A NEW GENERALIZATION OF HERMITE'S RECIPROCITY LAW

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Abstract. Given a partition λ of n, the *Schur functor* $\mathbb{S}_\lambda$ associates to any complex vector space V, a subspace $\mathbb{S}_\lambda(V)$ of $V^{\otimes n}$. Hermite's reciprocity law, in terms of the Schur functor, states that $\mathbb{S}_{(p)}\left(\mathbb{S}_{(q)}(\mathbb{C}^2)\right) \simeq \mathbb{S}_{(q)}\left(\mathbb{S}_{(p)}(\mathbb{C}^2)\right)$. We extend this identity to many other identities of the type $\mathbb{S}_\lambda\left(\mathbb{S}_\delta(\mathbb{C}^2)\right) \simeq \mathbb{S}_\mu\left(\mathbb{S}_\epsilon(\mathbb{C}^2)\right)$.

1. Introduction

Hermite's reciprocity law states that

$$\mathrm{Sym}^p\left(\mathrm{Sym}^q(\mathbb{C}^2)\right) \simeq \mathrm{Sym}^q\left(\mathrm{Sym}^p(\mathbb{C}^2)\right)$$

as $\mathrm{GL}(2,\mathbb{C})$-modules, for any pair of non-negative integers p and q, (see e.g. [FH], Exercise 6.18). Here $\mathrm{Sym}^n(V)$ is the homogeneous component of degree n in the symmetric algebra of V. This identity can also be stated in terms of the *Schur functor*. Recall that given any partition λ of n, the Schur functor $\mathbb{S}_\lambda$ associates to any complex vector space V, a subspace (also known as the *Weyl module*) $\mathbb{S}_\lambda(V)$ of $V^{\otimes n}$ (see e.g. §6.1 in [FH]. We give some details in subsection §2.2). For instance, if $\lambda = (n)$ then $\mathbb{S}_\lambda(V) \simeq \mathrm{Sym}^n(V)$, and if $\lambda = (1^n)$ then $\mathbb{S}_\lambda(V) \simeq \Lambda^n(V)$.

Thus, in terms of Schur functors, Hermite's reciprocity law states that

$$\mathbb{S}_{(p)}\left(\mathbb{S}_{(q)}(\mathbb{C}^2)\right) \simeq \mathbb{S}_{(q)}\left(\mathbb{S}_{(p)}(\mathbb{C}^2)\right).$$

This reciprocity law has been extended to more general plethysms involving rectangle partitions by L. Manivel in [M]. More precisely a proof of Hermite's reciprocity law can be obtained from the Cayley-Silvester formula ([Sp]); this formula was extended by M. Brion in [B] and Manivel used it to prove the following extension of Hermite's reciprocity law, valid for all positive integers n, k, d:

$$\begin{array}{ccccc}
\mathbb{S}_{(n^k)}\left(\mathbb{S}_{(d+k-1)}(\mathbb{C}^2)\right) & \simeq & \mathbb{S}_{(d^n)}\left(\mathbb{S}_{(k+n-1)}(\mathbb{C}^2)\right) & \simeq & \mathbb{S}_{(k^d)}\left(\mathbb{S}_{(n+d-1)}(\mathbb{C}^2)\right) \\
\wr\| & & \wr\| & & \wr\| \\
\mathbb{S}_{(n^d)}\left(\mathbb{S}_{(d+k-1)}(\mathbb{C}^2)\right) & \simeq & \mathbb{S}_{(d^k)}\left(\mathbb{S}_{(k+n-1)}(\mathbb{C}^2)\right) & \simeq & \mathbb{S}_{(k^n)}\left(\mathbb{S}_{(n+d-1)}(\mathbb{C}^2)\right)
\end{array}$$

where the isomorphisms are now only as $\mathrm{SL}(2,\mathbb{C})$-modules.

It is now natural to ask for other solutions to the following plethysm equation

$$\mathbb{S}_\lambda\left(\mathbb{S}_\delta(\mathbb{C}^2)\right) \simeq \mathbb{S}_\mu\left(\mathbb{S}_\epsilon(\mathbb{C}^2)\right) \tag{1.1}$$

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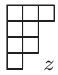
considering the partitions λ, δ, μ and ϵ as unknowns and the isomorphism either as $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{GL}(2,\mathbb{C})$-modules.

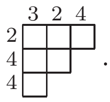
In this paper, we obtain new solutions to the plethysm equation (1.1) involving partitions of arbitrary number of 'steps'. Manivel's result (involving rectangular partitions) turns out to be our one-step case. In addition, we address the question of when an $\mathrm{SL}(2,\mathbb{C})$-isomorphism is (or can twisted to obtain) an $\mathrm{GL}(2,\mathbb{C})$-isomorphism.

Main results. Let us denote $\mathbb{S}_\lambda\left(\mathbb{S}_{(d)}(\mathbb{C}^2)\right)$ by Y_{d+1} where Y is the Young diagram of λ (recall that $\dim \mathbb{S}_{(d)}(\mathbb{C}^2) = d+1$). For instance



$$= \mathbb{S}_\lambda\left(\mathbb{S}_{(z-1)}(\mathbb{C}^2)\right), \quad \lambda = (3, 2^2, 1).$$

We add labels to a Young diagram to indicate the width and hight of the boxes. For instance, if $\lambda = (9^2, 5^4, 3^4)$, its Young diagram is



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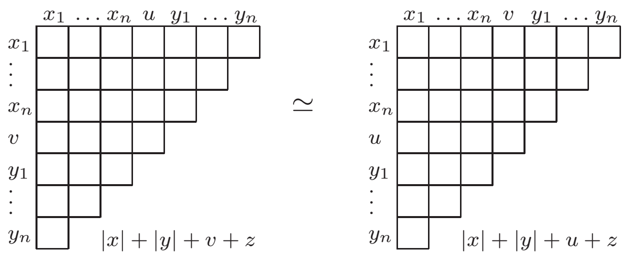
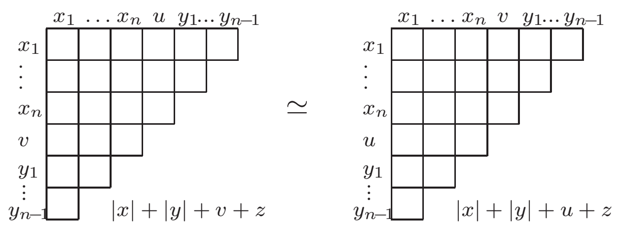
One of the main results of the paper is the following theorem (see Theorem 3.10).

