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Solution of a functional equation related to the Pythagorean Proposition

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Abstract. The functional equation

$$f(x+y) = f(x) + f(y) + 2f(\Phi(x,y)),\ x, y > 0,$$

is solved for pairs (f, Φ) constituted by a strictly monotonic function f and a sufficiently regular Lagrangian mean Φ. Some related questions stated in a recent paper by R. Ger ([5]) are answered.

Resumen. La ecuación funcional

$$f(x+y) = f(x) + f(y) + 2f(\Phi(x,y)),\ x, y > 0,$$

es resuelta para pares (f, Φ) constituidos por una función estrictamente monótona y un Lagrangiano suficientemente regular Φ. Algunas preguntas formuladas en un reciente artículo de R. Ger ([5]) son respondidas.

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1 Introduction

The composite functional equation

$$f(x+y) = f(x) + f(y) + 2f(\Phi(x,y)),\ x, y > 0, \tag{1}$$

for the unknown pair of functions (f, Φ) was settled in [3] in connection with a functional approach to the Pythagorean Proposition. Indeed, the equation (1) admits the pair

$$f(x) = cx^2,\ (c \neq 0), \quad \Phi(x,y) = \sqrt{xy} \tag{2}$$

as a distinguished solution and then, the associative operation $\triangle$ defined on $\mathbb{R}^+$ by

$$x \triangle y = f^{-1}(f(x) + f(y)) \tag{3}$$

gives the length $\sqrt{x^2+y^2}$ of the hypotenuse of a right triangle in terms of the lengths x, y of its legs. Theorem 1 in [3] can be stated in the following way.

Theorem 1. *The pair (2) is the unique solution to equation (1) in the class of pairs* (f,Φ) *consisting of a* reflexive *function* Φ *(i.e.,* $\Phi(x,x)=x,\ x>0$*) and a continuous and strictly monotonic function* f *such that the operation* $\triangle$ *defined by (3) is* homogeneous *(i.e.* $\lambda x \triangle \lambda y = \lambda(x \triangle y),\ x,y,\lambda>0$*).*

The study of the equation (1) was deepen by R. Ger in [5]. In this article, equation (1) is extracted of its originating geometric frame and then, suitable analytic hypotheses on its solutions are freely imposed to characterize the pair (2). Several orders of differentiability and existence of the limit $\lim_{x\downarrow 0} f(x)/x^2$ are among the assumptions made on the function f. As a matter of fact, once suppressed the hypothesis of homogeneity on the operation $\triangle$, the reflexivity of the function Φ turns out to be an insufficient condition to characterize the pair (2) and it must be consequently reinforced. To this purpose, the functional form

$$\Phi(x,y)=h^{-1}\left(\frac{h(x)+h(y)}{2}\right) \tag{4}$$

with $h:\mathbb{R}^+\to\mathbb{R}$ a strictly monotonic and continuous function is prescribed to the function Φ by R. Ger, so that the equation (1) becomes

$$f(x+y)=f(x)+f(y)+2f\left(h^{-1}\left(\frac{h(x)+h(y)}{2}\right)\right),\ x,y>0. \tag{5}$$

Before continuing it will be useful to remind some basic terminology and facts concerning means. Let I be a real interval. A function $M: I\times I\to I$ is said to be a *mean* (defined on the interval I) when the inequalities

$$\min\{x,y\}\le M(x,y)\le \max\{x,y\},\ x,y\in I,$$

are satisfied by M. Note that a mean M is a reflexive function. M is said *symmetric* provided that $M(y,x)=M(x,y),\ x,y\in I$. By a *strict mean* (defined on the interval I) is understood a continuous function $M: I\times I\to I$ which is reflexive and strictly increasing in both variables. The strict inequalities

$$\min\{x,y\}< M(x,y)< \max\{x,y\},\ x,y\in I,\, x\neq y,$$

are clearly satisfied by a strict mean M; whence a strict mean is a (continuous) mean.

A function Φ of the form (4) belongs to a particular class of strict (symmetric) means named *quasiarithmetic means* (cf. Chap. 4 of [4], for example). The function h in (4) is said to be the *generator function* of the mean Φ. Most of the classical means are quasiarithmetic; nevertheless, there exist strict symmetric means that are not quasiarithmetic: the logarithmic mean

$$L(x,y) = \begin{cases} \frac{x-y}{\ln x - \ln y}, & x \neq y \\ x, & x = y \end{cases}$$

is a remarkable example of this fact. $L(x,y)$ belongs instead to another important class of strict symmetric means: the class of Lagrangian means. Given a continuous and strictly increasing function h defined on the interval I, the *Lagrangian mean* with generator function h is the strict symmetric mean defined (cf. [4], pg. 403 and ff.) by

$$\Lambda(x,y) = \begin{cases} h^{-1}\left(\frac{1}{y-x}\int_x^y h(\xi)\,d\xi\right), & x \neq y \\ x, & x = y \end{cases} .$$

Returning to the main exposition, let us quote a result by R. Ger (Theorem 3 in [5]) furnishing the strictly monotonic solutions to equation (5).

