STABILITY OF LOGARITHMIC DIFFERENTIAL ONE-FORMS.

FERNANDO CUKIERMAN,

JAVIER GARGIULO ACEA,

CÉSAR MASSRI.

Abstract. This article deals with the irreducible components of the space of codimension one foliations in a projective space defined by logarithmic forms of a certain degree. We study the geometry of the natural parametrization of the logarithmic components and we give a new proof of the stability of logarithmic foliations, obtaining also that these irreducible components are reduced.

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1. Introduction.

We consider differential one-forms of logarithmic type $\omega = F\ \sum_{i=1}^{m} \lambda_i\ dF_i/F_i$ where, for $i = 1, \dots, m$, F_i is a homogeneous polynomial of a fixed degree d_i in variables $x_0, \dots, x_n$, with complex coefficients, $F = \prod_j F_j$, and λ_i are complex numbers such that $\sum_i d_i \lambda_i = 0$. Such an ω defines a global section of $\Omega^1_{\mathbb{P}^n}(d)$ for $d = \sum_i d_i$. Also, ω satisfies the Frobenius integrability condition $\omega \wedge d\omega = 0$.

Fixing $\mathbf{d} = (m; d_1, \dots, d_m)$ denote $L_n(\mathbf{d}) \subset H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$ the collection of all such logarithmic one-forms and $\mathcal{L}_n(\mathbf{d}) \subset \mathbb{P}H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)) = \mathbb{P}^N$ the corresponding closed projective variety. It is easy to see that $\mathcal{L}_n(\mathbf{d})$ is an irreducible algebraic variety. Also, $\mathcal{L}_n(\mathbf{d})$ is contained in the subvariety $\mathcal{F}_n(d) \subset \mathbb{P}^N$ of integrable one-forms of degree d. Here the motivating problem is to describe the irreducible components of $\mathcal{F}_n(d)$.

It was proved by Omegar Calvo in [2] that, for any $\mathbf{d}$, the variety of logarithmic forms $\mathcal{L}_n(\mathbf{d})$ is an irreducible component of the moduli space $\mathcal{F}_n(d)$ of codimension one algebraic foliations of degree d in $\mathbb{P}^n(\mathbb{C})$. In other words, the logarithmic one-forms enjoy a stability condition among integrable forms. Actually, the results of [2] hold for more general ambient varieties than projective spaces.

In this article we will provide another proof of O. Calvo's theorem, in case the ambient space is a complex projective space. Our strategy will be to calculate the tangent space $T(\omega)$ of $\mathcal{F}_n(d)$ at a general point $\omega \in \mathcal{L}_n(\mathbf{d})$. The main results are stated in Theorems 24 and 25.

This method is completely algebraic and provides further information, especially the fact that $\mathcal{F}_n(d)$ results *generically reduced* along the irreducible component $\mathcal{L}_n(\mathbf{d})$.

The logarithmic components are the closure of the image of a multilinear map ρ, defined in Section 4, from a product of projective spaces into a projective space. We describe the base locus of ρ in Section 5, and study its generic injectivity in Section 6. Our proof requires a detailed analysis of the derivative of ρ, started in Section 7. Another important ingredient is the resolution of the ideal of various strata of the singular scheme of a logarithmic form; this is carried out in Section 8. The end of the proof is achieved in Section 9, where we distinguish two cases, depending on whether or not $\mathbf{d}$ is balanced.

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2. Notation.

We shall use the following notations:

$\mathbb{C}^{n+1}$ = complex affine space of dimension $n+1$.

$\mathbb{P}^n$ = complex projective space of dimension n.

$S_n = \mathbb{C}[x_0, \dots, x_n]$ = graded ring of polynomials with complex coefficients in $n+1$ variables.
When n is understood we denote $S_n = S$.

$S_n(d)$ = homogeneous elements of degree d in S_n.
When n is understood we denote $S_n(d) = S(d)$.
Recall that one has $S_n(d) = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$.

Ω^q_X = sheaf of algebraic differential q-forms on an algebraic variety X.
$\Omega^q(X)$ = the set of rational q-forms on X (with X an irreducible variety).
It is a vector space over the field $\mathbb{C}(X)$ of rational functions of X.

$\Omega^q_n = H^0(\mathbb{C}^{n+1}, \Omega^q_{\mathbb{C}^{n+1}})$.
A typical element of Ω^1_n is $\omega = \sum_{i=0}^n a_i\ dx_i$ with $a_i \in S_n$.
More generally, a typical element of Ω^q_n may be written in the usual way as $\sum_{|J|=q} a_J\ dx_J$ with $a_J \in S_n$ and $dx_J = dx_{j_1} \wedge \cdots \wedge dx_{j_q}$ where $J = \{j_1, \dots, j_q\}$ with $j_1 < \cdots < j_q$.
When n is understood we denote $\Omega^q_n = \Omega^q$.
Ω^q_n is a graded S_n-module with homogeneous piece of degree d defined by $\Omega^q_n(d) = \{\sum_{|J|=q} a_J\ dx_J,\ a_J \in S_n(d-q)\}$.
In particular, dx_i is homogeneous of degree one.
The exterior derivative is an operator of degree zero, i. e. it preserves degree.

$H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$ = projective one-forms of degree d.
It follows from the Euler exact sequence that $\omega = \sum_i a_i dx_i \in \Omega^1_n(d)$ is projective if and only if it contracts to zero with the Euler or radial vector field $R = \sum_{i=0}^n x_i \frac{\partial}{\partial x_i}$, that is, if $\sum_i a_i x_i = 0$.

$\mathbb{P}^n(d) = \mathbb{P}(H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)))$.

$F_n(d) = \{\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)) / \omega \wedge d\omega = 0\}$ = the set of integrable projective one-forms in $\mathbb{P}^n$ of degree d, and
$\mathcal{F}_n(d) \subset \mathbb{P}^n(d)$ the projectivization of $F_n(d)$.

$\mathbb{P}^n(\mathbf{d}) = \mathbb{P}\Lambda(\mathbf{d}) \times \prod_{i=1}^m \mathbb{P}S_n(d_i)$.

3. Logarithmic one-forms.

1. **Definition.** *Fix natural numbers n, d and m. Let*

$$\mathbf{d} = (m; d_1, \ldots, d_m)$$

be a partition of d into m parts, that is, for $i = 1, \ldots, m$ each d_i is a natural number and $\sum_{i=1}^m d_i = d$. Let us normalize so that $d_i \geq d_{i+1}$ for all $i < m$. We denote

$$P(m, d)$$

the set of all such partitions of d into m parts.

2. **Definition.** *Fix $\mathbf{d} = (m; d_1, \ldots, d_m) \in P(m, d)$. A differential one-form $\omega \in \Omega^1_n$ is* logarithmic of type $\mathbf{d}$ *if*

$$\omega = (\prod_{j=1}^m F_j) \sum_{i=1}^m \lambda_i \ dF_i/F_i = \sum_{i=1}^m \lambda_i \ (\prod_{j \neq i} F_j) \ dF_i$$

where $F_i \in S_n(d_i)$ is a non-zero homogeneous polynomial of degree d_i and the λ_i are complex numbers.

3. **Definition.** *It will be convenient to use the following notation. For $\mathbf{d}$ and $F_i \in S_n(d_i)$ as above,*

$$\mathbf{F} = (F_1, \ldots, F_m), \qquad F = \prod_{j=1}^m F_j,$$

$$\hat{F}_i = \prod_{j \neq i} F_j = F/F_i, \qquad \hat{F}_{ij} = \prod_{k \neq i, k \neq j} F_k = F/F_iF_j, \ (i \neq j),$$

or, more generally, for a subset $A \subset \{1, \ldots, m\}$ we write

$$\hat{F}_A = \prod_{j \notin A} F_j$$

Hence a logarithmic one-form may be written

$$\omega = F \ \sum_{i=1}^m \lambda_i \ dF_i/F_i = \sum_{i=1}^m \lambda_i \ \hat{F}_i \ dF_i. \tag{3.1}$$

We denote $\hat{d}_i = \sum_{j \neq i} d_j$ the degree of $\hat{F}_i$ and, more generally, $\hat{d}_A = \sum_{j \notin A} d_j$ the degree of $\hat{F}_A$.

4. **Proposition.** *For ω a logarithmic one-form as above,*
a) ω is homogeneous of degree $d = \sum_{i=1}^m d_i$.
b) ω is integrable.
c) $< R, \omega >= (\sum_{i=1}^m d_i \lambda_i) F$. In particular, ω is projective if and only if

$$\sum_{i=1}^m d_i \lambda_i = 0.$$

Proof. a) Since the exterior derivative is of degree zero, each term in the sum $\sum_{i=1}^m \lambda_i \hat{F}_i \, dF_i$ is homogeneous of degree d, hence the claim.

b) For each polynomial G, the rational one-form dG/G is closed. It follows that $\omega/F = \sum_{i=1}^m \lambda_i \, dF_i/F_i$ is closed, hence integrable. A short calculation shows that the product of a rational function with an integrable rational one-form is an integrable rational one-form. Therefore, $\omega = F \, \omega/F$ is integrable.

c) Euler's formula implies that $< R, dG >= eG$ for $G \in S_n(e)$. By linearity of contraction we have $< R, \omega >=< R, \sum_i \lambda_i \hat{F}_i \, dF_i >= \sum_i d_i \lambda_i \hat{F}_i F_i = (\sum_i d_i \lambda_i) F$.

□

5. **Proposition.** *Suppose ω is logarithmic as in 3.1. Then,*

a) $d\omega = (dF/F) \wedge \omega = \sum_{1 \le i,j \le m} \lambda_j \hat{F}_{ij} \, dF_i \wedge dF_j = \sum_{1 \le i < j \le m} (\lambda_j - \lambda_i) \hat{F}_{ij} \, dF_i \wedge dF_j$.

b) F is an integrating factor of ω: $d(\omega/F) = 0$, or, equivalently, $F d\omega - dF \wedge \omega = 0$.

c) Each hypersurface $F_i = 0$ is an algebraic leaf of ω, that is, $dF_i/F_i \wedge \omega$ is a regular 2-form (i. e. without poles). Hence $dF_i \wedge \omega = 0$ on the hypersurface $F_i = 0$.