Theorem. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be two sequences in $\mathbb{Z}_{\geq 0}$, set $|x| = \sum x_i$, $|y| = \sum y_i$, and let $u, v, z \in \mathbb{Z}_{\geq 0}$. Then the following $\mathrm{SL}(2,\mathbb{C})$-isomorphism holds:



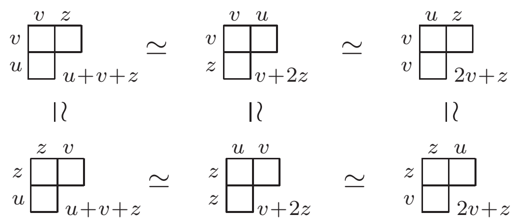
Although the diagrams in the above isomorphism have an odd number of steps, it is immediate to derive from it (taking $u = 0$) the following analogous isomorphism for even number of steps:



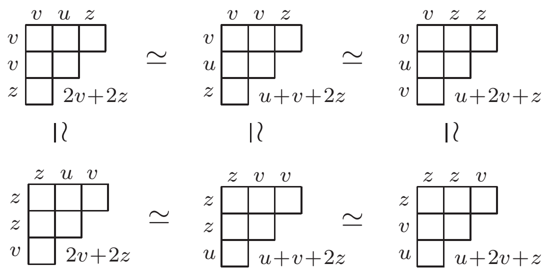
Let's say that two pairs (λ, d) and (μ, e) are equivalent if $\mathbb{S}_\lambda\left(\mathbb{S}_{(d)}(\mathbb{C}^2)\right) \simeq \mathbb{S}_\mu\left(\mathbb{S}_{(e)}(\mathbb{C}^2)\right)$. From the above isomorphism it is possible to obtain another isomorphism by using the fact that an $\mathrm{SL}(2,\mathbb{C})$-module is isomorphic to its dual module. This, in general, yields an equivalence class of four different pairs (λ, d). If we additionally assume in the previous theorem that $x_i = v$

and $y_i = z$ for all $i = 1, \ldots, n$, then we can make use of its result twice, and obtain an equivalence class of six different pairs (λ, d). In the odd case with $n = 0$, this equivalence class of six pairs corresponds to Manivel's Theorem.

The even and odd cases with $n = 1$ state that



and



These, and other corollaries, are obtained in §4.

Recall that given a partition λ and a number $d \geq 0$ the *hook length* of λ and the *d-content* of λ are, respectively, the following polynomials

$$\mathbf{h}_\lambda(q) = \prod [h(u)]_q, \qquad \mathbf{c}^d_\lambda(q) = \prod [d + 1 + c(u)]_q,$$

where $[a]_q$ is the q-analog of a, h and c are, respectively, the hook and the content functions and both products run over the entries of the Young diagram of λ. It is known (see e.g. [St, Ch. 7]) that the $\mathrm{SL}(2, \mathbb{C})$-character of $\mathbb{S}_\lambda\left(\mathbb{S}_\delta(\mathbb{C}^2)\right)$ is, up to a power of q, equal to

$$P^d_\lambda(q) = \frac{\mathbf{c}^d_\lambda(q)}{\mathbf{h}_\lambda(q)}$$

where $d = \delta_1 - \delta_2$.

The following theorem translates the plethysm equation (1.1) in terms of P. Although the results stated in this theorem might be known, we did not find an explicit reference to it, thus we prove it in §3 (see Theorem 3.1). If λ is a partition, then $|\lambda|$ denotes the sum of its parts.

Theorem. Let $\delta = (\delta_1, \delta_2)$, $\epsilon = (\epsilon_1, \epsilon_2)$ and $d = \delta_1 - \delta_2$, $e = \epsilon_1 - \epsilon_2$. Let λ, μ be partitions with $\ell(\lambda) \leq d + 1$ and $\ell(\mu) \leq e + 1$. Then

(1) $\mathbb{S}_\lambda\left(\mathbb{S}_\delta(\mathbb{C}^2)\right) \simeq \mathbb{S}_\mu\left(\mathbb{S}_\epsilon(\mathbb{C}^2)\right)$ as $\mathrm{SL}(2, \mathbb{C})$-modules if and only if

$$P^d_\lambda = P^e_\mu$$

and in this case $|\lambda|d - |\mu|e$ is even.

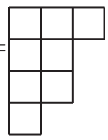
(2) $\mathbb{S}_\lambda\left(\mathbb{S}_\delta(\mathbb{C}^2)\right) \simeq \mathbb{S}_\mu\left(\mathbb{S}_\epsilon(\mathbb{C}^2)\right)$ as $\mathrm{GL}(2, \mathbb{C})$-modules if and only if, in addition to $P^d_\lambda = P^e_\mu$, it also holds

$$|\delta||\lambda| = |\epsilon||\mu|.$$

2. Technical background

2.1. **Partitions.** A *partition* λ of n is an ordered sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq ...$ with $|\lambda| = n$, where $|\lambda| = \sum \lambda_i$. The λ_i's are called the parts of the partition and the length $\ell(\lambda)$ of λ is the number of non zero parts. If $k \geq \ell(\lambda)$ then λ will be denoted as $\lambda = (\lambda_1, \dots, \lambda_k)$ or by indicating multiplicities with exponential notation, for instance $(4,4,3,1,1,1) = (4^2, 3, 1^3)$. If λ and μ are two partitions, we denote by $\lambda + \mu$ the partition whose parts are $(\lambda + \mu)_i = \lambda_i + \mu_i$.

To each partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n we associate its *Young diagram* $Y(\lambda)$ and its *standard tableau* $T(\lambda)$: $Y(\lambda)$ is the graphical arrangement consisting of k left-justified rows of boxes, with λ_i boxes in the i-th row, and $T(\lambda)$ is the assigment of the integers $1, 2, \dots, n$ to the n boxes of $Y(\lambda)$ obtained by writing the numbers $1, 2, \dots, n$ starting on the first row and increasing to the right and then continuing on the second row, etc. For example, if $\lambda = (3, 2, 2, 1)$ then

$Y(\lambda) =$ 

$$T(\lambda) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array}$$

The *transpose* of a partition is the partition λ^t whose Young diagram is the transpose of the Young diagram of λ. For example the transpose of the partition $(3, 2, 2, 1)$ is the partition $(4, 3, 1)$ as can be seen by the drawing above.

2.2. **Schur functor.** If λ is a partition of n, two subgroups of the symmetric group $\mathfrak{S}_n$ are associated to $T(\lambda)$:

$$P_\lambda = \{\sigma \in \mathfrak{S}_n : \sigma \text{ preserves each row of } T(\lambda)\},$$
$$Q_\lambda = \{\sigma \in \mathfrak{S}_n : \sigma \text{ preserves each column of } T(\lambda)\}.$$

Following [FH] we denote by a_λ, b_λ, c_λ the following elements of the group algebra $\mathbb{C}[\mathfrak{S}_n]$:

$$a_\lambda = \sum_{\sigma \in P_\lambda} \sigma, \qquad b_\lambda = \sum_{\sigma \in Q_\lambda} \operatorname{sgn}(\sigma)\sigma, \qquad c_\lambda = a_\lambda b_\lambda.$$

The element c_λ is called the *Young symmetrizer* associated to λ. The permutation group $\mathfrak{S}_n$ acts naturally on $V^{\otimes n}$ by $\sigma.(v_1 \otimes ... \otimes v_n) = v_{\sigma(1)} \otimes ... \otimes v_{\sigma(n)}$. This action is naturally extended to an action of its group algebra $\mathbb{C}[\mathfrak{S}_n]$. The image of $V^{\otimes n}$ under the action of c_λ is denoted $\mathbb{S}_\lambda(V)$ and the map $V \mapsto \mathbb{S}_\lambda(V)$ is called the *Schur functor*.

For instance:

- $\mathbb{S}_{(n)}(V) \simeq \operatorname{Sym}^n(V)$,
- $\mathbb{S}_{(1^n)}(V) \simeq \Lambda^n(V)$
- $\mathbb{S}_\lambda(V) = 0$ if λ has more than $\dim(V)$ parts.

