Front propagation and quasi-stationary distributions for one-dimensional Lévy processes

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Abstract

We jointly investigate the existence of quasi-stationary distributions for one dimensional Lévy processes and the existence of traveling waves for the Fisher-Kolmogorov-Petrovskii-Piskunov (F-KPP) equation associated with the same motion. Using probabilistic ideas developed by S. Harris [13], we show that the existence of a traveling wave for the F-KPP equation associated with a centered Lévy processes that branches at rate r and travels at velocity c is equivalent to the existence of a quasi-stationary distribution for a Lévy process with the same movement but drifted by $-c$ and killed at zero, with mean absorption time $1/r$. This also extends the known existence conditions in both contexts. As it is discussed in [12], this is not just a coincidence but the consequence of a relation between these two phenomena.

Keywords: quasi-stationary distributions, traveling waves, branching random walk, branching Lévy proceses.

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1 Introduction

Let $\mathcal{L}$ be the generator of a centered one-dimensional Lévy process (precise definitions and assumptions are be given below) and consider the (generalized) F-KPP equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{L}^* u + r(u^2 - u), \quad x \in \mathbb{R},\, t > 0, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1}$$

Here $\mathcal{L}^*$ denotes the adjoint of $\mathcal{L}$. Both Fisher and Kolmogorov, Petrovskii and Piskunov considered this equation for $\mathcal{L} = \frac{d^2}{dx^2}$ and proved independently that in this case this equation admits traveling wave solutions of the form $u(t,x) = w_c(x - ct)$ that travel at velocity c for every $c \geq \sqrt{2r}$, [11, 15].

It is well known [5, 6, 18, 24] that a large class of equations describing the propagation of a front into an unstable region have properties similar to (1). These equations admit traveling-wave solutions for any velocity c larger than a minimal velocity c^* and the front moves with this minimal velocity c^* for any initial data with "light enough" tails.

For the Brownian case $\mathcal{L} = \frac{d^2}{dx^2}$ we have $c^* = \sqrt{2r}$ and for more general $\mathcal{L}$ the minimal velocity can be computed in terms of the Legendre transform of the process (see Theorem 1.1 below). This was essentially done by Kyprianou [16] using the seminal McKean's representation

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[20] for the solutions of (1). We complete this characterization in this note to arrive to our main theorem.

The theory of quasi-stationary distributions has its own counterpart. It is a typical situation that there is an infinite number of quasi-stationary distributions while the *Yaglom limit* (the limit of the conditioned evolution of the process started from a deterministic initial condition) selects the minimal one, i.e. the one with minimal expected time of absorption [7, 10, 23].

To be more precise, consider a Lévy process $(X_t - ct)_{t\geq 0}$ with generator $\mathcal{L} - c\frac{d}{dx}$ killed at the origin defined in certain filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with expectation denoted by $\mathbb{E}$. The absorption time is defined by $\tau = \inf\{t > 0 \colon X_t - ct = 0\}$. The conditioned evolution at time t is defined by

$$\mu_t^\gamma(\cdot) := \mathbb{P}^\gamma(X_t - ct \in \cdot | \tau > t).$$

Here γ denotes the initial distribution of the process and $\mathbb{P}^\gamma(\cdot) = \mathbb{P}(\cdot|X_0 \sim \gamma)$. A probability measure ν is said to be a quasi-stationary distribution (QSD) if $\mu_t^\nu = \nu$ for all $t \geq 0$.

The *Yaglom limit* is a probability measure ν defined by

$$\nu := \lim_{t\to\infty} \mu_t^{\delta_x},$$

if the limit exists and does not depend on x. It is known that if the Yaglom limit exists, then it is a QSD. A general principle is that the Yaglom limit *selects* the minimal QSD, i.e. the Yaglom limit is the QSD with minimal mean absorption time. This fact has been proved for a wide class of processes that include birth and death process, subcritical Galton-Watson processes, drifted random walks and Brownian motion among others, but the conjecture is still open for a much wider class of processes.

