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Hopf lemma for the fractional diffusion operator and its application to a fractional free-boundary problem

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ABSTRACT

We consider a one-dimensional moving-boundary problem for the time-fractional diffusion equation, where the time-fractional derivative of order $\alpha \in (0, 1)$ is taken in the Caputo sense. A generalization of the Hopf lemma is proved and then used to prove a monotonicity property for the free-boundary when a fractional free-boundary Stefan problem is investigated.

1. Introduction

The development of fractional calculus dates from the XIX century. Mathematicians as Lacroix, Abel, Liouville, Riemann and Letnikov proposed several definitions of fractional derivatives. While the definition given by Caputo in 1967 [6] motivated the physical applications, the previous definitions enabled a great theoretical progress.

The study of fractional differential equations started to develop at the end of 50's, and in the past decades many authors pointed out that derivatives and integrals of non-integer order are very useful to describe the properties of various real-world materials such as polymers or some types of non-homogeneous solids. The trend indicates that the new fractional order models are more suitable than integer order models previously used, since fractional derivatives constitute an excellent tool to describe properties of memory and heritage of various materials and processes. Works in this direction are e.g. [1,7,10,15,26].

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The fractional derivative in the Caputo sense of arbitrary order $\alpha > 0$ is defined by

$$ {}_aD^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n \\ f^{(n)}(t), & \alpha = n, \end{cases} $$

where $n \in \mathbb{N}$ and Γ is the Gamma function defined by $\Gamma(x) = \int_0^\infty w^{x-1} e^{-w} dw$.

This paper deals with the fractional diffusion equation (hereinafter FDE), obtained from the standard diffusion equation by replacing the first order time-derivative by a fractional derivative of order $\alpha \in (0,1)$ in the Caputo sense:

$$ {}_0D_t^\alpha u(x,t) = \lambda^2 u_{xx}(x,t), \quad -\infty < x < \infty,\ t > 0,\ 0 < \alpha < 1. $$

The FDE has been investigated by a number of authors (see [11,16,19,21,23]) and several applications were considered. In particular, Mainardi in [22] studied the application to the theory of linear viscoelasticity.

Eberhard F. Hopf was an Austrian mathematician who made significant contributions in differential equations, topology and ergodic theory. One of his most famous works is related to the strong maximum principle for partial differential equations of elliptic type. In [13,14] the proof of an important theorem related to the sign of the outside directional derivative of a solution to an elliptic partial differential inequality is given. This theorem was extended later for partial differential operators of parabolic type by A. Friedman [9] and R. Viborni [27] separately. A one-dimensional version of this theorem can be found in [5], under the name of Hopf Lemma, and its generalization for the FDE is the aim of this work.

That is to say, under certain assumptions that will be given in detail later, we can prove that $u_x(s_2(t_0), t_0) > 0$; provided that u attains its maximum at $(s_2(t_0), t_0)$ and

$$ \begin{array}{lll} (i) & {}_0D_t^\alpha u(x,t) = \lambda^2 u_{xx}(x,t), & s_1(t) < x < s_2(t),\ 0 < t \le T,\ 0 < \alpha < 1, \\ (ii) & u(s_1(t),t) = g(t), & 0 < t \le T, \\ (iii) & u(s_2(t),t) = h(t), & 0 < t \le T, \\ (iv) & u(x,0) = f(x), & a \le x \le b, \quad s_1(0) = a, \quad s_2(0) = b, \end{array} \tag{1} $$

where s_1 and s_2 are given functions.

2. Fractional Hopf lemma

Consider the moving-boundary problem for the FDE defined in (1) assuming the hypotheses below:

(H1) s_1 is given and it is an upper Lipschitz continuous function in $[0,T]$.[1]
(H2) s_2 is given and it is a lower Lipschitz continuous function in $[0,T]$.
(H3) $s_1(0) = a$, $s_2(0) = b$, where $a \le b$, and condition (iv) of problem (1) is not considered if $a = b$.
(H4) $s_1(t) < s_2(t)$ for all $t \in (0,T)$.
(H5) f is a non-negative continuous function defined in $[a,b]$.
(H6) g and h are non-negative continuous functions defined in $(0,T]$.