2.3. **Schur polynomials.** If λ is a partition of n and $k \geq \ell(\lambda)$, the *Schur polynomial* in k variables associated to λ is

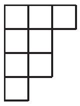
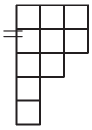
$$s_\lambda(x_1, \dots, x_k) = \frac{\det(x_j^{\lambda_i + k - i})}{\det(x_j^{k-i})},$$

This is a symmetric polynomial in k variables of degree n for any $k \geq \ell(\lambda)$.

The Schur polynomial has an interesting property that will be useful later: given a partition λ and $k \geq \ell(\lambda)$ we will denote by λ' the partition whose Young diagram is the complement of $Y(\lambda)$ in the $(k \times \lambda_1)$-rectangle. (This definition depends on k, though this fact is not indicated in the notation). That is,

$$\lambda' = (\lambda_1 - \lambda_k, \dots, \lambda_1 - \lambda_2)$$

For example, for $k = 6$ we have that

if $Y(\lambda) =$  then $Y(\lambda') =$ 

It is not difficult to prove (see Exercise 7.41 of [St]) that

$$(x_1 \dots x_k)^{\lambda_1} s_\lambda(x_1^{-1}, \dots, x_k^{-1}) = s_{\lambda'}(x_1, \dots, x_k). \tag{2.1}$$

2.4. **Polynomial representations of GL(V) and SL(V).** Let V be a finite dimensional complex vector space of dimension k. A *polynomial representation* of $\mathrm{GL}(V)$ is a finite dimensional representation of $\mathrm{GL}(V)$ such that the matrix entries (associated to a given basis) are given by polynomial functions on V. It is well known that every polynomial representation of $\mathrm{GL}(V)$ can be decomposed into irreducible subrepresentations. In particular, $\mathbb{S}_\lambda(V)$ is an irreducible $\mathrm{GL}(V)$-subrepresentation of $V^{\otimes n}$ for all partitions λ of n. The *highest weight theorem* states that $\lambda \mapsto \mathbb{S}_\lambda(V)$ establishes a one-to-one correspondence between the set of equivalence classes of irreducible polynomial representations of $\mathrm{GL}(V)$ and the set of partitions λ with $\ell(\lambda) \leq k$, see for instance §6 in [FH].

Moreover $\lambda \mapsto \mathbb{S}_\lambda(V)$ also establishes a one-to-one correspondence between the set of equivalence classes of irreducible polynomial representations of $\mathrm{SL}(V)$ and the set of partitions λ with $\ell(\lambda) \leq k-1$. This follows from the following fact: if

$$\tilde{\lambda} = \lambda - (\lambda_k^k) = (\lambda_1 - \lambda_k, \dots, \lambda_{k-1} - \lambda_k)$$

then $\mathbb{S}_\lambda(V) \simeq \mathbb{S}_{(\lambda_k^k)}(V) \otimes \mathbb{S}_{\tilde{\lambda}}(V)$ as $\mathrm{GL}(V)$-modules. But since $\mathbb{S}_{(r^k)}(V)$ is the 1-dimensional $\mathrm{GL}(V)$-module corresponding to $\det^r$, then we obtain that $\mathbb{S}_\lambda(V) \simeq \mathbb{S}_{\tilde{\lambda}}(V)$ as $\mathrm{SL}(V)$-modules. Note that $\tilde{\lambda}$ now has at most $k-1$ parts.

2.5. **Characters of GL(V)-modules.** If π is a polynomial representation of $\mathrm{GL}(V)$, the *character* of π is the function $\chi_\pi : \mathrm{GL}(V) \to \mathbb{C}$ defined by $\chi_\pi(g) = \mathrm{tr}(\pi(g))$. If π_1 and π_2 are two polynomial representations of $\mathrm{GL}(V)$ then $\pi_1 \simeq \pi_2$ if and only if they have the same character. Similarly, $\pi_1 \simeq \pi_2$ as $\mathrm{SL}(V)$-modules if and only if $\chi_{\pi_1}|_{\mathrm{SL}(V)} = \chi_{\pi_2}|_{\mathrm{SL}(V)}$. If $g \in \mathrm{GL}(V)$ has eigenvalues $\theta_1, \dots, \theta_k$ (counted with multiplicities), then it is known that

$$\chi_{\mathbb{S}_\lambda(V)}(g) = s_\lambda(\theta_1, \dots, \theta_k) \tag{2.2}$$

for any partition λ with $\ell(\lambda) \leq k$. (See for instance [FH]).

Let $\delta = (\delta_1, \delta_2)$ be a partition with at most two parts and let $d = \delta_1 - \delta_2$. We know that $\dim \mathbb{S}_\delta\left(\mathbb{C}^2\right) = d+1$ and, as a representation of

$\mathrm{SL}(2,\mathbb{C})$, $\mathbb{S}_\delta(\mathbb{C}^2)$ corresponds to the irreducible representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ of highest weight d.

If $g \in \mathrm{GL}(2,\mathbb{C})$ has eigenvalues x_1 and x_2 then it follows from (2.2) that

$$\chi_{\mathbb{S}_\delta(\mathbb{C}^2)}(g) = s_\delta(x_1,x_2) = x_1^{\delta_1}x_2^{\delta_2} + x_1^{\delta_1-1}x_2^{\delta_2+1} + \cdots + x_1^{\delta_2}x_2^{\delta_1}.$$

Hence the eigenvalues of g in $\mathbb{S}_\delta\left(\mathbb{C}^2\right)$ are $\{x_1^{\delta_1}x_2^{\delta_2},\dots,x_1^{\delta_2}x_2^{\delta_1}\}$ (all with multiplicity 1) and thus, if λ is a partition with $\ell(\lambda) \le d+1$, then the character of $\mathbb{S}_\lambda\left(\mathbb{S}_\delta(\mathbb{C}^2)\right)$ is the plethysm

$$\chi_{\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2))}(g) = s_\lambda(x_1^{\delta_1}x_2^{\delta_2}, x_1^{\delta_1-1}x_2^{\delta_2+1},\dots,x_1^{\delta_2}x_2^{\delta_1}). \tag{2.3}$$

In particular, if $g \in \mathrm{SL}(2,\mathbb{C})$ with eigenvalues x_1 and x_1^{-1} then

$$\chi_{\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2))}(g) = \chi_{\mathbb{S}_\lambda\left(\mathbb{S}_{(d)}(\mathbb{C}^2)\right)}(g) = s_\lambda(x_1^d, x_1^{d-2},\dots,x_1^{-d}).$$

This identity and (2.1) imply that if λ' is as in §2.3 (with $k=d+1$) then $\chi_{\mathbb{S}_\lambda\left(\mathbb{S}_{(d)}(\mathbb{C}^2)\right)}$ and $\chi_{\mathbb{S}_{\lambda'}\left(\mathbb{S}_{(d)}(\mathbb{C}^2)\right)}$ coincide in $\mathrm{SL}(2,\mathbb{C})$ and therefore we obtain:

Theorem 2.1.

$$\mathbb{S}_\lambda\left(\mathbb{S}_{(d)}(\mathbb{C}^2)\right) \simeq \mathbb{S}_{\lambda'}\left(\mathbb{S}_{(d)}(\mathbb{C}^2)\right)$$

as $SL(2,\mathbb{C})$*-modules.*

This corresponds to the fact that $\mathbb{S}_{\lambda'}\left(\mathbb{S}_{(d)}(\mathbb{C}^2)\right)$ and $\mathbb{S}_\lambda\left(\mathbb{S}_{(d)}(\mathbb{C}^2)\right)$ are dual to each other as $\mathrm{SL}(2,\mathbb{C})$-modules and every polynomial representation of $\mathrm{SL}(2,\mathbb{C})$ is isomorphic to its dual.