In the last decades, a great deal of attention has been given to establish on the one hand conditions for the existence of quasi-stationary measures of Lévy processes (see for instance [17, 19]) and on the other hand to the existence of traveling waves for (1) [16]. The purpose of this note is to show that given parameters $r, c > 0$, the existence of a traveling wave for (1) with velocity c is equivalent to the existence of a QSD ν for $\mathcal{L} - c\frac{d}{dx}$ with expected absorption time $\mathbb{E}_\nu(\tau) = 1/r$. Moreover, minimal velocity TWs are in a one-to-one correspondence with minimal absorption time QSDs with the same parameters. Note that when dealing with traveling-waves the branching rate r is an input while the velocity c is chosen by the system, while when dealing with QSDs the velocity c is the input and r is chosen by the system.

Although our proof consists in showing that the conditions for the existence of TW and QSD coincide, in a companion paper [12] we show that these is not just a coincidence but that the two phenomena are essentially two faces of the same coin.

All in all, our main result reads.

Theorem 1.1. *Under assumption* **A** *(stated below), the following are equivalent:*

1. *There exists a non trivial traveling wave for* (1) *with velocity c, i.e. a solution to*
$$\mathcal{L}^* w + cw' + rw(w-1) = 0. \tag{2}$$
2. *There exists an (absolutely continuous) QSD for $\mathcal{L} - c\frac{d}{dx}$ with expected absorption time $1/r$, i.e. a solution to,*
$$\mathcal{L}^* v + cv' + rv = 0. \tag{3}$$
3. *$r \leq \Gamma(c)$, where Γ is the Legendre transform of the Laplace exponent of $\mathcal{L}$.*
4. *A branching Lévy process driven by $\mathcal{L} - c\frac{d}{dx}$, absorbed in 0 gets almost surely extinct.*

Moreover, c is a minimal velocity for $(\mathcal{L}^*, r)$ if and only if $1/r$ is a minimal mean absorption time for $\mathcal{L} - c\frac{d}{dx}$.

Remark 1.2. *In* (2) *the domain is $\mathbb{R}$ and the boundary conditions are $w(+\infty) = 1 - w(-\infty) = 1$, while in* (3) *the domain is $(0, +\infty)$ and also $v \geq 0$, $v(0) = 0$, $\int v = 1$ is imposed.*

2 Preliminaries

Let $X = (X_t)_{t\geq 0}$ be a Lévy process with values in $\mathbb{R}$, defined on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and Laplace exponent $\psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathbb{E}(e^{\theta X_t}) = e^{\psi(\theta)t},$$

such that

$$\psi(\theta) = b\theta + \sigma^2 \frac{\theta^2}{2} + g(\theta),$$

where $b \in \mathbb{R}$, $\sigma > 0$ (which ensures that X is non-lattice) and g is defined in terms of the jump measure Π supported in $\mathbb{R} \setminus \{0\}$ by

$$g(\theta) = \int_{\mathbb{R}} (e^{\theta x} - 1 - \theta x \mathbf{1}_{\{|x|<1\}})\Pi(dx), \qquad \int_{\mathbb{R}} (1 \wedge x^2)\Pi(dx) < \infty.$$

Let $\theta_+^\star = \sup\{\theta \colon |\psi(\theta)| < \infty\}$, $\theta_-^\star = \inf\{\theta \colon |\psi(\theta)| < \infty\}$ and recall that ψ is strictly convex in $(\theta_-^\star, \theta_+^\star)$ and by monotonicity $\psi(\theta_\pm^\star) = \psi(\theta^\star\mp)$ and $\psi'(\theta_\pm^\star) = \psi'(\theta^\star\mp)$ are well defined as well as the derivative at zero $\psi'(0) = \mathbb{E}(X_1)$, that we assume to be zero. We also assume that $\theta_\pm^\star > 0$. The generator of X applied to a function $f \in C_0^2$, the class of compactly supported functions with continuous second derivatives, gives

$$\mathcal{L}f(x) = \frac{1}{2}\sigma^2 f''(x) + bf'(x) + \int_{\mathbb{R}} (f(x+y) - f(x) - yf'(x)\mathbf{1}\{|y| \leq 1\})\Pi(dy).$$

