On the Chern-Ricci flow and its solitons for Lie groups

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This paper is concerned with Chern-Ricci flow evolution of left-invariant hermitian structures on Lie groups. We study the behavior of a solution, as t is approaching the first time singularity, by rescaling in order to prevent collapsing and obtain convergence in the pointed (or Cheeger-Gromov) sense to a Chern-Ricci soliton. We give some results on the Chern-Ricci form and the Lie group structure of the pointed limit in terms of the starting hermitian metric and, as an application, we obtain a complete picture for the class of solvable Lie groups having a codimension one normal abelian subgroup. We have also found a Chern-Ricci soliton hermitian metric on most of the complex surfaces which are solvmanifolds, including an unexpected shrinking soliton example.

1 Introduction

The *Chern-Ricci flow* (CRF) is the evolution equation for a one-parameter family $\omega(t)$ of hermitian metrics on a fixed complex manifold (M, J) defined by

$$\frac{\partial}{\partial t}\omega = -2p, \qquad \text{or equivalently,} \qquad \frac{\partial}{\partial t}g = -2p(\cdot, J\cdot), \tag{1}$$

where $p = p(J, \omega(t))$ is the Chern-Ricci form and $g = \omega(\cdot, J\cdot)$ (see [6, 21, 22, 7]). This paper is concerned with CRF-flow evolution of (compact) hermitian manifolds (M, J, ω) whose universal cover is a Lie group G and such that if $\pi : G \longrightarrow M$ is the covering map, then π^*J and $\pi^*\omega$ are left-invariant. This is in particular the case of invariant structures on a quotient $M = G/\Gamma$, where Γ is a cocompact discrete subgroup of G (e.g. solvmanifolds and nilmanifolds). A CRF-flow solution on M is obtained by pulling down the corresponding CRF-flow solution on the Lie group G, which by diffeomorphism invariance stays left-invariant. Equation (1) therefore becomes an ODE for a non-degenerate 2-form $\omega(t)$ on the Lie algebra $\mathfrak{g}$ of G and thus short-time existence (forward and backward) and uniqueness of the solutions are always guaranteed (see [14]). We therefore study, more in general, left-invariant solutions on Lie groups which may or may not admit a cocompact discrete subgroup.

Let (G, J) be a Lie group endowed with a left-invariant complex structure. Since on Lie groups the Chern-Ricci form p depends only on J (see (3)), we obtain that along the CRF-solution starting at a left-invariant hermitian metric ω_0, $p(t) \equiv p_0 := p(J, \omega_0)$. This implies that $\omega(t)$ is simply given by

$$\omega(t) = \omega_0 - 2tp_0.$$

If P_0 is the Chern-Ricci operator of ω_0 (i.e. $p_0 = \omega_0(P_0\cdot, \cdot)$), then

$$\omega(t) = \omega_0((I - 2tP_0)\cdot, \cdot),$$

and so the solution exists as long as the hermitian map $I - 2tP_0$ is positive, say on a maximal interval (T_-, T_+), which can be easily computed in terms of the extremal eigenvalues of the symmetric operator P_0.

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We aim to understand the behavior of a CRF-solution $(G, \omega(t))$, as t is approaching $T_\pm$, in the same spirit as in [16, Section 3], where the long-time behavior of homogeneous type-III Ricci flow solutions is studied. In order to prevent collapsing and obtain as a limit a manifold of the same dimension as G, the question is whether we can find a hermitian manifold $(M, J_\pm, \omega_\pm)$, bi-holomorphic embeddings $\phi(t) : M \longrightarrow G$ and a scaling function $a(t) > 0$ so that $a(t)\phi(t)^*\omega(t)$ converges smoothly to $\omega_\pm$, as $t \to T_\pm$. Sometimes it is only possible to obtain this along a subsequence $t_k \to T_\pm$ and the diffeomorphisms $\phi(t_k)$ may be only defined on open subsets Ω_k exhausting M and so M might be non-diffeomorphic and even non-homeomorphic to G. This is called *pointed* or *Cheeger-Gromov* convergence of $(G, a(t)\omega(t))$ toward $(G_\pm, \omega_\pm)$.

It is proved in [14] that given any CRF-solution $(G, \omega(t))$, there is always a pointed limit $(G_\pm, J_\pm, \omega_\pm)$ as above, where $G_\pm$ is a Lie group (possibly non-isomorphic to G) and the hermitian structure $(J_\pm, \omega_\pm)$ is left-invariant. Moreover, $(J_\pm, \omega_\pm)$ is a *CR-soliton*, i.e.

$$p(J_\pm, \omega_\pm) = c\omega_\pm + \mathcal{L}_X\omega_\pm,$$

for some $c \in \mathbb{R}$ and a complete holomorphic vector field X on $G_\pm$, or equivalently, the CRF-flow solution $\widetilde{\omega}(t)$ starting at $\omega_\pm$ is self-similar, in the sense that

$$\widetilde{\omega}(t) = (-2ct + 1)\varphi(t)^*\omega_\pm,$$

for some bi-holomorphic diffeomorphisms $\varphi(t)$ of $(G_\pm, J_\pm)$. Actually, $\varphi(t)$ can be chosen to be a one-parameter group of automorphisms of $G_\pm$. In many cases, the rescaling considered to obtain a pointed limit is the usual one given by $\omega(t)/t$.

After some preliminaries, we give in Section 2 an alternative proof of the fact that any hermitian nilmanifold (i.e. G nilpotent) is Chern-Ricci flat (see [1, Lemma 2.2]) and so a fixed point for CRF. In Sections 3 and 4, we give an overview on the bracket flow approach and a structural result on CR-solitons from [14] and then give a construction procedure for CR-solitons, including a characterization of those which are Kähler-Ricci solitons.

We study in Section 5 to what extent the Chern-Ricci form and the Lie group structure of the pointed limit $(G_\pm, \omega_\pm)$ are determined by the starting hermitian metric (G, ω_0). For instance, we proved the following:

- If $P_0 \leq 0$ (i.e. $T_+ = \infty$) and $\mathfrak{k} := \operatorname{Ker} P_0$ is an abelian ideal of $\mathfrak{g}$, then $\omega(t)/t$ converges in the pointed sense, as $t \to \infty$, to a Chern-Ricci soliton (G_+, ω_+) with Lie algebra $\mathfrak{g}_+ = \mathfrak{k}^\perp \ltimes \mathfrak{k}$ and Lie bracket $[\cdot,\cdot]_+$ such that $[\mathfrak{k}, \mathfrak{k}]_+ = 0$. The Chern-Ricci operator of (G_+, ω_+) is given by $P_+|_{\mathfrak{k}^\perp} = -I$, $P_+|_{\mathfrak{k}} = 0$.

- If the eigenspace $\mathfrak{g}_m$ of the maximum positive eigenvalue of P_0 is a nonzero Lie subalgebra of $\mathfrak{g}$, then $T_+ < \infty$ and $\omega(t)/(T_+ - t)$ converges in the pointed sense, as $t \to T_+$, to a Chern-Ricci soliton (G_+, ω_+) with Lie algebra $\mathfrak{g}_+ = \mathfrak{g}_m \ltimes \mathfrak{g}_m^\perp$ and Lie bracket $[\cdot,\cdot]_+$ satisfying $[\mathfrak{g}_m^\perp, \mathfrak{g}_m^\perp]_+ = 0$ and whose Chern-Ricci operator equals $P_+|_{\mathfrak{g}_m} = \frac{1}{2}I$, $P_+|_{\mathfrak{g}_m^\perp} = 0$.

In Section 6, we apply the above mentioned results on convergence and CR-solitons to the class of solvable Lie groups having a codimension one normal abelian subgroup.

Finally, we deal with complex surfaces in Section 7. The family of 4-dimensional solvable Lie groups admitting a left-invariant complex structure is quite large. It consists of 19 groups, although six of them are actually continuous pairwise non-isomorphic families (see Table 1). Moreover, many of them admit more than one complex structure up to equivalence and one of them does admit a two-parameter continuous family of complex structures (see Table 2). This classification was obtained in [17]. We found a CR-soliton hermitian metric for each of these complex structures, with the exceptions of only seven structures. Most of them are either expanding or steady (i.e. $c \leq 0$), but one of the groups does admit an unexpected shrinking (i.e. $c > 0$) CR-soliton (see Example 7.3). Recall that this is in clear contrast to the behavior of other curvature flows on solvmanifolds like the Ricci flow (see [12]) and the symplectic curvature flow (see [15]). We were able to prove the non-existence of a CR-soliton in only one of the seven cases; namely for $\mathfrak{r}_{4,1}$. In this case, we found the non-isomorphic CR-soliton (G_+, ω_+) where all CRF-solutions on $\mathfrak{r}_{4,1}$ are converging to (see Example 7.2). The CR-soliton metrics and their respective Chern-Ricci operators are given in Table 3.