Theorem 2. *Let f, $h : \mathbb{R}^+ \to \mathbb{R}$ be two strictly monotonic functions and let h be continuous. Then the pair (f,h) yields a solution to the equation (5) if and only if there exist a, b, $c \in \mathbb{R}$, $a \neq 0 \neq c$, such that*

$$f(x) = cx^2 \text{ and } h(x) = a\ln x + b,\ x > 0.$$

Note that the geometric mean is the quasiarithmetic mean generated by the functions $h(x) = a\ln x + b$ of the theorem.

In the final section of [5], the author asks for solutions to equation (1) in the case in which Φ is a general (symmetric, non quasiarithmetic) mean defined on $\mathbb{R}^+$. A response to this question for regular Lagrangian means is given in the next section, where the the pairs (f,h) of strictly monotonic functions solving the equation

$$f(x+y) = f(x) + f(y) + 2f\left(h^{-1}\left(\frac{1}{y-x}\int_x^y h(\xi)\,d\xi\right)\right),\ x,y > 0, \tag{6}$$

are determined for a sufficiently regular generator function h. Concretely, it is established the following:

Theorem 3. *A pair (f,h) of strictly monotonic functions with $h \in \mathcal{C}^4$ solves equation (6) if and only if there exist a, b, $c \in \mathbb{R}$, $a \neq 0 \neq c$, such that*

$$f(x) = cx^2 \text{ and } h(x) = ax^{-2} + b,\ x > 0. \tag{7}$$

It should be noted that $h(x) = ax^{-2} + b$ is the Lagrangian generator corresponding to the geometric mean. As a consequence of this theorem the equation

(1) with $\Phi = L(x, y)$, the logarithmic mean, does not admit strictly monotonic solutions. This observation answer a question raised in the remarks at the end of [5]. Another consequence of a different order is the following: others classes of means share with the class of quasiarithmetic means the property of making collapse to a unique pair the solutions to equation (1). Thus, the consideration without qualifications of the case in which Φ is quasiarithmetic as an "optimal setting" for solving equation (1) (as made by R. Ger in [5]) may be not adjusted.

The next section contains a proof of Theorem 3. Some complementary discussion of equation (1) is presented in the following Section 3. The tedious computations involved in the proof of Theorem 3 are detailed in the final Appendix.

2 Proof of Theorem 3

Let us begin by making an elementary observation. By setting $y = x$ in equation (1) and taking into account the reflexivity of Φ, it is seen that the Schröder equation

$$f(2x) = 4f(x), \; x > 0, \tag{8}$$

is satisfied by every solution f to (1). This fact together with

$$f^{(n)}(2x) = 2^{2-n} f^{(n)}(x), \; x > 0, \, (n \in \mathbb{N}), \tag{9}$$

which is inductively obtained from (8) by differentiation, will be often used in what follows.

A regularity result on monotonic solutions to equation (1) is now given.

Theorem 4. *Let f be a monotonic solution to equation (1) in which Φ is assumed to be a strict mean of class $\mathcal{C}^k$, $(k \geq 1)$, defined on $\mathbb{R}^+$. Then, f is a function belonging to the class $\mathcal{C}^k$.*

If $k \geq 2$ then, the theorem holds under the weaker assumption of measurability on f (it is a corollary of Theorem 1.30 in [6], pg. 35). In the present context only strictly monotonic solutions to equation (1) are to be considered, so that this improvement is, for our purpose, unsubstantial.

Proof. By the Lebesgue's differentiability theorem of monotonic functions, f turns out to be almost everywhere differentiable. Then, Theorem 1.26 in [6], pg. 35, applies to show that $f \in \mathcal{C}^k$. □

Now, consider a pair of functions (f, h) solving (6). Two propositions furnishing relationships among the derivatives of the functions f and h are established below. The proof of both propositions depends on routine computations which are gathered together in the Appendix.

Proposition 5. *Let (f, h) be a pair of strictly monotonic functions with $h \in \mathcal{C}^2$. If equation (6) is solved by (f, h); then, the differential relationship*

$$3\frac{f''(x)}{f'(x)} = -\frac{h''(x)}{h'(x)}, \ x > 0, \tag{10}$$

holds.

Proof. Assume that the equation (6) is solved by a pair (f, h) of strictly monotonic functions f and h, $h \in \mathcal{C}^2$. Theorem 4 (with $k = 2$) ensures that $f \in \mathcal{C}^2$ as well. Denoting by Φ the Lagrangian mean with generator function h, it turns out to be $\Phi \in \mathcal{C}^2$. Now, in view of the strict monotonicity of f and h, the equalities

$$3\frac{f''(x)}{f'(x)} = 12\Phi_{xy}(x, x) = -\frac{h''(x)}{h'(x)},$$

are quickly obtained from the equalities (28) and (40) (see Appendix). $\square$

As stated by the next result, a second differential relationship among f and h can be proved under further regularity assumptions on h.