The adjoint of $\mathcal{L}$ is also well defined in C_0^2 and has the form

$$\mathcal{L}^* f(x) = \frac{1}{2}\sigma^2 f''(x) - bf'(x) + \int_{\mathbb{R}} (f(x-y) - f(x) + yf'(x)\mathbf{1}\{|y| \leq 1\})\Pi(dy).$$

It is immediate to see that the Laplace exponent of $(X_t - ct)_{t\geq 0}$ is given by $\psi_c(\theta) = \psi(\theta) - c\theta$ for $\theta \in [\theta_-^\star, \theta_+^\star]$ and that C_0^2 is contained in the domain of the generator $\mathcal{L} - c\frac{d}{dx}$. We denote by Γ the Legendre transform of ψ, i.e.,

$$\Gamma(\alpha) = \sup_{\theta \in \mathbb{R}} \alpha\theta - \psi(\theta).$$

Similarly we will denote $\bar{\Gamma}$ the Legendre transform of the Laplace exponent of the dual process $(-X_t)_{t\geq 0}$,

$$\bar{\Gamma}(\alpha) = \sup_{\theta \in \mathbb{R}} \alpha\theta - \psi(-\theta).$$

Observe that since $\sigma > 0$, Γ as well as $\bar{\Gamma}$ are defined in $\mathbb{R}$. To summarize, hereafter we assume

(**A**) $\sigma > 0$, $\theta_\pm^\star > 0$ and $\mathbb{E}(X_1) = 0$.

Recall that the backward Kolmogorov equation for X is given by

$$\frac{d}{dt}\mathbb{E}^x(f(X_t)) = \mathcal{L}f(x),$$

while the forward Kolmogorov (or Fokker-Plank) equation for the density u (which exists since $\sigma > 0$) is given by

$$\frac{d}{dt}u(t,x) = \mathcal{L}^* u(t,\cdot)(x).$$

We will consider on the one hand Lévy processes with generator $\mathcal{L}$ (or $\mathcal{L}^*$) that evolve in $\mathbb{R}$ and on the other hand Lévy processes with generator $\mathcal{L} - c\frac{d}{dx}$, killed at zero. A probability measure in $\mathbb{R}_+$ with density v is a QSD for the process $(X_t - ct)_{t\geq 0}$ killed at 0, if and only if, v is a positive solution of (3).

We will need the following.

Lemma 2.1 (Girsanov theorem for Lévy processes)**.** *Let $M_t^\theta := \exp(\theta X_t - \psi_c(\theta)t)$ and the measure $\tilde{\mathbb{Q}}$ be defined by*

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = M_t^\theta, \qquad t \in [0, +\infty). \tag{4}$$

Then $(M_t^\theta)_{t\geq 0}$ is a martingale and under $\tilde{\mathbb{Q}}$, $(X_t)_{t\geq 0}$ is a Lévy process with drift $\mathbb{E}_{\tilde{\mathbb{Q}}}(X_1) = \psi_c'(\theta) = \psi'(\theta) - c$, variance σ^2, and jump measure $e^{\theta x} d\pi(x)$.

2.1 Some useful results on branching Lévy processes

Consider a continuous time branching process with binary branching at rate $r > 0$. Each individual performs independent Lévy processes with generator $\mathcal{L}$ started at the position of his ancestor at her birth-time. Details on the construction of this process can be found in [16]. Call N_t the number of individuals in the process at time t and $(\zeta_t^i, 1 \leq i \leq t)$ the positions of the individuals that are alive at time t. We call $Z_t = (\zeta_t^1, \ldots, \zeta_t^{N_t})$ and $Z = (Z_t)_{t\geq 0}$ a branching Lévy process (BLP) driven by $\mathcal{L}$. For some results, we need to consider BLP killed at some barrier $x \in \mathbb{R}$, the extension of the definition to this situation is straightforward.

The following proposition is proved in [1, 2]. See also [4, Theorem 4.17] for an alternative proof with spines and a setting closer to ours.