2.6. **The Hook-content formula.** Given a natural number a, let

$$[a] = [a]_q = \frac{1-q^a}{1-q} = 1 + q + \cdots + q^{a-1}$$

be the q-analog of a. If $u=(i,j)$ is a box of the Young diagram of λ let $c(u) = j-i$ and let $h(u)$ be the number of boxes directly below or directly to the right of u, including u once. For example, we indicate in the following diagrams the values of c and h respectively:

0	1	2
-1	0	
-2	-1	
-3		

6	4	1
4	2	
3	1	
1		

Given a partition λ and a number d we define the *hook length* of λ and the d*-content* of λ as the following polynomials:

$$\mathbf{h}_\lambda(q) = \prod_{u\in Y(\lambda)} [h(u)]_q \qquad \mathbf{c}_\lambda^d(q) = \prod_{u\in Y(\lambda)} [d+1+c(u)]_q.$$

Let $\delta = (\delta_1,\delta_2)$ be a partition with at most two parts and let $d = \delta_1 - \delta_2$. Let λ be a partition with $\ell(\lambda) \le d+1$. Since s_λ is homogeneous of degree $|\lambda|$ it follows that

$$s_\lambda(x_1^{\delta_1}x_2^{\delta_2}, x_1^{\delta_1-1}x_2^{\delta_2+1},\dots,x_1^{\delta_2}x_2^{\delta_1}) = (x_1^{\delta_1}x_2^{\delta_2})^{|\lambda|} s_\lambda(1,q,q^2,\dots,q^d),$$

where $q = x_1^{-1}x_2$. If $b(\lambda) = \sum (i-1)\lambda_i$ and

$$P_\lambda^d(q) = \frac{\mathbf{c}_\lambda^d(q)}{\mathbf{h}_\lambda(q)}$$

then Theorem 7.21.2 in [St] states that

$$s_\lambda(1, q, ..., q^d) = q^{b(\lambda)} P_\lambda^d(q). \tag{2.4}$$

This identity is known as the *Hook-content formula*, see the notes in Ch. 7 of [St] for more information about it.

It follows from (2.3) that if x_1 and x_2 are the eigenvalues of $g \in \mathrm{GL}(2, \mathbb{C})$, then

$$\chi_{\mathbb{S}_\lambda(\mathbb{S}_\delta(\mathbb{C}^2))}(g) = (x_1^{\delta_1} x_2^{\delta_2})^{|\lambda|} q^{b(\lambda)} P_\lambda^d(q). \tag{2.5}$$

3. Main results

3.1. **Equation** (1.1) **and the Hook-content formula.** The following theorem expresses the isomorphism condition of (1.1) in terms of the function P.

Theorem 3.1. *Let $\delta = (\delta_1, \delta_2)$, $\epsilon = (\epsilon_1, \epsilon_2)$ and $d = \delta_1 - \delta_2$, $e = \epsilon_1 - \epsilon_2$. Let λ, μ be partitions with $\ell(\lambda) \leq d+1$ and $\ell(\mu) \leq e+1$. Then*

(1) $\mathbb{S}_\lambda\left(\mathbb{S}_\delta(\mathbb{C}^2)\right) \simeq \mathbb{S}_\mu\left(\mathbb{S}_\epsilon(\mathbb{C}^2)\right)$ as SL$(2, \mathbb{C})$-modules if and only if

$$P_\lambda^d = P_\mu^e \tag{3.1}$$

and in this case $|\lambda|d - |\mu|e$ is even.

(2) $\mathbb{S}_\lambda\left(\mathbb{S}_\delta(\mathbb{C}^2)\right) \simeq \mathbb{S}_\mu\left(\mathbb{S}_\epsilon(\mathbb{C}^2)\right)$ as GL$(2, \mathbb{C})$-modules if and only if in addition to (3.1) *it also holds*

$$|\delta||\lambda| = |\epsilon||\mu|. \tag{3.2}$$

Proof. On the one hand, it follows from (2.5) that $\mathbb{S}_\lambda\left(\mathbb{S}_\delta(\mathbb{C}^2)\right) \simeq \mathbb{S}_\mu\left(\mathbb{S}_\epsilon(\mathbb{C}^2)\right)$ as representations of $\mathrm{GL}(2, \mathbb{C})$ if and only if

$$(x_1^{\delta_1} x_2^{\delta_2})^{|\lambda|} q^{b(\lambda)} P_\lambda^d(q) = (x_1^{\epsilon_1} x_2^{\epsilon_2})^{|\mu|} q^{b(\mu)} P_\mu^e(q) \tag{3.3}$$

and since the identity $x_1 x_2 = 1$ holds in $\mathrm{SL}(2, \mathbb{C})$, it follows that $q = x_1^{-1} x_2 = x_2^2$ and hence $\mathbb{S}_\lambda\left(\mathbb{S}_\delta(\mathbb{C}^2)\right) \simeq \mathbb{S}_\mu\left(\mathbb{S}_\epsilon(\mathbb{C}^2)\right)$ as $\mathrm{SL}(2, \mathbb{C})$-modules if and only if

$$x_2^{-d|\lambda|+2b(\lambda)} P_\lambda^d(x_2^2) = x_2^{-e|\mu|+2b(\mu)} P_\mu^e(x_2^2) \tag{3.4}$$

as a function of x_2.

On the other hand, since s_λ is symmetric, it follows from (2.3) and (2.5) that

$$x_1^{\delta_1|\lambda|-b(\lambda)} x_2^{\delta_2|\lambda|+b(\lambda)} P_\lambda^d(q) = x_2^{\delta_1|\lambda|-b(\lambda)} x_1^{\delta_2|\lambda|+b(\lambda)} P_\lambda^d(q^{-1})$$

and thus

$$\begin{aligned} \frac{P_\lambda^d(q)}{P_\lambda^d(q^{-1})} &= x_2^{(\delta_1-\delta_2)|\lambda|-2b(\lambda)} x_1^{(\delta_2-\delta_1)|\lambda|+2b(\lambda)} \\ &= q^{d|\lambda|-2b(\lambda)}. \end{aligned}$$

A similar identity holds for μ and ϵ instead of λ and δ.

We now assume condition (3.1). This and the above identities imply that

$$d|\lambda| - 2b(\lambda) = e|\mu| - 2b(\mu) \tag{3.5}$$

and therefore (3.4) holds and thus $\mathbb{S}_\lambda\left(\mathbb{S}_\delta(\mathbb{C}^2)\right) \simeq \mathbb{S}_\mu\left(\mathbb{S}_\epsilon(\mathbb{C}^2)\right)$ as representations of $\mathrm{SL}(2,\mathbb{C})$. It also follows from (3.5) that $|\lambda|d - |\mu|e$ is even.