We consider the following two regions:

[1] We say that f is an upper (respectively, lower) Lipschitz continuous function in $[0,T]$ if there exists a constant $c > 0$ such that $f(t_2) - f(t_1) \ge -c(t_2 - t_1)$, $0 \le t_1 < t_2 \le T$ (respectively, $f(t_2) - f(t_1) \le -c(t_2 - t_1)$, $0 \le t_1 < t_2 \le T$).

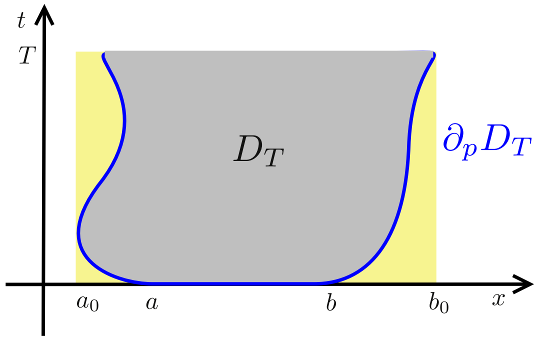

Fig. 1. Region $[a_0, b_0] \times [0, T]$ where u is defined.

$$D_T = \{(x,t)/s_1(t) < x < s_2(t),\ 0 < t \le T\};$$
$$\partial_p D_T = \{(s_1(t),t),\ 0 < t \le T\} \cup \{(s_2(t),t),\ 0 < t \le T\} \cup \{(x,0),\ a \le x \le b\},$$

where the latter is called parabolic boundary.

Definition 1. A function $u = u(x,t)$ is a solution of problem (1) if

1. u is defined in $[a_0, b_0] \times [0, T]$, where $a_0 := \min\{s_1(t),\ t \in [0,T]\}$ and $b_0 := \max\{s_2(t),\ t \in [0,T]\}$.
2. $u \in CW_{D_T} := C(D_T) \cap W_t^1((0,T)) \cap C_x^2(D_T)$, where $W_t^1((0,T)) := \{f(x,\cdot) \in C^1((0,T)) \cap L^1(0,T)$ for every fixed $x \in [a_0, b_0]\}$.
3. u is continuous in $D_T \cup \partial_p D_T$ except perhaps at $(a,0)$ and $(b,0)$ where

$$0 \le \liminf_{(x,t)\to(a,0)} u(x,t) \le \limsup_{(x,t)\to(a,0)} u(x,t) < +\infty$$

and

$$0 \le \liminf_{(x,t)\to(b,0)} u(x,t) \le \limsup_{(x,t)\to(b,0)} u(x,t) < +\infty.$$

4. u satisfies the conditions in (1).

Remark 1. We request u to be defined in $[a_0, b_0] \times [0, T]$ since the fractional derivative ${}_0D_t^\alpha u(x,t)$ involves values $u_t(x,\tau)$ for all τ in $[0,t]$. See Fig. 1.

Remark 2. This kind of problem has not yet been deeply studied. Nevertheless, taking into account the results obtained in [24] and [25], we can assert that the following problem:

$$\begin{aligned}
&{}_0D_t^\alpha u(x,t) = u_{xx}(x,t), && 0 < x < t^{\alpha/2}, \quad 0 < t \le T, \quad 0 < \alpha < 1,\\
&u(0,t) = B, && 0 < t \le T,\\
&u(t^{\alpha/2},t) = C, && 0 < t \le T,
\end{aligned}$$

where B and C are constants, admits the solution given by

$$u(x,t) = B + \frac{C - B}{1 - W\left(-1, -\frac{\alpha}{2}, 1\right)} \left[1 - W\left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right]$$

where $W\left(\cdot, -\frac{\alpha}{2}, 1\right)$ is the *Wright function* with the parameters $\rho = -\frac{\alpha}{2}$ and $\beta = 1$,

$$W(z, \rho, \beta) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\rho k + \beta)}, \quad z \in \mathbb{C}, \quad \rho > -1, \quad \beta \in \mathbb{R}.$$

