

# Fundamental limits for cooling of linear quantum refrigerators

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We study the asymptotic dynamics of arbitrary linear quantum open systems which are periodically driven while coupled with generic bosonic reservoirs. We obtain exact results for the heat current flowing into the network, which are valid beyond the usual weak coupling or Markovian approximations. We prove the validity of the dynamical third law of thermodynamics (Nernst unattainability principle) and estimate the minimum achievable temperature for any given cooling strategy. We show that the ultimate limit for cooling is imposed by a fundamental heating mechanism which becomes dominant at low temperatures: the non resonant creation of pairs of excitations in the reservoirs induced by the driving field. This non perturbative quantum effect restores the unattainability principle whose validity was recently challenged.

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*Introduction:* Is there a fundamental limit for cooling? The third law of thermodynamics formulated by Nernst [?] (the unattainability principle) establishes that it is not possible to reach zero temperature in finite time. But this is not a sacred principle and, in fact, its validity has been challenged [?] on the basis of new results obtained in the emergent field of “quantum thermodynamics” [?]. Challenging macroscopic thermodynamical laws is not a luxury but a necessity when studying energy exchange in a regime dominated by quantum fluctuations. Using modern quantum technologies, systems evolving in these conditions (often out of equilibrium and far from the thermodynamical limit), are now regularly built and manipulated [?]. Thus, these technologies enabled a new generation of machines, operating as engines or refrigerators in the quantum domain. In fact, to understand devices such as the single ion engine[?], the quantum absorption refrigerators[?], the phononic coolers[?], or the quantum thermal transistor[?], quantum thermodynamics is essential.

In this Letter we show that, contrary to previous claims [?], the unattainability principle is indeed valid for a wide class of quantum refrigerators which includes the most prominent cases mentioned above (phononic and ionic coolers, for example). Most importantly, we show that the ultimate limit for cooling is imposed by a fundamental quantum process which is non perturbative in the coupling between the system and the reservoirs and becomes dominant at sufficiently low temperatures. It consists of the non resonant creation of excitation pairs in the reservoirs, a which is closely related with the dynamical Casimir effect [?]. As this process is not of leading order in the coupling between the system and the reservoirs, its fundamental role was missed in all previous works that were based on a weak coupling approximation. On the basis of those analysis, potential violations of the third law were suggested [?]. Instead, the unavoidable heating process we characterize enforces the third law.

We present exact results for quantum refrigerators formed by networks of oscillators whose frequencies and couplings are driven by external sources. Their Hamiltonian is  $H_S(t) = (P^T M^{-1} P + X^T V(t) X)/2$  ( $X$  and  $P$  are vectors whose components are position and momentum operators satisfying canonical commutation relations, being  $M$  and  $V(t)$  positive and symmetric matrices. The network is coupled with arbitrary bosonic reservoirs  $E_\alpha$  and the total Hamiltonian is  $H_T = H_S(t) + \sum_\alpha H_{E_\alpha} + \sum_\alpha H_{int,\alpha}$ . Each reservoir  $E_\alpha$  is formed by independent oscillators whose positions (momenta) are  $q_{\alpha,j}$  ( $\pi_{\alpha,j}$ ). Thus,  $H_{E,\alpha} = \sum_j (\pi_{\alpha,j}^2/2m + m\omega_{\alpha,j}^2 q_{\alpha,j}^2/2)$  and the interaction is described by the bilinear Hamiltonian  $H_{int,\alpha} = \sum_{j,k} C_{\alpha,jk} X_j q_{\alpha,k}$  (with coupling constants  $C_{\alpha,jk}$ ).

Work and heat, the two central thermodynamical concepts are naturally defined once we notice that the expectation value of the energy of  $S$ ,  $\langle H_S \rangle$ , changes for two reasons. The variation induced by the external driving is associated with the work  $W$ , while the change induced by the coupling with each reservoir  $E_\alpha$  is associated with heat  $Q_\alpha$ . Thus, we can write  $d\langle H_S \rangle/dt = \dot{W} + \sum_\alpha \dot{Q}_\alpha$ , where work and heat rates are, respectively,  $\dot{W} = \langle \frac{\partial}{\partial t} H_S \rangle$  and  $\dot{Q}_\alpha = \langle [H_S, H_{int,\alpha}] \rangle / i$  (both work and heat are positive when energy flows into the network).

We restrict to consider a periodic driving for which there is a stable stationary regime (see below). In this regime, the quantum state of  $S$  is periodic and the average of  $d\langle H_S \rangle/dt$  vanishes. Then, the asymptotic value of the average work in a cycle, defined as  $\bar{\dot{W}} = \lim_{n \rightarrow \infty} \int_n^{(n+1)\tau} \dot{W}(t) dt / \tau$  and the analogously defined averaged heat rate  $\bar{\dot{Q}}_\alpha$ , satisfy the identity  $\bar{\dot{W}} + \sum_\alpha \bar{\dot{Q}}_\alpha = 0$ . This is nothing but the first law of thermodynamics for a cyclic process. Notice that heat could be defined differently: Thus, the rate of change of the energy of  $E_\alpha$  is  $\dot{Q}'_\alpha = \langle [H_{E,\alpha}, H_{int,\alpha}] \rangle / i$ . As the interaction  $H_{int}$  does not commute with the total Hamiltonian  $H_T$ , the energy

lost by  $E_\alpha$  is not equal to the one absorbed by  $S$  (and therefore  $\dot{Q}_\alpha + \dot{Q}'_\alpha \neq 0$ ). However, as shown in SM B, in the stationary regime, in average, no energy is stored in the interaction terms and the identity  $\dot{Q}_\alpha + \dot{Q}'_\alpha = 0$  holds.

*The solution:* We sketch the method we use and leave most details for the Supplementary Material (SM). The effect of the environments on the network depends on the so called spectral density  $I(\omega) = \sum_\alpha I_\alpha(\omega)$  where, for each reservoir, we define the matrix  $[I_\alpha(\omega)]_{j,k} = \sum_l \frac{1}{m\omega} C_{\alpha,jl} C_{\alpha,kl} \delta(\omega - \omega_{\alpha,l})$ . For example, the effect of the environment on the retarded Green function  $g(t, t')$ , that determines the response of the network to an impulse and is a central ingredient in our analysis, shows up in its evolution equation which reads as  $\dot{g}(t, t') + V_R(t)g(t, t') + \int_0^t d\tau \gamma(t - \tau)\dot{g}(\tau, t') = \delta(t - t')\mathbb{1}$ , where a dot denotes the derivative with respect to the first temporal argument. The reservoirs determine the damping kernel  $\gamma(t)$  which is  $\gamma(t) = \int_0^\infty d\omega I(\omega) \cos(\omega t)/\omega$  and the renormalized potential  $V_R(t) = V(t) - \gamma(0)$ .

We will make the following assumptions: a) The external driving is periodic with period  $\tau = 2\pi/\omega_d$  (i.e.  $V_R(t) = \sum_{k=-\infty}^{+\infty} V_k e^{ik\omega_d t}$ ) and that  $\omega_d$  is such that the driving does not induce instabilities via parametric resonance. b) The coupling with the reservoirs is such that the Green function decays exponentially with  $(t - t')$ . In such conditions there is a stationary regime, where all relevant quantities are also  $\tau$ -periodic, and Floquet theory enables the exact solution of the problem. Details are shown in the Supplementary Material (SM) A, but the crucial observation is that, for a periodic driving, the function  $p(\omega, t) = \int_0^t g(t, t') e^{i\omega(t-t')} dt'$  is also  $\tau$ -periodic for large  $t$  and therefore can be expanded as  $p(\omega, t) = \sum_k A_k(\omega) e^{ik\omega_d t}$ . The Fourier coefficients  $A_k(\omega)$  satisfy a set of equations obtained directly from the one for  $g(t, t')$ , which read:

$$\hat{g}(i(\omega + k\omega_d))^{-1} A_k(\omega) + \sum_{j \neq k} V_j A_{k-j}(\omega) = \mathbb{1} \delta_{k,0}. \quad (1)$$

Here,  $\hat{g}(s) = (Ms^2 + V_0 + s\hat{\gamma}(s))^{-1}$ , is the Laplace transform of the Green function of the undriven network and  $\hat{\gamma}(s) = \int_0^\infty d\omega I(\omega)/\omega s/(\omega^2 + s^2)$  is the transform of  $\gamma(t)$  (see [? ?] for a related approach). Finally, we assume: c) that the initial state has no correlations between  $S$  and  $E$  and that each reservoir is initially in a thermal state with temperature  $T_\alpha$ . Then, independently of the initial conditions, the asymptotic state of  $S$  has a Gaussian density matrix that is fully characterized by a  $\tau$ -periodic covariance matrix whose coefficients admit a Fourier series expansion. For example, position-momentum correlation function, defined as  $\sigma^{xp}(t) = \langle XP^T + PX^T \rangle(t)/2$  can be expressed as  $\sigma^{xp}(t) = \text{Im} \left[ \sum_{j,k} \sigma_{j,k}^{xp} e^{i\omega_d(j-k)t} \right]$ . Here, the Fourier components can be written, as shown in SM A, as:

$$\sigma_{j,k}^{xp} = \frac{1}{2} \int_0^\infty d\omega (\omega + k\omega_d) A_j(\omega) \tilde{\nu}(\omega) A_k^\dagger(\omega), \quad (2)$$

where  $\tilde{\nu}(\omega) = \sum_\alpha I_\alpha(\omega) \coth\left(\frac{\omega}{2T_\alpha}\right)$  is the Fourier transform of the so called noise kernel ( $\hbar = k_b = 1$ ).

*Heat rate:* In the asymptotic regime,  $\dot{Q}_\alpha$  is the temporal average of  $\text{Tr} [P_\alpha V(t) \sigma^{xp}(t) M^{-1}]$ , where  $P_\alpha$  is a projector over the sites of the network in contact with  $E_\alpha$  (see SM B). Using this, we find an important exact result:  $\dot{Q}_\alpha$  is the sum of three terms

$$\dot{Q}_\alpha = \dot{Q}_\alpha^{\text{RP}} + \dot{Q}_\alpha^{\text{RH}} + \dot{Q}_\alpha^{\text{NRH}}. \quad (3)$$

The first term,  $\dot{Q}_\alpha^{\text{RP}}$ , is responsible for the resonant pumping (RP) of energy from (or into)  $E_\alpha$ . It reads:

$$\begin{aligned} \dot{Q}_\alpha^{\text{RP}} = & \sum_{\beta \neq \alpha} \sum_k \int_{0'}^\infty d\omega \omega p_{\beta,\alpha}^{(k)}(\omega) N_\alpha(\omega) - \\ & - \sum_{\beta \neq \alpha} \sum_k \int_{0'}^\infty d\omega (\omega + k\omega_d) p_{\alpha,\beta}^k(\omega) N_\beta(\omega) \end{aligned} \quad (4)$$

where  $N_\alpha(\omega) = (e^{\omega/T_\alpha} - 1)^{-1}$  is the Planck distribution and  $p_{\alpha,\beta}^{(k)}(\omega) = \frac{\pi}{2} \text{Tr}[I_\alpha(|\omega + k\omega_d|) A_k(\omega) I_\beta(\omega) A_k^\dagger(\omega)]$  is a positive number, proportional to the probability for the network to couple the mode with frequency  $\omega$  in  $E_\beta$  with the one with frequency  $|\omega + k\omega_d|$  in  $E_\alpha$ . The first term in Eq. (4) is positive and accounts for energy flowing out of  $E_\alpha$ : a quantum of energy  $\omega$  is lost in  $E_\alpha$  and excites the mode  $|\omega + k\omega_d|$  in  $E_\beta$  after absorbing energy  $k\omega_d$  from the driving. The second term corresponds to the opposite effect: A quantum of energy  $\omega$  is lost from  $E_\beta$  and dumped into the the mode  $|\omega + k\omega_d|$  in  $E_\alpha$  after absorbing energy  $k\omega_d$  from the driving.