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2 Chern-Ricci form

Let (M, J, ω, g) be a $2n$-dimensional hermitian manifold, where $\omega = g(J\cdot, \cdot)$. The *Chern connection* is the unique connection ∇ on M which is hermitian (i.e. $\nabla J = 0$, $\nabla g = 0$) and its torsion satisfies $T^{1,1} = 0$. In terms of the Levi Civita connection D of g, the Chern connection is given by

$$g(\nabla_X Y, Z) = g(D_X Y, Z) - \tfrac{1}{2} d\omega(JX, Y, Z).$$

We refer to e.g. [24, (2.1)], [5, (2.1)] and [21, Section 2] for different equivalent descriptions. Note that $\nabla = D$ if and only if (M, J, ω, g) is Kähler.

The *Chern-Ricci form* $p = p(J, \omega, g)$ is defined by

$$p(X, Y) = \sum_{i=1}^{n} g(R(X, Y)e_i, Je_i),$$

where $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ is the curvature tensor of ∇ and $\{e_i, Je_i\}_{i=1}^n$ is a local orthonormal frame for g. It follows that p is closed, of type $(1, 1)$ (i.e. $p = p(J\cdot, J\cdot)$), locally exact and in the Kähler case coincides with the Ricci form $\operatorname{Rc}(J\cdot, \cdot)$.

Consider now a left-invariant (almost-) hermitian structure (J, ω, g) on a Lie group with Lie algebra $\mathfrak{g}$. The integrability condition can be written as

$$[JX, JY] = [X, Y] + J[JX, Y] + J[X, JY], \qquad \forall X, Y \in \mathfrak{g}. \tag{2}$$

It is proved in [24, Proposition 4.1] (see also [18]) that the Chern-Ricci form of (J, ω, g) is given by

$$p(X, Y) = -\tfrac{1}{2} \operatorname{tr} J \operatorname{ad}[X, Y] + \tfrac{1}{2} \operatorname{tr} \operatorname{ad} J[X, Y], \qquad \forall X, Y \in \mathfrak{g}. \tag{3}$$

We note that, remarkably, p only depends on J. The *Chern-Ricci operator* $P \in \operatorname{End}(\mathfrak{g})$, defined by

$$p = \omega(P\cdot, \cdot), \tag{4}$$

is a symmetric and hermitian map with respect to (J, g) which vanishes on the center of $\mathfrak{g}$.

It follows from [24, Proposition 4.2] that p vanishes if J is *bi-invariant* (i.e. $[J\cdot, \cdot] = J[\cdot, \cdot]$) or J is *abelian* (i.e. $[J\cdot, J\cdot] = [\cdot, \cdot]$) and $\mathfrak{g}$ unimodular. On the other hand, it follows from [1, Lemma 2.2] that hermitian nilmanifolds are all Chern-Ricci flat. We now give a proof of this fact for completeness, which is based on the proof of that lemma and it is a bit shorter.

Proposition 2.1 *The Chern-Ricci form vanishes for any left-invariant hermitian structure on a nilpotent Lie group.*

P r o o f. It is sufficient to prove that $\operatorname{tr}(J \operatorname{ad}_X) = 0$ for any $X \in \mathfrak{g}$ (see (3)), or equivalently, $\operatorname{tr}(J^c \operatorname{ad}_X) = 0$, for any $X \in \mathfrak{g}_{\mathbb{C}}$, where $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ is the complexification of $\mathfrak{g}$ and $J^c : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ is given by $J^c(X + iY) = JX + iJY$. Consider now the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ in $\pm i$-eigenspaces of J^c. Since J is integrable and $\mathfrak{g}$ is nilpotent, we have that $\mathfrak{g}^{1,0}$ is a (complex) nilpotent Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$. It follows that if $\{X_1, \dots, X_n\}$ is a basis of $\mathfrak{g}^{1,0}$, then $\beta = \{X_1, \dots, X_n, \overline{X}_1, \dots, \overline{X}_n\}$ is a basis of $\mathfrak{g}_{\mathbb{C}}$ and the matrix of ad_{X_k} relative to β has the form

$$\begin{bmatrix} A_k & * \\ 0 & B_k \end{bmatrix}.$$

Since $\operatorname{tr} A_k = 0$ and $\operatorname{tr} \operatorname{ad}_{X_k} = 0$ by nilpotency, we obtain that $\operatorname{tr} B_k = 0$. On the other hand, as the matrix of J^c relative to β is given by

$$\begin{bmatrix} iId & 0 \\ 0 & -iId \end{bmatrix},$$

it follows that the matrix of $J^c \operatorname{ad}_{X_k}$ is of the form

$$\begin{bmatrix} iA_k & * \\ 0 & -iB_k \end{bmatrix},$$

and so it has zero trace. A similar argument gives that $\operatorname{tr}(J^c \operatorname{ad}_{\overline{X}_k}) = 0$, concluding the proof. □

3 Chern-Ricci flow

Let (M, J) be a complex manifold. The *Chern-Ricci flow* (CRF) is the evolution equation for a one-parameter family $\omega(t)$ of hermitian metrics defined by

$$\frac{\partial}{\partial t}\omega = -2p, \qquad \text{or equivalently,} \qquad \frac{\partial}{\partial t}g = -2p(\cdot, J\cdot), \tag{5}$$

where $p = p(J, \omega(t))$ is the Chern-Ricci form and $g = \omega(\cdot, J\cdot)$. We refer to [6, 21, 22, 7] and the references therein for further information on this flow. If the starting metric ω_0 is Kähler, then CRF becomes the Kähler-Ricci flow (KRF).

Let (G, J) be a Lie group endowed with a left-invariant complex structure. Given a left-invariant hermitian metric ω_0, it follows from the diffeomorphism invariance of equation (5) that the CRF-solution starting at ω_0 stays left-invariant and so it can be studied on the Lie algebra. Indeed, the CRF becomes the ODE system

$$\frac{d}{dt}\omega = -2p, \tag{6}$$

where $\omega(t), p(t) \in \Lambda^2\mathfrak{g}^*$, as all the tensors involved are determined by their value at the identity of the group. Thus short-time existence (forward and backward) and uniqueness of the solutions are always guaranteed.

Since on Lie groups the Chern-Ricci form p depends only on J (see (3)), we obtain that along the CRF-solution starting at ω_0, $p(t) \equiv p_0 := p(J, \omega_0)$, and so $\omega(t)$ is simply given by

$$\omega(t) = \omega_0 - 2tp_0, \qquad \text{or equivalently,} \qquad g(t) = g_0 - 2tp_0(\cdot, J\cdot). \tag{7}$$

If P_0 is the Chern-Ricci operator of ω_0 (see (4)), then

$$\omega(t) = \omega_0((I - 2tP_0)\cdot, \cdot),$$

and so the solution exists as long as the hermitian map $I - 2tP_0$ is positive. It follows that the maximal interval of time existence (T_-, T_+) of $\omega(t)$ is given by

$$T_+ = \begin{cases} \infty, & \text{if } P_0 \leq 0, \\ 1/(2p_+), & \text{otherwise,} \end{cases} \qquad T_- = \begin{cases} -\infty, & \text{if } P_0 \geq 0, \\ 1/(2p_-), & \text{otherwise,} \end{cases} \tag{8}$$

where p_+ is the maximum positive eigenvalue of the Chern-Ricci operator P_0 of ω_0 (see (4)) and p_- is the minimum negative eigenvalue.

Bracket flow

Given a left-invariant hermitian metric ω_0 on a simply connected Lie group (G, J) endowed with a left-invariant complex structure, one has that the new metric

$$\omega = h^*\omega_0 := \omega_0(h\cdot, h\cdot),$$

is also hermitian for any $h \in \mathrm{GL}(\mathfrak{g}, J) \simeq \mathrm{GL}_n(\mathbb{C})$. Moreover, the corresponding holomorphic Lie group isomorphism

$$\widetilde{h} : (G, J, \omega) \longrightarrow (G_\mu, J, \omega_0), \qquad \text{where} \qquad \mu = h \cdot [\cdot, \cdot] := h[h^{-1}\cdot, h^{-1}\cdot],$$

is an equivalence of hermitian manifolds. Here $[\cdot, \cdot]$ denotes the Lie bracket of the Lie algebra $\mathfrak{g}$ and so μ defines a new Lie algebra (isomorphic to $(\mathfrak{g}, [\cdot, \cdot])$) with same underlying vector space $\mathfrak{g}$. We denote by G_μ the simply connected Lie group with Lie algebra $(\mathfrak{g}, \mu)$. This equivalence suggests the following natural question:

What if we evolved μ rather than ω?

