0<A<\infty$ and $f(x)=v(x)^{-p'}$, if $x\in(b,c)$ and $f(x)=0$, if $x\notin(b,c)$. Then

$$\frac{A^{p'/q}}{(b-c)^{1-\alpha}}=\frac{1}{(b-c)^{1-\alpha}}\int_b^c v^{-p'}(y)dy \leq 2^{1-\alpha}M^+_\alpha f(x)$$

for all $x\in(a,b)$. Therefore, $(a,b)\subset\{x\in\mathbb{R}: M^+_\alpha f(x) > \frac{A^{p'/q}}{2(2(b-c))^{1-\alpha}}\}$ and

$$\begin{aligned}\int_a^b u(x)^q dx &\leq u^q\left(\left\{x\in\mathbb{R}: M^+_\alpha f(x) > \frac{A^{p'/q}}{2(2(b-c))^{1-\alpha}}\right\}\right)\\ &\leq \|M^+_\alpha\|^q_{L^p(v^p)\to L^{q,\infty}(u^q)}\frac{2(2(b-c))^{(1-\alpha)q}}{A^{p'}}\left(\int_{\mathbb{R}} f(x)^p v(x)^p dx\right)^{q/p}\\ &\leq \|M^+_\alpha\|^q_{L^p(v^p)\to L^{q,\infty}(u^q)}\frac{2(2(b-c))^{(1-\alpha)q}}{A^{p'}}\left(\int_b^c v^{-p'}(x)dx\right)^{q/p}\\ &\leq \|M^+_\alpha\|^q_{L^p(v^p)\to L^{q,\infty}(u^q)}2(2(b-c))^{(1-\alpha)q}A^{-1}.\end{aligned}$$

Multiplying by $(b-c)^{-(1-\alpha)q}A$ both sides of the inequality, and as $1+q/p'=(1-\alpha)q$, we obtain

$$\left(\frac{1}{b-a}\int_a^b u^q(x)dx\right)\left(\frac{1}{b-c}\int_b^c v^{-p'}(x)dx\right)^{q/p'} \leq 2^{(1-\alpha)q}2\|M^+\|^q_{L^p(v^p)\to L^{q,\infty}(u^q)},$$

which proves the statement for $p>1$.

For $p=1$, observe that for every pair of intervals (a,b) and (b,c) such that $a<b<c$ and $b-a=c-b$ if

$$\frac{1}{b-a}\int_a^b u(y)^q dy \leq 4\|M\|^q_{L^1(v)\to L^{q,\infty}(u^q)} \operatorname*{ess\,inf}_{x\in(b,c)} v(x)^q, \tag{6.1}$$

then for almost every $x \in \mathbb{R}$ and $h > 0$, we have

$$\frac{1}{h}\int_{x-h}^{x} u(y)^q dy \leq 4\|M\|^q_{L^1(v)\to L^{q,\infty}(u^q)} v(x)^q.$$

So, $\|(u,v)\|_{A^+_{1,q}} \leq 4\|M\|^q_{L^1(v)\to L^{q,\infty}(u^q)}$.

As $(1-\alpha)q = 1$, it is enough to prove (6.1). Fix (a,b) and (b,c) such that $a < b < c$ and $b-a = c-b$. If ess inf $_{x\in(b,c)} v(x)^q = \infty$, then (6.1) holds. If ess inf $_{x\in(b,c)} v(x)^q < \infty$, for all $\epsilon > 0$, there exists a medible set $E \subset (b,c)$ such that $|E| > 0$ and $v(x) < \epsilon +$ ess inf $_{y\in(b,c)} v(y)$ for all $x \in E$. Let $f(x) = \chi_E(x)$. Then

$$\frac{|E|}{(b-c)^{1-\alpha}} = \frac{1}{(b-c)^{1-\alpha}}\int_b^c \chi_E(y)dy \leq 2^{1-\alpha}M^+_\alpha f(x)$$