The second term in Eq. (3) is responsible for the resonant heating (RH) of  $E_\alpha$  and reads:

$$\dot{Q}_\alpha^{\text{RH}} = - \sum_{\beta \neq \alpha} \sum_k \int_{0'}^\infty d\omega k\omega_d p_{\alpha,\alpha}^k(\omega) N_\alpha(\omega). \quad (5)$$

Its interpretation is analogous to the previous one except for the fact that in this case energy is transferred, through the network, between modes  $\omega$  and  $|\omega + k\omega_d|$  in  $E_\alpha$ . In all cases  $E_\alpha$  can gain or loose energy  $k\omega_d$ , depending on the sign of  $k$ . However, when  $N_\alpha(\omega)$  monotonically decreases with  $\omega$  (as it does for the Planck distribution) the upwards flow of energy is more probable than the downwards one and  $E_\alpha$  heats up. A subtlety should be noticed: The lower limit of the frequency integral in both resonant terms (RP and RH) is not  $\omega = 0$  but  $\omega = 0' = \max\{0, -\omega_d k\}$ . This is because processes with negative  $k$ , which correspond to the emission into the driving, can only take place if  $\omega + k\omega_d > 0$ . The role of the low frequency modes (with  $\omega < -k\omega_d$ ) is crucial, as we now discuss.

The term  $\dot{Q}_\alpha^{\text{NRH}}$  is corresponds to non resonant heating

(NRH) effect and reads:

$$\begin{aligned}
\dot{Q}_\alpha^{\text{NRH}} = & - \sum_{k>0} \int_0^{k\omega_d} d\omega k\omega_d p_{\alpha,\alpha}^{-k}(\omega) (N_\alpha(\omega) + \frac{1}{2}) - \\
& - \sum_{\beta \neq \alpha} \sum_{k>0} \int_0^{k\omega_d} d\omega (k\omega_d - \omega) p_{\alpha,\beta}^{-k}(\omega) (N_\beta(\omega) + \frac{1}{2}) - \\
& - \sum_{\beta \neq \alpha} \sum_{k>0} \int_0^{k\omega_d} d\omega \omega p_{\beta,\alpha}^{-k}(\omega) (N_\alpha(\omega) + \frac{1}{2})
\end{aligned} \tag{6}$$

(strictly, this formula is valid if the driving is time reversal invariant, i.e. if  $V(t) = V(-t)$ ; the general expression is included in SM).

The physical process that gives rise to  $\dot{Q}_\alpha^{\text{NRH}}$  is rather different from the resonant ones discussed above. In the non resonant case a pair of excitations is created. One of them has energy  $\omega$  while the other one has energy  $k\omega_d - \omega$  (these values add up to the driving energy  $k\omega_d$ , notice that only  $k > 0$  enters in the above expression). As opposed to the resonant case, in NRH, excitations are not transferred between modes but created in pairs from the driving. The three terms in Eq. (6) respectively correspond to the following three cases: i) both excitations are created in  $E_\alpha$ ; ii) mode  $\omega$  is excited in  $E_\alpha$  and mode  $k\omega_d - \omega$  in  $E_\beta$ ; iii) mode  $k\omega_d - \omega$  is excited in  $E_\alpha$  while mode  $\omega$  is excited in  $E_\beta$ . In all cases  $E_\alpha$  gains energy: in the first case the net gain is  $k\omega_d$  while in the last two  $E_\alpha$  gains, respectively,  $\omega$  and  $k\omega_d - \omega$ . As  $\dot{Q}_\alpha^{\text{NRH}} < 0$ ,  $E_\alpha$  heats up. Noticeably, the non resonant heating term is the only one surviving when all reservoirs are at zero temperature (in that case both  $\dot{Q}_\alpha^{\text{RP}}$  and  $\dot{Q}_\alpha^{\text{RH}}$  vanish).

In the absence of driving we have  $\dot{Q}_\alpha^{\text{NRH}} = \dot{Q}_\alpha^{\text{RH}} = 0$  and only  $\dot{Q}_\alpha^{\text{RP}}$  survives. In this case, only  $k = 0$  contributes to Eq. (4) and the transition probabilities are symmetric (i.e.,  $p_{\alpha,\beta}^{(0)}(\omega) = p_{\beta,\alpha}^{(0)}(\omega)$ ). Then,  $\dot{Q}_\alpha \geq 0$  when  $E_\alpha$  is the hottest reservoir (accordingly,  $\dot{Q}_\alpha \leq 0$  for the coldest one). This is nothing but Kelvin's version of the second law (i.e., "heat flows from hot to cold" [? ]). In the driven case, Eq (3) can be used to derive the most general form of the second law, that reads  $\sum_\alpha \dot{Q}_\alpha / T_\alpha \leq 0$ . We omit the proof here (but include it in SM C for completeness) and focus on the derivation of the third law.

*Cooling and the third law:* Since both  $\dot{Q}_\alpha^{\text{RH}} \leq 0$  and  $\dot{Q}_\alpha^{\text{NRH}} \leq 0$ , the only way to pump energy out of  $E_\alpha$  (and therefore cool it) is to design a process such that  $\dot{Q}_\alpha^{\text{RP}} > 0$ . In fact, to pump more out of  $E_\alpha$  than the one flowing into it, requires us to impose spatial (or temporal) asymmetries to the driving or to the coupling with the reservoirs [? ? ?]. However, to prove the validity of the third law it is not necessary to go into the details of cooling processes. For this, one should simply notice that both resonant terms vanish at zero temperature while the non resonant heating still survives. This alone is enough

to establish the validity of Nernst unattainability principle: heating dominates at sufficiently low temperatures. Moreover, this also implies that for any cooling process there is a minimum achievable temperature which can be estimated as the one for which the cooling term in  $\dot{Q}_\alpha^{\text{RP}}$  becomes comparable to the  $\dot{Q}_\alpha^{\text{NRH}}$ . The minimal temperature is not universal and its properties for various cooling mechanisms will be analyzed elsewhere. Here we focus on analyzing an important example. We estimate the minimal temperature that can be achieved by the cooling strategy proposed in [?] to violate Nernst unattainability principle.

To gain some intuition on the behavior of the heat currents we use the following simplifying assumptions: a) We assume that the driving is weak (i.e. if  $|V_k|/|V_0| \ll 1$ ). Then, using perturbation theory we find that  $A_k(\omega) = -\hat{g}(i(\omega + k\omega_d))V_k\hat{g}(i\omega)$ , for  $k \neq 0$ . b) We assume that the coupling with the environments is also weak. Then,  $\hat{g}(s)$  (the Green function of the undriven network) can be expressed as a sum over the normal modes of the isolated network, whose eigenfrequencies we denote as  $\Omega_\alpha$ . Using this, the frequency integrals in Eqs. (4) and (5) can be performed since the integrand is strongly peaked around the eigenfrequencies and their sidebands  $\Omega_\alpha \pm k\omega_d$  (the peaks in  $p_{\alpha,\beta}^k(\omega)$  arise because of their dependence on  $A_k(\omega)$ ). In this way, as shown in SM D) the resonant heat currents can be expressed as a sum over all normal modes and sidebands. Finally, in the low temperature limit, this sum is dominated by the term with the lowest frequency ( $\Omega_0$ ) because the Planck distribution enforces a natural cutoff. c) To simplify the analysis we will consider all reservoirs at the same temperature  $T_\alpha$  (the most favorable condition for cooling), and use the harmonic driving  $V(t) = V_0 + 2V_1 \cos(\omega_d t)$  (for which only  $k = \pm 1$  appear in the weak driving limit). In this case we obtain

$$\begin{aligned}
\dot{Q}_\alpha^{\text{RP}} = & \frac{e^{-\frac{\Omega_0 - \omega_d}{T_\alpha}} |V_1^0|^2 (\pi^2/8)}{\Gamma_0 \Omega_0^2 (\Omega_0^2 - (\Omega_0 - \omega_d)^2)^2} \sum_{\beta \neq \alpha} I_\beta^0(\Omega_0) I_\alpha^0(\Omega_0 - \omega_d) \\
& \times \left\{ (\Omega_0 - \omega_d) - \Omega_0 \frac{I_\alpha^0(\Omega_0) I_\beta^0(\Omega_0 - \omega_d)}{I_\beta^0(\Omega_0) I_\alpha^0(\Omega_0 - \omega_d)} \right\}
\end{aligned} \tag{7}$$

where  $\Gamma_0$  is the decay width of the mode  $\Omega_0$  (see below) and  $M^0$  denotes the matrix element of  $M$  in the normal mode with frequency  $\Omega_0$ . The above equation is revealing: when the spectral densities are identical the heat current is negative and  $E_\alpha$  absorbs energy. However, if the condition  $I_\alpha^0(\Omega_0) \ll I_\beta^0(\Omega_0)$  is satisfied  $\dot{Q}_\alpha^{\text{RP}} > 0$  and the reservoir loses energy. This cooling condition simply states that the cooling process (i.e., extracting energy  $\Omega_0 - \omega_d$  from  $E_\alpha$  and dumping it in the mode  $\Omega_0$  of  $E_\beta$ ) has a higher rate than the heating process (taking energy  $\Omega_0 - \omega_d$  from  $E_\beta$  and dumping it in mode  $\Omega_0$  in  $E_\alpha$ ). The reduction in the heating rate arises because the density of final states is small.

Even if the cooling condition is satisfied, the heat current rapidly decreases with the temperature. This is be-

cause the thermal factor appearing Eq. (7) decreases with temperature faster than any power law. However, there is an interesting strategy that we could use to maximize the heat current. For this, as suggested in [? ], we can use an adaptive method by slowly adjusting the driving frequency  $\omega_d$  in such a way that  $\Omega_0 - \omega_d = T_\alpha$ . In this case, for  $T_\alpha \ll \Omega_0$ , the heat current is

$$\dot{Q}_\alpha^{\text{RP}} = \frac{\pi^2}{8e} \frac{|V_1^0|^2}{\Omega_0^6} T_0 I_\alpha^0(T_0) \sum_{\beta \neq \alpha} \frac{I_\beta^0(\Omega_0)}{\Gamma_0} \quad (8)$$

It is clear that the ratio  $\sum_{\beta} I_\beta(\Omega_0)/\Gamma_0$  is of zero-th order in the coupling strength between the system and the environment. Therefore, the above identity shows that  $\dot{Q}_\alpha^{\text{RP}}$  is of first order in the constants appearing in the spectral densities. We will use this expression to estimate the minimal temperature that can be achieved by this cooling protocol. For this, we must study cooling as a dynamical process.