We consider for a family $\mu(t) \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ of Lie brackets the following evolution equation:

$$\frac{d}{dt}\mu = \delta_\mu(P_\mu), \qquad \mu(0) = [\cdot,\cdot], \tag{9}$$

where $P_\mu \in \mathrm{End}(\mathfrak{g})$ is the Chern-Ricci operator of the hermitian manifold (G_μ, J, ω_0) and $\delta_\mu : \mathrm{End}(\mathfrak{g}) \longrightarrow \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ is defined by

$$\delta_\mu(A) := \mu(A\cdot,\cdot) + \mu(\cdot, A\cdot) - A\mu(\cdot,\cdot) = -\frac{d}{dt}|_{t=0} e^{tA} \cdot \mu, \qquad \forall A \in \mathrm{End}(\mathfrak{g}).$$

This evolution equation is called the *bracket flow* and has been proved in [14] to be equivalent to the CRF. Note that since J is fixed, the algebraic subset

$$\left\{\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g} : \mu \text{ satisfies the Jacobi identity and } J \text{ is integrable on } G_\mu\right\},$$

is invariant under the bracket flow; indeed, $\mu(t) \in \mathrm{GL}_n(\mathbb{C}) \cdot [\cdot,\cdot]$ for all t.

For a given simply connected hermitian Lie group (G, J, ω_0) with Lie algebra $\mathfrak{g}$, we may therefore consider the following two one-parameter families of hermitian Lie groups:

$$(G, J, \omega(t)), \qquad (G_{\mu(t)}, J, \omega_0), \tag{10}$$

where $\omega(t)$ is the CRF (6) starting at ω_0 and $\mu(t)$ is the bracket flow (9) starting at the Lie bracket $[\cdot,\cdot]$ of $\mathfrak{g}$.

Theorem 3.1 [14, Theorem 5.1] *There exist time-dependent holomorphic Lie group isomorphisms $h(t) : G \longrightarrow G_{\mu(t)}$ such that*

$$\omega(t) = h(t)^*\omega_0, \qquad \forall t,$$

which can be chosen such that their derivatives at the identity, also denoted by $h = h(t)$ (in particular, $\mu(t) = h(t) \cdot [\cdot,\cdot]$), is the solution to any of the following systems of ODE's:

(i) $\frac{d}{dt}h = -hP, \quad h(0) = I.$

(ii) $\frac{d}{dt}h = -P_\mu h, \quad h(0) = I.$

The maximal interval of time existence (T_-, T_+) is therefore the same for both flows, as it is the behavior of any kind of curvature along the flows.

It is easy to see that the Chern-Ricci operator of $(G, J, \omega(t))$ equals

$$P(t) = (I - 2tP_0)^{-1}P_0,$$

from which it follows that the family $h(t) \in \mathrm{GL}(\mathfrak{g})$ is given by $h(t) = (I - 2tP_0)^{1/2}$. The solution to the bracket flow is therefore given by

$$\mu(t) = (I - 2tP_0)^{1/2} \cdot [\cdot,\cdot],$$

and hence relative to any orthonormal basis $\{e_1, \dots, e_{2n}\}$ of eigenvectors of P_0, say with eigenvalues $\{p_1, \dots, p_{2n}\}$, the structure coefficients of $\mu(t)$ are

$$\mu_{ij}^k(t) = \left(\frac{1 - 2tp_k}{(1 - 2tp_i)(1 - 2tp_j)}\right)^{1/2} c_{ij}^k, \tag{11}$$

where c_{ij}^k are the structure coefficients of the Lie bracket $[\cdot,\cdot]$ of $\mathfrak{g}$ (i.e. $[e_i, e_j] = \sum c_{ij}^k e_k$).

The Chern scalar curvature is therefore given by

$$\mathrm{tr}\, P(t) = \sum_{i=1}^{2n} \frac{p_i}{1 - 2tp_i}.$$

Thus $\mathrm{tr}\, P(t)$ is strictly increasing unless $P(t) \equiv 0$ (i.e. $\omega(t) \equiv \omega_0$) and the integral of $\mathrm{tr}\, P(t)$ must blow up at a finite-time singularity $T_+ < \infty$. However, $\mathrm{tr}\, P(t) \leq \frac{C}{T_+ - t}$ for some constant $C > 0$, which is the claim of a well-known general conjecture for the Kähler-Ricci flow (see e.g. [20, Conjecture 7.7]).

4 Chern-Ricci solitons

In this section, we deal with self-similar CRF-solutions on Lie groups. It follows from Proposition 2.1 that $p = 0$ if $\mathfrak{g}$ is nilpotent, and thus any left-invariant hermitian structure on a nilpotent Lie group (and consequently, on any compact nilmanifold) is a fixed point for the CRF. However, we will show in Section 7 that several 4-dimensional solvable Lie groups do admit Chern-Ricci solitons which are not fixed points (i.e. $p \neq 0$), including the covers of Inoue surfaces.

Definition 4.1 [14, (39)] (G, J, ω) is said to be a *Chern-Ricci soliton* (CR-soliton) if its Chern-Ricci operator satisfies

$$P = cI + \tfrac{1}{2}(D + D^t), \qquad \text{for some} \quad c \in \mathbb{R}, \quad D \in \operatorname{Der}(\mathfrak{g}), \quad DJ = JD.$$

This is equivalent to have

$$p(J,\omega) = c\omega + \tfrac{1}{2}(\omega(D\cdot,\cdot) + \omega(\cdot, D\cdot)) = c\omega - \tfrac{1}{2}\mathcal{L}_{X_D}\omega,$$

where X_D is the vector field on the Lie group defined by the one-parameter subgroup of automorphisms φ_t with derivative $e^{tD} \in \operatorname{Aut}(\mathfrak{g})$ and $\mathcal{L}_{X_D}$ denotes Lie derivative. The CRF-solution starting at a CR-soliton (G, J, ω) is given by

$$\omega(t) = (-2ct + 1)\left(e^{s(t)D}\right)^* \omega, \tag{12}$$

where $s(t) := \frac{\log(-2ct+1)}{-2c}$ if $c \neq 0$ and $s(t) = t$ when $c = 0$.

The following structural result for Chern-Ricci solitons, which in particular holds for Kähler-Ricci solitons, provides a starting point for approaching the classification problem.

Proposition 4.2 [14, Proposition 8.2] *Let (G, J, ω) be a hermitian Lie group with Lie algebra $\mathfrak{g}$ and Chern-Ricci operator $P \neq 0$. Then the following conditions are equivalent.*

(i) *ω is a Chern-Ricci soliton with constant c.*

(ii) *$P = cI + D$, for some $D \in \operatorname{Der}(\mathfrak{g})$.*

(iii) *The eigenvalues of P are all either equal to 0 or c, the kernel $\mathfrak{k} = \operatorname{Ker} P$ is an abelian ideal of $\mathfrak{g}$ and its orthogonal complement $\mathfrak{k}^\perp$ (i.e. the c-eigenspace of P) is a Lie subalgebra of $\mathfrak{g}$ (in particular, $\mathfrak{g}$ is the semidirect product $\mathfrak{g} = \mathfrak{k}^\perp \ltimes \mathfrak{k}$ and c is always nonzero).*

The following corollary essentially follows from the observation that J must leave $\mathfrak{k}^\perp$ and $\mathfrak{k}$ invariant, as it commutes with P.

Corollary 4.3 *Any Chern-Ricci soliton can be constructed as $(\mathfrak{g} = \mathfrak{g}_1 \ltimes \mathfrak{g}_2, J, \omega)$, with $J = \begin{bmatrix} J_1 & \\ & J_2 \end{bmatrix}$, $\omega = \omega_1 \oplus \omega_2$, from the following data:*

- *a hermitian Lie algebra $(\mathfrak{g}_1, J_1, \omega_1)$;*
- *a hermitian abelian Lie algebra $(\mathfrak{g}_2, J_2, \omega_2)$;*
- *and a representation $\theta : \mathfrak{g}_1 \longrightarrow \operatorname{End}(\mathfrak{g}_2)$;*

such that the following conditions hold:

- *$[\theta(J_1X), J_2] = J_2[\theta(X), J_2]$, for all $X \in \mathfrak{g}_1$;*
- *the Chern-Ricci operator P_1 of $(\mathfrak{g}_1, J_1, \omega_1)$ equals*

$$P_1 = cI - P_\theta,$$

where $P_\theta \in \operatorname{End}(\mathfrak{g}_1)$ is defined by

$$\omega_1(P_\theta X, Y) = -\tfrac{1}{2}\operatorname{tr} J_2\theta([X,Y]) + \tfrac{1}{2}\operatorname{tr}\theta(J_1[X,Y]), \qquad \forall X, Y \in \mathfrak{g}_1.$$

The Chern-Ricci operator of $(\mathfrak{g}, J, \omega)$ *is given by* $P|_{\mathfrak{g}_1} = cI$, $P|_{\mathfrak{g}_2} = 0$.