Proposition 2.2. *Let Z be a BLP driven by $\mathcal{L}$ and R_t the position of the maximum of Z_t. Then*

$$\lim_{t\to\infty} \frac{R_t}{t} = \Gamma^{-1}(r).$$

By means of this proposition we obtain the following partial extension of Theorem 1 in [3].

Proposition 2.3. *Let $\mathring{Z}$ be a BLP driven by $\mathcal{L} - c\frac{d}{dx}$ started at $x > 0$ and killed at the origin.*

- *(i) If $r \leq \Gamma(c)$, then $\mathring{Z}$ gets extinct with probability* 1.
- *(ii) If $r > \Gamma(c)$, then for any interval $A \subset \mathbb{R}^+$, $\mathbb{P}(\sum_{i=1}^{N_t} \mathbf{1}_{\{\mathring{\zeta}_t^i \in A\}} \to \infty) > 0$.*

Proof. Observe that $\mathring{Z}$ can be constructed straightforward with the trajectories of a non-absorbed process driven by the same generator. We just need to delete all the paths that touched the negative semi-axes at some time. In the case $r < \Gamma(c)$, we can directly use the previous proposition to see that the maximum of the non-absorbed branching process satisfies $\frac{R_t}{t} \to \Gamma^{-1}(r) - c < 0$ which implies that R_t is almost surely negative after some finite time. This in turn implies extinction of $\mathring{Z}$. For the critical case, we need to slightly refine the arguments given in [4].

Consider the branching Lévy process Z driven by $\mathcal{L}$ (without killing at 0) defined in the same filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and define the martingale

$$Z_t^\theta = \sum_{i=1}^{N_t} \exp(\theta\zeta_t^i - (\psi_c(\theta) + r)t),$$

as well as the change of measure,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z_t^\theta.$$

On some suitably augmented filtration $\tilde{\mathcal{F}}_t \supset \mathcal{F}_t$, the new process can be seen as a branching process with a spine $(S_t)_{t\geq 0}$ which branches at rate $2r$ and follows a motion given by the change of measure (4), i.e., a Lévy process with drift $\psi_c'(\theta) = \psi'(\theta) - c$, variance σ^2, and jump measure $e^{\theta x} d\pi(x)$. The other particles follow the usual process X. See [4] for details on this construction.

Since we assumed $r = \Gamma(c)$, we can define θ_c such that $\psi_c(\theta_c) = -\Gamma(c)$ and so $\psi_c'(\theta_c) = 0$. From now on we choose $\theta = \theta_c$ in the change of measure and hence, the spine (S_t) is centered.

As a consequence, it is recurrent (as a non trivial Lévy process). It follows that $\limsup_t S_t = \infty$. Now bounding $Z_t^{\theta_c}$ by the contribution of the spine, we have

$$\limsup Z_t^{\theta_c} \geq \limsup \exp(\theta_c S_t - (\psi_c(\theta_c) + r)t) = \exp(\theta_c S_t).$$

Since $1/Z^{\theta_c}$ is a positive super-martingale (under $\mathbb{Q}$), it converges $\mathbb{Q}-$almost surely and so does Z^{θ_c}. Hence,

$$\lim_{t\to\infty} Z_t^{\theta_c} = \infty, \quad \mathbb{Q} - \text{a.s.}$$

Observe that if $B \in \mathcal{F}_\infty$ we have

$$\mathbb{Q}(B) = \int_B \limsup_{t\to\infty} Z_t^\theta \, d\mathbb{P} + \mathbb{Q}(B \cap \{\limsup_{t\to\infty} Z_t^\theta = \infty\}).$$

It then follows that if $\lim Z_t^{\theta_c} = \infty$, under $\mathbb{Q}$, then $\lim Z_t^{\theta_c} = 0$, under $\mathbb{P}$. Finally, let

$$R_t := \max_{1\leq i\leq N_t} \zeta_t^i - ct,$$

and observe that $\exp(\theta_c R_t) \leq Z_t^{\theta_c}$, which implies that $\exp(\theta_c R_t)$ tends to 0 $\mathbb{P}-$a.s. and hence R_t tends to $-\infty$. As before, this implies extinction of $\mathring{Z}$.