Proposition 6. *Let (f, h) be as in Proposition 5. Moreover, assume that $h \in \mathcal{C}^4$; then, besides of (10), f and h satisfy the relationship:*

$$\begin{aligned}\frac{f^{(4)}(x)}{f'(x)} - 6\frac{f''(x)f'''(x)}{\left(f'(x)\right)^2} + \frac{\left(f''(x)\right)^3}{\left(f'(x)\right)^3} &= \\ &-\frac{7}{15}\frac{h^{(4)}(x)}{h'(x)} + 2\frac{h''(x)h'''(x)}{\left(h'(x)\right)^2} - \frac{5}{3}\frac{\left(h''(x)\right)^3}{\left(h'(x)\right)^3}.\end{aligned} \tag{11}$$

Proof. First of all, note that $f \in \mathcal{C}^4$ by Theorem 4 (with $k = 4$). Introducing in (41) the values of $\Phi_{xy}(x, x)$, $\Phi_{x^2y}(x, x)$ and $\Phi_{x^2y^2}(x, x)$ as given, respectively, by (28), (29) and (37), the following equality is obtained:

$$\begin{aligned}\frac{1}{16}f^{(4)}(x) &= \frac{1}{2}f'''(x)\frac{1}{4}\frac{f''(x)}{f'(x)} + f''(x)\left[\frac{\frac{1}{4}f'(x)f'''(x) - \frac{1}{16}\left(f''(x)\right)^2}{\left(f'(x)\right)^2}\right] \\ &\quad + f'(x)\frac{1}{48}\frac{1}{h'(x)}\left[-\frac{7}{5}h^{(4)}(x) + 6\frac{h''(x)h'''(x)}{h'(x)} - 5\frac{\left(h''(x)\right)^3}{\left(h'(x)\right)^2}\right];\end{aligned}$$

from which, after some simplifications, the relationship (11) is derived. $\square$

A proof of Theorem 3 is now presented.

Proof of Theorem 3. A direct substitution shows that the pair (f,h) given by (7) is a solution to the equation (6). Conversely, assume that (f,h) with $h\in\mathcal{C}^4$ solves equation (6); then, by Propositions 5 and 6, f and h simultaneously satisfy the differential relationships (10) and (11). Now, by setting

$$\varphi(x)=\frac{h''(x)}{h'(x)}=-3\frac{f''(x)}{f'(x)},\ x>0, \tag{12}$$

and, correspondingly,

$$\begin{aligned}\tfrac{h'''(x)}{h'(x)}&=\varphi'(x)+\varphi^2(x), & \tfrac{f'''(x)}{f'(x)}&=-\tfrac13\varphi'(x)+\tfrac19\varphi^2(x),\\ \tfrac{h^{(4)}(x)}{h'(x)}&=\varphi''(x)+3\varphi(x)\varphi'(x) & \tfrac{f^{(4)}(x)}{f'(x)}&=-\tfrac13\varphi''(x)+\tfrac13\varphi(x)\varphi'(x)\\ &\quad+\varphi^3(x), & &\quad-\tfrac1{27}\varphi^3(x),\end{aligned}$$

the relationship (11) is transformed as follows:

$$\begin{aligned}-\frac13\varphi''(x)+\frac13\varphi(x)\varphi'(x)-\frac1{27}\varphi^3(x)-6\left(-\frac13\varphi(x)\right)\left(-\frac13\varphi'(x)+\frac19\varphi^2(x)\right)\\ +\left(-\frac13\varphi(x)\right)^3\\ =-\frac{7}{15}\left(\varphi''(x)+3\varphi(x)\varphi'(x)+\varphi^3(x)\right)+2\varphi(x)\left(\varphi'(x)+\varphi^2(x)\right)-\frac53\varphi^3(x),\end{aligned}$$

from which, after some algebraic simplifications, the equation

$$\varphi''(x)-7\varphi(x)\varphi'(x)+\frac{19}{9}\varphi^3(x)=0 \tag{13}$$

is deduced.

A partial integration of equation (13) is made in the Appendix, where it is shown that the equality

$$\frac{\left|\varphi'(x)-\frac{19}{6}\varphi^2(x)\right|^{19}}{\left|\varphi'(x)-\frac13\varphi^2(x)\right|^{2}}=K \tag{14}$$

with a constant $K,\ 0\le K\le+\infty$, must be satisfied by a solution φ to (13). This will be sufficient for our purpose. In fact, taking into account (9) for $n=1$ and $n=2$, the equalities

$$\varphi(2x)=\frac12\varphi(x),\quad \varphi'(2x)=\frac14\varphi'(x),\ x>0, \tag{15}$$

turn out to be an immediate consequence of (12). Thus, a replacement of the variable x by $2x$ in (14) yields

$$\begin{aligned} K &= \frac{\left|\varphi'(2x) - \frac{19}{6}\varphi^2(2x)\right|^{19}}{\left|\varphi'(2x) - \frac{1}{3}\varphi^2(2x)\right|^{2}} \\ &= \frac{4^{-19}\left|\varphi'(x) - \frac{19}{6}\varphi^2(x)\right|^{19}}{4^{-2}\left|\varphi'(x) - \frac{1}{3}\varphi^2(x)\right|^{2}} \\ &= 4^{-17}K, \end{aligned}$$

and therefore, $K = 0$ or $K = +\infty$, the limit cases in which φ must be (see the Appendix) a solution to any one of the equations