The function $1 - W(-\cdot, -\frac{\alpha}{2}, 1)$ is the “fractional error function”, which satisfies

$$\lim_{\alpha \nearrow 1} 1 - W(-x, -\frac{\alpha}{2}, 1) = \operatorname{erf}\left(\frac{x}{2}\right)$$

(see [24, Theorem 4.1]).

Hereinafter we take $\lambda = 1$, and denote by D^α the fractional derivative in the Caputo sense with starting point $a = 0$, ${}_0D_t^\alpha$, and by L^α the operator associated with the FDE, $L^\alpha := \frac{\partial^2}{\partial x^2} - D^\alpha$.

Proposition 1. *If u is a function with $L^\alpha[u] > 0$ in D_T, then u does not attain its maximum at D_T.*

Proof. Suppose that there exists $(x_0, t_0) \in D_T$ (that is, $s_1(t_0) < x_0 < s_2(t_0)$, $0 < t_0 \leq T$), such that u attains its maximum at (x_0, t_0). Due to the extremum principle for the Caputo derivative (see [20]), we have $D_t^\alpha u(x_0, t_0) \geq 0$. Moreover, since $u \in C_x^2(D_T)$, we have $\frac{\partial^2 u}{\partial x^2}(x_0, t_0) \leq 0$. Then, $L^\alpha[u](x_0, t_0) \leq 0$ which is a contradiction. □

The following corollary is an immediate consequence of previous proposition.

Corollary 1. *If u is a function with $L^\alpha[u] < 0$ in D_T, then u does not attain its minimum in D_T.*

The results obtained in [18] can be adapted to the moving-boundary problem (1). For this reason we omit the proof of the following assertion.

Theorem 1. *Let $u \in CW_{D_T}$ be a solution of (1). Then either*

$$u(x, t) \geq 0 \quad \text{for all } (x, t) \in \overline{D_T} \quad \text{or} \quad u \text{ attains its negative minimum on } \partial_p D_T.$$

Let us state the main result of this paper.

Theorem 2. *Let $u \in CW_{D_T}$ be a solution of problem (1) satisfying the hypotheses (H1)–(H6).*

1. *If there exist $t_0 > 0$ and $\delta > 0$ such that*

$$u(s_2(t_0), t_0) = M = \sup_{\partial_p D_T} u, \tag{2}$$

$$|s_1(t_0) - s_2(t_0)| \geq \delta \quad \text{and} \quad u(x, t_0) < M \text{ for every } x \in (s_2(t_0) - \delta, s_2(t_0)), \tag{3}$$

then

$$\liminf_{x \nearrow s_2(t_0)} \frac{u(x, t_0) - u(s_2(t_0), t_0)}{x - s_2(t_0)} > 0. \tag{4}$$

If u_x exists at $(s_2(t_0), t_0)$, then

$$u_x(s_2(t_0), t_0) > 0. \tag{5}$$

2. *If there exist $t_0 > 0$ and $\delta > 0$ such that*

$$u(s_2(t_0), t_0) = m = \inf_{\partial_p D_T} u,$$

$$|s_1(t_0) - s_2(t_0)| \geq \delta \quad \text{and} \quad u(x, t_0) > m \text{ for every } x \in (s_2(t_0) - \delta, s_2(t_0)),$$

then

$$\limsup_{x \nearrow s_2(t_0)} \frac{u(x, t_0) - u(s_2(t_0), t_0)}{x - s_2(t_0)} < 0.$$

If u_x exists at $(s_2(t_0), t_0)$, then

$$u_x(s_2(t_0), t_0) < 0.$$

Proof. We prove only point 1. The proof of point 2 is analogous.