To prove *(ii)*, denote $\mathring{Z}_t(A) := \sum_{i=1}^{N_t} \mathbf{1}_{\{\mathring{\zeta}_t^i \in A\}}$. We use the many-to-one lemma to get

$$\mathbb{E}(\mathring{Z}_t(A)) = e^{rt}\mathbb{P}(X_t - ct \in A, \min_{0\leq s\leq t} X_s - cs \geq 0). \tag{5}$$

To compute the last probability we can discretize the time variable and consider the random walk $S_n^\delta = X_{n\delta} - c\delta n$. Following [22, Theorem 4] and [14, Theorem 2.1] we obtain that the decay parameter for the process $(X_t - ct)$ killed at zero is given by $\Gamma(c)$ and hence for every $r > \Gamma(c)$ the r.h.s of (5) grows to infinity. So, for any $x > 0$ we can choose t^* large enough to guarantee $\mathbb{E}^x(\mathring{Z}_{t^*}(A)) > 1$. Let $x = \inf A$. We can assume $x > 0$ without loss of generality. Consider the (discrete time) Galton-Watson process with offspring distribution $\mathring{Z}_{t^*}(A)$, started with one individual at x. This process at time n bounds from below $\mathring{Z}_{nt^*}(A)$ and since it is supercritical we have that $\mathring{Z}_{nt^*}(A)$ grows exponentially fast as $n \to \infty$ with positive probability. Now,

$$\mathbb{P}\Big(\mathring{Z}_s(A) \leq \frac{\mathring{Z}_{nt^*}(A)}{2} \text{ for some } nt^* \leq s \leq (n+1)t^* \Big| \mathring{Z}_{nt^*}(A)\Big) \leq \mathbb{P}^x(X_s - cs \leq 0 \text{ for some } 0 \leq s \leq t^*)^{\mathring{Z}_{nt^*}(A)/2}$$

and the conditional Borel-Cantelli lemma [9, p. 207] implies the result.

■

3 Quasi-stationary distributions and traveling waves

In this section we prove the equivalence between existence of traveling waves and quasi-stationary distributions. The proof boils down to show that both are equivalent to the absorption of a BLP driven by $\mathcal{L} - c\frac{d}{dx}$ and killed at the origin.

3.1 Existence of Quasi-stationary disributions

We first deal with the quasi-stationary distributions.

Proposition 3.1. *The following are equivalent*

1. *There exists a QSD for $\mathcal{L} - c\frac{d}{dx}$ killed at 0 with mean absorption time $1/r$.*
2. $r \leq \Gamma(c)$.

Remark 3.2. *The existence of a QSD for $r = \Gamma(c)$ has been established in [17] under stronger assumptions on the Lévy process.*

Proof. 1) $\Longrightarrow$ 2) (Non-existence). Assume there exists a non-trivial QSD ν and suppose $r > \Gamma(c)$. Since $\sigma > 0$, there necessarily exists a density v being the Radon-Nikodym derivative of ν with respect to the Lebesgue measure on $\mathbb{R}_+$. Note that $v(0) = 0$ and on $\mathbb{R}_+$ we have

$$\mathcal{L}^* v + cv' + rv = 0.$$

Let $\mathring{\bar{Z}} = (\mathring{\bar{\zeta}}^1_t, \ldots, \mathring{\bar{\zeta}}^{N_t}_t)$ be a branching Lévy process driven by $\mathcal{L}^* + c\frac{d}{dx}$ killed at 0 and started at $x > 0$. The process

$$M_t = \sum_{i=1}^{N_t} v(\mathring{\bar{\zeta}}_i(t)),$$

is a martingale. On the other hand, for every $A \subset \mathbb{R}^+$,

$$\mathbb{E}^x(M_t) \geq (\inf_A v)\mathbb{E}^x \sum_{i=1}^{N_t} \mathbf{1}_{\{\mathring{\bar{\zeta}}^i_t \in A\}} = (\inf_A v) e^{rt} \mathbb{P}^x(-X_t + ct \in A, \min_{0 \leq s \leq t} -X_s + cs \geq 0). \tag{6}$$