Moreover, $(\mathfrak{g}, J, \omega)$ *is Kähler (and so a Kähler-Ricci soliton) if and only if* ω_1 *is closed (i.e.* $(\mathfrak{g}_1, J_1, \omega_1)$ *Kähler) and* $\theta(\mathfrak{g}_1) \subset \mathfrak{sp}(\mathfrak{g}_2, \omega_2)$ *(i.e.* $\theta(X)^t = J_2\theta(X)J_2$ *for all* $X \in \mathfrak{g}_1$*).*

Proof. It is easy to check that the first condition which must hold is equivalent to J being integrable, and the second one comes from the fact that the Chern-Ricci operator of $(\mathfrak{g}, J, \omega)$ is given by $P = \left[\begin{smallmatrix} P_1+P_\theta & 0 \\ 0 & 0 \end{smallmatrix}\right]$. The last claim on Kähler property easily follows from the closedness condition for ω. □

Example 4.4 We therefore obtain a Chern-Ricci soliton from any hermitian Lie algebra $(\mathfrak{g}_1, J_1, \omega_1)$ with $P_1 = cI$ (i.e. $p_1 = c\omega_1$) and a representation $\theta : \mathfrak{g}_1 \longrightarrow \mathfrak{sl}(\mathfrak{g}_2, J_2)$ (i.e. $\operatorname{tr}\theta(X) = 0$ and $[\theta(X), J_2] = 0$ for all $X \in \mathfrak{g}_1$); note that $P_\theta = 0$ under such conditions. If in addition $(\mathfrak{g}_1, J_1, \omega_1)$ is Kähler-Einstein and

$$\theta(\mathfrak{g}_1) \subset \mathfrak{sl}(\mathfrak{g}_2, J_2) \cap \mathfrak{sp}(\mathfrak{g}_2, \omega_2) = \mathfrak{su}(\dim \mathfrak{g}_2/2),$$

then what we obtain is a Kähler-Ricci soliton, which is actually isometric to the direct product $G_1 \times \mathbb{R}^{\dim \mathfrak{g}_2}$.

5 Convergence

We study in this section the possible limits of bracket flow solutions under diverse rescalings.

If a rescaling $c(t)\mu(t)$, $c(t) \in \mathbb{R}$, of a bracket flow solution converges to λ, as $t \to T_\pm$, and $\varphi(t) : G \longrightarrow G_{c(t)\mu(t)}$ is the isomorphism with derivative $\frac{1}{c(t)}h(t)$, where $h(t)$ is as in Theorem 3.1, then it follows from [13, Corollary 6.20] that (after possibly passing to a subsequence) the Riemannian manifolds $\left(G, \frac{1}{c(t)^2}\omega(t)\right)$ converge in the pointed (or Cheeger-Gromov) sense to (G_λ, ω_0), as $t \to T_\pm$. We note that G_λ may be non-isomorphic, and even non-homeomorphic, to G (see [14, Section 5.1]).

Recall also that all the limits obtained by any of such rescalings are automatically CR-solitons (see [14, Section 7.1]).

Two rescalings will be considered, the one given by the bracket norm $\mu(t)/|\mu(t)|$, which always converges, and $|2t+1|^{1/2}\mu(t)$, which corresponds according to the observation above to the standard rescaling $\omega(t)/(2t+1)$ of the original CRF-solution in the forward case. We note that $\omega(t)/(2t+1)$ is, up to reparametrization in time, the solution to the *normalized Chern-Ricci flow*

$$\frac{\partial}{\partial t}\widetilde{\omega} = -2p(\widetilde{\omega}) - 2\widetilde{\omega}, \qquad \widetilde{\omega}(0) = \omega_0, \tag{13}$$

which is the one preserving the volume in the case when M is compact, ω is Kähler and $[\omega] = -c_1(M)$. This normalization has also been used in the general hermitian case (see e.g. [21, Theorem 1.7] and [6]).

Let (G, J, ω_0) be a hermitian Lie group with Lie algebra $\mathfrak{g}$ and Chern-Ricci operator P_0. A straightforward analysis using (11) gives that $\mu(t)$ converges as $t \to T_\pm$ if and only if $T_\pm = \pm\infty$ (i.e. $\pm P_0 \leq 0$) and $\operatorname{Ker} P_0$ is a Lie subalgebra of $\mathfrak{g}$. Moreover, the following conditions are equivalent in the case $T_\pm = \pm\infty$:

- $\mu(t) \to 0$, as $t \to \pm\infty$.
- $\operatorname{Ker} P_0$ is an abelian ideal of $\mathfrak{g}$.
- $|2t+1|^{1/2}\mu(t)$ converges as $t \to \pm\infty$.

Remark 5.1 Any statement as the above ones, involving the $\pm$ sign, must always be understood as two separate statements, one for the $+$ sign and the other for the $-$ sign.

In the case $\pm T_\pm < \infty$, it follows that $|T_\pm - t|^{1/2}\mu(t)$ converges as $t \to T_\pm$ if and only if $\mathfrak{g}_\pm$ is a Lie subalgebra of $\mathfrak{g}$, where $\mathfrak{g}_\pm$ is the eigenspace of P_0 of eigenvalue $p_\pm$ (see (8)).

Lemma 5.2 *If* $\mu(t) \to \lambda$*, as* $t \to \pm\infty$*, then* (G_λ, J, ω_0) *is Chern-Ricci flat.*

Proof. We have that λ is a fixed point and so the solution starting at λ is defined on $(-\infty, \infty)$, which implies that $P_\lambda = 0$ by (8). □

We now explore in which way is the limit of the normalization $\mu(t)/|\mu(t)|$ related to the starting point (G, J, ω_0). The norm $|\mu|$ of a Lie bracket will be defined in terms of the canonical inner product on $\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ given by

$$\langle \mu, \lambda \rangle := \sum g_0(\mu(e_i, e_j), \lambda(e_i, e_j)) = \sum \mu_{ij}^k \lambda_{ij}^k, \tag{14}$$

where $\{e_i\}$ is any orthonormal basis of $(\mathfrak{g}, g_0)$ and $g_0 = \omega_0(\cdot, J\cdot)$. A natural inner product on $\operatorname{End}(\mathfrak{g})$ is also determined by g_0 by $\langle A, B \rangle := \operatorname{tr} AB^t$.

Proposition 5.3 *Let (G, J, ω_0) be a hermitian Lie group with Lie algebra $\mathfrak{g}$ and Chern-Ricci operator P_0, and let $\mathfrak{k}$, $\mathfrak{g}_+$ and $\mathfrak{g}_-$ denote the eigenspaces of P_0 of eigenvalues 0, p_+ and p_-, respectively (see* (8)*).*

(i) *The normalized Chern-Ricci bracket flow $\mu(t)/|\mu(t)|$ always converges, as $t \to T_\pm$, to a nonabelian Lie bracket $\lambda_\pm$ such that $(G_{\lambda_\pm}, J, \omega_0)$ is a Chern-Ricci soliton, say with Chern-Ricci operator $P_{\lambda_\pm}$.*

(ii) *If $\pm P_0 \leq 0$ (i.e. $\pm T_\pm = \infty$), then $P_{\lambda_\pm}|_{\mathfrak{k}^\perp} = c_\pm I$, $P_{\lambda_\pm}|_{\mathfrak{k}} = 0$, with $\pm c_\pm < 0$ if and only if $\mathfrak{k}$ is an abelian ideal of $\mathfrak{g}$. Otherwise, $P_{\lambda_\pm} = 0$.*

(iii) *In the case when $\pm T_\pm < \infty$, one has $P_{\lambda_\pm}|_{\mathfrak{g}_\pm} = c_\pm I$, $P_{\lambda_\pm}|_{(\mathfrak{g}_\pm)^\perp} = 0$, with $\pm c_\pm > 0$ if and only if $\mathfrak{g}_\pm$ is a Lie subalgebra of $\mathfrak{g}$. Otherwise, $P_{\lambda_\pm} = 0$.*

Remark 5.4 In particular, the only way to obtain in the limit the Einstein-like condition $p_{\lambda_\pm} = c\omega_0$ with $c \neq 0$, is precisely when $\pm P_0 < 0$.

Remark 5.5 It follows from the last paragraph in [14, Section 5.1] that when $P_{\lambda_\pm} = 0$, it actually holds that $P_\nu = 0$ for any limit $\nu = \lim\limits_{t\to T_\pm} c(t)\mu(t)$ and any rescaling of the form $c(t)\mu(t)$, with $c(t) \in \mathbb{R}$.

P r o o f. It follows from (14) and (11) that

$$\frac{\mu_{rs}^l}{|\mu|} = \frac{c_{rs}^l}{\left(\sum\limits_{i,j,k} \frac{(1-2tp_k)(1-2tp_r)(1-2tp_s)}{(1-2tp_i)(1-2tp_j)(1-2tp_l)} \left(c_{ij}^k\right)^2\right)^{1/2}} \underset{t\to T_\pm}{\longrightarrow} (\lambda_\pm)_{rs}^l. \tag{15}$$

Since each of the terms in the sum above converges, as $t \to T_\pm$, to either a nonnegative real number or ∞, we obtain that $\mu(t)/|\mu(t)|$ always converges, and so part (i) follows.