Consider

$$w_\alpha(x, t) = \epsilon \left[1 - \exp\{-\mu(x - s_2(t_0))\} \frac{E_\alpha(\mu A t^\alpha)}{E_\alpha(\mu A t_0^\alpha)}\right] + M$$

where A, μ and ϵ will be determined, M is defined in (2) and E_α is the Mittag–Leffler function defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}, \quad \alpha > 0.$$

Note that $w_\alpha = M$ for every point in the curve

$$\exp\{-\mu(x - s_2(t_0))\} \frac{E_\alpha(\mu A t^\alpha)}{E_\alpha(\mu A t_0^\alpha)} = 1 \tag{6}$$

and that curve (6) is the graph of the function

$$f(t) = \frac{1}{\mu} \ln\left(\frac{E_\alpha(\mu A t^\alpha)}{E_\alpha(\mu A t_0^\alpha)}\right) + s_2(t_0), \quad t \in (0, t_0].$$

Clearly,

$$f(t_0) = s_2(t_0) \quad \text{and} \quad f \text{ is an increasing function if } \mu > 0. \tag{7}$$

Furthermore, there exists $t_1 < t_0$ such that $f(t) < s_2(t)$, $t \in (t_1, t_0)$. In fact, we know from assumption (H2) that s_2 is a lower Lipschitz continuous function. Then there exists a constant $L > 0$ such that $s_2(t) \geq L(t - t_0) + s_2(t_0)$, for every $0 \leq t \leq t_0$.

Besides, taking into account that $E_\alpha(\mu A t^\alpha) = \sum_{k=0}^{\infty} \frac{(\mu A t^\alpha)^k}{\Gamma(\alpha k+1)}$ is a uniform convergent series over compact sets, and $z\Gamma(z) = \Gamma(z + 1)$ for all $z \in \Omega = \mathbb{C} - \{z = -n,\ n \in \mathbb{N}_0\}$, we have

$$[E_\alpha(\mu A t^\alpha)]' = \sum_{k=1}^{\infty} \frac{(\mu A)^k \alpha k t^{\alpha k - 1}}{\Gamma(\alpha k + 1)} = \sum_{k=0}^{\infty} \frac{(\mu A)^{k+1} t^{\alpha k + \alpha - 1}}{\Gamma(\alpha k + \alpha)} = \mu A t^{\alpha - 1} E_{\alpha,\alpha}(\mu A t^\alpha),$$

where the function $E_{\alpha,\alpha}$ is the generalized Mittag–Leffler function with the parameters $\rho = \beta = \alpha$,

$$E_{\rho,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \beta)}, \quad z \in \mathbb{C}, \rho > 0, \quad \beta \in \mathbb{C}.$$

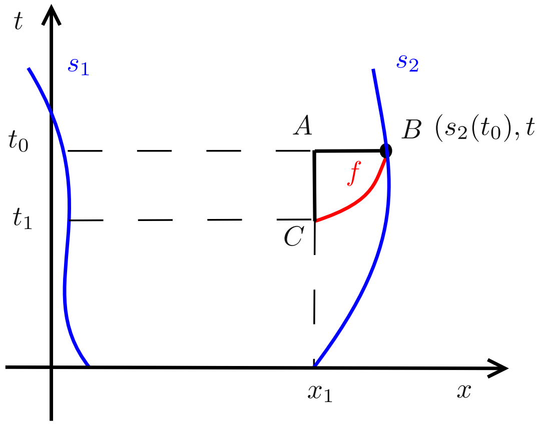

Fig. 2. Region $\mathcal{R}$.