Proof. These follow by straightforward calculations, left to the reader. □

4. The logarithmic components and their parametrization.

As before, we fix natural numbers n, d and m and a partition $\mathbf{d} = (m; d_1, \dots, d_m)$ of d.

For a complex vector space V we denote $\mathbb{P}V = V - \{0\}/\mathbb{C}^*$ the corresponding projective space of one-dimensional subspaces of V. Let $\pi : V - \{0\} \to \mathbb{P}V$ be the canonical projection. If $X \subset V$ we call $\mathbb{P}X = \pi(X - \{0\}) \subset \mathbb{P}V$ the projectivization of X.

As in Section 2, we denote

$$\mathbb{P}^n(d) = \mathbb{P}H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$$

the projective space of sections of $\Omega^1_{\mathbb{P}^n}(d)$. This is the ambient projective space that contains the set of integrable forms $\mathcal{F}_n(d)$ and the logarithmic components that we will investigate.

6. **Definition.** *Let $L_n(\mathbf{d}) \subset H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$ denote the set of all logarithmic projective one-forms of type $\mathbf{d}$ in $\mathbb{P}^n$, and $\mathbb{P}L_n(\mathbf{d}) \subset \mathbb{P}^n(d)$ its projectivization. We denote*

$$\mathcal{L}_n(\mathbf{d}) \subset \mathbb{P}^n(d)$$

the Zariski closure of $\mathbb{P}L_n(\mathbf{d})$.

If ω is a non-zero logarithmic form, the corresponding projective point $\pi(\omega)$ will be denoted simply by ω when the danger of confusion is small.

Let

$$\Lambda(\mathbf{d}) = \{(\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m / \sum_{i=1}^m d_i \lambda_i = 0\}$$

which is a hyperplane in $\mathbb{C}^m$.

7. **Definition.** *Consider the map*

$$\mu : V_n(\mathbf{d}) := \Lambda(\mathbf{d}) \times \prod_{i=1}^{m} S_n(d_i) \to H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$$

such that

$$\mu((\lambda_1, \dots, \lambda_m), (F_1, \dots, F_m)) = \sum_{i=1}^{m} \lambda_i \ \hat{F}_i \ dF_i$$

and

$$\rho : \mathbb{P}^n(\mathbf{d}) := \mathbb{P}\Lambda(\mathbf{d}) \times \prod_{i=1}^{m} \mathbb{P}S_n(d_i) \dashrightarrow \mathbb{P}^n(d) = \mathbb{P}H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$$

such that

$$\rho(\pi(\lambda_1, \dots, \lambda_m), (\pi(F_1), \dots, \pi(F_m))) = \pi(\sum_{i=1}^{m} \lambda_i \ \hat{F}_i \ dF_i).$$

8. **Remark.** *a) μ is a multi-linear map. By Proposition 4, the image of μ is $L_n(\mathbf{d})$.*
b) The induced map ρ from a product of projective spaces into a projective space is only a rational map. Later we will determine the base locus $B(\rho) = \{(\pi(\lambda), \pi(F))/\mu(\lambda, F) = 0\}$ of ρ. Anyway, it is clear that the image of ρ is $\mathbb{P}L_n(\mathbf{d})$. Hence $\mathcal{L}_n(\mathbf{d})$ is the closure of the image of ρ. Therefore, $\mathcal{L}_n(\mathbf{d})$ is a projective irreducible variety.

5. Base locus.

Let $B(\mu) = \mu^{-1}(0)$. Then $B(\mu) \subset V_n(\mathbf{d})$ is an affine algebraic set, and we intend to describe its irreducible components.

Let us remark that the multilinearity of μ implies that $B(\mu)$ is stable under the natural action of $(\mathbb{C}^*)^{m+1}$ on $V_n(\mathbf{d})$.

From the multilinearity of μ it follows that $Z = \{(\lambda, \mathbf{F}) \in V_n(\mathbf{d})/\lambda = 0 \text{ or } F_i = 0 \text{ for some } i\}$ is contained in $B(\mu)$. We denote $B = B(\mu) - Z$ and

$$B(\rho) = \pi(B) \subset \mathbb{P}^n(\mathbf{d})$$

the base locus of ρ.

An example of a point in the base locus is the following. Suppose $d_1 = \cdots = d_m$. It is then clear that if $F_1 = \cdots = F_m$ then $(\lambda, \mathbf{F}) \in B(\mu)$. More generally, each string of equal d_i's gives elements of $B(\mu)$: if $d_i = d_j$ for all $i, j \in A$, where $A \subset \{1, \dots, m\}$, then taking $F_i = F_j$ for all $i, j \in A$, $\sum_{i \in A} d_i \lambda_i = 0$, $\lambda_j = 0$ for $j \notin A$, we obtain that $(\lambda, \mathbf{F}) \in B(\mu)$.

These examples generalize as follows: suppose our d_i's may be written as

$$d_i = \sum_{j=1}^{m'} e_{ij} d'_j, \quad i = 1, \dots, m, \tag{5.1}$$

where $m' \in \mathbb{N}$, $d'_j \geq 1$ and $e_{ij} \geq 0$ are integers. Let $\lambda \in \Lambda_n(\mathbf{d})$ such that $\sum_{i=1}^{m} e_{ij}\lambda_i = 0$ for $j = 1, \ldots, m'$, and take $\mathbf{F}$ such that

$$F_i = \prod_{j=1}^{m'} G_j^{e_{ij}} \tag{5.2}$$

for some $G_j \in S_n(d'_j)$, $j = 1, \ldots, m'$. Then,

$$\sum_{i=1}^{m} \lambda_i \; dF_i/F_i = \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{m'} e_{ij} \; dG_j/G_j = \sum_{j=1}^{m'} (\sum_{i=1}^{m} \lambda_i e_{ij}) \; dG_j/G_j = 0 \tag{5.3}$$

and we obtain elements in the base locus.

We will see now that this construction accounts for all the irreducible components of the base locus.

9. **Definition.** *We denote $F(\mathbf{d})$ the collection of all decompositions of $\mathbf{d}$ as in 5.1, that is, let*

$F(\mathbf{d}) = \{(m', e, \mathbf{d}')/ \; m' \in \mathbb{N}, \; e \in \mathbb{N}^{m \times m'}, \; \mathbf{d}' \in (\mathbb{N}-\{0\})^{m'}, \; \mathbf{d} = e\,\mathbf{d}', \; e$ without zero columns $\}$

In 5.1, for each i there exists j such that $e_{ij} > 0$; that is, all rows of e are non-zero. This follows from $d_i > 0$. If the j-th column of e is zero then in the decomposition 5.1 the terms $e_{ij}d'_j$ are zero and do not contribute, so this zero column may be disregarded.

Let us remark that $F(\mathbf{d})$ is finite: we have, $d = \sum_i d_i = \sum_{i,j} e_{ij}d'_j \geq \sum_j d'_j \geq m'$, hence m' is bounded. Also, 5.1 implies $e_{ij} \leq d_i/d'_j \leq d_i$, so all e_{ij} are also bounded.

For $\varphi = (m', e, \mathbf{d}') \in F(\mathbf{d})$ denote the (Segre-Veronese) map

$$\nu_\varphi : \prod_{j=1}^{m'} S_n(d'_j) \to \prod_{i=1}^{m} S_n(d_i)$$

$$\nu_\varphi(G_1, \ldots, G_{m'}) = (F_1, \ldots, F_m)$$

such that $F_i = \prod_{j=1}^{m'} G_j^{e_{ij}}$. Also, let

$$\Lambda(e) = \{\lambda \in \Lambda(\mathbf{d}) / \lambda \; e = 0\}$$

which is a linear subspace of $\mathbb{C}^m$ of dimension $m - \operatorname{rank}(e)$.
Notice that $\lambda \; e = 0$ implies $\lambda \; \mathbf{d} = 0$. For $\varphi \in F(\mathbf{d})$ let

$$B_\varphi = \Lambda(e) \times \operatorname{im} \nu_\varphi \subset V_n(\mathbf{d})$$

By the calculation 5.3 we know that $B_\varphi \subset B(\mu)$ for all $\varphi \in F(\mathbf{d})$.
Each B_φ is clearly irreducible. Next we will see, first, that $B(\mu) = Z \cup \bigcup_{\varphi \in F(\mathbf{d})} B_\varphi$. And, second, we will determine when there are inclusions among the B_φ's, thus characterizing the irreducible components of the base locus.
Let us first recall from [14], Lemme 3.3.1, page 102, the following

10. **Proposition.** *Let $F_i \in S_n(d_i)$, $i = 1, \ldots, m$, be irreducible distinct (modulo multiplicative constants) homogeneous polynomials. If $\lambda_i \in \mathbb{C}$ are such that*

$$\sum_{i=1}^{m} \lambda_i \; dF_i/F_i = 0$$

then $\lambda_i = 0$ for all i. That is, the rational one-forms $dF_1/F_1, \ldots, dF_m/F_m$ are linearly independent over $\mathbb{C}$.