Only a finite reservoir can be cooled. Thus, we assume that the  $E_\alpha$  has a finite heat capacity  $C_v$ . As discussed in [? ],  $C_v$  depends on the dimensionality ( $d$ ) of the reservoir and scales with temperature as  $C_v \propto T_\alpha^d$ . If the rate at which energy flows away from  $E_\alpha$  is sufficiently small, one can think that the environment has a time dependent temperature  $T_\alpha(t)$  that satisfies the equation  $\dot{T}_\alpha = -\dot{Q}_\alpha/C_v$ . To solve this equation we need an expression for the heat current. We could use Eq. (8) provided that rate of change of  $T_\alpha$  is smaller than the driving (since in that case we can instantaneously satisfy the adaptive condition  $\omega_d = \Omega_0 - T_\alpha$ ). In this way, we obtain that  $T_\alpha(t)$  satisfies

$$\frac{dT_\alpha}{dt} = -\frac{1}{C_v} \dot{Q}_\alpha \propto -\gamma_0 \eta T_\alpha^{1+\lambda_\alpha-d} \quad (9)$$

where we used that for low frequencies the spectral density  $I_\alpha^0(\omega) \propto \gamma_0 \omega^{\lambda_\alpha}$  (where  $\gamma_0$  is a relaxation rate scaling quadratically with the coupling constants between the system and the environment while the exponent  $\lambda_\alpha$  characterizes the environment being  $\lambda_\alpha = 1$  the one corresponding to a ohmic reservoir). Above,  $\eta$  is a constant depending on  $\Omega_0$  and  $V_1$ . The solutions of this equation approach  $T_\alpha = 0$  in finite time when  $1 + \lambda_\alpha - d < 1$ , which leads us to the surprising conclusion that the unattainability principle could be violated. However, this argument, presented in [? ], is not correct, because the above equation for  $T_\alpha$  stops being valid at sufficiently low temperatures. In that case,  $\dot{Q}_\alpha^{\text{NRH}}$ , which was not taken into account so far, becomes dominant.

The non resonant heating cannot be neglected in spite of the fact that (for  $\omega_d \leq \Omega_0$ ) it is proportional to  $\gamma_0^2$ . This scaling with  $\gamma_0$  makes this term invisible to any treatment based on the weak coupling limit (such as the master equation used in [? ], which is valid to first order in  $\gamma_0$ ). On the contrary, the weak coupling limit captures the resonant term given in Eq. (8), which is first order in  $\gamma_0$  (in fact, such expression is equivalent to the heat current used in [? ]).

Our analysis, which is non perturbative, shows that the non resonant term given in Eq. (6) will end up stopping any cooling. Moreover, it enables us to estimate the minimum achievable temperature by estimating when resonant and non resonant contributions become comparable. As  $\dot{Q}_\alpha^{\text{NRH}} \propto \gamma_0^2$  (and is roughly independent of  $T_\alpha$  for sufficiently low temperatures) and Eq. (8) shows that the cooling term scales as  $\dot{Q}_\alpha^{\text{RP}} \propto \gamma_0 T_\alpha^{1+\lambda_\alpha}$ . Therefore, both terms become comparable for temperatures scaling as  $T_\alpha \propto \gamma_0^{1/(1+\lambda_\alpha)}$ . In Figure x we see that this naive scaling argument is confirmed by a detailed numerical evaluation of both resonant and non resonant heat currents (the minimal temperature is estimated for various reservoirs characterized by different values of  $\lambda_\alpha$  and plotted as a function of  $\gamma_0$ ). Thus, the dynamical third law (Nernst unattainability principle) has been restored.

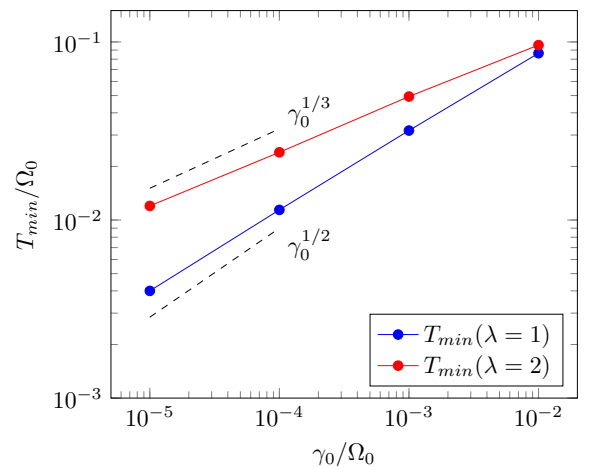


FIG. 1: Minimum temperature  $T_{min}$  as a function of the coupling strength  $\gamma_0$ .  $T_{min}$  is numerically obtained as the lowest temperature for which  $\dot{Q}_\alpha^{\text{RP}} > |\dot{Q}_\alpha^{\text{NRH}}|$ . Results are shown for the adaptive cooling strategy considered in [? ]. The system consists of a single harmonic oscillator of frequency  $\Omega_0$  in contact to two reservoirs with spectral densities satisfying the cooling condition  $I_\alpha(\Omega_0) \ll I_\beta(\Omega_0)$ . Heat currents are obtained through an exact numerical evaluation (see details in SM).

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*Summary:* We presented a complete description of linear quantum refrigerators and demonstrated the validity of the third law. Assuming linearity (i.e. a quadratic Hamiltonian) is not essential for our analysis about the validity of such law. In fact, for very low temperatures the discrete lower sector of the spectrum of the system is the only relevant piece of information. Therefore, our proof of the third law is general. The physical process responsible for cooling is the excitation of the network ground state by the driving, that ends up creating excitation pairs in the reservoirs, as in the dynamical Casimir effect. This work is supported by grants from Ubacyt, Anpcyt and Conicet (Argentina).

## Appendix A: Model and analytic solution

We consider the following Hamiltonian,

$$H = H_S + H_E + H_{int}, \quad (\text{A1})$$

where  $H_S$  is the Hamiltonian of an arbitrary network of quantum harmonic oscillators, and  $H_E$  describes a set of uncoupled bosonic reservoirs.  $H_{int}$  is the interaction between system and reservoirs. Therefore,

$$H_S = \frac{1}{2}P^T M^{-1}P + \frac{1}{2}X^T V(t)X, \quad (\text{A2})$$

where  $X$  and  $P$  are vectors whose components are position and momentum operators satisfying the usual commutation relations,  $[X_i, X_j] = [P_i, P_j] = 0$  and  $[X_i, P_j] = i\hbar\delta_{i,j}$ . The matrices  $M$  and  $V$  are symmetric and positive definite, and  $V$  can be time-dependent. The environmental Hamiltonian is

$$H_E = \sum_{\alpha} H_{E,\alpha} \quad H_{E,\alpha} = \sum_{j=1}^{N_{\alpha}} \frac{\pi_{\alpha,j}^2}{2m} + \frac{m\omega_{\alpha,j}^2}{2} q_{\alpha,j}^2$$

That is, the environment consists of several independent reservoirs formed in turn by a collection of quantum harmonic oscillators of mass  $m$ . The operator  $q_{\alpha,j}$  is the position operator of the  $j$ -th oscillator in the  $\alpha$ -th environment, and  $\pi_{\alpha,j}$  its associate momentum. We consider a bilinear interaction between system and reservoirs through the position coordinates

$$H_{int} = \sum_{\alpha} \sum_{j,k} C_{\alpha,jk} X_j q_{\alpha,k}, \quad (\text{A3})$$

where  $C_{\alpha,jk}$  are time-independent interaction constants.

### 1. Equations of motion

We now derive, working in the Heisenberg picture, the equations of motion for all the operators involved in the Hamiltonian of Eq. (A1). For the system operators the motion equations are

$$\dot{X} = M^{-1}P \quad (\text{A4a})$$

$$\dot{P} = -V(t)X - \sum_{\alpha} C_{\alpha}q_{\alpha}, \quad (\text{A4b})$$

where  $q_{\alpha}$  and  $\pi_{\alpha}$  are vectors formed with the position and momentum operators of the  $\alpha$ -th reservoir, respectively. Similarly, the matrix  $C_{\alpha}$  has as elements the interaction constants  $C_{\alpha,jk}$ . If the system has  $N$  degrees of freedom and the  $\alpha$ -th reservoir is formed by  $N_{\alpha}$  oscillators, then the matrix  $C_{\alpha}$  has dimensions  $N \times N_{\alpha}$ .

Turning to a description in the phase space, we define the  $2N$ -component vectors  $Z = (X, P)^T$  and  $z_{\alpha} = (q_{\alpha}, \pi_{\alpha})^T$ . In terms of  $Z$ , the Eqs. (A4a) and (A4b) can be written as:

$$\dot{Z} + a_s(t)Z = \sum_{\alpha} C_{\alpha}z_{\alpha}, \quad (\text{A5})$$

where the  $2N \times 2N$  matrix  $a_s(t)$  is defined as

$$a_s(t) = \begin{pmatrix} 0 & -M^{-1} \\ V(t) & 0 \end{pmatrix}, \quad (\text{A6})$$

and the  $2N \times N_L$  matrix  $C_L$  is

$$C_{\alpha} = \begin{pmatrix} 0 & 0 \\ -C_{\alpha} & 0 \end{pmatrix} \quad (\text{A7})$$

For the operators corresponding to the  $\alpha$ -th reservoir the equations of motion in phase space are:

$$\dot{z}_{\alpha} + a_{\alpha}(t)z_{\alpha} = \sum_{\alpha} \bar{C}_{\alpha}Z \quad (\text{A8})$$

In this case  $a_\alpha(t)$  is the  $2N_\alpha \times 2N_\alpha$  matrix

$$a_\alpha(t) = \begin{pmatrix} 0 & -\mathbb{1}_{N_\alpha}/m \\ m\Omega_\alpha^2 & 0 \end{pmatrix}, \quad (\text{A9})$$

where  $\Omega_\alpha^2$  is a diagonal matrix containing the squared frequencies of the oscillators of the  $\alpha$ -th reservoir. Finally,  $\bar{C}_\alpha$  is:

$$\bar{C}_\alpha = \begin{pmatrix} 0 & 0 \\ -[C_\alpha]^T & 0 \end{pmatrix} \quad (\text{A10})$$

In summary, if we are interested in the dynamics of the system, the following set of linear coupled differential equations must be solved for  $Z$ :

$$\begin{cases} \dot{Z} + a_s(t)Z = \sum_\alpha C_\alpha z_\alpha, \\ \dot{z}_\alpha + a_\alpha(t)z_\alpha = \bar{C}_\alpha Z \end{cases} \quad (\text{A11})$$

with the initial conditions  $Z(t=0) = (X(0), P(0))^T$  and  $z_\alpha(t=0) = (q_\alpha(0), \pi_\alpha(0))^T$ .