Then

$$f'(t) = \frac{1}{\mu}\frac{1}{E_\alpha(\mu A t^\alpha)}\frac{\mu A}{t^{1-\alpha}}E_{\alpha,\alpha}(\mu A t^\alpha) = \frac{A}{t^{1-\alpha}}\frac{E_{\alpha,\alpha}(\mu A t^\alpha)}{E_\alpha(\mu A t^\alpha)}. \tag{8}$$

Define the function $H\colon \mathbb{R}_0^+ \to \mathbb{R}$ such that $H(t) = \frac{E_{\alpha,\alpha}(\mu A t^\alpha)}{E_\alpha(\mu A t^\alpha)}$.

H is positive and continuous in $[0,\infty)$. $H(0) = \frac{1}{\Gamma(\alpha)} > 0$ since $0 < \alpha < 1$. $H(+\infty) = C > 0$ because it is a quotient of continuous functions with equal order in ∞ (see [12]). Then, there exists $m_0 > 0$ such that

$$H(t) \geq m_0 \quad \text{for all } t \geq 0, \text{ for every } A, \mu > 0. \tag{9}$$

From (8) and (9), $f'(t_0) \geq \frac{A}{t_0^{1-\alpha}} m_0$. Selecting $A > 0$ such that $\frac{A}{t_0^{1-\alpha}} m_0 > L$ we can assure that $f'(t_0) > L$.

Finally, let ρ be a positive number such that $f'(t_0) - \rho > L$. Due to the differentiability of f at t_0 there exists $t_1 < t_0$ such that for every $t \in (t_1, t_0)$,

$$L < f'(t_0) - \rho < \frac{f(t) - f(t_0)}{t - t_0} \;\Rightarrow\; f(t) < L(t - t_0) + f(t_0) = L(t - t_0) + s_2(t_0) \leq s_2(t).$$

Note that assumption (3) and condition (7) imply that we can select t_1 so that $s_1(t) < f(t) < s_2(t)$ for all $t \in (t_1, t_0)$.

Now, consider the points $A(x_1, t_0)$ (where $x_1 = f(t_1)$), $B(s_2(t_0), t_0)$ and $C(x_1, t_1)$. Hypothesis (3) allows to set t_1 again such that $x_1 \in (s_2(t_0) - \delta, s_2(t_0))$ and $u < M$ in $\overline{AC}$.

Let $\mathcal{R}$ be the region limited by $\overline{AB}$, $\overline{AC}$ and the portion of the graph of f from B to C, which we call $\widehat{CB}$ (see Fig. 2). The region $\mathcal{R}_{t_0} = \mathcal{R}^\circ \cup (\overline{AB} - \{A, B\})$ and its parabolic boundary $\partial_p\mathcal{R} = \overline{AC} \cup \widehat{CB}$ will be considered.

Next, we analyze the behavior of u and w_α in the parabolic boundary $\partial_p\mathcal{R}$. Let $M_0 = \max\limits_{t_1 \leq t \leq t_0} u(x_1, t)$. Because of the continuity of u, assumption (3) and the possibility to reset t_1 if it is necessary, we have $M_0 < M$. Calling $\eta = M - M_0$, it yields

$$u \leq M - \eta \quad \text{in } \overline{AC} \quad \text{and} \quad u \leq M \quad \text{in } \widehat{CB}. \tag{10}$$

Moreover,

$$w_\alpha \geq -\eta + M \quad \text{in } \overline{AC} \quad \text{and} \quad w_\alpha = M \quad \text{in } \widehat{CB}. \tag{11}$$

In fact, $E_\alpha(\mu A t^\alpha)$ is an increasing function in $\overline{AC}$, then

$$w_\alpha(x_1,t) = \epsilon\left[1 - \exp\{-\mu(x_1 - s_2(t_0))\}\frac{E_\alpha(\mu A t^\alpha)}{E_\alpha(\mu A t_0^\alpha)}\right] + M \geq$$
$$\geq \epsilon\left[1 - \exp\{-\mu(x_1 - s_2(t_0))\}\right] + M \geq -\eta + M,$$

if $\epsilon = \dfrac{\eta}{\exp\{-\mu(x_1 - s_2(t_0))\} - 1}$.