If we additionally assume that condition (3.2) holds, then adding and substracting (3.5) and (3.2) we obtain

$$\begin{aligned} \delta_1|\lambda| - b(\lambda) &= \epsilon_1|\tau| - b(\tau) \\ \delta_2|\lambda| + b(\lambda) &= \epsilon_2|\tau| + b(\tau), \end{aligned}$$

and taking into account that $q = x_1^{-1}x_2$, (3.3) follows and thus $\mathbb{S}_\lambda\left(\mathbb{S}_\delta(\mathbb{C}^2)\right) \simeq \mathbb{S}_\mu\left(\mathbb{S}_\epsilon(\mathbb{C}^2)\right)$ as representations of $\mathrm{GL}(2,\mathbb{C})$.

For the converse statements, we first observe that $q = 0$ is neither a root nor a pole of the rational function $P_\lambda^d(q) = \frac{\mathbf{c}_\lambda^d(q)}{\mathbf{h}_\lambda(q)}$. Therefore, if $\mathbb{S}_\lambda\left(\mathbb{S}_\delta(\mathbb{C}^2)\right) \simeq \mathbb{S}_\mu\left(\mathbb{S}_\epsilon(\mathbb{C}^2)\right)$ as representations of $\mathrm{SL}(2,\mathbb{C})$ then it follows from (3.4) that $P_\lambda^d = P_\mu^e$. If the isomorphism also holds as representations of $\mathrm{GL}(2,\mathbb{C})$, then we obtain (3.2) by specializing (3.3) at $x_1 = x_2$. □

3.2. **$\mathbf{GL}(2,\mathbb{C})$-isomorphisms from $\mathbf{SL}(2,\mathbb{C})$-isomorphisms.** Let $\delta = (\delta_1, \delta_2)$, $d = \delta_1 - \delta_2$, and let λ be partition with $\ell(\lambda) \le d+1$. Since $d+1 = \dim(\mathbb{S}_{(d)}(\mathbb{C}^2))$ it follows from the discusion in §2.4, that if

$$\tilde{\lambda} = (\lambda_1 - \lambda_{d+1}, \dots, \lambda_d - \lambda_{d+1})$$

then $\mathbb{S}_\lambda(\left(\mathbb{S}_{(d)}(\mathbb{C}^2)\right)) \simeq \mathbb{S}_{\tilde{\lambda}}(\left(\mathbb{S}_{(d)}(\mathbb{C}^2)\right))$ as $\mathrm{SL}(2,\mathbb{C})$-modules. Thus, in order to study the plethysm equation (1.1) as $\mathrm{SL}(V)$-modules it is enough to consider the problem of finding d, e, λ and μ with $\ell(\lambda) \le d$, $\ell(\mu) \le e$, such that

$$\mathbb{S}_\lambda\left(\mathbb{S}_{(d)}(\mathbb{C}^2)\right) \simeq \mathbb{S}_\mu\left(\mathbb{S}_{(e)}(\mathbb{C}^2)\right) \tag{3.6}$$

as representations of $\mathrm{SL}(2,\mathbb{C})$.

On the other hand, if (3.6) holds, part (2) of Theorem 3.1 says that the isomorphism also holds as $\mathrm{GL}(2,\mathbb{C})$-modules if and only if $|\lambda|d = |\mu|e$.

If this is not the case, a natural question to ask is whether there exist $l, m, x, y \in \mathbb{Z}_{\ge 0}$ such that

$$\mathbb{S}_{\lambda+(l^{d+1})}\left(\mathbb{S}_{(d+x,x)}(\mathbb{C}^2)\right) \simeq \mathbb{S}_{\mu+(m^{e+1})}\left(\mathbb{S}_{(e+y,y)}(\mathbb{C}^2)\right)$$

as representations of $\mathrm{GL}(2,\mathbb{C})$.

According to part (2) of Theorem 3.1 the answer is positive if and only if

$$(|\lambda| + l(d+1))(d+2x) = (|\mu| + m(e+1))(e+2y)$$

wich is equivalent to

$$\big(|\mu|+m(e+1)\big)y - \big(|\lambda|+l(d+1)\big)x = \frac{|\lambda|d - |\mu|e}{2} + l\binom{d+1}{2} - m\binom{e+1}{2}. \tag{3.7}$$

From part (1) of Theorem 3.1 we know that the right hand side of (3.7) is an integer number. In addition, there exist $l, m, x, y \in \mathbb{Z}_{\ge 0}$ satisfying (3.7) if and only if the there exist $l, m \in \mathbb{Z}_{\ge 0}$ such that

(3.8)
$$\gcd\left\{\big(|\mu| + m(e+1)\big), \big(|\lambda| + l(d+1)\big)\right\} \,\Big|\, \frac{|\lambda|d - |\mu|e}{2} + l\binom{d+1}{2} - m\binom{e+1}{2}.$$

Such l and m do not always exist but in many cases they do. Concretely

Theorem 3.2. *If n is an integer number, let $\nu_2(n)$ be the exponent of the highest power of the prime 2 that divides n.*

Then there exist l and m such that (3.8) *holds unless $\nu_2(|\mu|) \neq \nu_2(|\lambda|)$ and $0 < \min\{\nu_2(|\mu|), \nu_2(|\lambda|)\} < \min\{\nu_2(e+1), \nu_2(d+1)\}$.*

Since this is a side issue with respect to the main thrust of this paper, and the proof, while not difficult, is slightly complicated, we will prove the above theorem in another article.

3.3. **Equation** (1.1) **as** $\mathbf{SL}(2,\mathbb{C})$**-modules.**

Notation 3.3. In order to write the proofs easier, if we have a rectangular array of q numbers:

$$i\left\{\overbrace{\begin{matrix} [x+i+j-2] & [x+i+j-3] & \dots & [x+i-1] \\ [x+i+j-3] & [x+i+j-3] & \dots & [x+i-2] \\ \vdots & & \dots & \vdots \\ [x+j-1] & [x+j-2] & \dots & [x] \end{matrix}}^{j}\right.$$

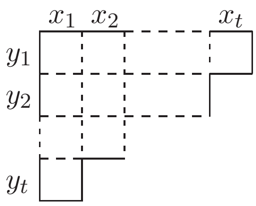
in which all the columns and rows decrease by 1, we will denote the product of all the elements $[*]$ in that rectangle by $\rho_{i,j}(x)$. Clearly $\rho_{i,j}(x) = \rho_{j,i}(x)$ and if $k > j$ then $\rho_{i,k}(x) = \rho_{i,k-j}(x+j)\rho_{i,j}(x)$.