We want to show that the r.h.s in (6) goes to infinity. Observe that if we take $A = \mathbb{R}^+$ we know the asymptotic behavior of the probability on the r.h.s of (6), but since $\inf_{R^+} v = 0$ this is useless. So we need to choose a smaller A. Irreducibility implies that $\inf_A v > 0$ for every $A \subset \mathbb{R}_+$ bounded and at a positive distance from the origin. We are going to choose $A = [\frac{1}{n}, n]$ for an adequate $n > 0$. Consider the process $X^n = (X^n_t)_{t \geq 0}$ with generator $\mathcal{L} - c\frac{d}{dx}$ killed at $\frac{1}{n}$ and n and call $p_n(x, t, B) = \mathbb{P}^x(X^n_t \in B)$ the transition semigroup and λ_n its decay parameter ([22, Theorem 6]) such that for every interval B

$$-\lim_{t \to \infty} \frac{1}{t} \log p_n(x, t, B) = \lambda_n.$$

We use p_∞, λ_∞, etc. when we deal with the process in $\mathbb{R}^+$ killed at the origin. We will show that $\lambda_n \searrow \lambda_\infty = \Gamma(c)$ and hence, since $r > \Gamma(c)$ we can choose n such that $r - \lambda_n > 0$ and the r.h.s of (6) goes to infinity. A contradiction to the fact that M_t is a martingale. Here we are using the fact that the exit problem from $[\frac{1}{n}, n]$ for a process with generator $\mathcal{L}^* + c\frac{d}{dx}$ started at x is equivalent to the exit problem from the same interval for a process with generator $\mathcal{L} - c\frac{d}{dx}$ started at $y = n - x + \frac{1}{n}$.

Since (λ_n) is decreasing in n, we only need to show $\lim \lambda_n \leq \lambda_\infty$. By means of time-discretization, using the splitting technique (which allows us to assume that X^n_t has an atom) and the subadditive ergodic theorem [22, Section 4], it can be shown that there exists a sequence of times $t_k \nearrow \infty$, $\varepsilon > 0$ and a constant $c > 0$, both depending on x and ε but not on n such that

$$-\frac{1}{t_k} \log p_n(y, t_k, (y - \varepsilon, y + \varepsilon)) + \frac{c}{t_k} \geq \lambda_n.$$

For fixed $t_k < \infty$ we can take $n \to \infty$ to obtain

$$-(1/t_k) \log p_\infty(y, t_k, B) + \frac{c}{t_k} \geq \lim_{n \to \infty} \lambda_n.$$

Now we let $k \to \infty$ to get $\lambda_\infty \geq \lim_n \lambda_n$. The fact that $\lambda_\infty = \Gamma(c)$ was already shown in the course of the proof of Proposition 2.3.

2) $\Longrightarrow$ 1) (Existence). As before, note that ν is a QSD with density v if and only if

$$\int f(\mathcal{L}^* v + cv' + rv) = 0, \tag{7}$$

for all $f \in \mathcal{D}$ where $\mathcal{D}$ is a subset of the domain of the generator with killing, i.e. the original generator but with domain composed of functions vanishing at 0, with the property that for every measurable set $A \subset \mathbb{R}_*^+$, there exists a sequence f_n in $\mathcal{D}$, uniformly bounded and converging pointwise to 1_A. Let $\theta > 0$ and denote by $e_{-\theta}$ the function $x \mapsto e^{-\theta x}$ and $v(x) = e^{-\theta x} h(x)$. The function $h\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ will be determined later. Let $(X_t)_{t\geq 0}$ be a Lévy process with generator $\mathcal{L}$. We compute