for all $x \in (a,b)$. Then $(a,b) \subset \{x \in \mathbb{R} : M^+_\alpha f(x) > \frac{|E|}{2(2(b-c))^{1-\alpha}}\}$ and

$$\begin{aligned}\int_a^b u(x)^q dx &\leq u^q\left(\left\{x \in \mathbb{R} : M^+_\alpha f(x) > \frac{|E|}{2(2(b-c))^{1-\alpha}}\right\}\right)\\ &\leq \|M^+_\alpha\|^q_{L^1(v)\to L^{q,\infty}(u^q)}\frac{4(b-c)}{|E|^q}\left(\int_E v(x)dx\right)^q\\ &\leq \|M^+_\alpha\|^q_{L^1(v)\to L^{q,\infty}(u^q)}4(b-c)\Big(\epsilon + \operatorname*{ess\,inf}_{y\in(b,c)} v(y)\Big).\end{aligned}$$

Using that $c-b = b-a$, we get

$$\frac{1}{b-a}\int_a^b u(x)^q dx \leq 4\|M^+_\alpha\|^q_{L^1(v)\to L^{q,\infty}(u^q)}\Big(\epsilon + \operatorname*{ess\,inf}_{y\in(b,c)} v(y)\Big)$$

for all $\epsilon > 0$, then (6.1) is true.

Now let us show the dependence of the norm $\|I^+_\alpha\|_{L^p(w^p)\to L^{q,\infty}(w^q)}$ respect to the constant $\|w\|_{A^+_{p,q}}$ of the weight in Theorem 2.10 is sharp. For $p \geq 1$, observe that (2.5) is equivalent to

$$\|I^+_\alpha f\|_{L^{q,\infty}(w^q)} \leq C\|w^q\|^{1-\alpha}_{A^+_{1+q/p'}}\|f\|_{L^p(w^p)},$$

and if $w^q \in A^+_1$, then

$$\|I^+_\alpha f\|_{L^{q,\infty}(w^q)} \leq C\|w^q\|^{1-\alpha}_{A^+_1}\|f\|_{L^p(w^p)}.$$

Let $u = w^q$. Putting $u^\alpha f = w^{q\alpha}f$, the last equation is equivalent to

$$\|I^+_\alpha(u^\alpha f)\|_{L^{q,\infty}(u)} \leq C\|u\|^{1-\alpha}_{A^+_1}\|f\|_{L^p(u)}.$$

We will prove that in the last inequality the exponent is sharp. This will imply that the exponent in (2.5) is also sharp. Let $u_\delta(x) = |x|^{\delta-1}$, observe that $u_\delta \in A^+_1$ with

$$\|u_\delta\|_{A^+_1} \leq \frac{1}{\delta}. \tag{6.2}$$

Let $f_\delta = \chi_{[0,1]}$, it is easy to check that

$$\|f_\delta\|_{L^p(u_\delta)} = \left(\frac{1}{\delta}\right)^{1/p}. \tag{6.3}$$

Let $0 < \xi < 1$ be a number to be choose later. Then

$$\|I^+_\alpha(u^\alpha_\delta f_\delta)\|_{L^{q,\infty}(u_\delta)} \geq \sup_{\lambda>0}\lambda\bigg(u_\delta\Big\{0 < x < \xi : \int_x^1 \frac{y^{(\delta-1)\alpha}}{(y-x)^{1-\alpha}}dy > \lambda\Big\}\bigg)^{1/q}$$