Lemma 3.4. *If λ^t is the transpose of λ (see §2.1), then:*

$$\mathbf{h}_\lambda = \mathbf{h}_{\lambda^t}$$

Proof. Let $x_1, \dots, x_t, y_1, \dots, y_t$ be such that the Young diagram of λ is:

$Y(\lambda) =$

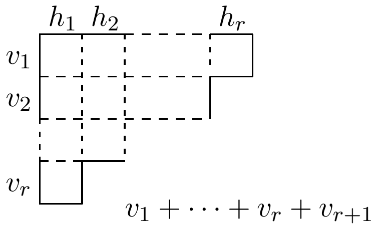


Then $\mathbf{h}_\lambda$ is the product of all the $\rho_{y_i,x_j}(1 + y_{i+1} + \cdots + y_t + x_{j+1} + \cdots + x_t)$. Since that product is obviously symmetric on the x's and y's, then we obtain the result. □

Notation 3.5. Let $h_1, \dots, h_t, v_1, \dots, v_{t+1}$ be positive integers. The notation $\langle h_1, \dots, h_t || v_1, \dots, v_t, v_{t+1}\rangle$ will mean the $\mathrm{SL}(2,\mathbb{C})$-module $\mathbb{S}_\lambda\left(\mathbb{S}_{(w)}(\mathbb{C}^2)\right)$ where $w = v_1 + \cdots v_{r+1} - 1$ and

$$\lambda = \left((h_1 + \cdots + h_{t-1} + h_t)^{v_1}, (h_1 + \cdots + h_{t-1})^{v_2}, \dots, (h_1 + h_2)^{v_{t-1}}, h_1^{v_t}\right),$$

In order to simplify this notation, given a sequence $x_1, x_2, \dots, x_t$, we will denote by $\overrightarrow{x}$ the sequence $x_1, x_2, \dots, x_t$ and by $\overleftarrow{x}$ the sequence $x_t, x_{t-1}, \dots, x_1$. That is:

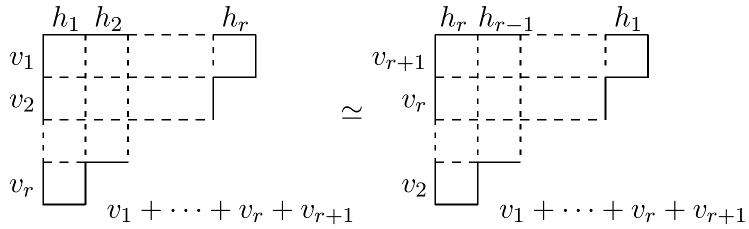
$\langle \overrightarrow{h} \mid\mid \overrightarrow{v} \rangle =$ 

If In this notation, the SL$(2,\mathbb{C})$-modules isomorphism given in Theorem 2.1 becomes

Theorem 3.6.

$$\langle \overrightarrow{h} \mid\mid \overrightarrow{v} \rangle \simeq \langle \overleftarrow{h} \mid\mid \overleftarrow{v} \rangle$$

In pictures:



Theorem 3.7. *Let $s \geq 0$ and $t = s$ or $t = s+1$. Let $x_1, \ldots, x_s$ and $y_1, \ldots, y_t$ be two sequences of positive integers, u, v, z three positive integers. Let $\mid \overrightarrow{x} \mid$ denote $\sum_i x_i$.*

a) The following $SL(2,\mathbb{C})$-isomorphisms hold:

$$\begin{array}{ccc} \langle \overrightarrow{x}, u, \overrightarrow{y} \mid\mid z, \overrightarrow{x}, v, \overrightarrow{y} \rangle & \simeq & \langle \overrightarrow{x}, v, \overrightarrow{y} \mid\mid z, \overrightarrow{x}, u, \overrightarrow{y} \rangle \\ \wr\wr & & \wr\wr \\ \langle \overleftarrow{y}, u, \overleftarrow{x} \mid\mid \overleftarrow{y}, v, \overleftarrow{x}, z \rangle & \simeq & \langle \overleftarrow{y}, v, \overleftarrow{x} \mid\mid \overleftarrow{y}, u, \overleftarrow{x}, z \rangle \end{array}$$

b) Let $S = \mid \overrightarrow{x} \mid^2 + 2\sum_{i,j:i+j=t} x_i y_j$. If $z(z-1) = S + \mid \overrightarrow{x} \mid (u+v)$ in the case $t = s$, or $z(z-1) = S + \mid \overrightarrow{x} \mid (u+v) + uv$ in the case $t = s+1$ then the first row is a $GL(2,\mathbb{C})$-isomorphism.

c) Except in the trivial case $u = v$, the second row and both columns are never $GL(2,\mathbb{C})$ isomorphisms.

Proof. a) The horizontal isomorphisms reveal a symmetry between u and v. Since the vertical isomorphisms follow from Theorem 3.6, we only need to prove one of the horizontal ones. We will prove the second one, i.e., we will show that $\langle \overleftarrow{y}, u, \overleftarrow{x} \mid\mid \overleftarrow{y}, v, \overleftarrow{x}, z \rangle$ is symmetric on u and v.

Let us call $\lambda_{u,v}$ the subyacent partition in $\langle \overleftarrow{y}, u, \overleftarrow{x} \mid\mid \overleftarrow{y}, v, \overleftarrow{x}, z \rangle$ Since $\lambda_{v,u} = \lambda_{u,v}^t$, then by Lemma 3.4 we have $\mathbf{h}_{\lambda_{u,v}} = \mathbf{h}_{\lambda_{v,u}}$.

Now let us see $\mathbf{c}$.

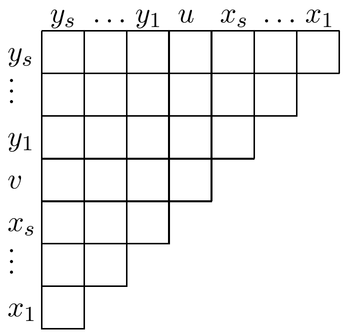
We need to compute $\mathbf{c}^w_{\lambda_{u,v}}$, where $w = \mid \overleftarrow{x} \mid + v + \mid \overleftarrow{y} \mid + z - 1$. Note that w depends on v but not u.

In this case the product $\rho_{i,j}(k)$ arises from an array of the form:

$$i\left\{\overbrace{\begin{matrix} [k+i-1] & [k+i] & \ldots & [k+i+j-2] \\ \vdots & & \ldots & \vdots \\ [k+1] & [k+2] & \ldots & [k+j] \\ [k] & [k+1] & \ldots & [k+j-1] \end{matrix}}^{j}\right.$$

Let's consider first the case $t = s$. The partition is then:

$Y(\lambda_{u,v})$:



We see from $Y(\lambda_{u,v})$ that $\mathbf{c}^{w}_{\lambda_{u,v}}$ is the product of

(1) $\rho_{|\overleftarrow{y}|+v,|\overleftarrow{y}|+u}(w+2-|\overleftarrow{y}|-v)$. Note that $w+2-|\overleftarrow{y}|-v = |\overleftarrow{x}|+z+1$, this item is $\rho_{|\overleftarrow{y}|+v,|\overleftarrow{y}|+u}(|\overleftarrow{x}|+z+1)$, thus symmetric in u, v.

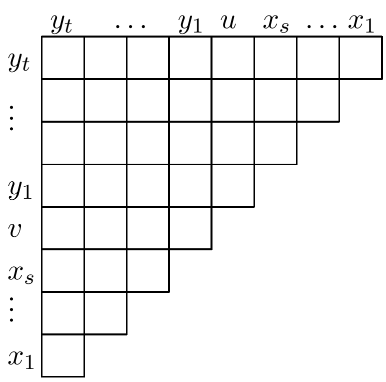
(2) ρ's from the part of the table below the horizontal v line, which are independent of u, v.

(3) ρ's from the part of the table to the right of vertical u column. These are of the form $\rho_{y_i,x_j}(*)$ and $*$ is of the form $w+2+|\overleftarrow{y}|+u+$ some x's $-$ some y's, i.e. $w+u+$ other stuff. Note that since w depends on v and not u, then $w+u$ is symmetric on u, v.