## 2. An integro-differential equation for the system

In this section we derive an integro-differential equation describing the dynamics of the system only. We start by considering the Green's function  $g_\alpha(t, t')$  for the homogeneous equation of motion of the  $\alpha$ -th reservoir. Such function satisfies:

$$\frac{d}{dt}g_\alpha(t, t') + a_\alpha(t)g_\alpha(t, t') = \mathbb{1}_{N_\alpha} \delta(t - t'). \quad (\text{A12})$$

With initial conditions  $g_\alpha(t^-, t') = 0$ , the function  $g_\alpha(t, t')$  encodes the response of the  $\alpha$ -th reservoir to a delta impulse at time  $t'$ . For this simple case it just represents a rotation in phase space:  $g_\alpha(t, t') = \theta(t - t')e^{-a_\alpha(t-t')}$  where  $\theta$  is the Heaviside step function and

$$e^{-a_\alpha t} = \begin{pmatrix} \cos(\Omega_\alpha t) & \sin(\Omega_\alpha t)(m\Omega_\alpha)^{-1} \\ -\sin(\Omega_\alpha t)m\Omega_\alpha & \cos(\Omega_\alpha t) \end{pmatrix} \quad (\text{A13})$$

The function  $g_\alpha(t, t')$  is an homogeneous solution of Eq. (A8) for  $t > t'$ . A particular solution is  $z_\alpha^p(t) = \int_0^t g_\alpha(t, t')\bar{C}_\alpha z(t')dt'$ . Therefore, the complete solution of Eq. (A8) is

$$z_\alpha(t) = g_\alpha(t, 0)z_\alpha(0) + \int_0^t g_\alpha(t, t')\bar{C}_\alpha z(t')dt', \quad (\text{A14})$$

which satisfies the required initial condition.

The solution for  $z_\alpha$  of Eq. (A14) can now be inserted in Eq. (A5), the differential equation for the system coordinates. Doing this we obtain:

$$\dot{Z} + a_s(t)Z - \int_0^t \left[ \sum_\alpha C_\alpha g_\alpha(t, t')\bar{C}_\alpha \right] Z(t')dt' = \sum_\alpha C_\alpha g_\alpha(t, 0)z_\alpha(0) \quad (\text{A15})$$

This is a non-Markovian equation of motion for the system with a source term depending on the operators of the environment at the initial time. Since the equation is linear a general solution can be obtained in terms of the Green's function of the homogeneous system, as we did before for each reservoir. Before doing that, we define the dissipation kernel.

## 3. Dissipation kernel

The quantity multiplying  $Z$  in the integrand of Eq. (A15) is the dissipation kernel, to which we will refer as  $\eta(t, t') = \sum_\alpha \eta_\alpha(t, t')$ , where  $\eta_\alpha(t, t') = C_\alpha g_\alpha(t, t')\bar{C}_\alpha$ . It can be written in a more convenient way in terms of the spectral densities of the reservoirs, which are defined below. Explicitly, we have:

$$\eta_\alpha(t, t') = \theta(t - t') \begin{pmatrix} 0 & 0 \\ \eta_\alpha^{xx}(t - t') & 0 \end{pmatrix} \quad (\text{A16})$$

where the matrix  $\eta_\alpha^{xx}$  is defined as:

$$\eta_\alpha^{xx}(t) = \int_0^\infty I_\alpha(\omega) \sin(\omega t) d\omega \quad (\text{A17})$$

The function  $I_\alpha(\omega)$  is the spectral density associated to the  $\alpha$ -th reservoir. It is defined as follows:

$$[I_\alpha(\omega)]_{j,k} = \sum_{p=1}^{N_\alpha} \frac{1}{m\omega} [C_\alpha]_{jp} [C_\alpha]_{kp} \delta(\omega - \omega_{\alpha,p}) \quad (\text{A18})$$

#### 4. Solution of the equation of motion

Using the previously defined dissipation kernel the equation of motion for the system is:

$$\dot{Z} + a_s(t)Z - \int_0^t \eta(t, t') Z(t') dt' = \sum_\alpha C_\alpha g_\alpha(t, 0) z_\alpha(0) \quad (\text{A19})$$

We consider the Green function  $G(t, t')$  associated with the previous equation. It is such that:

$$\frac{\partial}{\partial t} G(t, t') + a_s(t)G(t, t') - \int_0^t \eta(t, \tau) G(\tau, t') d\tau = \mathbf{1}_{2N} \delta(t - t') \quad (\text{A20})$$

with initial conditions  $G(t'^-, t') = 0$ . The function  $G(t, t')$  is therefore the response of the system to an impulse at time  $t'$ . It fully takes into account the non-Markovian nature of the dynamics and the dissipation induced by the environment. In some cases the function  $G(t, t')$  can be computed analytically. In general only a numerical approach is possible. In any case, if  $G(t, t')$  is known, the complete solution to Eq. (A19) can be obtained. In fact, it is easy to verify that the expression

$$Z(t) = G(t, 0)Z(0) + \int_0^t G(t, t') \left[ \sum_\alpha C_\alpha g_\alpha(t', 0) z_\alpha(0) \right] dt' \quad (\text{A21})$$

is a solution of Eq. (A19) and satisfies the required initial condition.

#### 5. Renormalization and damping kernel

It is useful to rewrite the integro-differential equation in Eq. (A20) and express the non-Markovian integral term as a functional of the velocity in phase space,  $\frac{\partial}{\partial t} G(t, t')$ , instead of  $G(t, t')$ . For that purpose a partial integration must be performed, with the following result:

$$\frac{\partial}{\partial t} G(t, t') + a_R(t)G(t, t') + \int_0^t \gamma(t, \tau) \frac{\partial}{\partial \tau} G(\tau, t') d\tau = \mathbf{1}_{2N} \delta(t - t') \quad (\text{A22})$$

Note that the matrix  $a_s(t)$ , that describes the unitary dynamics of the system, has been renormalized to  $a_R(t) = a_s(t) - \gamma(0)$ . The function  $\gamma(t, t')$  is such that  $\eta(t, t') = \frac{\partial}{\partial t'} \gamma(t, t')$ , and is known as the damping kernel. For  $t > t'$  it can be calculated as follows:

$$\gamma_\alpha(t, t') = \begin{pmatrix} 0 & 0 \\ \gamma_\alpha^{xx}(t - t') & 0 \end{pmatrix} \quad (\text{A23})$$

where the matrix  $\gamma_\alpha^{xx}$  is defined as:

$$\gamma_\alpha^{xx}(t) = \int_0^\infty \frac{I_\alpha(\omega)}{\omega} \cos(\omega t) d\omega \quad (\text{A24})$$

## 6. Evolution of the covariance matrix

The covariance matrix of the system at time  $t$  is defined as

$$C(t) = \text{Re} [\langle Z(t)Z(t)^T \rangle] - \langle Z(t) \rangle \langle Z(t)^T \rangle \quad (\text{A25})$$

where  $\langle A(t) \rangle = \text{Tr}(\rho^0 A(t))$ , and  $\rho^0$  is the initial state of the system and reservoirs. We will consider initial states such that  $\langle Z(0) \rangle = \langle z_l(0) \rangle = 0$  and therefore, according to Eq. (A21),  $\langle Z(t) \rangle = 0$  for all  $t$ . Inserting Eq. (A21) in Eq. (A25) the following expression is obtained:

$$C(t) = G(t, 0)C(0)G(t, 0)^T + G(t, 0) \text{Re} [\langle Z(0)\beta(t)^T \rangle] + \text{Re} [\langle \beta(t)Z(0)^T \rangle] G(t, 0)^T + \text{Re} [\langle \beta(t)\beta(t)^T \rangle] \quad (\text{A26})$$

where  $\beta(t)$  is the integral term of Eq. (A21),

$$\beta(t) = \sum_{\alpha} \beta_{\alpha}(t) z_{\alpha}(0) \quad (\text{A27a})$$

$$\beta_{\alpha}(t) = \int_0^t G(t, t') C_{\alpha} g_{\alpha}(t', 0) dt' \quad (\text{A27b})$$

The first term in Eq. (A26) is the deterministic propagation of the initial covariance matrix given by the phase space flow  $G(t, 0)$ . The second and third terms are the propagation of the initial correlations between system and reservoirs. The last term correspond to the noise and diffusion induced by the environment on the system, and in a stable system will dominate the long term behaviour. If there are no system-reservoirs correlations in the initial state, i.e, if  $\text{Re} [\langle Z(0)z_{\alpha}(0)^T \rangle] = 0$  for all  $\alpha$ , then the second and third terms of Eq. (A26) vanish for all  $t$ . In that case the evolution of the covariance matrix is just:

$$C(t) = G(t, 0)C(0)G(t, 0)^T + \text{Re} [\langle \beta(t)\beta(t)^T \rangle] \quad (\text{A28})$$

Using Eq. (A27a) we find the following expression for the diffusive term of the covariance matrix:

$$\text{Re} [\langle \beta(t)\beta(t)^T \rangle] = \int_0^t \int_0^t G(t, t_1) \left[ \sum_{\alpha, \beta} C_{\alpha} g_{\alpha}(t_1, 0) \text{Re} [\langle z_{\alpha}(0)z_{\beta}(0)^T \rangle] g_{\beta}(t_2, 0)^T C_{\beta}^T \right] G(t, t_2)^T dt_1 dt_2 \quad (\text{A29})$$

Now we introduce the condition that in the initial state the reservoirs are in thermal states and uncorrelated with each other. In that case:

$$\text{Re} [\langle z_{\alpha}(0)z_{\beta}(0)^T \rangle] = \delta_{\alpha, \beta} \frac{\hbar}{2} \begin{pmatrix} (m\Omega_{\alpha})^{-1} \coth\left(\frac{\hbar\Omega_{\alpha}}{2k_B T_{\alpha}}\right) & 0 \\ 0 & (m\Omega_{\alpha}) \coth\left(\frac{\hbar\Omega_{\alpha}}{2k_B T_{\alpha}}\right) \end{pmatrix}, \quad (\text{A30})$$

Where  $T_{\alpha}$  is the temperature of the  $\alpha$ -th reservoir and  $k_B$  is the Boltzmann constant. Introducing Eq. (A30) in Eq. (A29) the following final expression is obtained:

$$\text{Re} [\langle \beta(t)\beta(t)^T \rangle] = \frac{\hbar}{2} \int_0^t \int_0^t G(t, t_1) \nu(t_1 - t_2) G(t, t_2)^T dt_1 dt_2 \quad (\text{A31})$$

The matrix function  $\nu(t) = \sum_{\alpha} \nu_{\alpha}(t)$  is the noise kernel, with

$$\nu_{\alpha}(t) = \begin{pmatrix} 0 & 0 \\ 0 & \nu_{\alpha}^{xx}(t) \end{pmatrix} \quad (\text{A32})$$

where:

$$\nu_{\alpha}^{xx}(t) = \int_0^{\infty} I_{\alpha}(\omega) \cos(\omega t) \coth\left(\frac{\hbar\omega}{2k_B T_{\alpha}}\right) d\omega \quad (\text{A33})$$



## 7. Asymptotic state for stable systems

In this section we characterize the asymptotic state for driven systems that are exponentially stable, i.e., systems with a Green's function  $G(t, t')$  decaying exponentially with  $t - t'$ . The previous condition is not always fulfilled for driven systems, even in the presence of strong dissipation, since it is possible, for example, to induce a divergent dynamics by the phenomenon of parametric resonance.