$$\varphi'(x) - \frac{19}{6}\varphi^2(x) = 0,\ x > 0,$$

or

$$\varphi'(x) - \frac{1}{3}\varphi^2(x) = 0,\ x > 0.$$

Now, the general solution to these equation are respectively given by $\varphi(x) = \left(C - \frac{19}{6}x\right)^{-1}$ and $\varphi(x) = \left(C - \frac{1}{3}x\right)^{-1}$, with C a suitable real constant which, in view of the first condition in (15), must vanish, so that

$$\varphi(x) = -\frac{6}{19}x^{-1} \text{ or } \varphi(x) = -3x^{-1}. \tag{16}$$

Finally, the functions f and h are recovered from equality (12). In particular, (12) together with (16) give for f the following equations

$$\frac{f''(x)}{f'(x)} = \frac{2}{19}x^{-1} \text{ or } \frac{f''(x)}{f'(x)} = x^{-1},\ x > 0,$$

which are quickly integrated to give

$$f(x) = cx^{\frac{21}{19}} + d \text{ or } f(x) = cx^2 + d,\ x > 0, \tag{17}$$

with $c, d \in \mathbb{R}$. The condition (8) is not satisfied by the first solution in (17), so that it must be discarded. For the remaining solution, the same condition together the strict monotonicity of f prescribe $c \neq 0,\ d = 0$, and then $f(x) = cx^2,\ c \neq 0$. Correspondingly, the equation

$$\frac{h''(x)}{h'(x)} = -\frac{1}{3}x^{-1},\ x > 0,$$

has $h(x) = ax^{-2} + b$ with $a,\ b \in \mathbb{R},\ a \neq 0$, as the unique strictly monotonic solutions.

Summarizing, the pair (f, h) with $f(x) = cx^2$ and $h(x) = ax^{-2} + b$ ($a,\ b,\ c \in \mathbb{R},\ a \neq 0 \neq b$) is the unique solution to the equation (6) in the class of pairs of strictly monotonic functions with $h \in \mathcal{C}^4$. □

3 Final remarks

The general solution to the Schröder equation (8) is given (see, for instance, [1]) by

$$f(x) = x^2 P\left(\frac{\ln x}{\ln 2}\right),\ x > 0,$$

where $P : \mathbb{R} \to \mathbb{R}$ is an arbitrary periodic function of period 1. The function P clearly reduces to a constant provided that the limit

$$\lim_{x\downarrow 0} \frac{f(x)}{x^2} \left(= \lim_{x\downarrow 0} P\left(\frac{\ln x}{\ln 2}\right) = \lim_{x\downarrow -\infty} P(x) \right) = c \in \mathbb{R}.$$

This fact is registered as a theorem (Theorem 1) in [5]. An identical conclusion is obtained by assuming that

$$\lim_{x\uparrow +\infty} \frac{f(x)}{x^2} = c.$$

Other assumptions not involving boundary points of the domain can be made in order that $P = c$. For example, it can be supposed that $x \to f(x)/x^2 = p(x),\ x > 0$, is a monotonic or convex (concave) function in an interval of the form $(\delta, 4\delta),\ \delta > 0$. In this case, taking into account that $p(\delta) = p(2\delta) = p(4\delta) = c$, it turns out to be $p(x) = c,\ x \in [\delta, 4\delta]$. From this and the fact that $p(2x) = p(x),\ x > 0$, the equality $p(x) = c,\ x > 0$, follows. A possible response to a question stated by R. Ger at the final of [5] is given by this simple observation.

Another observation is, perhaps, more interesting. By adding the hypothesis

$$\Phi(x, 4x) = 2x,\ x > 0, \tag{18}$$

to the reflexivity of Φ, the substitution $y = 4x$ in the equation (1) yields

$$f(5x) = f(x) + f(4x) + 2f(2x)$$

or, in view of (8),

$$f(5x) = f(x) + 16f(x) + 8f(x) = 25f(x),\ x > 0.$$

In this situation, the following theorem holds.

Theorem 7. *The general monotonic solution to the equation (1) with a reflexive function Φ satisfying (18) is given by*

$$f(x) = cx^2,\ x > 0, \tag{19}$$

where c is a real constant.

Proof. It was seen above that, under the hypotheses made on Φ, a solution f to the equation (1) must solve simultaneously the equations

$$f(2x) = 2^2 f(x), \quad f(5x) = 5^2 f(x), \ x > 0.$$

The general monotonic solution to this system of Schröder equations is given by (19) (cf. [8], Cor. 2, or also [7],Theor. 9.5.1). $\square$

A similar result is obtained if, instead of (18), it is assumed that there exists $n \in \mathbb{N}$ such that

$$\Phi(x, 4^n x) = 2^n x, \ x > 0.$$

To end this section, let us remark that the reduction to differential equations used in solving equation (1) for quasiarithmetic or Lagrangian means (like in Theorem 2 of [5] or in Theorem 3 of this article) can be successfully employed in other situations. The case in which Φ is a symmetric mean with a finite number of generator functions seems to be specially tractable by this method. Cauchy means

$$\lambda_h^{(g)}(x,y) = \begin{cases} h^{-1}\left(\frac{1}{g(y)-g(x)} \int_x^y h(\xi)\, dg(\xi)\right), & x \neq y \\ x, & x = y \end{cases},$$

(cf. [2] or [4], pg. 405 and ff.) and the symmetric Bajraktarević means (cf. [4], pg. 310 and ff.)

$$M_{h,g}(x,y) = h^{-1}\left(\frac{g(x)h(x) + g(y)h(y)}{g(x) + g(y)}\right),$$

are both examples of means with two generators whose study in connection with equation (1) would serve to generalize Theorem 3 as well as the results in [5].