11. **Corollary.** *Let $(\lambda, \mathbf{F}) \in V_n(\mathbf{d})$ with the F_i distinct and irreducible, and $\lambda \neq 0$. Then $(\lambda, \mathbf{F}) \notin B(\mu)$.*

12. **Proposition.** *With the notations above, we have $B(\mu) = Z \cup \bigcup_{\varphi \in F(\mathbf{d})} B_\varphi$.*

Proof. Let $(\lambda, \mathbf{F}) \in B = B(\mu) - Z$. Write each F_i as a product of distinct irreducible homogeneous polynomials:

$$F_i = \prod_{j=1}^{m'} G_j^{e_{ij}}$$

We allow some $e_{ij} = 0$. Denote d'_j the degree of G_j. Taking degree we obtain $\mathbf{d} = e\ \mathbf{d}'$. Repeating the calculation of 5.3 we have

$$0 = \sum_{i=1}^{m} \lambda_i \; dF_i/F_i = \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{m'} e_{ij} \; dG_j/G_j = \sum_{j=1}^{m'} (\sum_{i=1}^{m} \lambda_i e_{ij}) \; dG_j/G_j \tag{5.4}$$

Since the G_j are irreducible, Proposition 10 implies that $\sum_{i=1}^{m} \lambda_i e_{ij} = 0$ for all $j = 1, \ldots, m'$. Therefore, $(\lambda, \mathbf{F}) \in B_\varphi$ with $\varphi = (m', e, \mathbf{d}') \in F(\mathbf{d})$, as claimed. □

Regarding possible inclusions among the B_φ's, we make the following

13. **Definition.** *For $\varphi_1 = (m_1, e_1, \mathbf{d}_1)$, $\varphi_2 = (m_2, e_2, \mathbf{d}_2) \in F(\mathbf{d})$ we write $\varphi_2 \leq \varphi_1$ if* $\operatorname{rank}(e_1) = \operatorname{rank}(e_2)$ *and there exists $e_3 \in \mathbb{N}^{m_1 \times m_2}$ such that $e_2 = e_1\ e_3$.*

Then we have

14. **Proposition.** *For $\varphi_1, \varphi_2 \in F(\mathbf{d})$, $B_{\varphi_2} \subset B_{\varphi_1}$ if and only if $\varphi_2 \leq \varphi_1$.*

Proof. Suppose $B_{\varphi_2} \subset B_{\varphi_1}$. Choose an element $(\lambda, \mathbf{F}) \in B_{\varphi_2}$, that is, $\lambda\ e_2 = 0$ and $F_i = \prod_{k=1}^{m_2} H_k^{e_{2ik}}$ for all i, for some H_k. We may take this element so that the H_k's are irreducible. By our hypothesis, $(\lambda, \mathbf{F}) \in B_{\varphi_1}$ and we also have $F_i = \prod_{j=1}^{m_1} G_j^{e_{1ij}}$ for all i, for some G_j. By unique factorization and the irreducibility of the H_k, $G_j = \prod_{k=1}^{m_2} H_k^{e_{3jk}}$ for some $e_{3jk} \in \mathbb{N}$. A simple calculation now gives $e_2 = e_1\ e_3$.

Also, the equality $e_2 = e_1\ e_3$ just obtained easily implies $\Lambda(e_1) \subset \Lambda(e_2)$. Since we are assuming $B_{\varphi_2} \subset B_{\varphi_1}$, we also have $\Lambda(e_2) \subset \Lambda(e_1)$. Hence $\Lambda(e_1) = \Lambda(e_2)$, and therefore $\operatorname{rank}(e_1) = \operatorname{rank}(e_2)$.

Conversely, suppose $\varphi_2 \leq \varphi_1$. Then $e_2 = e_1\ e_3$ and $\operatorname{rank}(e_1) = \operatorname{rank}(e_2)$ imply, as before, that $\Lambda(e_1) = \Lambda(e_2)$. Also, the condition $e_2 = e_1\ e_3$ easily implies that $\operatorname{im} \nu_{\varphi_2} \subset \operatorname{im} \nu_{\varphi_1}$. Hence $B_{\varphi_2} \subset B_{\varphi_1}$. □

15. **Corollary.** *The irreducible components of* $B(\rho)$ *are the* $\pi(B_\varphi)$ *for* φ *a maximal element of the finite ordered set* $(F(\mathbf{d}), \leq)$.

6. Generic injectivity.

Suppose $(\lambda, \mathbf{F}), (\lambda', \mathbf{F}') \in V_n(\mathbf{d})$ are such that $\mu(\lambda, \mathbf{F}) = \mu(\lambda', \mathbf{F}') \neq 0$, that is,

$$F \sum_{i=1}^{m} \lambda_i \, dF_i/F_i = \omega = F' \sum_{i=1}^{m} \lambda'_i \, dF'_i/F'_i.$$

Next we discuss conditions that imply that $(\lambda, \mathbf{F}) = (\lambda', \mathbf{F}')$.

Let's observe that if the partition $\mathbf{d}$ contains repeated d_i *'s* then the generic injectivity may hold only *up to order*. More precisely, suppose $A \subset \{1, \dots, m\}$ is such that $d_i = d_j$ for all $i, j \in A$. For each permutation $\sigma \in \mathbb{S}_m$ such that $\sigma(j) = j$ for $j \notin A$, clearly we have $\mu(\lambda, \mathbf{F}) = \mu(\sigma.\lambda, \sigma.\mathbf{F})$ for all $(\lambda, \mathbf{F}) \in V_n(\mathbf{d})$. For $e \in \mathbb{N}$ let $A_e = \{i/d_i = e\}$. Then the non-empty A_e form a partition of $\{1, \dots, m\}$. Let $\mathbb{S}(e) = \{\sigma \in \mathbb{S}_m/\sigma(j) = j, \forall j \notin A_e\}$ and $\mathbb{S}(\mathbf{d}) = \prod_e \mathbb{S}(e)$. Then the subgroup $\mathbb{S}(\mathbf{d}) \subset \mathbb{S}_m$ acts on $V_n(\mathbf{d})$ and μ is constant on its orbits. By injectivity *up to order* we will of course mean injectivity of the induced map with domain $V_n(\mathbf{d})/\mathbb{S}(\mathbf{d})$.

16. **Proposition.** *The rational map*

$$\rho : \mathbb{P}^n(\mathbf{d}) \dashrightarrow \mathcal{L}_n(\mathbf{d}) \subset \mathbb{P}^n(d)$$

as in Definition 7, is generically injective (up to order).

Proof. We will prove the existence of a non-empty Zariski open $U \subset X$ such that $\rho|_U$ is injective morphism (up to order). It is easy to see, using that ρ is a dominant map of irreducible varieties, that the existence of such a U implies that there exists a non-empty Zariski open $V \subset \mathcal{L}_n(\mathbf{d})$ such that $\rho : \rho^{-1}(V) \to V$ is injective (up to order).

Consider the Zariski open $\mathbb{S}(\mathbf{d})$-stable $U \subset V_n(\mathbf{d})$ of points $(\lambda, \mathbf{F})$ such that the F_i are irreducible and all distinct. Hence, for $(\lambda, \mathbf{F}), (\lambda', \mathbf{F}') \in U$ distinct (up to order), $F = \prod_i F_i \neq F' = \prod_i F'_i$. Suppose $\mu(\lambda, \mathbf{F}) = \omega = \mu(\lambda', \mathbf{F}') \neq 0$. Then ω has two integrating factors F and F', and therefore has a rational first integral $f = F/F'$. It follows that ω has infinitely many algebraic leaves (the fibers of f).

On the other hand, if $(\lambda_1 : \dots : \lambda_m) \in \mathbb{P}^{m-1}(\mathbb{C}) - \mathbb{P}^{m-1}(\mathbb{Q})$, Proposition (3.7.8) from [14] implies that ω has only finitely many algebraic leaves.

Let $U_0 = \{(\lambda, \mathbf{F}) \in U/\lambda \in \mathbb{P}^{m-1}(\mathbb{C}) - \mathbb{P}^{m-1}(\mathbb{Q})\}$.

Consider the restriction $\rho : U \to \mathcal{L}_n(\mathbf{d})$ and $\tilde{\rho} : U/\mathbb{S}(\mathbf{d}) \to \mathcal{L}_n(\mathbf{d})$ the induced map.

We obtain that if $\omega = \mu(\lambda, \mathbf{F})$ with $(\lambda, \mathbf{F}) \in U_0$ then $\tilde{\rho}^{-1}(\omega) = \{(\lambda, \mathbf{F})\}$.

This implies, first, that since ρ has a fiber of dimension zero, $\dim(U) = \dim(\mathcal{L}_n(\mathbf{d}))$ and the general fiber of ρ is finite. Also, since the (open analytic) set U_0 is Zariski dense in U (because $\mathbb{C} - \mathbb{Q}$ is dense in $\mathbb{C}$), U_0 is not contained in the branch divisor of $\tilde{\rho}$ and hence $\tilde{\rho}$ has degree one, and therefore is birational, as claimed.

□

7. Derivative of the parametrization.

With the notation of Definition 7, let

$$(\lambda, \mathbf{F}) = ((\lambda_1, \dots, \lambda_m), (F_1, \dots, F_m)) \in V_n(\mathbf{d})$$

be a point in the vector space $V_n(\mathbf{d})$ domain of μ.

Let $(\lambda', \mathbf{F}') = ((\lambda'_1, \dots, \lambda'_m), (F'_1, \dots, F'_m)) \in V_n(\mathbf{d})$ represent a tangent vector

$$(\lambda, \mathbf{F}) + \epsilon(\lambda', \mathbf{F}'), \quad \epsilon^2 = 0,$$

to $V_n(\mathbf{d})$ at $(\lambda, \mathbf{F})$.

From the multilinearity of μ we easily obtain the following formula for its derivative:

$$d\mu(\lambda, \mathbf{F}) : V_n(\mathbf{d}) \to H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$$

$$d\mu(\lambda, \mathbf{F})(\lambda', \mathbf{F}') = \sum_i \lambda'_i \; \hat{F}_i \; dF_i + \sum_{i \neq k} \lambda_i \; F'_k \; \hat{F}_{ik} \; dF_i + \sum_i \lambda_i \; \hat{F}_i \; dF'_i \tag{7.1}$$

17. **Remark.** *By Proposition 4 b), the image of μ is contained in the variety of integrable projective forms $F_n(d) \subset H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$. Hence for each $(\lambda, \mathbf{F}) \in V_n(\mathbf{d})$ we have an inclusion of vector spaces*

$$\operatorname{im} d\mu(\lambda, \mathbf{F}) \subset T_{F_n(d)}(\omega) = \{\alpha \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)) / \; \omega \wedge d\alpha + \alpha \wedge d\omega = 0\} \tag{7.2}$$

where $\omega = \mu(\lambda, \mathbf{F})$ and $T_{F_n(d)}(\omega)$ denotes de tangent space of $F_n(d)$ at the point ω.

Our main task in Section 9 will be to show that this inclusion is actually an equality, for a sufficiently general $(\lambda, \mathbf{F}) \in V_n(\mathbf{d})$.