From Eq. (A28) it is clear that for stable systems the asymptotic covariance matrix is:

$$C(t) = \text{Re} [\langle \beta(t) \beta(t)^T \rangle] = \begin{bmatrix} \sigma^{xx}(t) & \sigma^{xp}(t) \\ \sigma^{px}(t) & \sigma^{pp}(t) \end{bmatrix} \quad (\text{A34})$$

since  $G(t, 0) \rightarrow 0$  for large  $t$ . Also, Eq. (A31) is equivalent to the following expressions for each block of the asymptotic covariance matrix:

$$\sigma^{xx}(t) = \frac{\hbar}{2} \int_0^t \int_0^t g(t, t_1) M^{-1} \nu^{xx}(t_1 - t_2) M^{-1} g(t, t_2)^T dt_1 dt_2 \quad (\text{A35a})$$

$$\sigma^{xp}(t) = \frac{\hbar}{2} \int_0^t \int_0^t g(t, t_1) M^{-1} \nu^{xx}(t_1 - t_2) M^{-1} \frac{\partial}{\partial t} g(t, t_2)^T M dt_1 dt_2 \quad (\text{A35b})$$

$$\sigma^{pp}(t) = \frac{\hbar}{2} \int_0^t \int_0^t M \frac{\partial}{\partial t} g(t, t_1) M^{-1} \nu^{xx}(t_1 - t_2) M^{-1} \frac{\partial}{\partial t} g(t, t_2)^T M dt_1 dt_2 \quad (\text{A35c})$$

In the previous expressions, the function  $g(t, t')$  is the Green's function of the system in the configuration space, and satisfies the following second order integro-differential equation for  $t > t'$ :

$$M \frac{\partial^2}{\partial t^2} g(t, t') + V_R(t) g(t, t') + \int_0^t \gamma^{xx}(t - \tau) \frac{\partial}{\partial \tau} g(\tau, t') d\tau = 0 \quad (\text{A36})$$

with initial conditions  $g(t = t', t') = 0$  and  $\frac{\partial}{\partial t} g(t = t', t') = \mathbf{1}$ . Also,  $V_R(t) = V(t) - \gamma^{xx}(0)$  is the renormalized potential. We now introduce the spectral decomposition of the noise kernel (see Eq. (A33)):

$$\nu^{xx}(t_1 - t_2) = \text{Re} \left[ \int_0^\infty \tilde{\nu}(\omega) e^{i\omega t_1} e^{-i\omega t_2} d\omega \right] \quad (\text{A37})$$

where  $\tilde{\nu}(\omega)$  is the Fourier transform of  $\nu^{xx}(t)$ :

$$\tilde{\nu}(\omega) = \sum_\alpha I_\alpha(\omega) \coth \left( \frac{\hbar\omega}{2k_b T_\alpha} \right) \quad (\text{A38})$$

Introducing Eq. (A38) into Eqs. (A35a - A35c) we obtain:

$$\sigma^{xx}(t) = \frac{\hbar}{2} \text{Re} \left[ \int_0^\infty q(t, \omega) \tilde{\nu}(\omega) q(t, \omega)^\dagger d\omega \right] \quad (\text{A39a})$$

$$\sigma^{xp}(t) M^{-1} = \frac{\hbar}{2} \text{Re} \left[ \int_0^\infty q(t, \omega) \tilde{\nu}(\omega) \frac{\partial}{\partial t} q(t, \omega)^\dagger d\omega \right] \quad (\text{A39b})$$

$$M^{-1} \sigma^{pp}(t) M^{-1} = \frac{\hbar}{2} \text{Re} \left[ \int_0^\infty \frac{\partial}{\partial t} q(t, \omega) \tilde{\nu}(\omega) \frac{\partial}{\partial t} q(t, \omega)^\dagger d\omega \right] \quad (\text{A39c})$$

where the function  $q(t, \omega)$  is defined as:

$$q(t, \omega) = \int_0^t g(t, t') e^{i\omega t'} dt' \quad (\text{A40})$$

So far we have only given alternatives expressions for the asymptotic covariance matrix valid when the system is stable. We now analyze the asymptotic properties of the function  $q(t, \omega)$  for the case in which the driving is periodic.

### 8. Periodic driving

The potential energy matrix  $V(t)$  is assumed to be  $\tau$ -periodic:  $V(t + \tau) = V(t)$ . From the integro-differential equation defining  $g(t, t')$  (Eq. (A36)) it follows that:

$$g(t + \tau, t' + \tau) = g(t, t'), \quad (\text{A41})$$

since  $g(t + \tau, t' + \tau)$  and  $g(t, t')$  are both solution of Eq. (A36) with the same initial conditions. This observation implies that

$$q(t + \tau, \omega) = \int_0^{t+\tau} g(t + \tau, t') e^{i\omega t'} dt' = \int_{-\tau}^t g(t + \tau, t' + \tau) e^{i\omega(t'+\tau)} dt' = \left[ \int_{-\tau}^t g(t, t') e^{i\omega t'} dt' \right] e^{i\omega\tau}. \quad (\text{A42})$$

Now, if  $g(t, t')$  decays exponentially with  $(t - t')$  the following approximation holds for large  $t$ :

$$\int_{-\tau}^t g(t, t') e^{i\omega t'} dt' \simeq \int_0^t g(t, t') e^{i\omega t'} dt' = q(t, \omega) \quad (\text{A43})$$

Therefore, in the asymptotic limit, the function  $q(t, \omega)$  satisfies:

$$q(t + \tau, \omega) = q(t, \omega) e^{i\omega\tau} \quad (\text{A44})$$

from which it follows that the function

$$p(t, \omega) = q(t, \omega) e^{-i\omega t} \quad (\text{A45})$$

is  $\tau$ -periodic. In summary the function  $q(t, \omega)$ , from which the covariance matrix at time  $t$  can be obtained, can be expressed for sufficiently long times as  $q(t, \omega) = p(t, \omega) e^{i\omega t}$ , where  $p(t, \omega)$  is  $\tau$ -periodic. As a consequence, the asymptotic state for long times will also be  $\tau$ -periodic. To see that, as an example, we rewrite the asymptotic limit of  $\sigma^{xx}(t)$  in terms of  $p(t, \omega)$ :

$$\sigma^{xx}(t) = \frac{\hbar}{2} \text{Re} \left[ \int_0^\infty p(t, \omega) \tilde{\nu}(\omega) p(t, \omega)^\dagger d\omega \right] \quad (\text{A46})$$

similar expressions hold for  $\sigma^{xp}(t)$  and  $\sigma^{pp}(t)$ , in which time only enters through  $p(t, \omega)$  or its derivative.

To finish this section we note that since the functions  $V_R(t)$  and  $p(t, \omega)$  are  $\tau$ -periodic they are determined by their Fourier coefficients  $V_k$  and  $A_k(\omega, \omega_d)$ :

$$V_R(t) = \sum_{k=-\infty}^{+\infty} V_k e^{i\omega_d k t} \quad (\text{A47})$$

$$p(t, \omega) = \sum_{k=-\infty}^{+\infty} A_k(\omega, \omega_d) e^{i\omega_d k t} \quad (\text{A48})$$

where  $\omega_d = 2\pi/\tau$  is the fundamental angular frequency of the driving. In the following section we explain how to calculate the coefficients  $A_k$  given the driving coefficients  $V_k$ . However, if they are known, then the asymptotic correlations can be easily obtained as:

$$\sigma^{xx}(t) = \text{Re} \left[ \sum_{j,k} \sigma_{j,k}^{xx} e^{i\omega_d(j-k)t} \right] \quad (\text{A49})$$

where:

$$\sigma_{j,k}^{xx} = \frac{\hbar}{2} \int_0^\infty A_j(\omega, \omega_d) \tilde{\nu}(\omega) A_k^\dagger(\omega, \omega_d) d\omega \quad (\text{A50})$$

Similar expressions can be found for the correlations  $\sigma^{xp}(t)$  and  $\sigma^{pp}(t)$ .

### 9. Calculation of the function $p(t, \omega)$

An integro-differential equation for  $p(t, \omega)$  can be derived from the one defining  $g(t, t')$ , Eq. (A36). It reads:

$$M \left[ \frac{\partial^2}{\partial t^2} p(t, \omega) + 2(i\omega) \frac{\partial}{\partial t} p(t, \omega) + (i\omega)^2 p(t, \omega) \right] + V_R(t) P(t, \omega) + \int_0^t \gamma^{xx}(t-\tau) \left[ \frac{\partial}{\partial t} p(\tau, \omega) + (i\omega) p(\tau, \omega) \right] e^{-i\omega(t-\tau)} d\tau = \mathbb{1} \quad (\text{A51})$$

Inserting the Fourier decomposition of Eq. (A48) into the previous equation one obtains the following algebraic relation for the coefficients  $A_k(\omega, \omega_d)$ :

$$\hat{g}(i(\omega + \omega_d k))^{-1} A_k(\omega, \omega_d) + \sum_{j \neq k} V_j A_{k-j}(\omega, \omega_d) = \mathbb{1} \delta_{k,0} \quad (\text{A52})$$

where the matrix function  $\hat{g}(s)$  is the Laplace transform of the Green's function of the system without driving, which satisfies:

$$\hat{g}(s)^{-1} = M s^2 + V_R + s \hat{\gamma}(s) \quad (\text{A53})$$

In turn,  $\hat{\gamma}(s)$  is the Laplace transform of the damping kernel  $\gamma^{xx}(t)$ :

$$\hat{\gamma}(s) = \int_0^\infty \frac{I(\omega)}{\omega} \frac{s}{\omega^2 + s^2} d\omega \quad (\text{A54})$$

The infinite set of equations given in Eq. (A52) can be solved for any given value of  $\omega$  by standard techniques. For example, a finite linear system can be obtained by only considering coefficients  $A_k(\omega, \omega_d)$  with  $|k| \leq k_{max}$ , which is later solved by a regular matrix inversion. Alternatively, a perturbative approach can be employed. Thus, if the driving is weak (i.e., if  $|V_k| \ll |V_0|$  for all  $k \neq 0$ ), then up to second order in  $V_k$  we have

$$A_0(\omega, \omega_d) = \hat{g}(i\omega) + \sum_{k \neq 0} g(i\omega) V_k g(i(\omega - \omega_d k)) V_{-k} g(i\omega) \quad (\text{A55a})$$

$$A_k(\omega, \omega_d) = -\hat{g}(i(\omega + \omega_d k)) V_k \hat{g}(i\omega) \quad \text{for } k \neq 0 \quad (\text{A55b})$$

To finish this section, we note that the coefficients  $A_k(\omega, \omega_d)$  satisfy certain exact symmetries, which can be obtained by examining the linear system given by Eq. (A52). Thus, if  $\{A_k(\omega, \omega_d)\}$  are the solutions of Eq. (A52) for a given process  $V(t)$ , and  $\{A_k^r(\omega, \omega_d)\}$  are the solutions corresponding to the time reversed process  $V(-t)$ , we have:

$$A_k^r(\omega, \omega_d) = A_{-k}(\omega, -\omega_d) \quad (\text{A56a})$$

$$A_k^r(\omega, \omega_d) = A_{-k}^T(\omega + \omega_d k, \omega_d) \quad (\text{A56b})$$

$$A_k^*(\omega, \omega_d) = A_{-k}(-\omega, \omega_d) \quad (\text{A56c})$$

As will be clear in the next sections, if two reservoirs are connected to sites  $\alpha$  and  $\beta$  of the network, then the function  $|(A_k(\omega, \omega_d))_{\alpha, \beta}|^2$  is related to the rate at which a quantum of energy  $\hbar\omega$  is extracted from the reservoir at  $\beta$  while a quantum of energy  $\hbar(\omega + \omega_d)$  is dumped into the reservoir at  $\alpha$  (via absorption of  $k\hbar\omega_d$  energy from the driving, for  $k > 0$ ). Thus, the above relations express fundamental symmetries between energy exchange processes.