Considering the fact that $D^\alpha(E_\alpha(\mu A t^\alpha)) = \mu A E_\alpha(\mu A t^\alpha)$ (see [16]) and applying the operator L^α to the function w_α,

$$L^\alpha[w_\alpha](x,t) = \epsilon\exp\{-\mu(x - s_2(t_0))\}\frac{E_\alpha(\mu A t^\alpha)}{E_\alpha(\mu A t_0^\alpha)}(\mu A - \mu^2) < 0 \quad \text{if } \mu = A + 1. \tag{12}$$

Finally, we define $z = w_\alpha - u$ in $\mathcal{R}$ and analyze the behavior of z in the parabolic boundary $\partial_p\mathcal{R}$. From (10) and (11) we obtain

$$z \geq 0 \quad \text{in} \quad \overline{AC} \quad \text{and} \quad z \geq 0 \quad \text{in} \quad \widehat{CB}.$$

Also, from (12), we have $L^\alpha[z] = L^\alpha[w_\alpha] - L^\alpha[u] < 0$ in $\mathcal{R}_{t_0}$.

Applying Corollary 1, we can state that z does not attain its minimum at $\mathcal{R}_{t_0}$. Then $z \geq 0$ in $\mathcal{R}$. In particular,

$$z(x,t_0) = w_\alpha(x,t_0) - u(x,t_0) \geq 0, \quad \text{for all } x_1 \leq x \leq s_2(t_0). \tag{13}$$

Recall that $u(s_2(t_0),t_0) = w_\alpha(s_2(t_0,t_0)) = M$, so the next inequality is equivalent to (13):

$$\frac{u(x,t_0) - u(s_2(t_0),t_0)}{x - s_2(t_0)} \geq \frac{w_\alpha(x,t_0) - w_\alpha(s_2(t_0,t_0))}{x - s_2(t_0)}. \tag{14}$$

Then

$$\liminf_{x\nearrow s_2(t_0)} \frac{u(x,t_0) - u(s_2(t_0),t_0)}{x - s_2(t_0)} \geq \liminf_{x\nearrow s_2(t_0)} \frac{w_\alpha(x,t_0) - w_\alpha(s_2(t_0,t_0))}{x - s_2(t_0)}.$$

But w_α is a differentiable function at $(s(t_0),t_0)$, therefore

$$\liminf_{x\nearrow s_2(t_0)} \frac{w_\alpha(x,t_0) - w_\alpha(s_2(t_0,t_0))}{x - s_2(t_0)} = (w_\alpha)_x(s_2(t_0),t_0) = \epsilon\mu = \epsilon(A+1) > 0$$

and condition (4) holds.

Finally, if the derivative u_x exists at $(s_2(t_0),t_0)$, the inequality (14) implies that $u_x(s_2(t_0),t_0) \geq (w_\alpha)_x(s_2(t_0),t_0) > 0$ and hence the condition (5) holds. □

Analogous results hold if we consider s_1 instead of s_2.

Theorem 3. *Let $u \in CW_{D_T}$ be a solution of problem (1) satisfying the hypotheses (H1)–(H6).*

1. *If there exist $t_0 > 0$ and $\delta > 0$ such that*

$$u(s_1(t_0),t_0) = M = \sup_{\partial_p D_T} u,$$
$$|s_1(t_0) - s_2(t_0)| \geq \delta \quad \textit{and} \quad u(x,t_0) < M \textit{ for every } x \in (s_1(t_0), s_1(t_0) + \delta),$$

then

$$\limsup_{x\nearrow s_1(t_0)} \frac{u(x,t_0)-u(s_1(t_0),t_0)}{x-s_1(t_0)} < 0.$$