We will only prove the $+$-statements, the proofs for those with a $-$ sign are completely analogous. Since $P_{\mu/|\mu|} = \frac{1}{|\mu|^2} P \to P_{\lambda_+}$, as $t \to T_+$ (recall that $P_{\mu(t)} = P(t) = (I - 2tP_0)^{-1}P_0$), one can easily check that for each eigenvalue p_r of P_0,

$$\lim_{t\to T_+} \frac{p_r}{|\mu|^2(1-2tp_r)} = \lim_{t\to T_+} \frac{p_r}{\sum\limits_{i,j,k} \frac{(1-2tp_k)(1-2tp_r)}{(1-2tp_i)(1-2tp_j)} \left(c_{ij}^k\right)^2}$$

$$= \begin{cases} \dfrac{1}{\sum\limits_{p_i,p_j,p_k<0} \frac{p_k}{p_ip_j}\left(c_{ij}^k\right)^2 + 2\sum\limits_{p_i<0,p_j=p_k=0} \frac{1}{p_i}\left(c_{ij}^k\right)^2} < 0, & T_+ = \infty, \quad p_r < 0, \quad \mathfrak{k} \text{ abelian ideal;} \\[2ex] 0, & T_+ = \infty, \quad \text{otherwise;} \\[2ex] \dfrac{p_+}{\sum\limits_{i,j,k=+} \left(c_{ij}^k\right)^2 + 2\sum\limits_{i,k\neq+,j=+} \frac{p_+-p_k}{p_+-p_i}\left(c_{ij}^k\right)^2} > 0, & T_+ < \infty, \quad p_r = p_+, \quad \mathfrak{g}_+ \text{ subalgebra;} \\[2ex] 0, & T_+ < \infty, \quad \text{otherwise.} \end{cases}$$

This shows that the value of P_{λ_+} is as in parts (ii) and (iii), concluding the proof of the proposition. $\square$

Proposition 5.6 *Let (G, J, ω_0) be a hermitian Lie group as in the above proposition and consider $\lambda_\pm$, the limit of $\mu(t)/|\mu(t)|$ as $t \to T_\pm$.*

(i) *If $\pm P_0 \leq 0$ (i.e. $\pm T_\pm = \infty$) and $\mathfrak{k}$ is an abelian ideal of $\mathfrak{g}$, then $(\mathfrak{g}, \lambda_\pm) = \mathfrak{k}^\perp \ltimes \mathfrak{k}$ and $\lambda_\pm(\mathfrak{k}, \mathfrak{k}) = 0$. On the contrary, if $\mathfrak{k}$ is not an abelian ideal of $\mathfrak{g}$, then $\mathfrak{k}^\perp$ is an abelian ideal of $(\mathfrak{g}, \lambda_\pm)$. Moreover, if $\mathfrak{k}$ is not even a Lie subalgebra of $\mathfrak{g}$, then $\lambda_\pm$ is 2-step nilpotent and $\mathfrak{k}^\perp$ is contained in its center.*

(ii) *If $\pm T_\pm < \infty$ and $\mathfrak{g}_\pm$ is a Lie subalgebra of $\mathfrak{g}$, then $(\mathfrak{g}, \lambda_\pm) = \mathfrak{g}_\pm \ltimes \mathfrak{g}_\pm^\perp$ and $\lambda_\pm(\mathfrak{g}_\pm^\perp, \mathfrak{g}_\pm^\perp) = 0$. On the contrary, if $\mathfrak{g}_\pm$ is not a Lie subalgebra of $\mathfrak{g}$, then $\lambda_\pm$ is 2-step nilpotent and $\mathfrak{g}_\pm^\perp$ is contained in its center.*

Proof. The first claims in the items are both direct consequences of Proposition 4.2, (iii). As above, we only prove the $+$-statements.

If $\mathfrak{k}$ is not an abelian ideal of $\mathfrak{g}$, then there is a $c_{ij}^k \neq 0$ with either $p_i, p_j, p_k = 0$, or $p_i p_j = 0$ and $p_k < 0$. The corresponding term in the sum appearing in formula (15) therefore converges to ∞ for any triple (r, s, l) such that either $p_r, p_s, p_l < 0$, or $p_l = 0$, or $p_l < 0$ and at least one of p_r, p_s is negative. This implies that $\lambda_{rs}^l = 0$ for all such triples and hence $\lambda_+(\mathfrak{k}^\perp, \mathfrak{k}^\perp) = 0$ and $\lambda_+(\mathfrak{g}, \mathfrak{k}^\perp) \subset \mathfrak{k}^\perp$, respectively.

Assume now that $\mathfrak{k}$ is not a subalgebra of $\mathfrak{g}$. Thus there is a $c_{ij}^k \neq 0$ with $p_i, p_j = 0$ and $p_k < 0$. The corresponding term in (15) therefore converges to ∞ for any triple (r, s, l) such that either $p_l = 0$, or $p_l < 0$ and at least one of p_r, p_s is negative. This implies that $\lambda_+(\mathfrak{g}, \mathfrak{g}) \subset \mathfrak{k}^\perp$ and $\lambda_+(\mathfrak{g}, \mathfrak{k}^\perp) = 0$, respectively. The second claim in part (i) therefore follows.

It only remains to prove the second claim in part (ii). If $\mathfrak{g}_+$ is not a subalgebra of $\mathfrak{g}$, then there is a $c_{ij}^k \neq 0$ with $p_i, p_j = p_+$ and $p_l \neq p_+$. Thus the corresponding term in (15) does not converge to ∞ if and only if $p_r = p_s = p_+$ and $p_l \neq p_+$, that is, the only part of λ_+ which survives is $\lambda_+ : \mathfrak{g}_+ \times \mathfrak{g}_+ \longrightarrow \mathfrak{g}_+^\perp$, as was to be shown. □

We now study the rescaling $|2t+1|^{1/2}\mu(t)$, or equivalently $\omega(t)/(2t+1)$, corresponding to the normalized CRF given in (13). Recall that we always denote by $[\cdot, \cdot]$ the Lie bracket of the Lie algebra $\mathfrak{g}$ of the Lie group G.

Proposition 5.7 *Let (G, J, ω_0) be a hermitian Lie group as in the propositions above.*

(i) *If $\pm P_0 \leq 0$ (i.e. $\pm T_\pm = \infty$) and $\mathfrak{k}$ is an abelian ideal of $\mathfrak{g}$, then $|2t+1|^{1/2}\mu(t)$ converges, as $t \to \pm\infty$, to a Chern-Ricci soliton $\nu_\pm$ such that $(\mathfrak{g}, \nu_\pm) = \mathfrak{k}^\perp \ltimes \mathfrak{k}$, $\nu_\pm(\mathfrak{k}, \mathfrak{k}) = 0$ and with Chern-Ricci operator given by $P_{\nu_\pm}|_{\mathfrak{k}^\perp} = \mp I$, $P_{\nu_\pm}|_{\mathfrak{k}} = 0$.*

(ii) *If $\mathfrak{g}^\pm$ is a nonzero Lie subalgebra of $\mathfrak{g}$, then $\pm T_\pm < \infty$ and $|T_\pm - t|^{1/2}\mu(t)$ converges, as $t \to T_\pm$, to a Chern-Ricci soliton $\nu_\pm$ such that $(\mathfrak{g}, \nu_\pm) = \mathfrak{g}_\pm \ltimes \mathfrak{g}_\pm^\perp$, $\nu_\pm(\mathfrak{g}_\pm^\perp, \mathfrak{g}_\pm^\perp) = 0$ and with $P_{\nu_\pm}|_{\mathfrak{g}^\pm} = \pm\frac{1}{2}I$, $P_{\nu_\pm}|_{(\mathfrak{g}^\pm)^\perp} = 0$.*

Proof. One can prove this proposition in much the same way as Propositions 5.3 and 5.6, by using for the second statements that

$$\left(|2t+1|^{1/2}\mu(t)\right)_{rs}^l = \left(\frac{|2t+1|(1-2tp_l)}{(1-2tp_r)(1-2tp_s)}\right)^{1/2} c_{rs}^l \underset{t \to T_\pm}{\longrightarrow} (\nu_\pm)_{rs}^l,$$

and considering separately the cases $T_+ = \infty$ and $T_+ < \infty$. □

6 Almost-abelian Lie groups

We apply in this section the results obtained above on CR-solitons and convergence on a class of solvable Lie algebras, which are very simple from the algebraic point of view but yet geometrically very rich.