$$\begin{aligned}\left(\mathcal{L}^* + c\frac{d}{dx}\right) v(x) &= \frac{d}{dt}\mathbb{E}^0\left[e^{-\theta(x-X_t+ct)}h(x-X_t+ct)\right]_{t=0},\\ &= e^{-\theta x}\frac{d}{dt}\left[e^{\psi(\theta)t}\mathbb{E}^0\left(e^{\theta X_t-(c\theta+\psi(\theta))t}h(x-X_t+ct)\right)\right]_{t=0}\\ &= e^{-\theta x}\frac{d}{dt}\left[e^{(\psi(\theta)-c\theta)t}\tilde{\mathbb{E}}^0\left(h(x-X_t+ct)\right)\right]_{t=0}\\ &= e^{-\theta x}\left((\psi(\theta)-c\theta)h(x)+\tilde{\mathcal{L}}h(x)\right).\end{aligned}$$

here $\tilde{\mathbb{E}}$ denotes expectation under the measure $\tilde{\mathbb{Q}}$ defined by (4) and $\tilde{\mathcal{L}}$ is the generator of a Lévy process with drift $\tilde{\mathbb{E}}(\tilde{X}_1) = \tilde{\mathbb{E}}(-X_1) = c - \psi'(\theta)$, variance σ^2 and jump measure $e^{-\theta x}d\pi(-x)$ as in Lemma 2.1. Hence

$$\mathcal{L}^* v + cv' + rv = e^{-\theta x}\left((\psi(\theta) - c\theta + r)h(x) + \tilde{\mathcal{L}}h(x)\right).$$

We obtained that (7) is equivalent to the following equation

$$\int f e_{-\theta}(\tilde{\mathcal{L}}h + (r + \psi_c(\theta))h) = 0,$$

Note that since ψ is a convex function and $-\Gamma(c) \leq -r$, it is possible to choose θ such that $\psi(\theta) - c\theta = -r$. Hence (7) is equivalent to

$$\int f\tilde{\mathcal{L}}h = 0,$$

for all $f \in e_{-\theta}\mathcal{D} = \{g = e_{-\theta}u, u \in \mathcal{D}\}$. We then look for harmonic functions for the killed Lévy process $\tilde{X}$ with generator $\tilde{\mathcal{L}}$.

Define the renewal measure associated to $\tilde{X}$

$$h(x) = \mathbb{E}\int_0^\infty \mathbf{1}_{\{\tilde{H}_t \geq x\}}dt,$$

where $\tilde{H} = (\tilde{H}_t)_{t\geq 0}$ is the ladder process associated to $-\tilde{X}$.

Let θ_c be defined by $\Gamma(c) = c\theta_c - \psi(\theta_c)$. For $\theta \leq \theta_c$, the process $\tilde{X}$ does not drift to $-\infty$, since

$$\tilde{\mathbb{E}}^0(\tilde{X}_1) = c - \psi'(\theta) = -\psi_c'(\theta) \geq 0.$$

This implies that the function h is harmonic (see Lemma 1 in [8]) and since moreover $\tilde{X}_1$ has a finite mean, the renewal theorem implies that h is asymptotically equivalent to the identity and so

$$\int e_{-\theta}h < \infty.$$

Then v is the density of a QSD with absorption rate r. ■

3.2 Existence of Traveling waves

We now present the corresponding equivalence for the case of traveling waves which was actually the inspiration for the equivalence presented previously. Let us underline that these results are already known except in the critical case $r = \Gamma(c)$, [16]. The proof is included for completeness but follows the proof of [13] who himself quote the results of Neveu [21].

Proposition 3.3 ([13, 21])**.** *The following are equivalent*

1. There exists a solution to (2).

2. $r \le \Gamma(c)$.

Proof. 1) $\Longrightarrow$ *2) (Non-existence).* Assume the existence of a non-trivial traveling wave w. This allows us to define the multiplicative positive martingale

$$M_t = \prod_{i=1}^{N_t} w(\bar{\zeta}_t^i + ct).$$

Here $\bar{Z}_t = (\bar{\zeta}_t^1, \dots, \bar{\zeta}_t^{N_t})$ is a BLP driven by $\mathcal{L}^*$ (with no killing). This martingale being positive and bounded, it converges and its mean being $w(x)$, its limit is not 0. On the other hand, since $w \le 1$,