Note that in the case $s = t = 0$, the proof reduces to just the case (1).

Now consider the case $t = s + 1$. Now the partition is:

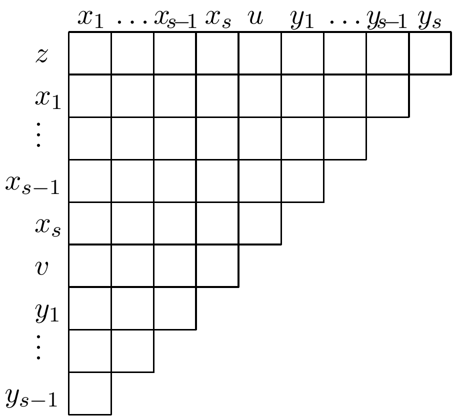
$Y(\lambda_{u,v})$:



As in the previous case, the ρ's from the part of the table below the horizontal v line are independent of u, v and the ρ's from the part of the table to the right of vertical u column depend on $u+v$ and thus are symmetric on u, v. So the only problem is the central part of the table, which, unlike the previous case is not a rectangle. Let's call Γ the product of the ρ's corresponding to $\mathbf{c}_{\lambda_{u,v}}$ of that part of the table. Here we simply observe that if we were to append an extra rectangle of height v and length u to the southeast corner of that part, then we would have a rectangle whose ρ, as in the previous case, would be symmetric in u, v. But the rectangle appended contributes with a $\rho_{v,u}$ of something that does not depend on u, v, hence it is symmetric on u, v. Therefore Γ is the quotient between two things symmetric on u, v, hence, symmetric itself.

b) Let $\mu_{u,v}$ now denote the subyacent partition in $\langle \vec{x}, u, \vec{y} \,||z, \vec{x}, v, \vec{y}\rangle$ and set $w_v = |\vec{x}| + v + |\vec{y}| + z - 1$. By part a) of this theorem and part b) of Theorem 3.1, in order to prove $\langle \vec{x}, u, \vec{y} \,||z, \vec{x}, v, \vec{y}\rangle \simeq \langle \vec{x}, v, \vec{y} \,||z, \vec{x}, u, \vec{y}\rangle$ as GL$(2, \mathbb{C})$-modules it suffices to see that $|\mu_{u,v}|w_v = |\mu_{v,u}|w_u$, i.e. it is enough to see that $|\mu_{u,v}|w_v$ is symmetric in u, v.

Let us start with the case $s = t$. The partition in this case is:



Thus $|\mu_{u,v}| = z(|\vec{x}| + u + |\vec{y}|) + |\vec{x}|^2 + |\vec{x}|(u + v) + 2\sum_{i,j:i+j=s} x_i y_j$. Since $S = |\vec{x}|^2 + 2\sum_{i,j:i+j=s} x_i y_j$. (because this is the case $t = s$). Hence

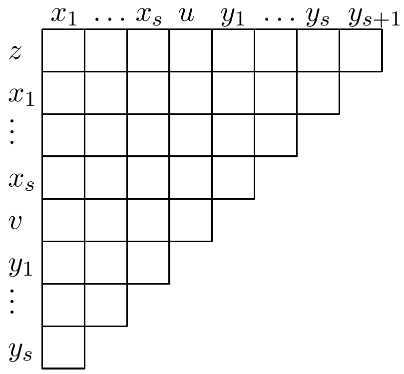
$$\begin{aligned}|\mu_{u,v}|w_v = &\Big((|\vec{x}| + u + |\vec{y}|)z + S + |\vec{x}|(u + v)\Big) . \Big(|\vec{x}| + v + |\vec{y}| + z - 1\Big)\\ &= (|\vec{x}| + u + |\vec{y}|)z(|\vec{x}| + v + |\vec{y}|) + (|\vec{x}| + |\vec{y}|)z(z - 1)+\\ &\quad + uz(z - 1) + Sv + S(|\vec{x}| + |\vec{y}| + z - 1) + |\vec{x}|(u + v)v+\\ &\quad + |\vec{x}|(u + v)(|\vec{x}| + |\vec{y}| + z - 1)\end{aligned}$$

The first, second, fifth and last terms are symmetric on u, v. The third, fourth and sixth, using $z(z - 1) = S + |\vec{x}|(u + v)$, are equal to:

$$uz(z - 1) + Sv + |\vec{x}|(u + v)v = (u + v)(S + |\vec{x}|(u + v))$$

which is symmetric in u, v.

Let us analyze now the case $t = s + 1$. The partition in this case is:



Thus in this case $|\mu_{u,v}| = z(|\vec{x}| + u + |\vec{y}|) + |\vec{x}|^2 + |\vec{x}|(u+v) + 2\sum_{i,j:i+j=s+1} x_i y_j + uv$ Since in this case $2\sum_{i,j:i+j=s+1} x_i y_j = 2\sum_{i,j:i+j=t} x_i y_j$ we have $|\mu_{u,v}| = z(|\vec{x}| + u + |\vec{y}|) + S + |\vec{x}|(u+v) + uv$ and:

$$\begin{aligned}
|\mu_{u,v}|w_v = &\Big((|\vec{x}| + u + |\vec{y}|)z + S + |\vec{x}|(u+v) + uv\Big) \cdot \Big(|\vec{x}| + v + |\vec{y}| + z - 1\Big) \\
= &(|\vec{x}| + u + |\vec{y}|)z(|\vec{x}| + v + |\vec{y}|) + (|\vec{x}| + |\vec{y}|)z(z-1) + \\
&+ uz(z-1) + Sv + S(|\vec{x}| + |\vec{y}| + z - 1) + |\vec{x}|(u+v)v + \\
&+ |\vec{x}|(u+v)(|\vec{x}| + |\vec{y}| + z - 1) + uv(|\vec{x}| + |\vec{y}| + z - 1) + uv^2
\end{aligned}$$

The first, second, fifth, seventh and eight terms are symmetric on u, v. The third, fourth, sixth and last term, using $z(z-1) = S + |\vec{x}|(u+v) + uv$, are equal to $(u+v)(S + |\vec{x}|(u+v) + uv)$, symmetric.

c) The previous second horizontal isomorphism never holds as a $\mathrm{GL}(2,\mathbb{C})$ isomorphism. (except in the trivial case $u = v$).

This follows since $\lambda_{v,u} = \lambda_{u,v}^t$, hence $|\lambda_{u,v}| = |\lambda_{v,u}|$ but $w_v \neq w_u$ hence $|\lambda_{u,v}|w_v \neq |\lambda_{v,u}|w_u$, so by Theorem 3.1 the $\mathrm{GL}(2,\mathbb{C})$ isomorphism does not hold. □

Remark 3.8. Note that by the first part of Theorem 3.1, $|\lambda_{u,v}|w_v - |\lambda_{v,u}|w_u$ must be even. Since that diference is $|\lambda_{u,v}|(v-u)$ then either $v-u$ is even or λ is even. This can also be verified directly.

Remark 3.9. Although in the statement and proof of Theorem 3.7 all the variables must be positive, let us see what happens if we set some of them equal to 0.