Let (G, J, ω) be a hermitian Lie group with Lie algebra $\mathfrak{g}$ and assume that $\mathfrak{g}$ has a codimension-one abelian ideal $\mathfrak{n}$. These Lie algebras are sometimes called *almost-abelian* in the literature (see e.g. [4]). It is easy to see that there exists an orthonormal basis $\{e_1, \dots, e_{2n}\}$ such that

$$\mathfrak{n} = \langle e_1, \dots, e_{2n-1}\rangle, \qquad \omega = e^1 \wedge e^{2n} + \dots + e^n \wedge e^{n+1}, \qquad Je_i = e_{2n+1-i} \quad (1 \leq i \leq n),$$

where $\{e^i\}$ denotes the dual basis. It follows from (2) that J is integrable if and only if

$$-[e_1, Je_i] = [e_{2n}, e_i] - J[e_1, e_i] + J[e_{2n}, Je_i], \qquad \forall i = 2, \dots, 2n-1,$$

and since the left-hand side and the middle term in the right-hand side both vanish, we obtain that J is integrable if and only if $\operatorname{ad} e_{2n}$ leaves the subspace $\langle e_2, \dots, e_{2n-1}\rangle$ invariant and commutes with the restriction of J on such subspace. The matrix of $\operatorname{ad} e_{2n}$ in terms of $\{e_i\}$ is therefore given by

$$\operatorname{ad} e_{2n} = \left[\begin{array}{c|c|c} c & 0 & 0 \\ \hline d_1 & & \\ \vdots & A & 0 \\ d_{2n-2} & & \\ \hline 0 & 0 & 0 \end{array}\right], \qquad A \in \mathfrak{gl}_{n-1}(\mathbb{C}). \tag{16}$$

We call $\mu = \mu_{A,c,d_1,\dots,d_{2n-2}}$ the Lie bracket on $\mathfrak{g}$ defined by (16) and the condition that $\mathfrak{n}$ is an abelian ideal. It is easy to prove that two of these Lie algebras are isomorphic if and only if the corresponding adjoint maps $\operatorname{ad} e_{2n}|_{\mathfrak{n}}$ are conjugate up to nonzero scaling.

Lemma 6.1 *Any hermitian Lie algebra $(\mathfrak{g}, J, \omega)$ with a codimension-one abelian ideal is equivalent to*

$$(\mathfrak{g}, \mu_{A,c,d_1,\dots,d_{2n-2}}, J, \omega), \qquad \textit{for some} \quad A \in \mathfrak{gl}_{n-1}(\mathbb{C}), \quad c \geq 0, \quad d_i \in \mathbb{R}.$$

The Chern-Ricci form and operator of this structure are respectively given by

$$p = -\tfrac{1}{2}c(2c + \operatorname{tr} A)e^1 \wedge e^{2n}, \qquad P = -\tfrac{1}{2}c(2c + \operatorname{tr} A)\left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline & & \\ 0 & 0 & 0 \\ & & \\ \hline 0 & 0 & 1 \end{array}\right].$$

Proof. It only remains to prove the formula for the Chern-Ricci form. We use formula (3) to compute p as follows:

$$\begin{aligned} p(e_{2n}, e_1) &= -\tfrac{1}{2}\operatorname{tr} J\left(c \operatorname{ad} e_1 + \sum_{i=1}^{2n-2} d_i \operatorname{ad} e_{i+1}\right) + \tfrac{1}{2}\operatorname{tr}\operatorname{ad} J(ce_1), \\ &= -\tfrac{1}{2}c\operatorname{tr} J\operatorname{ad} e_1 + \tfrac{1}{2}c\operatorname{tr}\operatorname{ad} e_{2n} \\ &= \tfrac{1}{2}c^2 + \tfrac{1}{2}c(c + \operatorname{tr} A) = \tfrac{1}{2}c(2c + \operatorname{tr} A) \end{aligned}$$

$$p(e_{2n}, e_i) = -\tfrac{1}{2}\operatorname{tr} J\operatorname{ad} Ae_i + \tfrac{1}{2}\operatorname{tr}\operatorname{ad} JAe_i = 0 + 0 = 0, \quad \forall i = 2, \dots, 2n-1,$$

concluding the proof of the lemma. □

It is proved in [15] that $(\mathfrak{g}, \mu, J, \omega)$ is Kähler (i.e. $d\omega = 0$) if and only if $d_i = 0$ for all i and $A \in \mathfrak{u}(n)$ (i.e. $A^t = -A$). In such a case, the metric is known to be isometric to $\mathbb{R}H^2 \times \mathbb{R}^{2n-2}$, where $\mathbb{R}H^2$ denotes the 2-dimensional real hyperbolic space (see e.g. [9, Proposition 2.5]). On the other hand, it is easy to prove that $(\mathfrak{g}, \mu, J, \omega)$ is bi-invariant if and only if $c = d_1 = \dots = d_{2n-2} = 0$, and abelian if and only if $A = 0$.

Proposition 6.2 *Let (G_μ, J, ω) be the hermitian Lie group with $\mu = \mu_{A,c,d_i}$.*

(i) *(G_μ, J, ω) is a CR-soliton if and only if either $p = 0$ or $p \neq 0$ and $d_i = 0$ for all i.*

(ii) *The maximal interval of time existence of the CRF-solution $\omega(t)$ starting at (G_μ, J, ω) is*

$$\begin{cases} (\frac{1}{e}, \infty), & e < 0, \\ (-\infty, \frac{1}{e}), & e > 0, \\ (-\infty, \infty), & e = 0, \end{cases} \qquad \textit{where} \quad e := -c(2c + \operatorname{tr} A).$$

$\mathfrak{g}$	**Lie Bracket**
$\mathfrak{rh}_3$	$[e_1, e_2] = e_3$
$\mathfrak{rr}_{3,0}$	$[e_1, e_2] = e_2$
$\mathfrak{rr}_{3,1}$	$[e_1, e_2] = e_2$, $[e_1, e_3] = e_3$
$\mathfrak{rr}'_{3,0}$	$[e_1, e_2] = -e_3$, $[e_1, e_3] = e_2$
$\mathfrak{rr}'_{3,\gamma}$	$[e_1, e_2] = \gamma e_2 - e_3$, $[e_1, e_3] = e_2 + \gamma e_3, \quad \gamma > 0$
$\mathfrak{r}_2\mathfrak{r}_2$	$[e_1, e_2] = e_2$, $[e_3, e_4] = e_4$
$\mathfrak{r}'_2$	$[e_1, e_3] = e_3$, $[e_1, e_4] = e_4$, $[e_2, e_3] = e_4$, $[e_2, e_4] = -e_3$
$\mathfrak{r}_{4,1}$	$[e_4, e_1] = e_1$, $[e_4, e_2] = e_2$, $[e_4, e_3] = e_2 + e_3$
$\mathfrak{r}_{4,\alpha,1}$	$[e_4, e_1] = e_1$, $[e_4, e_2] = \alpha e_2$, $[e_4, e_3] = e_3, \quad -1 < \alpha \leq 1, \alpha \neq 0$
$\mathfrak{r}_{4,\alpha,\alpha}$	$[e_4, e_1] = e_1$, $[e_4, e_2] = \alpha e_2$, $[e_4, e_3] = \alpha e_3, \quad -1 \leq \alpha < 1, \alpha \neq 0$
$\mathfrak{r}'_{4,\gamma,\delta}$	$[e_4, e_1] = e_1$, $[e_4, e_2] = \gamma e_2 - \delta e_3$, $[e_4, e_3] = \delta e_2 + \gamma e_3, \gamma \in \mathbb{R}, \quad \delta > 0$
$\mathfrak{d}_4$	$[e_1, e_2]_1 = e_3$, $[e_4, e_1]_1 = e_1$, $[e_4, e_2]_1 = -e_2$
	$[e_1, e_2]_2 = e_3$, $[e_4, e_1]_2 = e_1$, $[e_4, e_2]_2 = -e_2 + e_3$
$\mathfrak{d}_{4,1}$	$[e_1, e_2] = e_3$, $[e_4, e_1] = e_1$, $[e_4, e_3] = e_3$
$\mathfrak{d}_{4,\frac{1}{2}}$	$[e_1, e_2]_1 = e_3$, $[e_4, e_1]_1 = \frac{1}{2}e_1$, $[e_4, e_2]_1 = \frac{1}{2}e_2$, $[e_4, e_3]_1 = e_3$
	$[\cdot, \cdot]_2 = [\cdot, \cdot]_1$
	$[e_1, e_2]_3 = e_3$, $[e_4, e_1]_3 = e_1$, $[e_4, e_2]_3 = e_2$, $[e_4, e_3]_3 = 2e_3$
$\mathfrak{d}_{4,\lambda}$	$[e_1, e_2]_1 = \lambda e_3$, $[e_4, e_1]_1 = \lambda e_1$, $[e_4, e_2]_1 = (1-\lambda)e_2$, $[e_4, e_3]_1 = e_3, \quad \frac{1}{2} < \lambda \neq 1$
	$[e_1, e_2]_2 = (1-\lambda)e_3$, $[e_4, e_1]_2 = \lambda e_1$, $[e_4, e_2]_2 = (1-\lambda)e_2$, $[e_4, e_3]_2 = e_3, \quad \frac{1}{2} < \lambda < 1$
	$[e_1, e_2]_3 = (\lambda-1)e_3$, $[e_4, e_1]_3 = \lambda e_1$, $[e_4, e_2]_3 = (1-\lambda)e_2$, $[e_4, e_3]_3 = e_3, \quad 1 < \lambda$
$\mathfrak{d}'_{4,0}$	$[e_1, e_2] = e_3$, $[e_4, e_1] = -e_2$, $[e_4, e_2] = e_1$
$\mathfrak{d}'_{4,\delta}$	$[e_1, e_2] = e_3$, $[e_4, e_1] = \frac{1}{2}e_1 - \frac{1}{\delta}e_2$, $[e_4, e_2] = \frac{1}{\delta}e_1 + \frac{1}{2}e_2$, $[e_4, e_3] = e_3, \quad \delta > 0$
$\mathfrak{h}_4$	$[e_1, e_2] = e_3$, $[e_4, e_1] = e_1$, $[e_4, e_2] = \sqrt{10}e_1 + e_2$, $[e_4, e_3] = 2e_3$

Table 1 Solvable Lie algebras of dimension 4 admitting a complex structure.