### Appendix B: Definition of work and heat currents

In this section we give microscopic definitions for the work performed on the system by the driving and for the energy exchange with each thermal reservoir, i.e, for the heat currents. We begin by analyzing the variation in time of the system energy. From Eq. (A2) we have:

$$\langle H_S \rangle(t) = \frac{1}{2} \text{Tr} [M^{-1} \sigma^{pp}(t)] + \frac{1}{2} \text{Tr} [V(t) \sigma^{xx}(t)], \quad (\text{B1})$$

and therefore,

$$\frac{d}{dt} \langle H_S \rangle(t) = \frac{1}{2} \text{Tr} \left[ M^{-1} \frac{d}{dt} \sigma^{pp}(t) \right] + \frac{1}{2} \text{Tr} \left[ V(t) \frac{d}{dt} \sigma^{xx}(t) \right] + \frac{1}{2} \text{Tr} \left[ \frac{d}{dt} V(t) \sigma^{xx}(t) \right], \quad (\text{B2})$$

The last term in the previous equation is the rate at which energy is injected into or absorbed from the system by the driving. The remaining terms represent the variation of the system energy due to the interaction with the thermal reservoirs. In order to see that it is useful to rewrite  $\frac{d}{dt}\langle H_S \rangle(t)$  as:

$$\frac{d}{dt}\langle H_S \rangle(t) = \frac{1}{i\hbar}\langle [H_S, H] \rangle + \left\langle \frac{\partial}{\partial t} H_S \right\rangle = \frac{1}{i\hbar}\langle [H_S, H_{int}] \rangle + \frac{1}{2}\text{Tr} \left[ \frac{d}{dt} V(t) \sigma^{xx}(t) \right] \quad (\text{B3})$$

where  $H$  is the total Hamiltonian defined in Eq. (A1). Comparing Eqs. (B2) and (B3) we see that the first two terms in Eq. (B2) can be interpreted as the energy exchange with the reservoirs. Thus, we arrive at the following definitions for the work rate  $\dot{W}$  and total heat rate  $\dot{Q}(t)$ :

$$\dot{W}(t) = \frac{1}{2}\text{Tr} \left[ \frac{d}{dt} V(t) \sigma^{xx}(t) \right] \quad (\text{B4})$$

and,

$$\dot{Q}(t) = \frac{1}{i\hbar}\langle [H_S, H_{int}] \rangle = \frac{1}{2}\text{Tr} \left[ M^{-1} \frac{d}{dt} \sigma^{pp}(t) \right] + \frac{1}{2}\text{Tr} \left[ V(t) \frac{d}{dt} \sigma^{xx}(t) \right] \quad (\text{B5})$$

### 1. Local heat currents

Equation (B5) defines the total energy interchange between the system and all the thermal reservoirs. However, the local heat current corresponding to a particular reservoir is also of interest. A working definition for such local heat currents can be obtained by expanding Eq. (B5) using Eq. (A3):

$$\dot{Q} = \frac{1}{i\hbar}\langle [H_S, H_{int}] \rangle = \sum_{\alpha} \frac{1}{i\hbar}\langle [H_S, H_{int,\alpha}] \rangle \quad (\text{B6})$$

where  $H_{int,\alpha} = \sum_{j,k} C_{\alpha,jk} X_j q_{\alpha,k} = X^T C_{\alpha} q_{\alpha}$  is the Hamiltonian term describing the interaction between the system and the  $\alpha$ -th reservoir. We define

$$\dot{Q}_{\alpha} = \frac{1}{i\hbar}\langle [H_S, H_{int,\alpha}] \rangle \quad (\text{B7})$$

as the heat rate corresponding to the  $\alpha$ -th reservoir. In this way we obtain a set  $\{\dot{Q}_{\alpha}\}$  of local heat rates such that the total heat rate is  $\dot{Q} = \sum_{\alpha} \dot{Q}_{\alpha}$ . A direct calculation shows that

$$\dot{Q}_{\alpha} = -\langle P^T M^{-1} C_{\alpha} q_{\alpha} \rangle \quad (\text{B8})$$

We can use the motion equation (A4b) in order to eliminate the reservoir coordinates from Eq. (B8). Thus, if  $P_{\alpha}$  is a projector over the sites of the network in contact with the  $\alpha$ -th reservoir, then  $P_{\alpha} \dot{P} = -P_{\alpha} V(t) X - C_{\alpha} q_{\alpha}$  (this identity is valid in the case in which different reservoirs are coupled to different sites of the networks, i.e., we assume that  $P_{\alpha} P_{\beta} = \delta_{\alpha,\beta} P_{\alpha}$  and  $P_{\alpha} C_{\beta} = \delta_{\alpha,\beta} C_{\alpha}$ ). Therefore:

$$\dot{Q}_{\alpha} = \frac{1}{2}\text{Tr} \left[ P_{\alpha} \frac{d}{dt} \sigma^{pp}(t) M^{-1} \right] + \text{Tr} \left[ P_{\alpha} V(t) \sigma^{xp}(t) M^{-1} \right] \quad (\text{B9})$$

The previous definition for the local heat currents is not the only possible. Another natural definition for the heat currents is given by the rate of change of the energy of each reservoir:

$$\dot{Q}'_{\alpha} = \frac{1}{i\hbar}\langle [H_{E,\alpha}, H_{int,\alpha}] \rangle \quad (\text{B10})$$

If the interactions terms were energy conserving, i.e. if it were  $[H_S + H_{E,\alpha}, H_{int,\alpha}] = 0$ , then we would have  $\dot{Q}'_{\alpha} + \dot{Q}_{\alpha} = 0$  and the two definitions of heat currents would be equivalent. Although in our model the energy conserving condition is not fulfilled and in general  $\dot{Q}'_{\alpha} + \dot{Q}_{\alpha} \neq 0$ , it is easy to see that:

$$\dot{Q}'_{\alpha} + \dot{Q}_{\alpha} = \frac{d}{dt}\langle X^t P_{\alpha} (\dot{P} + V(t) X) \rangle \quad (\text{B11})$$

Since the asymptotic state is  $\tau$ -periodic, the right hand side of the last equation is the derivative of a  $\tau$ -periodic function. This observation implies that the average heat currents per cycle obtained with the two possible definitions are equivalent, as if the interaction terms were energy conserving. This is explained in the next section.

## 2. Work and heat in the asymptotic state

We have seen that if the system is periodically driven and stable the asymptotic state is also periodic, with the same period as the driving. It follows that the function  $\langle H_S \rangle(t)$  and its derivative  $\frac{d}{dt} \langle H_S \rangle(t)$  are also periodic. Thus, averaging Eq. (B2) in one period (for long times) we obtain:

$$0 = \dot{Q}^c + \dot{W}^c \quad (\text{B12})$$

where  $\dot{W}^c$  and  $\dot{Q}^c$  are the average work and total heat rates per cycle:

$$\dot{W}^c = \frac{1}{\tau} \lim_{n \rightarrow \infty} \int_{n\tau}^{(n+1)\tau} \dot{W}(t') dt' \quad \dot{Q}^c = \frac{1}{\tau} \lim_{n \rightarrow \infty} \int_{n\tau}^{(n+1)\tau} \dot{Q}(t') dt' \quad (\text{B13})$$

Equation (B12) is nothing more than the expression of the first law of thermodynamics for cyclic processes. In the last equation we considered the fact that the first term of the right hand side of Eq. (B5) is the derivate of a periodic function and therefore does not contribute to the integral over a period. In the same way we can define the local heat rates per cycle:

$$\dot{Q}_\alpha^c = \frac{1}{\tau} \lim_{k \rightarrow \infty} \int_{k\tau}^{(k+1)\tau} \dot{Q}_\alpha(t') dt' = \frac{1}{\tau} \lim_{k \rightarrow \infty} \int_{k\tau}^{(k+1)\tau} \text{Tr} [P_\alpha V(t') \sigma^{xp}(t') M^{-1}] dt' \quad (\text{B14})$$

This set of heat rates trivially satisfy:

$$\dot{Q}^c = \sum_\alpha \dot{Q}_\alpha^c \quad (\text{B15})$$

From Eq. (B11) and the fact that the right hand side is a derivative of  $\tau$ -periodic function for long times it follows that:

$$\dot{Q}'_\alpha + \dot{Q}_\alpha^c = 0 \quad (\text{B16})$$

Therefore, in a complete cycle the variation of energy of a given reservoir is equal (in absolute value) to the variation of the energy of the system due to the interaction with that reservoir. No energy is stored in the interaction terms. We stress that this is true only for the averaged heat rates.

## 3. Heat transfer matrix

From Eqs. (A39b), (A40) and (A48) it is straightforward to derive the following expression for the correlation between position and momentum in terms of the Fourier coefficients  $\{A_k(\omega, \omega_d)\}$ :

$$\sigma^{xp}(t) = \text{Im} \left[ \sum_{j,k} \sigma_{j,k}^{xp} e^{i\omega_d(j-k)t} \right] \quad (\text{B17})$$

where:

$$\sigma_{j,k}^{xp} = \frac{\hbar}{2} \int_0^\infty (\omega + k\omega_d) A_j(\omega, \omega_d) \tilde{\nu}(\omega) A_k^\dagger(\omega, \omega_d) d\omega \quad (\text{B18})$$

Introducing Eq. (B17) into the expression for the local heat currents of Eq. (B14), and performing the time integral, the following result is obtained:

$$\dot{Q}_\alpha^c = \frac{\hbar}{2} \int_0^\infty \text{Im} \left\{ \sum_{j,k} \text{Tr} [P_\alpha V_{k-j} A_j(\omega, \omega_d) \tilde{\nu}(\omega) A_k^\dagger(\omega, \omega_d)] (\omega + \omega_d k) \right\} d\omega \quad (\text{B19})$$

Now, expanding the Fourier transform of the noise kernel as in Eq. (A38), the local heat  $\dot{Q}_\alpha^c$  can be written as:

$$\dot{Q}_\alpha^c = \sum_\beta \int_0^\infty Q_{\alpha,\beta}(\omega) \coth \left( \frac{\hbar\omega}{2k_b T_\beta} \right) d\omega \quad (\text{B20})$$

where the functions  $Q_{\alpha,\beta}(\omega)$  are defined as

$$Q_{\alpha,\beta}(\omega) = \frac{\hbar}{2} \operatorname{Im} \left\{ \sum_{j,k} (\omega + \omega_d k) \operatorname{Tr} \left[ P_\alpha V_{k-j} A_j(\omega, \omega_d) I_\beta(\omega) A_k^\dagger(\omega, \omega_d) \right] \right\} \quad (\text{B21})$$

If the number of reservoirs is  $L$ , there are  $L^2$  functions  $Q_{\alpha,\beta}(\omega)$ , which are considered to be the elements of matrix called the heat transfer matrix. They specify how the heat per cycle corresponding to the  $\alpha$ -th reservoir is affected by the temperature of the  $\beta$ -th reservoir. The previous expression for  $Q_{\alpha,\beta}(\omega)$  can be simplified. In order to do that we note that one of the sums appearing in Eq. (B21) can be performed with the aid of the algebraic equation that the coefficients  $A_k(\omega, \omega_d)$  satisfy. Indeed, from Eq. (A52) it follows that

$$\sum_j V_{k-j} A_j(\omega, \omega_d) = \mathbf{1} \delta_{k,0} - \left[ \hat{g}(i(\omega + \omega_d k))^{-1} - V_0 \right] A_k(\omega, \omega_d) \quad (\text{B22})$$