- If we set one of the variables y_i or x_i equal to zero, what happens is that this gives rise to another configuration with all variables positive but both s and t decrease by 1. For example, if we set $x_1 = 0$, this eliminates an x variable, decreasing s by 1 but "joins" y_{s-1} and y_s to form a new variable with value $y_{s-1} + y_s$, thus decreasing s by 1 too. Hence the total number of variables decrease by two.
- If we set the variable $z = 0$, then we decrease the total number of variables by 1, and we switch from the $t = s$ case to the $t = s+1$ case

and viceversa, but we go for example from the upper isomorphsim of case $s=t$ to the lower isomorphism for case $t=s+1$.

Therefore we could state just one theorem, in the following form:

Theorem 3.10. *Let $s \geq 0$. Let $x_1,\dots,x_s$ and $y_1,\dots,y_s$ be two sequences of nonnegative integers, u,v,z three nonnegative integers . Then the following $SL(2,\mathbb{C})$-isomorphisms hold:*

$$\begin{array}{ccc} \langle \overrightarrow{x}, u, \overrightarrow{y} \,||z, \overrightarrow{x}, v, \overrightarrow{y}\rangle & \simeq & \langle \overrightarrow{x}, v, \overrightarrow{y} \,||z, \overrightarrow{x}, u, \overrightarrow{y}\rangle \\ \wr| & & \wr| \\ \langle \overleftarrow{y}, u, \overleftarrow{x} \,||\, \overleftarrow{y}, v, \overleftarrow{x}, z\rangle & \simeq & \langle \overleftarrow{y}, v, \overleftarrow{x} \,||\, \overleftarrow{y}, u, \overleftarrow{x}, z\rangle \end{array}$$

4. Some Corollaries

Here we obtain corollaries of Theorem 3.7.

Remark 4.1. Hermite's is a corollary of our theorem, since it is the case $s=t=0$, with $z=1$ which implies that the condition of part b) of Theorem 3.7 is satisfied, since $z(z-1)=0$ while $S=|\overrightarrow{x}|=0$ too. Hence we obtain the full statement of Hermite's, while from Manivel's result only the $\mathrm{SL}(2,\mathbb{C})$ isomorphism can be deduced.

Theorem 4.2. *Let v,z,u be positive integers. Let $s \geq 0$. Then the two following families of isomorphism hold:*

$$\text{(I)}\quad \begin{array}{ccccc} \langle z^s v^{s+1} || z^{s+1} u v^s\rangle & \simeq & \langle z^s u v^s || z^{s+1} v^{s+1}\rangle & \simeq & \langle z^{s+1} v^s || z^s u v^{s+1}\rangle \\ \wr| & & \wr| & & \wr| \\ \langle v^s z^{s+1} || v^{s+1} u z^s\rangle & \simeq & \langle v^s u z^s || v^{s+1} z^{s+1}\rangle & \simeq & \langle v^{s+1} z^s || v^s u z^{s+1}\rangle \end{array}$$

and

$$\text{(II)}\quad \begin{array}{ccccc} \langle z^s u v^{s+1} || z^{s+2} v^{s+1}\rangle & \simeq & \langle z^{s+1} v^{s+1} || z^{s+1} u v^{s+1}\rangle & \simeq & \langle z^{s+1} u v^s || z^{s+1} v^{s+2}\rangle \\ \wr| & & \wr| & & \wr| \\ \langle v^s u z^{s+1} || v^{s+2} z^{s+1}\rangle & \simeq & \langle v^{s+1} z^{s+1} || v^{s+1} u z^{s+1}\rangle & \simeq & \langle v^{s+1} u z^s || v^{s+1} z^{s+2}\rangle \end{array}$$

Proof. From Theore 3.7 we obtain:

$$\begin{array}{ccc} \langle z^s u v^s || z^{s+1} v^{s+1}\rangle & \simeq & \langle z^s v^{s+1} || z^{s+1} u v^s\rangle \\ \wr| & & \wr| \\ \langle v^s u z^s || v^{s+1} z^{s+1}\rangle & \simeq & \langle v^{s+1} z^s || v^s u z^{s+1}\rangle \end{array}$$

If we applly the lower isomorphism to $\langle z^s u v^s || z^{s+1} v^{s+1}\rangle$ we obtain:

$$\begin{array}{ccccc} \langle z^{s+1} v^s || z^s u v^{s+1}\rangle & \simeq & \langle z^s u v^s || z^{s+1} v^{s+1}\rangle & \simeq & \langle z^s v^{s+1} || z^{s+1} u v^s\rangle \\ & & \wr| & & \wr| \\ & & \langle v^s u z^s || v^{s+1} z^{s+1}\rangle & \simeq & \langle v^{s+1} z^s || v^s u z^{s+1}\rangle \end{array}$$

Applying theorem 3.6 to the top left, we get:

$$\begin{array}{ccccc}
\langle z^{s+1}v^s||z^suv^{s+1}\rangle & \simeq & \langle z^suv^s||z^{s+1}v^{s+1}\rangle & \simeq & \langle z^sv^{s+1}||z^{s+1}uv^s\rangle \\
\wr\| & & \wr\| & & \wr\| \\
\langle v^sz^{s+1}||v^{s+1}uz^s\rangle & \simeq & \langle v^suz^s||v^{s+1}z^{s+1}\rangle & \simeq & \langle v^{s+1}z^s||v^suz^{s+1}\rangle
\end{array}$$

Rearranging the top line we get the first result. The second result is similar: from Theorem 3.7 we get:

$$\begin{array}{ccc}
\langle z^{s+1}v^{s+1}||z^{s+1}uv^{s+1}\rangle & \simeq & \langle z^suv^{s+1}||z^{s+2}v^{s+1}\rangle \\
\wr\| & & \wr\| \\
\langle v^{s+1}z^{s+1}||v^{s+1}uz^{s+1}\rangle & \simeq & \langle v^{s+1}uz^s||v^{s+1}z^{s+2}\rangle
\end{array}$$

Again, applying the lower isomorphism to $\langle z^{s+1}v^{s+1}||z^{s+1}uv^{s+1}\rangle$, using theorem 3.6 and rearranging the top line we get the result. □

Remark 4.3. Note that the isomorphism I with $s = 0$ gives:

$$\begin{array}{ccccc}
\langle v||zu\rangle & \simeq & \langle u||zv\rangle & \simeq & \langle z||vu\rangle \\
\wr\| & & \wr\| & & \wr\| \\
\langle z||vu\rangle & \simeq & \langle u||vz\rangle & \simeq & \langle v||zu\rangle
\end{array}$$

i.e., it says that $\langle z||vu\rangle$ is symmetric in z, v, u. This is Manivel's result. It is not possible to say more because in order to apply 3.7 b) in this case, we would need $z(z-1) = 0$ which only happens when $z = 1$, and this is Hermite's.

Remark 4.4. The isomorphism II with $s = 0$ gives:

$$\begin{array}{ccccc}
\langle uv||zzv\rangle & \simeq & \langle zv||zuv\rangle & \simeq & \langle zu||zvv\rangle \\
\wr\| & & \wr\| & & \wr\| \\
\langle uz||vvz\rangle & \simeq & \langle vz||vuz\rangle & \simeq & \langle vu||vzz\rangle
\end{array}$$

and the topright isomorphism is a $\mathrm{GL}(2,\mathbb{C})$ isomorphism if $z(z-1) = uv$.

Remark 4.5. If $s \geq 1$ we cannot obtain $\mathrm{GL}(2,\mathbb{C})$ isomorphisms from the isomorphisms I or II.

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