The method of solving a functional equation by reduction to differential equations often requires to impose heavy hypothesis of regularity on the unknown functions. In many situations, these hypotheses turn out to be unnecessary. This seems to be the case with equation (6), and Theorem 3 is conjectured to hold even if the function h is continuous.

4 Appendix

4.1 Derivatives of a reflexive and symmetric function

A set of relationships holds among the values assumed on the diagonal $\Delta(I) = \{(x,x) : x \in I\}$ by the successive partial derivatives of sufficiently regular reflexive and symmetric function $F : I \times I \to \mathbb{R}$. These relationships are quickly obtained from a repeated differentiation of the identities $F(x,x) = x$

and $F(x,y)=F(y,x)$, and those ones involving derivatives up to fourth order are listed below.

$$\left\{\begin{array}{l} F_x(x,x)=F_y(x,x)=\frac{1}{2};\\ F_{x^2}(x,x)=F_{y^2}(x,x)=-F_{xy}(x,x);\\ F_{x^3}(x,x)=-3F_{x^2y}(x,x)=-3F_{xy^2}(x,x)=F_{y^3}(x,x);\\ F_{x^4}(x,x)=F_{y^4}(x,x),\quad F_{x^3y}(x,x)=F_{xy^3}(x,x)\\ F_{x^4}(x,x)=-4F_{x^3y}(x,x)-3F_{x^2y^2}(x,x)\end{array}\right. . \tag{20}$$

4.1.1 Expression of the successive derivatives of a Lagrangian mean

For a Lagrangian mean Φ generated by the function h, it can be written

$$(y-x)h(\Phi(x,y))=\int_x^y h(\xi)\,d\xi. \tag{21}$$

In the sequel, the values at the diagonal $\Delta(I)$ of some partial derivatives of Φ are to be expressed in terms of the (regular) generator function h and its successive derivatives. Let us begin by a repeated differentiation of (21). The symbol at the left of the arrow indicates the corresponding differential operator.

$$\partial_x \to \quad -h(\Phi(x,y))+(y-x)h'(\Phi(x,y))\Phi_x(x,y)=-h(x);$$

$$\begin{aligned}\partial_{xy} \quad \to \quad & -h'(\Phi(x,y))\left[\Phi_y(x,y)-\Phi_x(x,y)\right]\\ & +(y-x)\left[h''(\Phi(x,y))\Phi_x(x,y)\Phi_y(x,y)+h'(\Phi(x,y))\Phi_{xy}(x,y)\right]\\ = \quad & 0;\end{aligned} \tag{22}$$

$$\begin{aligned}\partial_{x^2y} \to \quad & -h''(\Phi(x,y))\left[2\Phi_x(x,y)\Phi_y(x,y)-\Phi_x^2(x,y)\right]\\ & -h'(\Phi(x,y))\left[2\Phi_{xy}(x,y)-\Phi_{x^2}(x,y)\right]+(y-x)\times\\ & \Big[h'''(\Phi(x,y))\Phi_x^2(x,y)\Phi_y(x,y)+h''(\Phi(x,y))\left[\Phi_{x^2}(x,y)\Phi_y(x,y)\right.\\ & \left.+2\Phi_x(x,y)\Phi_{xy}(x,y)\right]\\ & +h'(\Phi(x,y))\Phi_{x^2y}(x,y)\Big]=0;\end{aligned}$$

$$\partial_{x^2y^2} \to \quad A(x,y)+(y-x)B(x,y)=0, \tag{23}$$

where

$$\begin{aligned}A(x,y)=&-h'''(\Phi(x,y))2\Phi_x(x,y)\Phi_y(x,y)\left[\Phi_y(x,y)-\Phi_x(x,y)\right]\\ &-h''(\Phi(x,y))\left[4\Phi_{xy}(x,y)\left(\Phi_y(x,y)-\Phi_x(x,y)\right)\right.\\ &\left.+2\Phi_x(x,y)\Phi_{y^2}(x,y)-2\Phi_y(x,y)\Phi_{x^2}(x,y)\right]\\ &+h'(\Phi(x,y))2\left[\Phi_{x^2y}(x,y)-\Phi_{xy^2}(x,y)\right],\end{aligned} \tag{24}$$

and

$$\begin{aligned}B(x,y) = h^{(4)}(\Phi(x,y))\Phi_x^2(x,y)\Phi_y^2(x,y)\\ + h'''(\Phi(x,y))\left[4\Phi_x(x,y)\Phi_y(x,y)\Phi_{xy}(x,y)\right.\\ \left.+\Phi_x^2(x,y)\Phi_{y^2}(x,y) + \Phi_{x^2}(x,y)\Phi_y^2(x,y)\right]\\ + h''(\Phi(x,y))\left[2\Phi_{x^2y}(x,y)\Phi_y(x,y) + \Phi_{x^2}(x,y)\Phi_{y^2}(x,y)\right.\\ \left.+ 2\Phi_{xy}^2(x,y) + 2\Phi_x(x,y)\Phi_{xy^2}(x,y)\right]\\ + h'(\Phi(x,y))\Phi_{x^2y^2}(x,y).\end{aligned} \tag{25}$$