$\mathfrak{g}$	**Complex structures**			
$\mathfrak{rh}_3$	$Je_1 = e_2,\, Je_3 = e_4$			
$\mathfrak{rr}_{3,0}$	$Je_1 = e_2,\, Je_3 = e_4$			
$\mathfrak{rr}_{3,1}$	$Je_1 = e_4,\, Je_3 = e_2$			
$\mathfrak{rr}'_{3,0}$	$Je_1 = e_4,\, Je_2 = e_3$			
$\mathfrak{rr}'_{3,\gamma}$	$J_1e_1 = e_4,\, J_1e_3 = e_2 - 2\gamma e_3,\ \gamma > 0$	$J_2e_1 = e_4,\, J_2e_3 = 2\gamma e_3 - e_2,\ \gamma > 0$		
$\mathfrak{r}_2\mathfrak{r}_2$	$Je_1 = e_2,\, Je_3 = e_4$			
$\mathfrak{r}'_2$	$J_1e_1 = e_3,\, J_1e_2 = e_4$	$J_{s,t}e_2 = -\frac{1}{t}e_1 - \frac{s}{t}e_2,\, J_{s,t}e_3 = e_4,\ s \in \mathbb{R}, t \neq 0$		
$\mathfrak{r}_{4,1}$	$Je_1 = e_2,\, Je_4 = e_3$			
$\mathfrak{r}_{4,\alpha,1}$	$Je_1 = e_3,\, Je_4 = e_2$			
$\mathfrak{r}_{4,\alpha,\alpha}$	$Je_4 = e_1,\, Je_2 = e_3$			
$\mathfrak{r}'_{4,\gamma,\delta}$	$J_1e_4 = e_1,\, J_1e_2 = e_3$	$J_2e_4 = e_1,\, J_2e_3 = e_2$		
$\mathfrak{d}_4$	$J_1e_3 = e_1,\, J_1e_4 = e_2$	$J_2 = J_1$		
$\mathfrak{d}_{4,1}$	$Je_1 = e_4,\, Je_2 = e_3$			
$\mathfrak{d}_{4,\frac{1}{2}}$	$J_1e_1 = e_2,\, J_1e_4 = e_3$	$J_2e_2 = e_1,\, J_2e_4 = e_3$	$J_3e_4 = e_1,\, J_3e_3 = e_2$	
$\mathfrak{d}_{4,\lambda}$	$J_1e_1 = e_4,\, J_1e_2 = e_3$	$J_2e_1 = e_3,\, J_2e_4 = e_2$	$J_3e_1 = e_3,\, J_3e_2 = e_4$	
$\mathfrak{d}'_{4,0}$	$J_1e_1 = e_2,\, J_1e_3 = e_4$	$J_2e_1 = e_2,\, J_2e_4 = e_3$		
$\mathfrak{d}'_{4,\delta}$	$J_1e_2 = e_1,\, J_1e_4 = e_3$	$J_2e_1 = e_2,\, J_2e_3 = e_4$	$J_3e_1 = e_2,\, J_3e_4 = e_3$	$J_4e_2 = e_1,\, J_4e_3 = e_4$
$\mathfrak{h}_4$	$Je_1 = e_3,\, Je_4 = e_2$			

Table 2 Complex structures on 4-dimensional solvable Lie algebras.

(iii) *If $T_\pm = \pm\infty$ and $p \neq 0$ (i.e. $e \neq 0$), then the rescaled solution $\omega(t)/|2t+1|$ converges in the pointed sense, as $t \to \pm\infty$, to the CR-soliton (G_λ, J, ω), where $\lambda = \frac{1}{2}|e|^{1/2}\mu_{A,c,0}$.*

(iv) *If ω is not a CR-soliton, then, as t approaches any finite-time singularity, $c(t)\omega(t)$ converges in the pointed sense to $(H_3 \times \mathbb{R}) \times \mathbb{R}^{2n-4}$, where $H_3 \times \mathbb{R}$ is the universal cover of the Kodaria-Thurston manifold, for some rescaling $c(t) > 0$.*

Remark 6.3 Recall that in part (iii), if $\lambda \not\simeq \mu$, which never holds if c is not an eigenvalue of A, then the limit is a left-invariant hermitian metric on a different Lie group (see Example 7.2).

Proof. Part (i) follows from Proposition 4.2 and Lemma 6.1, by using that the image of any derivation must be contained in $\mathfrak{n}$, and part (ii) follows from (8). Since $\mathfrak{k}$ is always an abelian ideal, the limit $\nu_\pm$ from Proposition 5.7 equals $\mu_{A,c,0}$, up to a positive scaling, and so part (iii) holds. On the other hand, $\mathfrak{g}_\pm = \langle e_1, e_{2n}\rangle$ is never a Lie subalgebra if $d_i \neq 0$ for at least one i, in which case by Proposition 5.6, $\lambda_\pm$ is 2-step nilpotent with $\mathfrak{g}_\pm^\perp$ contained in its center. Thus $(\mathfrak{g}, \lambda_\pm)$ is isomorphic to $\mathfrak{h}_3 \oplus \mathbb{R}^{2n-3}$, where $\mathfrak{h}_3$ denotes the 3-dimensional Heisenberg algebra, from which part (iv) follows. $\square$

7 Lie groups of dimension 4

We now study the existence problem for CR-solitons on 4-dimensional solvable Lie groups. We have listed in Table 1 all 4-dimensional solvable Lie algebras admitting a complex structure and in Table 2 all the complex structures up to equivalence on each Lie algebra (see [17]). In order to obtain simpler forms for the matrices of the complex structures and the CR-soliton metrics, we decided to give more than one different (but isomorphic) Lie brackets $[\cdot,\cdot]_i$ for each Lie algebra $\mathfrak{d}_4$, $\mathfrak{d}_{4,\lambda}$ with $\lambda \neq 1$, in such a way that the pair $([\cdot,\cdot]_i, J_i)$ is integrable for any $i = 1, 2, 3$ (see (2)).

Let (G, J, ω) be a 4-dimensional hermitian Lie group with Lie algebra $\mathfrak{g}$.

Example 7.1 Assume that $\mathfrak{g}$ has a codimension-one abelian ideal $\mathfrak{n}$ (i.e. $\mathfrak{g}$ is any of the Lie algebras denoted with $\mathfrak{r}$ in Table 1 except $\mathfrak{r}_2\mathfrak{r}_2$ and $\mathfrak{r}_2'$). By Lemma 6.1, we can assume that in terms of an orthonormal basis $\{e_i\}$,

$$J = \begin{bmatrix} & & 0 & -1 \\ & & -1 & 0 \\ 0 & 1 & & \\ 1 & 0 & & \end{bmatrix}, \qquad \omega = e^1 \wedge e^4 + e^2 \wedge e^3,$$

and the Lie bracket of $\mathfrak{g}$, denoted by $\mu = \mu_{a,b,c,d,e}$, is given by

$$\operatorname{ad}_\mu e_4|_{\mathfrak{n}} = \begin{bmatrix} c & 0 & 0 \\ d & a & -b \\ e & b & a \end{bmatrix}, \qquad c \geq 0.$$

The Chern-Ricci form and operator are therefore given by

$$p = -c(c+a)e^1 \wedge e^4, \qquad P = -c(c+a) \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{bmatrix}.$$

In the case $p \neq 0$, μ is a CR-soliton if and only if $d = e = 0$ (see Proposition 6.2, (i)). These are precisely the long time pointed limits one obtains by rescaling CRF-solutions (see Proposition 6.2, (iii)). By giving different values to a, b, c, we have found a CR-soliton for any complex structure on any Lie algebra in this class (see Table 3), with the only exception of $\mathfrak{r}_{4,1}$ (see example below).

Example 7.2 We have that μ is isomorphic to $\mathfrak{r}_{4,1}$ if and only if $a = c \neq 0$, $b = 0$ and at least one of d, e is nonzero, from which it follows that $(\mathfrak{r}_{4,1}, J)$ does not admit any CR-soliton metric. It follows that $\nu_+ = \mu_{a,0,a,0,0} \simeq \mathfrak{r}_{4,1,1}$, and so the rescaled solution $\omega(t)/(2t+1)$ converges in the pointed sense, as $t \to \infty$, to the 4-dimensional real hyperbolic space $\mathbb{R}H^4$.