If u_x exists at $(s_1(t_0),t_0)$, then

$$u_x(s_1(t_0),t_0) < 0.$$

2. *If there exist $t_0 > 0$ and $\delta > 0$ such that*

$$u(s_1(t_0),t_0) = m = \inf_{\partial_p D_T} u,$$
$$|s_1(t_0)-s_2(t_0)| \geq \delta \quad \textit{and} \quad u(x,t_0) > m \textit{ for every } x \in (s_1(t_0), s_1(t_0)+\delta),$$

then

$$\liminf_{x\nearrow s_1(t_0)} \frac{u(x,t_0)-u(s_1(t_0),t_0)}{x-s_1(t_0)} > 0.$$

If u_x exists at $(s_1(t_0),t_0)$, then

$$u_x(s_1(t_0),t_0) > 0.$$

Remark 3. As we said before, this result can be found in [5] for the case $\alpha = 1$, where the author works only with exponential functions. It is well known that the Caputo derivative of the exponential function is not an exponential function. Therefore, we use the Mittag–Leffler function.

3. An application to fractional free-boundary Stefan problems

In this section we consider the following fractional free-boundary Stefan problem for the FDE

$$\begin{array}{lll}
(i) & D^\alpha u(x,t) = u_{xx}(x,t), & 0<x<s(t),\ 0<t<T,\ 0<\alpha<1,\ \lambda>0 \\
(ii) & u(x,0) = f(x), & 0 \leq x \leq b = s(0), \\
(iii) & u(0,t) = g(t), & 0 < t \leq T, \\
(iv) & u(s(t),t) = 0, & 0 < t \leq T, \\
(v) & D^\alpha s(t) = -k u_x(s(t),t), & 0 < t \leq T, \quad k > 0 \text{ constant},
\end{array} \tag{15}$$

where we have replaced the Stefan condition $\dfrac{ds(t)}{dt} = k u_x(s(t),t)$, $t > 0$, by the fractional Stefan condition

$$D^\alpha s(t) = -k u_x(s(t),t), \quad t > 0, \quad 0 < \alpha < 1.$$

Definition 2. A pair $\{u, s\}$ is a solution of problem (15) if

1. u is defined in $[0,b_0] \times [0,T]$ where $b_0 := \max\{s(t),\ 0 \leq t \leq T\}$.
2. $u \in CW_{D_T}$.
3. u is continuous in $D_T \cup \partial_p D_T$ except perhaps at $(0,0)$ and $(b,0)$ where

$$0 \leq \liminf_{(x,t)\to(0,0)} u(x,t) \leq \limsup_{(x,t)\to(0,0)} u(x,t) < +\infty$$

and

$$0 \leq \liminf_{(x,t)\to(b,0)} u(x,t) \leq \limsup_{(x,t)\to(b,0)} u(x,t) < +\infty.$$

4. s is a continuous function in $[0,T]$ such that $s \in W^1(0,T)$.
5. There exists $u_x(s(t),t)$ for all $t \in (0,T]$.
6. u and s satisfy (15).

This kind of problem has recently been treated in [3,8,17,24], and numerical solutions can be found in [4]. Our next goal is to prove the following assertion involving the monotonicity of the free boundary.

Theorem 4. *Let $\{u_1, s_1\}$ and $\{u_2, s_2\}$ be solutions of the fractional free-boundary Stefan problems (15) corresponding to the data $\{b_1, f_1, g_1\}$ and $\{b_2, f_2, g_2\}$, respectively. Suppose that $b_1 < b_2$, $0 \leq f_1 \leq f_2$ and $0 \leq g_1 \leq g_2$. Then $s_1(t) < s_2(t)$ for all $t \in [0,T)$.*

Proof. We know that s_1 and s_2 are continuous functions, and $s_1(0) = b_1 < b_2 = s_2(0)$. Suppose that the set $A = \{t \in [0,T] \mid (s_1 - s_2)(t) = 0\} \neq \emptyset$, and let be $t_0 = \min A$. Due to the continuity of s_1 and s_2, $s_1(t_0) = s_2(t_0)$, and t_0 is the first t for which $s_1(t_0) = s_2(t_0)$.