$$
\begin{aligned}
&\geq \sup_{\lambda>0} \lambda \left(u_\delta \left\{ 0 < x < \xi : \int_x^1 \frac{y^{(\delta-1)\alpha}}{(2y)^{1-\alpha}} dy > \lambda \right\} \right)^{1/q} \\
&= \sup_{\lambda>0} \lambda \left(u_\delta \left\{ 0 < x < \xi : \frac{2^{\alpha-1}}{\alpha\delta}(1 - x^{\delta\alpha}) > \lambda \right\} \right)^{1/q} \\
&\geq \frac{2^{\alpha-2}}{\alpha\delta} \left(u_\delta \left\{ 0 < x < \xi : \frac{2^{\alpha-1}}{\delta\alpha}(1 - x^{\delta\alpha}) > \frac{2^{\alpha-2}}{\alpha\delta} \right\} \right)^{1/q} \\
&\geq \frac{2^{\alpha-2}}{\alpha\delta} \left(u_\delta \left\{ 0 < x < \xi : \left(\frac{1}{2}\right)^{\frac{1}{\alpha\delta}} > x \right\} \right)^{1/q} \\
&\geq \frac{2^{\alpha-2}}{\alpha\delta} \left(u_\delta[0, \xi] \right)^{1/q},
\end{aligned}
$$

where the last inequality holds by choosing $\xi = \left(\frac{1}{2}\right)^{\frac{1}{\alpha\delta}}$. Then for $0 < \delta < 1$, we have

$$
\|I_\alpha^+(u_\delta^\alpha f_\delta)\|_{L^{q,\infty}(u)} \geq \frac{2^{\alpha-2}}{\alpha\delta} \left(\frac{\xi^\delta}{\delta} \right)^{1/q} = c_{\alpha,q} \left(\frac{1}{\delta} \right)^{1+1/q}. \tag{6.4}
$$

Finally combining (6.2)–(6.4), we get

$$
c_{\alpha,q} \left(\frac{1}{\delta} \right)^{1+1/q} \leq \|I_\alpha^+(u^\alpha f)\|_{L^{q,\infty}(u)} \leq C \|u\|_{A_1^+}^{1-\alpha} \|f\|_{L^p(u)} \leq C \left(\frac{1}{\delta} \right)^{1+1/q},
$$

so we get that the dependence of the norm $\|I_\alpha^+\|_{L^p(w^p)\to L^{q,\infty}(w^q)}$ respect to the constant $\|w\|_{A_{p,q}^+}$ of the weight is $\|w\|_{A_{p,q}^+}^{1-\alpha}$.

Finally, let us prove that the dependence of the norm $\|I_\alpha^+\|_{L^p(w^p)\to L^q(w^q)}$ respect to the constant $\|w\|_{A_{p,q}^+}$ of the weight is sharp. If $f \geq 0$ then $M_\alpha^+ f(x) \leq I_\alpha^+ f(x)$.

Now, consider $w_\delta = |x|^{(1-\delta)/p'}$ de $A_{p,q}^+$ and $f_\delta(x) = |x|^{\delta-1}\chi_{[-1,0]}(x)$. If $p'/q \geq 1$, then $(1-\alpha)\max\{1, p'/q\} = (1-\alpha)p'/q$, and we obtain

$$
C \left(\frac{1}{\delta} \right)^{1+1/q} \leq \|M_\alpha^+ f_\delta\|_{L^q(w_\delta^q)} \leq \|I_\alpha^+ f_\delta\|_{L^q(w_\delta^q)} \leq \|w_\delta\|_{A_{p,q}^+}^{p'/q(1-\alpha)} \|f_\delta\|_{L^p(w_\delta^p)} \leq \left(\frac{1}{\delta} \right)^{1+1/q},
$$

which shows that the dependence of the weight norm is sharp for the case $p'/q \geq 1$.

If $p'/q < 1$, we use a duality argument. Let $w \in A_{p,q}^+$. Then $w^{-1} \in A_{q',p'}^-$.

If we apply an analogous argument to the operator I_α^-, the adjoint operator of I_α^+, and Lemma 3.1, we get

$$
\|I_\alpha^+\|_{L^p(w^p)\to L^q(w^q)} = \|I_\alpha^-\|_{L^{q'}(w^{-q'})\to L^{p'}(w^{-p'})} \approx \|w^{-1}\|_{A_{q',p'}^-}^{(1-\alpha)q/p'} = \|w\|_{A_{p,q}^+}^{(1-\alpha)},
$$

where the dependence is sharp.

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