Example 7.3 For any $\gamma \in \mathbb{R}$, $\delta > 0$, consider the solvable Lie algebra $\mathfrak{r}'_{4,\gamma,\delta}$ with Lie bracket as defined in Table 1, which coincides with $\mu_{\gamma,-\delta,1,0,0}$ from Example 7.1:

$$\operatorname{ad}_\mu e_4|_{\mathfrak{n}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma & \delta \\ 0 & -\delta & \gamma \end{bmatrix}, \qquad \gamma \in \mathbb{R}, \qquad \delta > 0.$$

The canonical metric is therefore a CR-soliton for both complex structures J_1 and J_2, with $p = -(1+\gamma)e^1 \wedge e^4$, which is therefore expanding, steady and shrinking for $\gamma > -1$, $\gamma = -1$ and $\gamma < -1$, respectively. Moreover, (J, ω) is a Kähler-Ricci soliton if and only if $\gamma = 0$ (expanding) and for $\gamma = -\frac{1}{2}$, the corresponding Lie group admits a lattice giving rise to a hermitian metric on an Inoue surface of type S^0 which is an expanding CR-soliton when pulled back on its universal cover (see [8]).

We have found a compatible CR-soliton for each complex structure on a 4-dimensional solvable Lie group, with the exceptions of the following seven cases:

$$(\mathfrak{r}_2', J_1), \quad (\mathfrak{r}_{4,1}, J), \quad (\mathfrak{d}_4, J_2), \quad (\mathfrak{d}_{4,\frac{1}{2}}, J_2), \quad (\mathfrak{d}'_{4,\delta}, J_1), \quad (\mathfrak{d}'_{4,\delta}, J_2), \quad (\mathfrak{h}_4, J). \tag{17}$$

We were able to prove the non-existence of a CR-soliton only in the case of $(\mathfrak{r}_{4,1}, J)$ (see Example 7.2). The CR-soliton metrics $g = \omega(\cdot, J\cdot)$ and their respective Chern-Ricci operators P are given in Table 3 as diagonal matrices with respect to the basis $\{e_1, e_2, e_3, e_4\}$, together with the constant c and the derivation D such that

$\mathfrak{g}$	**J**	**Metric**	**P**	**c**	**D**	**K**
$\mathfrak{rh}_3$	J	Any	$(0,0,0,0)$	0	$(0,0,0,0)$	——
$\mathfrak{rr}_{3,0}$	J	$(1,1,1,1)$	$(-1,-1,0,0)$	-1	$(0,0,1,1)$	Yes
$\mathfrak{rr}_{3,1}$	J	Any	$(0,0,0,0)$	0	$(0,0,0,0)$	——
$\mathfrak{rr}'_{3,0}$	J	Any	$(0,0,0,0)$	0	$(0,0,0,0)$	Yes
$\mathfrak{rr}'_{3,\gamma}$	J_1	Any	$(0,0,0,0)$	0	$(0,0,0,0)$	——
	J_2	Any	$(0,0,0,0)$	0	$(0,0,0,0)$	——
$\mathfrak{r}_2\mathfrak{r}_2$	J	$(1,1,1,1)$	$(-1,-1,-1,-1)$	-1	$(0,0,0,0)$	Yes
$\mathfrak{r}'_2$	J_1	$(1,1,1,1)$	$(-2,2,-2,2)$	——	——	——
	$J_{s,t}$	Any	$(0,0,0,0)$	0	$(0,0,0,0)$	——
$\mathfrak{r}_{4,1}$	J	$(1,1,1,1)$	$(0,0,-2,-2)$	——	——	——
$\mathfrak{r}_{4,\alpha,1}$	J	$(1,1,1,1)$	$-\alpha(\alpha+1)(0,1,0,1)$ $-1<\alpha\leq 1, \alpha\neq 0$	$-\alpha(\alpha+1)$	$\alpha(\alpha+1)(1,0,1,0)$	——
$\mathfrak{r}_{4,\alpha,\alpha}$	J	$(1,1,1,1)$	$-(\alpha+1)(1,0,0,1)$ $-1\leq\alpha<1, \alpha\neq 0$	$-(\alpha+1)$	$(\alpha+1)(0,1,1,0)$	——
$\mathfrak{r}'_{4,\gamma,\delta}$	J_1	$(1,1,1,1)$	$-(\gamma+1)(1,0,0,1)$ $\gamma\in\mathbb{R}, \delta>0$	$-(\gamma+1)$	$(\gamma+1)(0,1,1,0)$	$\gamma=0$
	J_2	$(1,1,1,1)$	$-(\gamma+1)(1,0,0,1)$ $\gamma\in\mathbb{R}, \delta>0$	$-(\gamma+1)$	$(\gamma+1)(0,1,1,0)$	$\gamma=0$
$\mathfrak{d}_4$	J_1	$(1,1,1,1)$	$(0,-1,0,-1)$	-1	$(1,0,1,0)$	——
	J_2	$(1,1,1,1)$	$(0,-1,0,-1)$	——	——	——
$\mathfrak{d}_{4,1}$	J	$(1,1,1,1)$	$(-2,0,0,-2)$	-2	$(0,2,2,0)$	——
$\mathfrak{d}_{4,\frac{1}{2}}$	J_1	$(1,1,1,1)$	$-\frac{3}{2}(1,1,1,1)$	$-\frac{3}{2}$	$(0,0,0,0)$	Yes
	J_2	$(1,1,1,1)$	$\frac{3}{2}(1,1,-1,-1)$	——	——	——
	J_3	$(2,5,\frac{5}{4},2)$	$-3(1,0,0,1)$	-3	$3(0,1,1,0)$	——
$\mathfrak{d}_{4,\lambda}$	J_1	$\left(\frac{e}{\lambda^2},2,2,e\right)$	$(a,0,0,a)$ $\frac{1}{2}<\lambda\neq 1$	a	$(0,-a,-a,0)$	$\lambda=2$
	J_2	$\left(2,\frac{d}{(\lambda-1)^2},2,d\right)$	$(0,b,0,b)$ $\frac{1}{2}<\lambda<1$	b	$(-b,0,-b,0)$	——
	J_3	$\left(2,\frac{d}{(\lambda-1)^2},2,d\right)$	$(0,b,0,b), \quad 1<\lambda$	b	$(-b,0,-b,0)$	——
$\mathfrak{d}'_{4,0}$	J_1	Any	$(0,0,0,0)$	0	$(0,0,0,0)$	——
	J_2	Any	$(0,0,0,0)$	0	$(0,0,0,0)$	——
$\mathfrak{d}'_{4,\delta}$	J_1	$(1,1,1,1)$	$\frac{3\delta}{2}(1,1,-1,-1)$	——	——	——
	J_2	$(1,1,1,1)$	$\frac{3\delta}{2}(1,1,-1,-1)$	——	——	——
	J_3	$(1,1,1,1)$	$-\frac{3\delta}{2}(1,1,1,1)$	$-\frac{3\delta}{2}$	$(0,0,0,0)$	Yes
	J_4	$(1,1,1,1)$	$-\frac{3\delta}{2}(1,1,1,1)$	$-\frac{3\delta}{2}$	$(0,0,0,0)$	Yes

$P = cI + D$. For example, the metric for the complex Lie algebra $(\mathfrak{d}_{4,\frac{1}{2}}, [\cdot,\cdot]_3, J_3)$ is given by $g(e_i, e_j) = \delta_{ij}$ for all $i \neq j$ and

$$g(e_1, e_1) = 2, \quad g(e_2, e_2) = 5, \quad g(e_3, e_3) = 5/4, \quad g(e_4, e_4) = 2.$$

In the last column we added the condition under which the metric is Kähler, that is, a Kähler-Ricci soliton.

In the case of $\mathfrak{d}_{4,\lambda}$, in order to simplify the description of the metrics in Table 3, we have introduced the following notation:

$$a := -\lambda(\lambda+1), \qquad b := (1-\lambda)(\lambda-2), \qquad d := (\lambda-1)^2+1, \qquad e := \lambda^2+1.$$

Remark 7.4 The existence of a CR-soliton on the complex Lie groups listed in (17) other than $\mathfrak{r}_{4,1}$ is an open problem. Based on the structure results obtained in [12] for Ricci soliton solvmanifolds, we conjecture that $(\mathfrak{h}_4, J)$ does not admit any CR-soliton.

Remark 7.5 There is an infinite family plus five individual solvable Lie groups of dimension 4 admitting a left-invariant complex structure which also admit a lattice, giving rise to the compact complex surfaces which are solvmanifolds (see [8]). Their Lie algebras are:

- $\mathbb{R}^4$: Complex tori.
- $\mathfrak{rh}_3$: Primary Kodaira surfaces.
- $\mathfrak{rr}'_{3,0}$: Hyperelliptic surfaces.
- $\mathfrak{r}'_{4,-\frac{1}{2},\delta}$: Inoue surfaces of type S^0.
- $\mathfrak{d}_4$: Inoue surfaces of type $S^{\pm}$.
- $\mathfrak{d}'_{4,0}$: Secondary Kodaira surfaces.

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