Let $h(t) = (s_1 - s_2)(t)$, $t \in [0,t_0]$. This function has the following properties:

(h-1) $h \in C^1(0,t_0] \cap C[0,t_0]$ (due to Definition 2).
(h-2) $h(0) = b_1 - b_2 < 0$.
(h-3) h is a non-positive function and $h(t_0) = 0$.

From (h-1)–(h-3), h attains its maximum value at t_0.

Using the estimate [2, Eq. (12)], we obtain $D^\alpha h(t_0) \geq \dfrac{h(t_0) - h(0)}{t_0^\alpha \Gamma(1-\alpha)}$.

Then,

$$D^\alpha h(t_0) \geq \frac{b_2 - b_1}{t_0^\alpha \Gamma(1-\alpha)} > 0. \tag{16}$$

Taking into account the linearity of the Caputo fractional derivative and the fact that s_1 and s_2 satisfy the Stefan condition (15)-(v), from the inequality (16) we derive

$$u_{2x}(s_2(t_0), t_0) - u_{1x}(s_1(t_0), t_0) > 0. \tag{17}$$

Observe that $w(x,t) = u_2(x,t) - u_1(x,t)$ is a solution of the moving-boundary problem

$$\begin{aligned}
&D^\alpha w(x,t) = w_{xx}(x,t), && 0 < x < s_1(t),\ 0 < t \leq t_0,\ 0 < \alpha < 1,\\
&w(0,t) = (g_2 - g_1)(t) \geq 0, && 0 < t \leq t_0,\\
&w(s_1(t),t) = u_2(s_1(t),t), && 0 < t \leq t_0,\\
&w(x,0) = (f_2 - f_1)(x) \geq 0, && 0 \leq x \leq b_1 = s_1(0).
\end{aligned} \tag{18}$$

Applying Theorem 1 to u_2 in the region $\overline{D^2_{t_0}}$, where $D^2_{t_0} = \{(x,t) \mid 0 < t \leq t_0,\ 0 < x < s_2(t)\}$, we have $u_2(s_1(t),t) \geq 0$.

For w satisfying the problem (18), Theorem 1 gives the condition $w \geq 0$ in $\overline{D^1_{t_0}}$, where $D^1_{t_0} = \{(x,t) \mid 0 < t \leq t_0,\ 0 < x < s_1(t)\}$. Then w attains a minimum at $(s_1(t_0), t_0)$.

If there exists $\epsilon > 0$ such that $w(x, t_0) > 0$ for all $x \in (s_1(t_0) - \epsilon, s_1(t_0))$, applying Theorem 2-2 we can conclude that $w_x(s_1(t_0), 0) < 0$. And then $u_{2x}(s_2(t_0), t_0) - u_{1x}(s_1(t_0), t_0) < 0$, which contradicts the inequality (17).

If, by contrast, we have a sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ and, for every $n \in \mathbb{N}$ there exists $x_n \in (s_1(t_0) - \epsilon_n, s_1(t_0))$ such that $w(x_n, t_0) = 0$, then

$$\lim_{n\to\infty} \frac{w(x_n, t_0) - w(s_1(t_0), t_0)}{\epsilon_n} = 0.$$

But the derivative $w_x(s_1(t_0), t_0)$ exists by Definition 2-6. Then $w_x(s_1(t_0), t_0) = 0$. Therefore $u_{2x}(s_2(t_0), t_0) - u_{1x}(s_1(t_0), t_0) = 0$, which contradicts the inequality (17) again.

This contradiction comes from assuming that there exists $t_0 > 0$ such that t_0 is the first t for which $s_1(t_0) = s_2(t_0)$. Therefore $s_1(t) < s_2(t)$ for all $t \in [0, T)$. □

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