18. **Definition.** *It is convenient now to introduce the following notation:*

$\omega = \mu(\lambda, \mathbf{F}) = \sum_{i=1}^m \lambda_i \; \hat{F}_i \; dF_i$ (a logarithmic one-form),

$\eta = \omega/F = \sum_{i=1}^m \lambda_i \; dF_i/F_i$ (the corresponding rational logarithmic one-form),

$\alpha = d\mu(\lambda, \mathbf{F})(\lambda', \mathbf{F}') = \sum_i \lambda'_i \; \hat{F}_i \; dF_i + \sum_{i \neq k} \lambda_i \; F'_k \; \hat{F}_{ik} \; dF_i + \sum_i \lambda_i \; \hat{F}_i \; dF'_i$,

$\beta = \alpha/F = \sum_i \lambda'_i \; dF_i/F_i + \sum_{i \neq k} \lambda_i \; F'_k/F_k \; dF_i/F_i + \sum_i \lambda_i \; dF'_i/F_i$.

19. **Proposition.** *With the notations above, we have*

$$\beta = \eta' + (G/F)\eta + d(H/F)$$

where

$\eta' = \sum_{i=1}^m \lambda'_i \; dF_i/F_i$,

$G = \sum_{i=1}^m \hat{F}_i \; F'_i \in S_n(d)$, *and*

$H = \sum_{i=1}^m \lambda_i \; \hat{F}_i \; F'_i \in S_n(d)$.

Proof. We add and substract to β the sum $\sum_i \lambda_i \; {F'}_i/F_i^2 \; dF_i$. A straightforward calculation gives the proposed expression. □

8. Singular ideals of logarithmic one-forms and their resolution.

For $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$ denote $S(\omega) \subset \mathbb{P}^n$ the scheme of zeros of ω and $\mathcal{I} = \mathcal{I}_\omega \subset \mathcal{O}_{\mathbb{P}^n}$ the corresponding ideal sheaf. Considering ω as a morphism $\mathcal{O}_{\mathbb{P}^n} \to \Omega^1_{\mathbb{P}^n}(d)$, $\mathcal{I}$ is defined as the image of the dual morphism $T_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n}$. Also, if $\omega = \sum_{i=0}^n a_i dx_i$ then $\mathcal{I}$ corresponds to the homogeneous ideal generated by $a_0, \ldots, a_n \in S_n(d-1)$.

We keep the notation of Definitions 2 and 3.

Let $(\lambda, \mathbf{F}) \in V_n(\mathbf{d})$ and $\omega = F. \sum_{i=1}^m \lambda_i \; dF_i/F_i = \sum_{i=1}^m \lambda_i \; \hat{F}_i \; dF_i$ the corresponding logarithmic one-form.

We denote

$$X_i = \{x \in \mathbb{P}^n / F_i(x) = 0\}$$

the hipersurface defined by F_i.

For $i \neq j$,

$$X_{ij} = X_i \cap X_j = \{x \in \mathbb{P}^n / F_i(x) = F_j(x) = 0\}$$

and, more generally, for a subset $A \subset \{1, \ldots, m\}$,

$$X_A = \bigcap_{i \in A} X_i$$

For $1 \leq r \leq m$ we write

$$X^{(r)} = \bigcup_{|A|=r} X_A$$

and we shall use especially the following particular cases

$$X^{(1)} = \bigcup_{i=1}^m X_i, \qquad X^{(2)} = \bigcup_{i<j} X_{ij}, \qquad X^{(3)} = \bigcup_{i<j<k} X_{ijk}.$$

20. **Remark.** *For our purposes we will be able to assume that the $F_i \in S_n(d_i)$ are general. We shall assume, more precisely, that each F_i is smooth irreducible and that $X^{(1)}$ is a normal crossings divisor. Hence, each X_A is a smooth complete intersection of codimension $|A|$, and thus the strata $X^{(r)}$ are of codimension r, singular only along $X^{(r+1)}$.*

It is shown in [8] and [3] that for ω logarithmic as above, with all $\lambda_i \neq 0$,

$$S(\omega) = X^{(2)} \cup P$$

with $P \subset \mathbb{P}^n - X^{(1)}$ closed, and P is a finite set if ω is general. Let's revisit the argument, under the assumptions of Remark 20. First, since clearly $\hat{F}_i$ vanishes on $X^{(2)}$ for all i, we have $X^{(2)} \subset S(\omega)$. Since $\omega = \lambda_i \hat{F}_i dF_i$ on X_i, we see that $(X^{(1)} - X^{(2)}) \cap S(\omega) = \emptyset$. As for the zeros of ω in the complement of $X^{(1)}$, they are the same as the zeros of $\eta = \omega/F = \sum_{i=1}^m \lambda_i \, dF_i/F_i$, which is a section of the locally free sheaf $E = \Omega^1_{\mathbb{P}^n}(\log X^{(1)})$ of rank n (see [9], [12], [15], [11]). Considering the F_i (hence the divisor $X^{(1)}$) as fixed, the space of global sections of E has dimension $m-1$, and these sections correspond bijectively with the residues $(\lambda_1, \ldots, \lambda_m)$, satisfying $\sum_i d_i \lambda_i = 0$, as it follows from taking cohomology in the exact sequence ([9] or [11], p. 170):

$$0 \to \Omega^1_{\mathbb{P}^n} \to E \to \oplus_{i=1}^m \mathcal{O}_{X_i} \to 0.$$

For general $(\lambda_1, \ldots, \lambda_m)$ as above, the corresponding section η of E has a finite set P of simple zeros. Further, the cardinality of P (see [8]) is the degree of the top Chern class $c_n(E)$, computable from the exact sequence above.

Coming back to the study of the resolution of the ideal $\mathcal{I}_\omega$, let us denote

$$\mathcal{J}^{(r)} = \mathcal{I}(X^{(r)}) \subset \mathcal{O}_{\mathbb{P}^n}$$

the ideal sheaf of regular functions vanishing on $X^{(r)}$, and

$$J^{(r)} = \bigoplus_{k \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{J}^{(r)}(k)) \subset S_n$$

the corresponding saturated homogeneous ideal.

Our arguments to prove stability of logarithmic forms will rely on the following results regarding the ideals $J^{(2)}$.

21. **Proposition.** *Under the hypothesis of Remark 20,*

a) $J^{(2)}$ *is generated by* $\{\hat{F}_i,\ 1 \leq i \leq m\}$.

b) The relations among the generators of a) are generated by

$$F_j\ \hat{F}_j - F_i\ \hat{F}_i, \quad 1 \leq i < j \leq m,$$

and also by the subset

$$R_j = F_j\ \hat{F}_j - F_1\ \hat{F}_1, \quad 2 \leq j \leq m.$$

c) We have a resolution of $\mathcal{J}^{(2)}$

$$0 \to \mathcal{O}(-d)^{m-1} \xrightarrow{\delta_0} \bigoplus_{1 \leq i \leq m} \mathcal{O}(-\hat{d}_i) \xrightarrow{\delta_1} \mathcal{J}^{(2)} \to 0$$

where, denoting $\{e_i\}$ *the respective canonical basis,*

$$\delta_0(e_j) = F_j\ e_j - F_1\ e_1 \quad \text{for} \quad 2 \leq j \leq m,$$

$$\delta_1(e_i) = \hat{F}_i \quad \text{for} \quad 1 \leq i \leq m.$$

Proof. a) We are assuming that the F_i are generic. This implies in particular that each ideal $< F_i, F_j >$ is prime. Then, $J^{(2)} = \bigcap_{1 \leq i < j \leq m} < F_i, F_j >$. Let us denote $J =< \hat{F}_1, \ldots, \hat{F}_m >$. It is clear that $J \subset J^{(2)}$. We shall prove that $J^{(2)} \subset J$ by induction on m. The case $m = 2$ is trivial. The inductive hypothesis, applied to $F_1, \ldots, F_{m-1}$, may be written as $\bigcap_{1 \leq i < j \leq m-1} < F_i, F_j > \subset < \hat{F}_{1m}, \ldots, \hat{F}_{m-1m} >$. Take an element $G \in \bigcap_{1 \leq i < j \leq m} < F_i, F_j > = \bigcap_{1 \leq i < j \leq m-1} < F_i, F_j > \cap \bigcap_{1 \leq i < m} < F_i, F_m >$. Using the inductive hypothesis, we may write $G = \sum_{i<m} a_i \hat{F}_{im}$, and we also have $G \in < F_i, F_m >$ for $i < m$. Since $\hat{F}_{jm} \in < F_i, F_m >$ for $j \neq i$, it follows that $a_i \hat{F}_{im} \in < F_i, F_m >$ for $i < m$.

Since $< F_i, F_m >$ is prime, we have $a_i = b_iF_i + c_iF_m$. Then, $G = \sum_{i<m}(b_iF_i + c_iF_m)\hat{F}_{im} = \sum_{i<m}(b_i\hat{F}_m + c_i\hat{F}_i) \in J$, as wanted.

b) and c) Using the relations R_j of b) we write down the complex in c). The proof will be complete if we show that this complex is exact. The surjectivity of δ_1 follows from a). Looking at the matrix of δ_0 it is easy to see that the determinant of the minor obtained by removing row j is precisely $\hat{F}_j$, for $j = 1, \dots, m$. Then this complex is the one associated to the maximal minors of a matrix of size $m \times m - 1$. Since in our case, by a), the ideal of minors vanishes in codimension two, the complex is exact (see [1] (5), [10] (20.4)). □

22. **Remark.** *Let X be an algebraic variety, $\mathcal{J} \subset \mathcal{O}_X$ a sheaf of ideals, and E a locally free sheaf on X. Let $Y \subset X$ denote the subvariety corresponding to $\mathcal{J}$. Taking global sections on the exact sequence $0 \to E \otimes \mathcal{J} \to E \to E \otimes \mathcal{O}_Y = E|_Y \to 0$ we obtain an identification of $H^0(X, E \otimes \mathcal{J})$ with the global sections of E vanishing on Y, that is, with the kernel of the restriction map $H^0(X, E) \to H^0(Y, E|_Y)$.*

23. **Proposition.** *Let $\alpha \in \Omega^1_n(d)$ be a 1-form of degree d in $\mathbb{C}^{n+1}$. Denote $\tilde{X}^{(2)} \subset \mathbb{C}^{n+1}$ the cone over $X^{(2)}$.*

a) α vanishes on $\tilde{X}^{(2)}$ if and only if it may be written as

$$\alpha = \sum_{i=1}^{m} \hat{F}_i \alpha_i$$

for some $\alpha_i \in \Omega^1_n(d_i)$.

b) α is projective (see Section 2) and vanishes on $X^{(2)}$ if and only if it may be written as

$$\alpha = \sum_{i=1}^{m} \lambda'_i \hat{F}_i dF_i + \sum_{i=1}^{m} \hat{F}_i \gamma_i$$

where $\lambda'_i \in \mathbb{C}$, $\sum_{i=1}^m d_i\lambda'_i = 0$ and $\gamma_i \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d_i))$ are projective 1-forms of respective degrees d_i.