$$M_t \le w(\bar{L}_t + ct),$$

where $\bar{L}_t = \min_{1 \le i \le N_t} \bar{\zeta}_t^i$. Remark that the minimum of a BLP driven by $\mathcal{L}^*$ has the same law as $-\max_{1 \le i \le \bar{N}_t} \bar{\zeta}_t^i$, where $Z = ((\zeta_t^i)_{1 \le i \le N_t})_{t \ge 0}$ is a BLP driven by $\mathcal{L}$. Proposition 2.2 implies that if $r > \Gamma(c)$, $R_t - ct = \max_{1 \le i \le \bar{N}_t} \zeta_t^i - ct \to +\infty$. So $\bar{L}_t \to -\infty$ and hence M_t should have a null limit. A contradiction to the the assumption.

2) $\Longrightarrow$ *1) (Existence).* Neveu's method for proving the existence of traveling waves consists in constructing a multiplicative martingale from a Galton-Watson process obtained as follows.

Consider $\mathring{Z}$ a BLP driven by $\mathcal{L} - c\frac{d}{dx}$ with killing at the origin and started with one individual at $x > 0$ as in Proposition 2.3. Since $r \le \Gamma(c)$ the process is absorbed and then the total population size is finite a.s. We can construct this random number for every $x > 0$ using a unique BLP $\mathring{\bar{Z}}$ with generator $\mathcal{L}^* + c\frac{d}{dx}$ started with one individual at the origin and killing (freezing) at x. If we couple all the processes in this way and call $G_x < \infty$ the number of individuals of $\bar{Z}$ that have reached high x, we get that $(G_x)_{x \ge 0}$ is a continuous-time Galton-Watson process, [13, 21]. Define

$$f_x(s) = \mathbb{E}(s^{G_x}),$$

and for some fixed $s \in (0, 1)$

$$w(x) = f_x^{-1}(s).$$

Note that both quantities are strictly positive since $G_x < \infty$. For $y \ge 0$ define

$$M_y^x := w(x+y)^{G_y}.$$

It turns out that $(M_y^x)_{y \ge 0}$ is a convergent martingale. To see that, observe that the branching property gives us

$$\begin{aligned} \mathbb{E}[M_{y'}^x | \mathcal{F}_y] &= \mathbb{E}[w(x+y')^{G_{y'}} | \mathcal{F}_y], \\ &= (f_{y'-y}(w(x+y')))^{G_y}, \\ &= (f_{y'-y}(f_{y'-y}^{-1}(w(x+y))))^{G_y} = M_y^x. \end{aligned}$$

In addition, $(M_y^x)_{y \ge 0}$ is positive and bounded and hence, it does converge and is uniformly integrable. Following the arguments of [13], for fixed t and for all y large enough

$$G_y = \sum_{k=1}^{N_t} G_{y - \bar{\zeta}_t^i}^i,$$

where the $(G^i)_{i \ge 1}$ are independent copies of $G = (G_x)_{x \ge 0}$. Hence

$$M_y^x = \prod_{i=1}^{N_t} w(x+y)^{G_{y-\bar{\zeta}_t^i}^i} = \prod_{i=1}^{N_t} M_{y-\bar{\zeta}_t^i}^{x+\bar{\zeta}_t^i, i}.$$

and the limit of the martingale satisfies

$$M^x = \prod_{i=1}^{N_t} M^{x+\bar{\zeta}_t^i, i}.$$

taking expectations leads to

$$w(x) = \mathbb{E} \prod_{i=1}^{N_t} w(x + \bar{\zeta}_t^i),$$

which in turn implies (see Theorem 8 in [16]) that

$$\mathcal{L}^* w + cw' + r(w^2 - w) = 0.$$

■

3.3 Proof of Theorem 1.1

Observe that Proposition 3.3 gives us 1) $\iff$ 3) while Proposition 3.1 proves 2) $\iff$ 3). The equivlence 3) $\iff$ 4) is the content of Proposition 2.3. Finally since Γ is strictly increasing, minimality of $1/r$ (for a given c) as well as minimality of c, for a given r, reduces to

$$r = \Gamma(c).$$

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