If $x \neq y$; then, the equality (22) can be written in the form

$$\begin{gathered}-h'(\Phi(x,y))\left[\tfrac{\Phi_y(x,y)-\Phi_x(x,y)}{y-x}\right] + h''(\Phi(x,y))\Phi_x(x,y)\Phi_y(x,y)\\ +h'(\Phi(x,y))\Phi_{xy}(x,y) = 0,\end{gathered}$$

whence, by making $y \to x$ it is obtained:

$$\begin{gathered}-h'(\Phi(x,x))\left[\Phi_{y^2}(x,x) - \Phi_{xy}(x,x)\right] + h''(\Phi(x,x))\Phi_x(x,x)\Phi_y(x,x)\\ +h'(\Phi(x,x))\Phi_{xy}(x,x) = 0.\end{gathered} \tag{26}$$

The L´Hospital rule was applied to compute

$$\lim_{y\to x}\left(\frac{\Phi_y(x,y) - \Phi_x(x,y)}{y-x}\right) = \Phi_{y^2}(x,x) - \Phi_{xy}(x,x) = -2\Phi_{xy}(x,x). \tag{27}$$

From (26) and the relationships (20), it follows

$$3\Phi_{xy}(x,x)h'(x) + \frac{1}{4}h''(x) = 0,$$

whence, in view of the strict monotonicity of h, it is finally obtained

$$\Phi_{xy}(x,x) = -\frac{1}{12}\frac{h''(x)}{h'(x)}. \tag{28}$$

Now, differentiating (28) and using the relationships (20), it is derived

$$2\Phi_{x^2y}(x,x) = \Phi_{x^2y}(x,x) + \Phi_{xy^2}(x,x) = -\frac{1}{12}\left(\frac{h''(x)}{h'(x)}\right)',$$

and hence,

$$\Phi_{x^2y}(x,x) = -\frac{1}{24}\left(\frac{h''(x)}{h'(x)}\right)'. \tag{29}$$

Differentiating this last equality, it is obtained

$$\Phi_{x^3y}(x,x)+\Phi_{x^2y^2}(x,x)=-\frac{1}{24}\left(\frac{h''(x)}{h'(x)}\right)''. \tag{30}$$

In continuing, another relationship among $\Phi_{x^3y}(x,x)$ and $\Phi_{x^2y^2}(x,x)$ is to be derived. From (23) it is obtained

$$\lim_{y\to x}\frac{A(x,y)}{y-x}+B(x,x)=0. \tag{31}$$

Using (24) and (25), let us compute the terms of the left member of this equality. From (24), it follows that

$$\begin{aligned}\frac{A(x,y)}{y-x}&=-h'''(\Phi(x,y))2\Phi_x(x,y)\Phi_y(x,y)\left[\frac{\Phi_y(x,y)-\Phi_x(x,y)}{y-x}\right]\\&-h''(\Phi(x,y))\left[4\Phi_{xy}(x,y)\left(\frac{\Phi_y(x,y)-\Phi_x(x,y)}{y-x}\right)\right.\\&\left.+2\left(\frac{\Phi_x(x,y)\Phi_{y^2}(x,y)-\Phi_y(x,y)\Phi_{x^2}(x,y)}{y-x}\right)\right]\\&+h'(\Phi(x,y))2\left[\frac{\Phi_{x^2y}(x,y)-\Phi_{xy^2}(x,y)}{y-x}\right].\end{aligned} \tag{32}$$

The L´Hospital rule and the subsequent use of the relationships (20) yield

$$\lim_{y\to x}\left(\frac{\Phi_x(x,y)\Phi_{y^2}(x,y)-\Phi_y(x,y)\Phi_{x^2}(x,y)}{y-x}\right)=-2\left(\Phi_{x^2y}(x,x)+\Phi_{xy}^2(x,x)\right);$$

$$\lim_{y\to x}\left(\frac{\Phi_{x^2y}(x,y)-\Phi_{xy^2}(x,y)}{y-x}\right)=\Phi_{x^2y^2}(x,x)-\Phi_{xy^3}(x,x);$$

and therefore, taking into account the equality (27), a passage to the limit in (32) gives

$$\begin{aligned}\lim_{y\to x}\frac{A(x,y)}{y-x}&=h'''(x)\Phi_{xy}(x,x)+h''(x)4\left[3\Phi_{xy}^2(x,x)+\Phi_{x^2y}(x,x)\right]\\&+h'(x)2\left[\Phi_{x^2y^2}(x,x)-\Phi_{xy^3}(x,x)\right].\end{aligned} \tag{33}$$