Proof. a) By Remark 22, we need to determine $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d) \otimes \mathcal{J}^{(2)})$. The stated result then follows from Proposition 21 c), by tensoring with $\Omega^1_{\mathbb{P}^n}(d)$ and taking global sections.
b) Suppose α is also projective, that is, $< R, \alpha >= 0$, where R is the radial vector field. From a) we have

$$\sum_{i=1}^{m} \hat{F}_i < R, \alpha_i >= 0.$$

This is a relation among the $\hat{F}_i$ with coefficients $< R, \alpha_i >$ homogeneous of degrees d_i. By Proposition 21 c), by tensoring with $\mathcal{O}_{\mathbb{P}^n}(d)$ and taking global sections, this relation is a linear combination of the relations R_i of Proposition 21 b), that is,

$$(< R, \alpha_1 >, \ldots, < R, \alpha_m >) = \sum_{2 \leq i \leq m} a_i R_i.$$

This means that

$$< R, \alpha_1 >= (\sum_j a_j) F_1, \quad < R, \alpha_i >= -a_i F_i, \quad i = 2, \ldots, m.$$

Hence a_i has degree zero, i. e. $a_i \in \mathbb{C}$, for all i. Define $\lambda'_i = a_i/d_i$ for $i = 2, \ldots, m$, $\lambda'_1 = -(\sum_j a_j)/d_1$ and $\gamma_i = \alpha_i - \lambda'_i dF_i$. It follows that $< R, \gamma_i >= 0$ and hence α may be written as stated. □

9. Surjectivity of the derivative and main Theorem.

As in Remark 17 we denote the derivative of μ at the point $\mu(\lambda, \mathbf{F})$

$$d\mu(\lambda, \mathbf{F}) : V_n(\mathbf{d}) \to T(\omega) \tag{9.1}$$

where $\omega = \mu(\lambda, \mathbf{F})$ and

$$T(\omega) = T_{F_n(d)}(\omega) = \{\alpha \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)) / \ \omega \wedge d\alpha + \alpha \wedge d\omega = 0\} \tag{9.2}$$

denotes the Zariski tangent space of $F_n(d)$ at the point ω.

Our main objective is to prove the following:

24. **Theorem.** *Let n, d, m and $\mathbf{d} \in P(m, d)$ be as in Definition 1. Suppose $n \geq 3$. Then the derivative $d\mu(\lambda, \mathbf{F}) : V_n(\mathbf{d}) \to T(\omega)$ is surjective for $(\lambda, \mathbf{F}) \in V_n(\mathbf{d})$ general.*

Proof. The proof will be obtained through various steps, including several Propositions of independent interest. □

25. **Theorem.** *If $n \geq 3$, the set of logaritmic forms $\mathcal{L}_n(\mathbf{d}) \subset \mathcal{F}_n(d)$, as in Definition 6, is an irreducible component of $\mathcal{F}_n(d)$. Furthermore, the scheme $\mathcal{F}_n(d)$ is* reduced *generically along $\mathcal{L}_n(\mathbf{d})$.*

Proof. Follows from Theorem 24 by the same arguments as in [6] or [7]. □

Let us now start with several steps towards the proof of Theorem 24.

26. **Remark.** *A typical element α in the image of $d\mu(\lambda, \mathbf{F})$ as in 7.1*

$$\alpha = \sum_i \lambda'_i \ \hat{F}_i \ dF_i + \sum_{i \neq j} \lambda_i \ F'_j \ \hat{F}_{ij} \ dF_i + \sum_i \lambda_i \ \hat{F}_i \ dF'_i$$

may be written

$$\alpha = \sum_i \hat{F}_i \ (\lambda'_i \ dF_i + \lambda_i \ dF'_i) + \sum_{i \neq j} \lambda_i \ F'_j \ \hat{F}_{ij} \ dF_i$$

or

$$\alpha = \sum_i \hat{F}_i \ (\lambda'_i \ dF_i + \lambda_i \ dF'_i) + \sum_{i<j} \hat{F}_{ij} \ (\lambda_i \ F'_j \ dF_i + \lambda_j \ F'_i \ dF_j)$$

Let us observe that the first sum is zero on $X^{(2)}$ (hence on $X^{(3)}$) and the second sum is zero on $X^{(3)}$. The idea of our proofs, leading to Theorem 24, will be based on this observation.

Our strategy to characterize the elements $\alpha \in T(\omega)$ will be this: first we shall determine $\alpha|_{X^{(3)}}$, next we shall determine $\alpha|_{X^{(2)}}$, and finally we show that α may be written as in 7.1 for some λ' and $\mathbf{F}'$, and therefore α belongs to the image of $d\mu(\lambda, \mathbf{F})$.

In order to carry out this plan, let us start with some Propositions, some of them of independent interest.

27. **Proposition.** *For $\omega \in F_n(d)$ and $\alpha \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$, the following conditions are equivalent:*

a) $\omega \wedge d\alpha + \alpha \wedge d\omega = 0$, that is, $\alpha \in T(\omega)$.

b) $d\omega \wedge d\alpha = 0$.

Further, for ω logarithmic, *$\eta = \omega/F$ and $\beta = \alpha/F$,*

c) $\eta \wedge d\beta = 0$.

d) $d(\eta \wedge \beta) = 0$.

Proof. From a) one obtains b) by applying exterior derivative. Conversely, from b) one obtains a) by contracting with the radial vector field. The equivalence with c) follows from Proposition 5 by a straightforward calculation. The equivalence of c) and d) follows from the fact that η is closed. □

28. **Proposition.** *Let $\omega = \mu(\lambda, \mathbf{F})$ be a logarithmic form and $\alpha \in T(\omega)$. Assume that $X^{(1)}$ is normal crossings, with smooth irreducible components X_i, as in Remark 20. Then $\alpha|_{X^{(3)}} = 0$, that is, $\alpha(x) = 0$ for all $x \in X^{(3)}$.*

Proof. Let us denote, for $1 \le i < j \le m$,

$$U_{ij} := X_{ij} - X^{(3)} = \{x \in \mathbb{P}^n / F_i(x) = F_j(x) = 0, \ F_k(x) \neq 0 \text{ for } k \notin \{i, j\}\}$$

and, similarly, for $1 \le i < j < k \le m$,

$$U_{ijk} := X_{ijk} - X^{(4)}$$

Since the set of zeros of α is closed, it is enough to see that α is zero on $X^{(3)} - X^{(4)}$, which is the disjoint union of the U_{ijk}. Notice that dF_i, dF_j, dF_k are linearly independent on U_{ijk} because of the normal-crossings hypothesis. Since clearly $\omega|_{X^{(2)}} = 0$, the relation $\omega \wedge d\alpha + \alpha \wedge d\omega = 0$ reduces to $\alpha(x) \wedge d\omega(x) = 0$ for each $x \in X^{(2)}$. We may assume that $\lambda_i \neq \lambda_j$ for $i \neq j$ without losing generality. Then it follows from Proposition 5 a) that

$$\alpha \wedge dF_i \wedge dF_j = 0 \tag{9.3}$$

on U_{ij}, and hence on its closure X_{ij}. This means that

$$\alpha(x) \in \mathbb{C}.dF_i(x) + \mathbb{C}.dF_j(x) \subset \Omega^1_{\mathbb{P}^n}(x) \tag{9.4}$$

for $x \in X_{ij}$. Therefore, for $x \in U_{ijk}$ we have

$$\alpha(x) \in (\mathbb{C}.dF_i(x) + \mathbb{C}.dF_j(x)) \cap (\mathbb{C}.dF_i(x) + \mathbb{C}.dF_k(x)) \cap (\mathbb{C}.dF_j(x) + \mathbb{C}.dF_k(x)).$$

Due to the normal crossings hypothesis this last intersection of two-dimensional subspaces is zero, hence $\alpha(x) = 0$ for $x \in U_{ijk}$, as wanted. □

29. **Proposition.** *With the notation and hypothesis of Proposition 28, for each ordered pair (i, j) with $1 \leq i, j \leq m$ and $i \neq j$, there exists $A_{ij} \in S_n(d_j)$ such that*

$$\alpha = \hat{F}_{ij}\ (A_{ij}\ dF_i + A_{ji}\ dF_j) \text{ on } X_{ij}.$$

Proof. This will follow easily combining that X_{ij} is a smooth complete intersection of codimension two in a proyective space, and the fact that $\alpha|_{X^{(3)}} = 0$ that we just proved.

Suppose $J =< A, B >$ is the ideal generated by general homogenous polynomials A and B of respective degrees a and b. Let $Y \subset \mathbb{P}^n$ be the set of zeroes of J. We have an exact sequence ([13], II.8)

$$0 \to J/J^2 = \mathcal{O}_Y(-a) \oplus \mathcal{O}_Y(-b) \xrightarrow{\delta} \Omega^1_{\mathbb{P}^n}|_Y \to \Omega^1_Y \to 0$$

Tensoring with $\mathcal{O}_Y(d)$ and taking global sections we obtain that an element $\alpha|_Y \in H^0(Y, \Omega^1_{\mathbb{P}^n}(d)|_Y)$ which belongs to the image of $H^0(\delta)$, may be written as $A'dA + B'dB$ for $A' \in H^0(Y, \mathcal{O}_Y(d-a))$ and $B' \in H^0(Y, \mathcal{O}_Y(d-b))$. By [13], Ex. III (5.5), A' and B' are represented by homogeneous polynomials of respective degrees $d-a$ and $d-b$.