Another important relation is:

$$\operatorname{Im} \left\{ \hat{g}(i\omega)^{-1} - V_0 \right\} = \omega \operatorname{Re} \{ \hat{\gamma}(i\omega) \} = \frac{\pi}{2} I(\omega) \quad (\text{B23})$$

where in the last equality the fluctuation-dissipation theorem,  $\operatorname{Re} \{ \hat{\gamma}(i\omega) \} = \frac{\pi}{2} \frac{I(|\omega|)}{|\omega|}$ , was employed. The last equation is valid if the spectral density  $I(\omega)$ , originally defined only for positive frequencies, is extended in a odd way for negative frequencies, i.e, such that  $I(-\omega) = -I(\omega)$ . Taking into account Eqs. (B22) and (B23) it is possible to arrive at the following simplified expression for the non-diagonal terms of the heat transfer matrix:

$$Q_{\alpha,\beta}(\omega) = \frac{-\pi\hbar}{4} \sum_k (\omega + \omega_d k) \operatorname{Tr} \left[ I_\alpha(\omega + \omega_d k) A_k(\omega, \omega_d) I_\beta(\omega) A_k^\dagger(\omega, \omega_d) \right] \quad (\alpha \neq \beta) \quad (\text{B24})$$

Also, the sum over the first index can be expressed as:

$$\tilde{Q}_\beta(\omega) = \sum_\alpha Q_{\alpha,\beta}(\omega) = \frac{-\pi\hbar}{4} \sum_k (\omega_d k) \operatorname{Tr} \left[ I(\omega + \omega_d k) A_k(\omega, \omega_d) I_\beta(\omega) A_k^\dagger(\omega, \omega_d) \right] \quad (\text{B25})$$

The last two equations completely determine all the elements of the heat transfer matrix. These expressions can be compared to the ones obtained in [?] for the case without driving.

#### 4. Heat currents in terms of elementary processes

Eq. (B20) is a simple and compact expression for the heat currents. However, as we will see, it condenses in a single formula terms with very different physical origins, and therefore it is not possible to assign a clear physical interpretation to each coefficient  $Q_{\alpha,\beta}(\omega)$  of the heat transfer matrix. In this section we analyze Eqs. (B20), (B24) and (B25) and identify different mechanisms of heat generation and energy transport between reservoirs. We begin by using Eq. (B25) to rewrite Eq. (B20) as:

$$\dot{Q}_\alpha^c = \int_0^\infty d\omega \tilde{Q}_\alpha(\omega) \coth \left( \frac{\hbar\omega}{2k_b T_\beta} \right) + \sum_{\beta \neq \alpha} \int_0^\infty d\omega \left\{ Q_{\alpha,\beta}(\omega) \coth \left( \frac{\hbar\omega}{2k_b T_\beta} \right) - Q_{\beta,\alpha}(\omega) \coth \left( \frac{\hbar\omega}{2k_b T_\alpha} \right) \right\} \quad (\text{B26})$$

Expanding  $\tilde{Q}_\alpha(\omega)$  and  $Q_{\alpha,\beta}(\omega)$  using Eqs. (B24) and (B25) it is clear that some terms in the previous expression cancel out. Taking that into account, and using the identity  $\coth \left( \frac{\hbar\omega}{2k_b T_\beta} \right) = 2(N_\alpha(\omega) + 1/2)$ , where  $N_\alpha(\omega) = (e^{\hbar\omega/(kT)} - 1)^{-1}$  is the Planck distribution, we write:

$$\begin{aligned} \dot{Q}_\alpha^c &= - \sum_k \int_0^\infty d\omega \hbar k \omega_d \frac{\pi}{2} \operatorname{Tr} [ I_\alpha(\omega + k\omega_d) A_k(\omega, \omega_d) I_\alpha(\omega) A_k^\dagger(\omega, \omega_d) ] (N_\alpha(\omega) + 1/2) \\ &\quad - \sum_{\beta \neq \alpha} \sum_k \int_0^\infty d\omega \hbar(\omega + k\omega_d) \frac{\pi}{2} \operatorname{Tr} [ I_\alpha(\omega + k\omega_d) A_k(\omega, \omega_d) I_\beta(\omega) A_k^\dagger(\omega, \omega_d) ] (N_\beta(\omega) + 1/2) \\ &\quad + \sum_{\beta \neq \alpha} \sum_k \int_0^\infty d\omega \hbar\omega \frac{\pi}{2} \operatorname{Tr} [ I_\beta(\omega + k\omega_d) A_k(\omega, \omega_d) I_\alpha(\omega) A_k^\dagger(\omega, \omega_d) ] (N_\alpha(\omega) + 1/2) \end{aligned} \quad (\text{B27})$$

We now analyze the heat current  $\dot{Q}_\alpha^c|_{T=0}$  when all the reservoirs are at zero temperature, which corresponds to removing the Planck's distributions from the previous equation. In order to simplify the discussion, we also consider the case in which the driving protocol is invariant under time reversal (i.e, such that  $V(t) = V(-t)$ ). In that case the relation of Eq. (A56b) reads  $A_k(\omega, \omega_d) = A_{-k}^T(\omega + k\omega_d, \omega_d)$ . Using this symmetry and Eq. (A56c), via a straightforward change of variable in the integrals, it is possible to relate the terms with negative  $k$  to the ones with positive  $k$  in the previous equation (when all the functions  $N_\alpha(\omega)$  are equal to zero). Doing that, we obtain:

$$\begin{aligned} \dot{Q}_\alpha^c|_{T=0} = & - \sum_{k>0} \int_0^{k\omega_d} d\omega \hbar k \omega_d p_{\alpha,\alpha}^{-k}(\omega) (1/2) \\ & - \sum_{\beta \neq \alpha} \sum_{k>0} \int_0^{k\omega_d} d\omega \hbar \omega p_{\beta,\alpha}^{-k}(\omega) (1/2) \\ & - \sum_{\beta \neq \alpha} \sum_{k>0} \int_0^{k\omega_d} d\omega \hbar (k\omega_d - \omega) p_{\alpha,\beta}^{-k}(\omega) (1/2) \end{aligned} \quad (\text{B28})$$

where we have defined the functions  $p_{\alpha,\beta}^k(\omega) = \frac{\pi}{2} \text{Tr}[I_\alpha(|\omega + k\omega_d|)A_k(\omega, \omega_d)I_\beta(\omega)A_k^\dagger(\omega, \omega_d)]$ . This last expression can already be interpreted in terms of pairs creation as explained in the main text.

Note that for  $k < 0$  and  $\omega < |k|\omega_d$  one spectral density in each terms of Eq. (B27) is evaluated at a negative frequency (recall that extending the spectral densities to negative frequencies was necessary in Eq. (B23) in order to obtain compact expressions). Thus, we split the integrals for the  $k < 0$  case in the domains  $(0, |k|\omega_d)$  and  $(|k|\omega_d, +\infty)$ . The resulting expression having only the integrals in  $(0, |k|\omega_d)$  is the one given in Eq. (6) for  $\dot{Q}_\alpha^{NRH}$  in the main text. The remaining terms (without the 1/2 added to the Planck's distributions, since the  $T = 0$  contribution is already accounted by  $\dot{Q}_\alpha^{NRH}$ ) are  $\dot{Q}_\alpha^{RP} + \dot{Q}_\alpha^{RH}$  given in Eqs. (4) and (5) in the main text.

## Appendix C: Validity of the second law of thermodynamics

### 1. The Planck proposition

To investigate the validity of the second law in this context we first consider the Planck proposition: *“It is impossible to construct an engine which will work in a complete cycle, and produce no effect except the raising of a weight and cooling of a heat reservoir”*. Translated to our setting, the previous proposition means that if the temperatures of all the reservoirs are the same, i.e, if  $T_\alpha = T_0$  for all  $\alpha$ , then the work performed on the system must be positive, that is,  $\dot{W}^c \geq 0$ . In other words, it must be impossible to extract work from a single thermal reservoir. Thus, considering Eqs. (B12), (B15) and (B25), we should be able to show that:

$$\dot{W}^c = \frac{\pi\hbar}{4} \int_0^\infty \sum_{k=-\infty}^{+\infty} (\omega_d k) \text{Tr} \left[ I(\omega + \omega_d k) A_k(\omega, \omega_d) I(\omega) A_k^\dagger(\omega, \omega_d) \right] \coth \left( \frac{\hbar\omega}{2k_b T_0} \right) d\omega \geq 0 \quad (\text{C1})$$

The main difficulty in assessing the previous inequality is that the integrand in the left hand side has no definite sign. It is thus convenient to write:

$$\dot{W}^c = \frac{\pi\hbar}{4} \sum_{k>0} (\omega_d k) (I_k(\omega_d) - I_{-k}(\omega_d)) \quad (\text{C2})$$

where the functions  $I_k^\pm(\omega_d)$  are defined as:

$$I_k(\omega_d) = \int_0^\infty \text{Tr} \left[ I(\omega + \omega_d k) A_k(\omega, \omega_d) I(\omega) A_k^\dagger(\omega, \omega_d) \right] \coth \left( \frac{\hbar\omega}{2k_b T_0} \right) d\omega \quad (\text{C3})$$

Now, if  $I_k^-(\omega_d)$  is the function corresponding to the time reversed process  $V(-t)$  it is possible to show that  $I_k^-(\omega_d) \geq I_{-k}(\omega_d)$ , which we do as follows:

$$I_{-k} = \int_0^\infty \dots d\omega = \int_0^{\omega_d k} \dots d\omega + \int_{\omega_d k}^\infty \dots d\omega \quad (\text{C4})$$

The first integral in the right hand side is always negative, since  $I(\omega - \omega_d k)$  is negative in the interval  $(0, \omega_d k)$ . The second integral is:

$$\begin{aligned} & \int_{\omega_d k}^{\infty} \text{Tr} \left[ I(\omega - \omega_d k) A_{-k}(\omega, \omega_d) I(\omega) A_{-k}^\dagger(\omega, \omega_d) \right] \coth \left( \frac{\hbar \omega}{2k_b T_0} \right) d\omega = \\ & \int_0^{\infty} \text{Tr} \left[ I(\omega) A_{-k}(\omega + \omega_d k, \omega_d) I(\omega + \omega_d k) A_{-k}^\dagger(\omega + \omega_d k, \omega_d) \right] \coth \left( \frac{\hbar(\omega + \omega_d k)}{2k_b T_0} \right) d\omega \leq \\ & \int_0^{\infty} \text{Tr} \left[ I(\omega + \omega_d k) A_k^{rT}(\omega, \omega_d) I(\omega) A_k^{r*}(\omega, \omega_d) \right] \coth \left( \frac{\hbar \omega}{2k_b T_0} \right) d\omega = I_k^r \end{aligned} \quad (\text{C5})$$

The first step was a simple change of variable and in the second step we used the identity of Eq. (A56b) and the fact that  $\coth(\omega)$  is a decreasing function of  $\omega$ . Therefore, we proved that  $I_{-k} \leq I_k^r$ , as required. Also,  $I_{-k}^r \leq I_k$ . It follows that  $\dot{W}^c + \dot{W}^{c^r} \geq 0$ . Using Eq. (A56a) it is easy to see that as a function of  $\omega_d$  we have  $\dot{W}^{c^r}(\omega_d) = \dot{W}^c(-\omega_d)$ . In summary, we have shown that

$$\dot{W}^c(\omega_d) + \dot{W}^c(-\omega_d) \geq 0 \quad (\text{C6})$$

We now note the following three facts: (i) the function  $\dot{W}^c(\omega_d)$  is continuous, (ii) the only root of the function  $\dot{W}^c(\omega_d)$  is  $\omega_d = 0$  and (iii)  $\dot{W}^c(\omega_d) = \dot{W}^c(-\omega_d)$  for  $\omega_d \rightarrow \pm\infty$  (this last fact is physically obvious but can be verified using the approximate solutions of the coefficients  $A_k(\omega, \omega_d)$  for large  $\omega_d$ ). It follows that  $\dot{W}^c(\omega_d) \geq 0$  for all  $\omega_d$ .