On the other hand, the equality

$$\begin{aligned}B(x,x)&=\frac{1}{16}h^{(4)}(x)+\frac{1}{2}h'''(x)\Phi_{xy}(x,x)\\&+h''(x)\left[2\Phi_{x^2y}(x,x)+3\Phi_{xy}^2(x,x)\right]+h'(x)\Phi_{x^2y^2}(x,x);\end{aligned} \tag{34}$$

it is easily obtained from (25) so that, from (31), (33) and (34), the equality

$$\frac{1}{16}h^{(4)}(x)+\frac{3}{2}h'''(x)\Phi_{xy}(x,x)+h''(x)\left[6\Phi_{x^2y}(x,x)+15\Phi_{xy}^2(x,x)\right] \\ +h'(x)\left[3\Phi_{x^2y^2}(x,x)-2\Phi_{xy^3}(x,x)\right]=0,$$

is deduced. From this, the sought for additional relationship involving $\Phi_{x^3y}(x,x)$ and $\Phi_{x^2y^2}(x,x)$ is quickly deduced as follows:

$$3\Phi_{x^2y^2}(x,x)-2\Phi_{xy^3}(x,x)=-\frac{1}{h'(x)}\left[\frac{1}{16}h^{(4)}(x)+\frac{3}{2}h'''(x)\Phi_{xy}(x,x)\right. \\ \left.+h''(x)\left[6\Phi_{x^2y}(x,x)+15\Phi_{xy}^2(x,x)\right]\right] \tag{35}$$

Indeed, in view of (28) and (29), the equality (35) takes the form

$$3\Phi_{x^2y^2}(x,x)-2\Phi_{xy^3}(x,x)=\frac{1}{h'(x)}\left[-\frac{1}{16}h^{(4)}(x)+\frac{1}{8}\frac{h'''(x)h''(x)}{h'(x)}\right. \\ \left.+h''(x)\left[\frac{1}{4}\left(\frac{h''(x)}{h'(x)}\right)'-\frac{5}{48}\left(\frac{h''(x)}{h'(x)}\right)^2\right]\right]. \tag{36}$$

The value of $\Phi_{x^2y^2}(x,x)$ is finally found from (30) and (36) in the form

$$5\Phi_{x^2y^2}(x,x)=\frac{1}{h'(x)}\left[-\frac{1}{16}h^{(4)}(x)+\frac{1}{8}\frac{h'''(x)h''(x)}{h'(x)}\right. \\ \left.+h''(x)\left[\frac{1}{4}\left(\frac{h''(x)}{h'(x)}\right)'-\frac{5}{48}\left(\frac{h''(x)}{h'(x)}\right)^2\right]\right]-\frac{1}{12}\left(\frac{h''(x)}{h'(x)}\right)''$$

or, developing the derivatives of the quotient $h''(x)/h'(x)$, also in the form

$$\Phi_{x^2y^2}(x,x)=\frac{1}{48}\frac{1}{h'(x)}\left[-\frac{7}{5}h^{(4)}(x)+6\frac{h''(x)h'''(x)}{h'(x)}-5\frac{(h''(x))^3}{(h'(x))^2}\right]. \tag{37}$$

4.2 Successive derivatives of the equation (1)

Differentiating the functions on both sides of equation (1) produces

$$\partial_x\rightarrow\quad f'(x+y)=f'(x)+2f'(\Phi(x,y))\Phi_x(x,y);$$

$$\partial_{xy}\rightarrow\quad \frac{1}{2}f''(x+y)=f''(\Phi(x,y))\Phi_x(x,y)\Phi_y(x,y)+f'(\Phi(x,y))\Phi_{xy}(x,y); \tag{38}$$

$$\partial_{x^2y} \to \quad \frac{1}{2}f'''(x+y) = f'''(\Phi(x,y))\Phi_x^2(x,y)\Phi_y(x,y)$$
$$+f''(\Phi(x,y))\left[\Phi_{x^2}(x,y)\Phi_y(x,y) + 2\Phi_x(x,y)\Phi_{xy}(x,y)\right]+f'(\Phi(x,y))\Phi_{x^2y}(x,y);$$

$$\begin{aligned}\partial_{x^2y^2} \to \quad \frac{1}{2}f^{(4)}(x+y) &= f^{(4)}(\Phi(x,y))\Phi_x^2(x,y)\Phi_y^2(x,y)\\ &+ f'''(\Phi(x,y))\left[4\Phi_x(x,y)\Phi_y(x,y)\Phi_{xy}(x,y)\right.\\ &\left.+\Phi_x^2(x,y)\Phi_{y^2}(x,y) + \Phi_{x^2}(x,y)\Phi_y^2(x,y)\right]\\ &+ f''(\Phi(x,y))\left[2\Phi_{x^2y}(x,y)\Phi_y(x,y) + 2\Phi_x(x,y)\Phi_{xy^2}(x,y)\right.\\ &\left.+\Phi_{x^2}(x,y)\Phi_{y^2}(x,y) + 2\Phi_{xy}^2(x,y)\right]\\ &+ f'(\Phi(x,y))\Phi_{x^2y^2}(x,y). \end{aligned} \tag{39}$$