For each (i, j), $\alpha|_{X_{ij}}$ belongs to the image of the corresponding $H^0(\delta)$, by 9.4. Hence, we know that $\alpha = A'_{ij}\ dF_i + A'_{ji}\ dF_j$ on X_{ij}, for homogeneous polynomials A'_{ij} of degree $d - d_i$. Now, $\alpha|_{X^{(3)}} = 0$ by Proposition 28, and in particular $\alpha = 0$ on X_{ijk} for all k. Since dF_i and dF_j are linearly independent at all points of X_{ijk} by the normal crossings hypothesis, it follows that A'_{ij} and A'_{ji} are divisible by $\hat{F}_{ij}$ and we obtain the claim. □

30. **Corollary.** *With the notation of Proposition 29, define*

$$\alpha' = \sum_{i<j} \hat{F}_{ij}\ (A_{ij}\ dF_i + A_{ji}\ dF_j) \in \Omega^1_n(d)$$

Then $\alpha'|_{\tilde{X}^{(2)}} = \alpha|_{\tilde{X}^{(2)}}$.

(But notice that α' may not satisfy 7.2; see the Proof of Corollary 35).

Proof. Follows from Proposition 29 since $\hat{F}_{ij}$ vanishes on X_{hk} if $\{h,k\} \neq \{i,j\}$. $\square$

31. **Corollary.** *We keep the notation of Proposition 29. Then any $\alpha \in T(\omega)$ may be written as*

$$\begin{aligned}\alpha &= \sum_{i<j} \hat{F}_{ij}\ (A_{ij}\ dF_i + A_{ji}\ dF_j) + \sum_i \hat{F}_i\ \alpha_i \\ &= \sum_{i\neq j} \hat{F}_{ij}\ A_{ij}\ dF_i + \sum_i \hat{F}_i\ \alpha_i.\end{aligned}$$

for some $\alpha_i \in \Omega^1_n(d_i)$.

Proof. For $\alpha \in T(\omega)$, take α' as in Corollary 30. Then $\alpha - \alpha' \in \Omega^1_n(d)$ vanishes on $\tilde{X}^{(2)}$ and hence, by Proposition 23 a), may be written as $\sum_{i=1}^m \hat{F}_i\alpha_i$ for some $\alpha_i \in \Omega^1_n(d_i)$. $\square$

We would like to obtain further information on the A_{ij}'s and the α_i's. For this, we will use again that α satisfies $\omega \wedge d\alpha + \alpha \wedge d\omega = 0$ as in 7.2.

32. **Proposition.** *Suppose $n \geq 3$. With notation as in Corollary 31, for each $j = 1, \dots, m$ there exists $F'_j \in S_n(d_j)$ such that*

$$A_{ij} = \lambda_i\ F'_j \quad \text{on } X_{ij}$$

for all (i,j) with $1 \leq i,j \leq m$ and $i \neq j$.

Proof. The calculation is nicer working with the equivalent condition $d\beta \wedge \eta = 0$, where $\beta = \alpha/F$ and $\eta = \omega/F$, see Proposition 27 c). We have:

$$\beta = \sum_{i\neq j} \frac{A_{ij}}{F_j}\ \frac{dF_i}{F_i} + \sum_i \frac{\alpha_i}{F_i}$$

$$d\beta = \sum_{i\neq j} d(\frac{A_{ij}}{F_j}) \wedge \frac{dF_i}{F_i} + \sum_i d(\frac{\alpha_i}{F_i})$$

$$d\beta \wedge \eta \ = \sum_{i\neq j,k} \lambda_k\ d(\frac{A_{ij}}{F_j}) \wedge \frac{dF_i}{F_i} \wedge \frac{dF_k}{F_k} + \sum_{i,k} \lambda_k\ d(\frac{\alpha_i}{F_i}) \wedge \frac{dF_k}{F_k} =$$

$$\sum_{i\neq j\neq k} \lambda_k\ d(\frac{A_{ij}}{F_j}) \wedge \frac{dF_i}{F_i} \wedge \frac{dF_k}{F_k} + + \sum_{i\neq j} \lambda_j\ d(\frac{A_{ij}}{F_j}) \wedge \frac{dF_i}{F_i} \wedge \frac{dF_j}{F_j} +$$

$$\sum_{i\neq k} \lambda_k\ d(\frac{\alpha_i}{F_i}) \wedge \frac{dF_k}{F_k} + \sum_k \lambda_k\ d(\frac{\alpha_k}{F_k}) \wedge \frac{dF_k}{F_k} = 0$$

Let's replace

$$d(\frac{A_{ij}}{F_j}) = \frac{dA_{ij}}{F_j} - \frac{A_{ij}}{F_j}\frac{dF_j}{F_j}, \quad d(\frac{\alpha_i}{F_i}) = \frac{d\alpha_i}{F_i} - \frac{dF_i}{F_i} \wedge \frac{\alpha_i}{F_i}$$

and multiply by F^2. After some straightforward calculation we obtain:

$$\begin{aligned} F \sum_{i\neq j\neq k} \lambda_k\ \hat{F}_{ijk}\ dA_{ij} \wedge dF_i \wedge dF_k + \sum_{i\neq k} \lambda_k\ \hat{F}_k\ \hat{F}_{ik}\ dA_{ik} \wedge dF_i \wedge dF_k + \\ \sum_{i\neq j\neq k} \lambda_k\ \hat{F}_j\ \hat{F}_{ijk}\ A_{ij}\ dF_i \wedge dF_j \wedge dF_k + \\ F \sum_{j\neq k} \lambda_k\ \hat{F}_{jk}\ d\alpha_j \wedge dF_k + \sum_{k} \lambda_k\ \hat{F}_k^2\ d\alpha_k \wedge dF_k + \\ \sum_{j\neq k} \lambda_k\ \hat{F}_j\ \hat{F}_{jk}\ \alpha_j \wedge dF_j \wedge dF_k \ = \ 0 \end{aligned}$$

Now we choose r such that $1 \leq r \leq m$ and restrict to X_r, that is, we reduce modulo F_r. We get:

$$\begin{aligned} \hat{F}_r\ (\sum_{i\neq r} \lambda_r\ \hat{F}_{ir}\ dA_{ir} \wedge dF_i \wedge dF_r + \sum_{i\neq k\neq r} \lambda_k\ \hat{F}_{irk}\ A_{ir}\ dF_i \wedge dF_r \wedge dF_k + \\ \lambda_r\ \hat{F}_r\ d\alpha_r \wedge dF_r + \sum_{k\neq r} \lambda_k\ \hat{F}_{rk}\ \alpha_r \wedge dF_r \wedge dF_k) \ = \ 0 \end{aligned} \tag{9.5}$$

Since $\hat{F}_r$ is not zero on the irreducible variety X_r, we may cancel this factor out.

Next, choose s such that $1 \leq s \leq m$, $s \neq r$, and further restrict to $X_r \cap X_s = X_{rs}$ to obtain:

$$\begin{aligned} \lambda_r\ \hat{F}_{sr}\ dA_{sr} \wedge dF_s \wedge dF_r + \sum_{k\neq r\neq s} \lambda_k\ \hat{F}_{srk}\ A_{sr}\ dF_s \wedge dF_r \wedge dF_k + \\ \sum_{i\neq r\neq s} \lambda_s\ \hat{F}_{irs}\ A_{ir}\ dF_i \wedge dF_r \wedge dF_s \ + \ \lambda_s\ \hat{F}_{rs}\ \alpha_r \wedge dF_r \wedge dF_s \ = \ 0 \end{aligned} \tag{9.6}$$

And, once more, choose t such that $1 \leq t \leq m$, $t \neq s \neq r$. Restricting to $X_r \cap X_s \cap X_t = X_{rst}$ we get:

$$\hat{F}_{rst}(\lambda_t\ A_{sr} - \lambda_s\ A_{tr})\ dF_r \wedge dF_s \wedge dF_t = 0$$

By the genericity of the F_i's, X_{rst} is irreducible, and we may cancel out the factor $\hat{F}_{rst} \neq 0$. By the normal crossing hypothesis we may also cancel out $dF_r \wedge dF_s \wedge dF_t \neq 0$.

Therefore,

$$A_{sr}/\lambda_s = A_{tr}/\lambda_t \quad \text{on } X_{rst} \tag{9.7}$$

for all distinct $1 \leq r, s, t \leq m$.

Let us fix r, $1 \leq r \leq m$. We consider the natural restriction maps

$$S_n(d_r) = H^0(\mathbb{P}^n, \mathcal{O}(d_r)) \to H^0(X_r, \mathcal{O}(d_r)) \to H^0(X_{rs}, \mathcal{O}(d_r)) \to H^0(X_{rst}, \mathcal{O}(d_r)).$$

For $s = 1, \ldots, m$, $s \neq r$, the polynomials $A_{sr}/\lambda_s \in S_n(d_r)$ (all of the same degree d_r) define, by restriction to the hypersurfaces $X_{rs} \subset X_r$, sections $A_{sr}/\lambda_s \in H^0(X_{rs}, \mathcal{O}(d_r))$. By 9.7 these sections coincide on the pairwise intersections $X_{rs} \cap X_{rt} = X_{rst}$. Hence this collection defines a section of $\mathcal{O}(d_r)$ on the (reducible) variety $D_r = \cup_{s\neq r} X_{rs} \subset X_r$. By Lemma 33 below, with $X = X_r$ and $D = D_r$, there exists $F'_r \in S_n(d_r)$, such that $A_{sr}/\lambda_s = F'_r$ on X_{rs}, for each $s \neq r$, as claimed.

□

33. **Lemma.** *Let $n \geq 3$, and let $X \subset \mathbb{P}^n$ be a smooth irreducible hypersurface of degree e. For $m \geq 1$ and $i = 1, \ldots, m$ let $D_i \subset X$ be smooth irreducible distinct hypersurfaces. We consider the (reducible) hypersurface $D = \cup_{1\leq i\leq m} D_i \subset X$. Then the natural restriction map*

$$H^0(X, \mathcal{O}(e)) \to H^0(D, \mathcal{O}(e))$$

is surjective.