## 2. Irreversible entropy generation

A general derivation of the second law in our setting can be obtained without using the specific expression for the heat currents. Thus, we would like to show that in a complete cycle of the process the production of entropy is always positive, i.e.,

$$\sum_{\alpha} \frac{-\dot{Q}_{\alpha}^c}{T_{\alpha}} \geq 0 \quad (\text{C7})$$

This statement of the second law is equivalent to the Planck principle only in the adiabatic limit (only in that limit the heat transfer matrix is symmetric), as can be verified by direct calculation in the high temperature regime. Although it must be possible in principle to prove Eq. (C7) from the general expression for the heat transfer matrix, that is not the more economic or elegant approach. To prove Eq. (C7) we begin by considering the variations of the Von Neumann entropy of each reservoir and the system. We take  $S(t)$  and  $S_{\alpha}(t)$  as the entropies of the system and the  $\alpha$ -th reservoir at time  $t$ . Also,  $\Delta S(t) = S(t) - S(0)$ . Using that the initial global state is a product state, that the global dynamics is unitary and the subadditivity of the Von Neumann entropy it is easy to see that:

$$\Delta S(t) + \sum_{\alpha} \Delta S_{\alpha}(t) \geq 0 \quad (\text{C8})$$

We now divide the previous equation by the total time and evaluate in  $t = k\tau$ . Since the asymptotic state of the system is  $\tau$ -periodic, its entropy is also a  $\tau$ -periodic, and since it is continuous, it is also bounded. Therefore  $\lim_{k \rightarrow \infty} \Delta S(k\tau)/(k\tau) = 0$ . On the other hand  $\Delta S_{\alpha}(k\tau)/(k\tau)$  converges to  $\Delta S_{\alpha}^c(t)/\tau$  for  $k \rightarrow \infty$ , where  $\Delta S_{\alpha}^c(t)$  is the change in entropy of the  $\alpha$ -th reservoir per cycle in the asymptotic state. In this way, we obtain the following inequality for the variations per cycle of the entropy of the reservoirs in the asymptotic state:

$$\sum_{\alpha} \Delta S_{\alpha}^c(t) \geq 0 \quad (\text{C9})$$

We now take advantage of the fact that the initial state of the  $\alpha$ -th reservoir is thermal at temperature  $T_{\alpha}$ , and therefore it is the only minimum of the free energy function  $F_{\alpha}(\rho) = \text{Tr}(\rho H_{E,\alpha}) - k_b T_{\alpha} S(\rho)$ . As a consequence:

$$0 \leq \Delta F_{\alpha}(t) = \Delta E_{\alpha}(t) - k_b T_{\alpha} \Delta S_{\alpha}(t) \quad (\text{C10})$$

As before, dividing the previous equation by  $t$ , evaluating in  $t = k\tau$ , and taking the limit  $k \rightarrow \infty$ , we obtain:

$$\frac{\Delta E_{\alpha}^c}{k_b T_{\alpha}} \geq \Delta S_{\alpha}^c \quad (\text{C11})$$



where  $\Delta E_\alpha^c$  is the variation per cycle of the energy of the  $\alpha$ -th reservoir in the asymptotic state. It is not immediately obvious how this variation is related to the previously defined heat currents (since in our model the interaction terms are not energy conserving, i.e.  $[H_S + H_{E,\alpha}, H_{int,\alpha}] \neq 0$ ). However, in Appendix B we show that in the asymptotic state  $\Delta E_\alpha^c = -\tau \dot{Q}_\alpha^c$ , as expected. Inserting this last identity in Eq. (C11), summing over all reservoirs, and using Eq. (C9) we obtain Eq. (C7).

#### Appendix D: Weak coupling approximation

In the weak coupling regime the frequency integrals in Eq. (B27) the heat transfer matrix can be approximated by sums over the normal modes of the closed system. To see that we need an analytic expression for the coefficients  $A_k(\omega, \omega_d)$ . Thus, we employ the weak driving approximation of Eq. (A55b), which gives the solution for those coefficients in terms of the Laplace's transform of the Green's function  $\hat{g}(i\omega)$ . As explained in [?], in the weak coupling limit  $\hat{g}(i\omega)$  can be approximated as

$$\hat{g}(i\omega) = \sum_a \frac{q_a q_a^T}{\Omega_a^2 - (\omega - i\Gamma_a)^2} \quad (D1)$$

where  $\{\Omega_a\}$  and  $\{q_a\}$  are the normal frequencies and modes of the closed system and  $\Gamma_a$  is the dissipation rate of each normal mode. We assume for simplicity that the system is not degenerated. As an example, we write the expression for  $p_{\alpha,\beta}^{(k)}(\omega)$  using the previous approximations (for  $k \neq 0$ ):

$$p_{\alpha,\beta}^{(k)}(\omega) = \frac{\pi}{2} \sum_{a,b,c,d} \frac{(q_a^T V_k q_b)(q_d^T V_k^\dagger q_c)(q_c^T I_\alpha(|\omega + \omega_d k|) q_a)(q_b^T I_\beta(\omega) q_c)}{(\Omega_a^2 - (\omega + \omega_d k - i\Gamma_a)^2)(\Omega_b^2 - (\omega - i\Gamma_b)^2)(\Omega_c^2 - (\omega + \omega_d k + i\Gamma_c)^2)(\Omega_d^2 - (\omega + i\Gamma_d)^2)} \quad (D2)$$

In the weak coupling limit where  $\Gamma_a \ll \Omega_a, \omega_d$  and under the condition  $\omega_d < \min_{\Omega_a \neq \Omega_b} \{|\Omega_a - \Omega_b|/2\}$  the typical shape of the functions  $p_{\alpha,\beta}^{(k)}(\omega)$  is like the one depicted in figure 2. We see the expected resonances at frequencies  $\{\Omega_a\}$

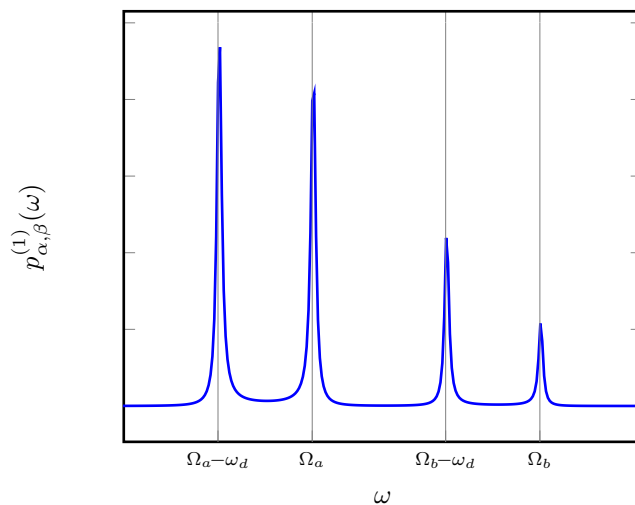


FIG. 2: Typical shape of the function  $p_{\alpha,\beta}^{(1)}(\omega)$  in the weak coupling limit

and  $\{\Omega_a - k\omega_d\}$ , which for the mentioned conditions are well defined and do not overlay. Therefore, an approximate solution for integrals of the form  $\int_0^\infty p_{\alpha,\beta}^{(k)}(\omega) N(\omega)$  can be obtained by dealing separately with each resonance peak, evaluating the remaining factors at the center of the peak. It is clear that the dominant terms are those with  $\Omega_c = \Omega_a$  and  $\Omega_d = \Omega_b$ . Also, for low temperatures the Planck distribution  $N(\omega)$  exponentially suppresses high frequencies, and therefore the dominant contribution to the integral is given by the peak at  $\omega = \Omega_0 - k\omega_d$ , where  $\Omega_0$  is the smallest normal frequency of the system. Using this method the following expression is found:

$$\int_0^\infty d\omega p_{\alpha,\beta}^{(k)}(\omega) N(\omega) \simeq \frac{\pi^2}{8} \frac{N(\Omega_0 - k\omega_d)}{\Omega_0^2 \Gamma_0} \sum_{b,c} \frac{(q_0^t V_k q_b)(q_b^t V_k^\dagger q_c)(q_b^t I_\beta(\Omega_a - k\omega_d) q_c)(q_0^t I_\alpha(\Omega_a) q_0)}{(\Omega_b^2 - (\Omega_0 - k\omega_d)^2)(\Omega_c^2 - (\Omega_0 - k\omega_d)^2)} \quad (D3)$$

Note that this expression is invariant upon time inversion of the process (i.e, invariant under complex conjugation of the Fourier coefficients  $V_k$ ). We now use the previous result to calculate the resonant contributions to the heat currents for the simple driving protocol  $V(t) = V_0 + 2V_1 \cos(\omega_d t)$  in a system connected to only two reservoirs,  $E_\alpha$  and  $E_\beta$ . Considering, for simplicity, only the contribution of the  $\Omega_0$  normal mode, we have:

$$\begin{aligned} \dot{Q}_\alpha^R = \dot{Q}_\alpha^{RP} + \dot{Q}_\alpha^{RH} &= \frac{\pi^2 N(\Omega_0 - \omega_d)}{8 \Omega_0^2 \Gamma_0} \frac{|V_1^0|^2}{(\Omega_0^2 - (\Omega_0 - \omega_d)^2)^2} \\ &\times \{(\Omega_0 - k\omega_d)I_\beta^0(\Omega_0)I_\alpha^0(\Omega_0 - \omega_d) - \Omega_0 I_\alpha^0(\Omega_0)I_\beta^0(\Omega_0 - \omega_d) - k\omega_d I_\alpha^0(\Omega_0)I_\alpha^0(\Omega_0 - \omega_d)\}, \end{aligned} \quad (\text{D4})$$

where we have defined  $M^0 = q_0^t M q_0$  for any matrix  $M$ . It is easy to see that if the two reservoirs are spectrally equivalent (if  $I_\alpha(\omega) = I_\beta(\omega)$ ), then the previous expression is always negative and therefore both reservoirs are heated. However, if the spectral densities satisfy  $I_\alpha(\Omega_0) \ll I_\beta(\Omega_0)$