Assuming that Φ is a reflexive and symmetric function and setting $y = x$, the equalities (38) and (39) respectively take (after (20)) the forms

$$\frac{1}{2}f''(2x) = \frac{1}{4}f''(x) + f'(x)\Phi_{xy}(x,x)$$

and

$$\begin{aligned}\frac{1}{2}f^{(4)}(2x) = \frac{1}{16}f^{(4)}(x) &+ \frac{1}{2}f'''(x)\Phi_{xy}(x,x)\\ &+ f''(x)\left[2\Phi_{x^2y}(x,x) + 3\Phi_{xy}^2(x,x)\right] + f'(x)\Phi_{x^2y^2}(x,x).\end{aligned}$$

In view of $f''(2x) = f''(x)$ and $f^{(4)}(2x) = \frac{1}{4}f^{(4)}(x)$, from the first equality it is obtained

$$f''(x) = 4f'(x)\Phi_{xy}(x,x), \tag{40}$$

while the second one gives

$$\begin{aligned}\frac{1}{16}f^{(4)}(x) = \frac{1}{2}f'''(x)\Phi_{xy}(x,x) &+ f''(x)\left[2\Phi_{x^2y}(x,x) + 3\Phi_{xy}^2(x,x)\right]\\ &+ f'(x)\Phi_{x^2y^2}(x,x).\end{aligned} \tag{41}$$

4.3 Partial integration of the equation (13)

The equation (13) is transformed in a first order equation by expressing φ' as a function of φ. In fact, once the substitution

$$\varphi' = p(\varphi), \quad \varphi'' = p'(\varphi)p(\varphi), \tag{42}$$

is made in (13), the following equation is obtained:

$$p'(\varphi)p(\varphi) - 7\varphi p(\varphi) + \frac{19}{9}\varphi^3 = 0.$$

Since this last equation is invariant under the scale transformation

$$\begin{cases} \varphi^* = \alpha\varphi \\ p^* = \alpha^2 p \end{cases},$$

the substitution

$$p(\varphi) = \varphi^2 u(\varphi), \quad p'(\varphi) = 2\varphi u(\varphi) + \varphi^2 u'(\varphi), \tag{43}$$

converts it in the following separable equation:

$$\varphi u(\varphi)u'(\varphi) + 2u^2(\varphi) - 7u(\varphi) + \frac{19}{9} = 0$$

which, written as a total differential, takes the form

$$\frac{u\,du}{2u^2 - 7u + \frac{19}{9}} + \frac{d\varphi}{\varphi} = 0.$$

An elementary integration of this equation gives

$$\frac{19}{34}\ln\left|u - \frac{19}{6}\right| - \frac{1}{17}\ln\left|u - \frac{1}{3}\right| + \ln\varphi = C$$

or

$$\varphi^{34}\frac{\left|u - \frac{19}{6}\right|^{19}}{\left|u - \frac{1}{3}\right|^2} = K, \tag{44}$$

with a constant $0 \le K \le +\infty$. After (42) and (43) it can be written

$$\begin{aligned} \frac{\left|\varphi' - \frac{19}{6}\varphi^2\right|^{19}}{\left|\varphi' - \frac{1}{3}\varphi^2\right|^2} &= \frac{\left|p(\varphi) - \frac{19}{6}\varphi^2\right|^{19}}{\left|p(\varphi) - \frac{1}{3}\varphi^2\right|^2} \\ &= \frac{\left|\varphi^2 u - \frac{19}{6}\varphi^2\right|^{19}}{\left|\varphi^2 u - \frac{1}{3}\varphi^2\right|^2} \\ &= \varphi^{34}\frac{\left|u - \frac{19}{6}\right|^{19}}{\left|u - \frac{1}{3}\right|^2}, \end{aligned}$$

so that the equality (44) can be equivalently written as follows:

$$\frac{\left|\varphi' - \frac{19}{6}\varphi^2\right|^{19}}{\left|\varphi' - \frac{1}{3}\varphi^2\right|^2} = K,$$

with $0 \leq K \leq +\infty$.

It is worth noticing that the limit cases in which $K = 0$ or $K = +\infty$ also provide solutions to equation (13). In fact, in these cases (44) respectively gives

$$\varphi'(x) - \frac{19}{6}\varphi^2(x) = 0 \text{ or } \varphi'(x) - \frac{1}{3}\varphi^2(x) = 0;$$

and if φ is a solution to, for example, the first equation, then the equalities

$$\varphi''(x) = \frac{361}{18}\varphi^3(x), \quad \varphi(x)\varphi'(x) = \frac{19}{6}\varphi^3(x)$$

hold, so that

$$\begin{aligned}\varphi''(x) - 7\varphi(x)\varphi'(x) + \frac{19}{9}\varphi^3(x) &= \left(\frac{361}{18} - 7 \times \frac{19}{6} + \frac{19}{9}\right)\varphi^3(x)\\ &= 0.\end{aligned}$$

The equalities

$$\varphi''(x) = \frac{2}{9}\varphi^3(x), \quad \varphi(x)\varphi'(x) = \frac{1}{3}\varphi^3(x),$$

hold for the second equation and a similar computation shows that a solution φ to this equation also solves (13).

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