Proof. In the exact sequence $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ we tensor by $\mathcal{O}_X(e)$ and take cohomology. Since $\mathcal{O}_X(-D)(e) = \mathcal{O}_X(-d)(e) = \mathcal{O}_X(e-d)$ for some d, and $H^1(X, \mathcal{O}_X(e-d)) = 0$ (see e. g. [13], Exercise III, (5.5)), we obtain the claim. □

34. **Corollary.** *Let $n \geq 3$. Any $\alpha \in T(\omega)$ may be written as*

$$\alpha = \sum_{i\neq j} \lambda_i \ \hat{F}_{ij} \ F'_j \ dF_i + \sum_i \hat{F}_i \ \alpha_i.$$

for some $F'_i \in S_n(d_i)$ and $\alpha_i \in \Omega^1_n(d_i)$.

Proof. Follows from Corollary 31 and Proposition 32. □

35. **Corollary.** *Let $n \geq 3$. Any $\alpha \in T(\omega)$ may be written as*

$$\alpha = \bar{\alpha} + \sum_i \hat{F}_i \ \gamma_i.$$

where $\bar{\alpha}$ belongs to the image of $d\mu(\lambda, \mathbf{F})$, $\gamma_i \in \Omega^1_n(d_i)$ and $\sum_i \hat{F}_i \ \gamma_i \in T(\omega)$.

Proof. Using Corollary 34, then adding and substracting $\sum_i \lambda_i \ \hat{F}_i \ dF'_i$, we have:

$$\begin{aligned}
\alpha &= \sum_{i\neq j} \lambda_i \ \hat{F}_{ij} \ F'_j \ dF_i + \sum_i \hat{F}_i \ \alpha_i \\
&= \sum_{i\neq j} \lambda_i \ \hat{F}_{ij} \ F'_j \ dF_i + \sum_i \lambda_i \ \hat{F}_i \ dF'_i + \sum_i \hat{F}_i \ (\alpha_i - \lambda_i \ dF'_i) \\
&= d\mu(\lambda, \mathbf{F})(0, \mathbf{F}') + \sum_i \hat{F}_i \ \gamma_i
\end{aligned}$$

taking $\gamma_i = \alpha_i - \lambda_i \ dF'_i$. Since $\alpha, \bar{\alpha} \in T(\omega)$, we have $\alpha - \bar{\alpha} = \sum_i \hat{F}_i \gamma_i \in T(\omega)$, as claimed. □

36. **Remark.** *Corollary 35 implies that to prove Theorem 24 we are reduced to showing that any $\alpha \in T(\omega)$ of the form $\alpha = \sum_i \hat{F}_i \gamma_i$, with $\gamma_i \in \Omega^1_n(d_i)$, belongs to the image of $d\mu(\lambda, \mathbf{F})$.*

To this end, let us first prove the following

37. **Proposition.** *Let $\alpha \in T(\omega)$ be of the form*

$$\alpha = \sum_j (\hat{F}_j)^e \ \gamma_j \tag{9.8}$$

with $e \in \mathbb{N}, e \geq 1$, and $\gamma_j \in \Omega^1_n(d - e\hat{d}_j)$. Then, for $1 \leq i, j \leq m$, $i \neq j$, there exist $\lambda'_j \in \mathbb{C}$, $D_{ij} \in S_n(d_j - e\hat{d}_j)$ and $\epsilon_j \in \Omega^1_n(d_j - e\hat{d}_j)$, such that

$$\gamma_j = \lambda'_j \ dF_j + \sum_{i \neq j} \hat{F}_{ij} \ D_{ij} \ dF_i + \hat{F}_j \ \epsilon_j$$

for $j = 1, \ldots, m$. In case $e \geq 2$, all $\lambda'_j = 0$.

Proof. Let us use once more that α satisfies 7.2 $\omega \wedge d\alpha + \alpha \wedge d\omega = 0$. We may apply to our present α the calculation in the Proof of Proposition 32, with $A_{ij} = 0$ and $\alpha_j = (\hat{F}_j)^{e-1} \ \gamma_j$, for all i, j. Then it follows from equation 9.6 that

$$\gamma_j \wedge dF_i \wedge dF_j \ = \ 0 \quad \text{ on } X_{ij}, \quad \text{for all } \ i \neq j,$$

since $\lambda_j \neq 0$, and $\hat{F}_{ij} \neq 0$ on X_{ij}. Then,

$$\gamma_j = B_{ij} dF_i + C_{ij} dF_j \quad \text{ on } X_{ij}$$

for some $B_{ij} \in S_n(d - e\hat{d}_j - d_i)$ and $C_{ij} \in S_n((1-e)\hat{d}_j)$. Notice that $C_{ij} \in S_n(0) = \mathbb{C}$ if $e = 1$, and $C_{ij} = 0$ if $e \geq 2$, since $(1-e)\hat{d}_j < 0$.
Now we fix j and vary $i \neq j$. On $X_{ij} \cap X_{kj} = X_{ijk}$ we have $B_{ij} dF_i + C_{ij} dF_j = B_{kj} dF_k + C_{kj} dF_j$. From the normal crossings hypothesis we obtain, for all $i \neq k$:

a) $B_{ij} = B_{kj} = 0$ on X_{ijk}, and
b) $C_{ij} = C_{kj}$

From b), C_{ij} does not depend on i and we may denote $C_{ij} = \lambda'_j$. As noticed above, $C_{ij} = \lambda'_j = 0$ in case $e \geq 2$.
On the other hand, a) implies that $B_{ij} = \hat{F}_{ij} D_{ij}$ on X_{ij} for some $D_{ij} \in S_n(d_j - e\hat{d}_j)$. Therefore,

$$\gamma_j = \lambda'_j dF_j + \hat{F}_{ij} D_{ij} dF_i \quad \text{ on } X_{ij}$$

for all j and all $i \neq j$. Let $\gamma'_j = \gamma_j - (\lambda'_j dF_j + \sum_{i \neq j} \hat{F}_{ij} D_{ij} dF_i) \in \Omega^1_n(d - e\hat{d}_j)$. Then γ'_j is zero on $D_j = \cup_{i \neq j} X_{ij} \subset X_j$, hence there exists $\epsilon_j \in \Omega^1_n(d_j - e\hat{d}_j)$ such that $\gamma'_j = \hat{F}_j \ \epsilon_j$ on X_j. Denoting $J_j \cong \mathcal{O}(-d_j)$ the ideal sheaf of X_j, we have $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d_j)(J_j)) \cong H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) = 0$. Therefore the equality $\gamma'_j = \hat{F}_j \ \epsilon_j$ holds in $\mathbb{P}^n$, and this implies our claim.

□

38. **Corollary.** *If $\alpha \in T(\omega)$ is divisible by $(\hat{F}_1)^e$, that is, $\alpha = (\hat{F}_1)^e\ \gamma_1$ for some $\gamma_1 \in \Omega^1_n(d - e\hat{d}_1)$, then there exist $\lambda'_1 \in \mathbb{C}$, $D_i \in S_n(d_1 - e\hat{d}_1)$, for $i > 1$, and $\epsilon_1 \in \Omega^1_n(d_1 - e\hat{d}_1)$, such that*

$$\alpha = (\hat{F}_1)^e(\lambda'_1\ dF_1 + \sum_{i>1} \hat{F}_{i1}\ D_i\ dF_i\ +\ \hat{F}_1\ \epsilon_1).$$

In case $e \geq 2$, $\lambda'_1 = 0$.

Proof. It follows immediately from Proposition 37 applied to the case $\gamma_j = 0$ for $j > 1$. □

9.1. End of the proof: balanced case.

39. **Definition.** *Let $\mathbf{d} = (m; d_1, \ldots, d_m) \in P(m, d)$. We say that $\mathbf{d}$ is* balanced *if $d_i < \sum_{j \neq i} d_j = \hat{d}_i$ for all $i = 1, \ldots, m$. Equivalently, if $2d_i < d$ for all i.*

Notice that if $\mathbf{d}$ is not balanced then there exists a *unique* i such that $2d_i \geq d$. Since we normalized $\mathbf{d}$ so that $d_1 \geq d_2 \geq \cdots \geq d_m$ (see Definition 1), it follows that $\mathbf{d}$ is balanced if and only if $2d_1 < d$.

40. **Theorem.** *Suppose $\mathbf{d} \in P(m, d)$ is balanced. Let $(\lambda, \mathbf{F}) \in V_n(\mathbf{d})$ be general and $\omega = \mu(\lambda, \mathbf{F})$. Then, for any $\alpha \in T(\omega)$ such that $\alpha = \sum_i \hat{F}_i\ \gamma_i$, with $\gamma_i \in \Omega^1_n(d_i)$, there exists $\lambda' = (\lambda'_1, \ldots, \lambda'_m) \in \mathbb{C}^m$, with $\sum_{i=1}^m d_i \lambda'_i = 0$, such that*

$$\alpha = \sum_{i=1}^{m} \lambda'_i\ \hat{F}_i\ dF_i.$$

In particular,

$$\alpha = d\mu(\lambda, \mathbf{F})(\lambda', 0)$$

belongs to the image of $d\mu(\lambda, \mathbf{F})$.

Proof. We apply Proposition 37 with $e = 1$. Since $\mathbf{d}$ is balanced, $d_j - \hat{d}_j < 0$ for all j and then $D_{ij} = 0$ and $\epsilon_j = 0$ for all i, j. Hence $\gamma_j = \lambda'_j\ dF_j$ for all j, as claimed. □

It follows from Remark 36 that the proof of Theorem 24 is now complete, if $\mathbf{d}$ is balanced.

9.2. End of the proof: general case.

When $\mathbf{d}$ is not balanced, Theorem 40 is not true; we may have an $\alpha \in T(\omega)$ such that $\alpha|_{X^{(2)}} = 0$ but α is not logarithmic as in Theorem 40. For example, take $F'_1 = G_1\ \hat{F}_1$ where G_1 is any homogeneous polynomial of degree $d_1 - \hat{d}_1 > 0$, and $F'_j = 0$ for $j > 1$. Then $\alpha = d\mu(\lambda, \mathbf{F})(0, F')$ satisfies this condition, as it easily follows from 7.1. Notice that this α is divisible by $\hat{F}_1$.

In Theorem 42 we will see that any $\alpha \in T(\omega)$ such that $\alpha|_{X^{(2)}} = 0$ may be written in a special form that still implies it belongs to the image of $d\mu(\lambda, \mathbf{F})$.

41. **Definition.** *Let* $\mathbf{d} \in P(m,d)$. *We define*

$$r(\mathbf{d}) = \max\ \{e \in \mathbb{N}/\ d_1 \geq e\ \hat{d}_1\} = [d_1/\hat{d}_1]$$

the integer part of $d_1/\hat{d}_1$.

Notice that $\mathbf{d}$ is balanced when $r(\mathbf{d}) = 0$.

42. **Theorem.** *Fix* $\mathbf{d} \in P(m,d)$. *Let* $(\lambda, \mathbf{F}) \in V_n(\mathbf{d})$ *be general and* $\omega = \mu(\lambda, \mathbf{F})$. *Then, any* $\alpha \in T(\omega)$ *such that* $\alpha = \sum_i\ \hat{F}_i \gamma_i$, *with* $\gamma_i \in \Omega^1_n(d_i)$, *may be written as*

$$\alpha = d\mu(\lambda, \mathbf{F})(\lambda', \mathbf{F}')$$

where $\lambda' \in \mathbb{C}^m$ *is such that* $\sum_{i=1}^m d_i \lambda'_i = 0$, $F'_j = 0$ *for* $j > 1$, *and*

$$F'_1 = \sum_{e=1}^{r(\mathbf{d})} G_e\ \hat{F_1}^e$$

where G_e *are homogeneous polynomials of respective degrees* $d_1 - e\hat{d}_1$, *for* $e = 1, \dots, r(\mathbf{d})$.

Proof. By Proposition 37 with $e = 1$,

$$\alpha = \sum_j \lambda'_j\ \hat{F}_j\ dF_j + \sum_{i \neq j} \hat{F}_{ij}\ \hat{F}_j D_{ij}\ dF_i + \sum_j \hat{F}_j\ \hat{F}_j \epsilon_j. \tag{9.9}$$

In the current unbalanced case, $d_1 - \hat{d}_1 \geq 0$ and $d_i - \hat{d}_i < 0$ for $i > 1$, as in Definition 9.2. Hence $D_{ij} = 0$ and $\epsilon_j = 0$ for $j > 1$. Also, since $\sum_j \lambda'_j\ \hat{F}_j\ dF_j = d\mu(\lambda, \mathbf{F})(\lambda', 0)$, it is enough to consider

$$\alpha = \alpha^{(1)} = \sum_{i>1} \hat{F}_{i1}\ \hat{F}_1 D_{i1}\ dF_i + \hat{F}_1\ \hat{F}_1 \epsilon_1 = \hat{F}_1\ (\sum_{i>1} \hat{F}_{i1}\ D_{i1}\ dF_i + \hat{F}_1\ \epsilon_1) \tag{9.10}$$

which is divisible by $\hat{F}_1$ (the last term is actually divisible by $\hat{F_1}^2$).
What we shall do is to express $\alpha^{(1)}$ as the sum of an element of the image of $d\mu(\lambda, \mathbf{F})$ (of the claimed shape) plus an $\alpha^{(2)} \in T(\omega)$ divisible by $\hat{F_1}^2$. Next we repeat the argument and express $\alpha^{(2)}$ as the sum of another element of the image of $d\mu(\lambda, \mathbf{F})$ plus an $\alpha^{(3)} \in T(\omega)$ divisible by $\hat{F_1}^3$. After at most $r(\mathbf{d})$ iterations this process ends, since $\alpha^{(r(\mathbf{d})+1)} = 0$ by degree reason, and hence we obtain the claimed expression for the original α.
The essential step is to pass from $\alpha^{(e)}$ to $\alpha^{(e+1)}$, for $1 \leq e \leq r(\mathbf{d})$.
To carry out this step, let us assume that α is divisible by $\hat{F_1}^e$, that is,

$$\alpha = \alpha^{(e)} = \hat{F_1}^e\ (\sum_{i>1} \hat{F}_{i1}\ D_{i1}\ dF_i + \hat{F}_1\ \epsilon_1). \tag{9.11}$$

as in Corollary 38.
Now we apply to α the calculation in the Proof of Proposition 32 with

$$A_{ij} = \hat{F_1}^e D_{ij}, \quad \alpha_j = \hat{F_1}^e \epsilon_j,$$

that is:

$$A_{i1} = \hat{F_1}^e D_{i1} \ \text{ for } i > 1, \quad \alpha_1 = \hat{F_1}^e \epsilon_1,$$

$$A_{ij} = 0, \quad \alpha_j = 0 \quad \text{for } j > 1.$$

From equation 9.5 with $r = 1$ we get

$$\hat{F}_1 \, (\sum_{i\neq 1} \lambda_1 \, \hat{F}_{i1} \, d(\hat{F}_1{}^e D_{i1}) \wedge dF_i \wedge dF_1 + \sum_{i\neq k\neq 1} \lambda_k \, \hat{F}_{i1k} \, \hat{F}_1{}^e D_{i1} \, dF_i \wedge dF_1 \wedge dF_k + \lambda_1 \, \hat{F}_1 \, d(\hat{F}_1{}^e \epsilon_1) \wedge dF_1 + \sum_{k\neq 1} \lambda_k \, \hat{F}_{1k} \, \hat{F}_1{}^e \epsilon_1 \wedge dF_1 \wedge dF_k) \;=\; 0 \quad (9.12)$$

We have $d(\hat{F}_1{}^e D_{i1}) = e\hat{F}_1{}^{e-1} D_{i1} d\hat{F}_1 + \hat{F}_1{}^e dD_{i1}$. Also, $d\hat{F}_1 \wedge dF_i = (\sum_{j\neq 1} \hat{F}_{j1} dF_j) \wedge dF_i = \sum_{j\neq 1, j\neq i} \hat{F}_{j1} dF_j \wedge dF_i$, so that $\hat{F}_{i1} d\hat{F}_1 \wedge dF_i = \sum_{j\neq 1, j\neq i} \hat{F}_{i1}\hat{F}_{j1} dF_j \wedge dF_i = \hat{F}_1 \sum_{j\neq 1, j\neq i} \hat{F}_{ij1} dF_j \wedge dF_i$. Replacing these into 9.12, we obtain, on X_1:

$$\hat{F}_1{}^{e+1}(\sum_{i\neq j\neq 1} e\lambda_1 \hat{F}_{ij1} D_{i1} \, dF_j \wedge dF_i \wedge dF_1 + \sum_{i\neq 1} \lambda_1 \hat{F}_{i1} \, dD_{i1} \wedge dF_i \wedge dF_1 + \sum_{i\neq j\neq 1} \lambda_j \hat{F}_{ij1} D_{i1} \, dF_i \wedge dF_1 \wedge dF_j + e\lambda_1 \, d\hat{F}_1 \wedge \epsilon_1 \wedge dF_1 + \lambda_1 \hat{F}_1 \, d\epsilon_1 \wedge dF_1 + \sum_{i\neq 1} \lambda_i \hat{F}_{1i} \, \epsilon_1 \wedge dF_1 \wedge dF_i) \;=\; 0 \qquad (9.13)$$

Now we cancel the factor $\hat{F}_1{}^{e+1}$ on X_1 and then restrict to X_{1st} for $1, s, t$ distinct. After straightforward calculation we obtain, on X_{1st}:

$$(e\lambda_1 + \lambda_s) D_{t1} = (e\lambda_1 + \lambda_t) D_{s1}$$

Then the collection $\{D_{s1}/(e\lambda_1 + \lambda_s) \in S_n(d_1 - e\hat{d}_1)\}_{s\neq 1}$ defines a section of $\mathcal{O}(d_1 - e\hat{d}_1)$ on $\cup_{s\neq 1} X_{1s} \subset X_1$. Hence, there exists $G_e \in S_n(d_1 - e\hat{d}_1)$ such that

$$D_{s1} = (e\lambda_1 + \lambda_s) G_e$$

on X_{1s} for all $s \neq 1$. Then, with the notation of 9.11,

$$\sum_{i>1} \hat{F}_{i1} \, D_{i1} \, dF_i \;+\; \hat{F}_1 \, \epsilon_1 \;-\; \sum_{i>1} \hat{F}_{i1} \, (e\lambda_1 + \lambda_i) G_e \, dF_i \;=\; 0$$

on $\cup_{s\neq 1} X_{1s} \subset X_1$, and hence is divisible by $\hat{F}_1$. We obtain

$$\alpha = \hat{F}_1{}^e \sum_{i>1} \hat{F}_{i1} \, (e\lambda_1 + \lambda_i) G_e \, dF_i \;+\; \hat{F}_1{}^{e+1} \, \bar{\epsilon}_1 \qquad (9.14)$$

for some $\bar{\epsilon}_1 \in \Omega^1_n(d_1 - e\hat{d}_1)$.
Denote $\mathbf{F}' = (\hat{F}_1{}^e \, G_e, 0, \ldots, 0)$. Combining 9.14 with

$$d\mu(\lambda, \mathbf{F})(0, \mathbf{F}') = \sum_{i>1} \lambda_i \, F_1{}^e \, G_e \, \hat{F}_{i1} \, dF_i + \lambda_1 \hat{F}_1 d(\hat{F}_1{}^e G_e)$$

(see 7.1), one immediately obtains

$$\alpha = d\mu(\lambda, \mathbf{F})(0, \mathbf{F}') + \alpha^{(e+1)}$$

with $\alpha^{(e+1)} = \hat{F}_1^{\,e+1}\,(\bar{\epsilon}_1 - \lambda_1 dG_e)$. Now, $\alpha^{(e+1)} \in T(\omega)$ because α and $d\mu(\lambda, \mathbf{F})(0, \mathbf{F}')$ belong to $T(\omega)$. Since $\alpha^{(e+1)}$ is divisible by $\hat{F}_1^{\,e+1}$, by Corollary 38, it may be written as in 9.11 with exponent $e+1$. Hence we may apply again the previous procedure to $\alpha^{(e+1)}$. This proves the essential iterative step and implies our statement. □

It follows from Remark 36 that the proof of Theorem 24 is now complete, for any $\mathbf{d}$.

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Universidad de Buenos Aires and CONICET.
Departamento de Matemática, FCEN.
Ciudad Universitaria.
(1428) Buenos Aires.
ARGENTINA.

Fernando Cukierman, fcukier@dm.uba.ar
Javier Gargiulo Acea, jngargiulo@gmail.com
César Massri, cmassri@dm.uba.ar