JORDAN-HÖLDER THEOREM FOR FINITE DIMENSIONAL HOPF ALGEBRAS

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Abstract. We show that a Jordan-Hölder theorem holds for appropriately defined composition series of finite dimensional Hopf algebras. This answers an open question of N. Andruskiewitsch. In the course of our proof we establish analogues of the Noether isomorphism theorems of group theory for arbitrary Hopf algebras under certain faithful (co)flatness assumptions. As an application, we prove an analogue of Zassenhaus' butterfly lemma for finite dimensional Hopf algebras. We then use these results to show that a Jordan-Hölder theorem holds as well for lower and upper composition series, even though the factors of such series may be not simple as Hopf algebras.

1. Introduction

Let G be a group. Composition series of G, when they exist, provide a tool to decompose the group G into a collection of simple groups: the composition factors of the series. Regarding this notion, a basic fundamental result in group theory is the Jordan-Hölder theorem, which asserts that the composition factors of a group G are in fact determined by G, independently of the choice of the composition series.

Let k be a field. Hopf algebra extensions play an important rôle in the classification problem of finite dimensional Hopf algebras over k. Indeed, suppose that H is a finite dimensional Hopf algebra over k and A is a normal Hopf subalgebra, that is, a Hopf subalgebra stable under the adjoint actions of H. Then the ideal HA^+, generated by the augmentation ideal A^+ of A, is a Hopf ideal of H and therefore $B = H/HA^+$ is a quotient Hopf algebra. In this way, A gives rise canonically to an exact sequence of Hopf algebras (see Section 2):

$$k \longrightarrow A \longrightarrow H \longrightarrow B \longrightarrow k.$$

As a consequence of the Nichols-Zoeller freeness theorem [12], any exact sequence of finite dimensional Hopf algebras is *cleft*, that is, it admits a convolution invertible B-colinear and A-linear section $B \to H$. This implies that H can be recovered from A and B plus some extra cohomological data. More precisely, H isomorphic as a Hopf algebra to a bicrossed product $A\#B$ with respect to suitable compatible data. See [15], [3].

These considerations motivated the question of deciding if an analogue of the Jordan-Hölder theorem of group theory does hold in the context of finite dimensional Hopf algebras, which was raised by N. Andruskiewitsch in [1, Question 2.1].

In this paper we show that this question has an affirmative answer. The following definition is proposed in [4]. Recall that a Hopf algebra is called *simple* if it contains no proper nontrivial normal Hopf subalgebra.

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Definition 1.1. A *composition series* of H is a sequence of finite dimensional simple Hopf algebras $\mathfrak{H}_1, \dots, \mathfrak{H}_n$ defined recursively as follows:

• If H is simple, we let $n = 1$ and $\mathfrak{H}_1 = H$.

• If $k \subsetneq A \subsetneq H$ is a normal Hopf subalgebra, and $\mathfrak{A}_1, \dots, \mathfrak{A}_m$, $\mathfrak{B}_1, \dots, \mathfrak{B}_l$, are composition series of A and $B = H/HA^+$, respectively, then we let $n = m + l$ and

$$\mathfrak{H}_i = \mathfrak{A}_i, \quad \text{if } 1 \leq i \leq m, \quad \mathfrak{H}_i = \mathfrak{B}_{i-m}, \quad \text{if } m < i \leq m + l.$$

The Hopf algebras $\mathfrak{H}_1, \dots, \mathfrak{H}_n$ will be called the *factors* of the series. The number n will be called the *length* of the series.

Every finite dimensional Hopf algebra admits a composition series. The next theorem is one of the main results of the paper:

Theorem 1.2. (Jordan-Hölder theorem for finite dimensional Hopf algebras.) *Let $\mathfrak{H}_1, \dots, \mathfrak{H}_n$ and $\mathfrak{H}'_1, \dots, \mathfrak{H}'_m$ be two composition series of H. Then there exists a bijection $f : \{1, \dots, n\} \to \{1, \dots, m\}$ such that $\mathfrak{H}_i \cong \mathfrak{H}'_{f(i)}$ as Hopf algebras.*

Theorem 1.2 is proved in Section 4. Its proof relies on appropriate analogues of the Noether isomorphism theorems of group theory that we establish, more generally, in the context of arbitrary Hopf algebras under suitable faithful (co)flatness assumptions (Theorems 3.2, 3.4 and 3.6).

As a consequence of Theorem 1.2 the *composition factors* and the *length* of H, defined, respectively, as the factors and the length of any composition series, are well-defined invariants of H. By definition, the composition factors of H are simple Hopf algebras. We prove some basic properties of these invariants, for instance, we show that they are additive with respect to exact sequences of Hopf algebras, and compatible with duality.

We also apply the isomorphism theorems to prove an analogue of the Jordan-Hölder theorem for lower and upper composition series of H; these are lower (respectively, upper) subnormal series which do not admit a proper refinement. See Definition 5.1 and Theorem 5.9. This allows us to introduce the lower and upper composition factors of H and its lower and upper lengths, which are also well-defined invariants of H. We remark that, in contrast with the case of the composition factors, the lower or upper composition factors are not necessarily simple as Hopf algebras (see Example 5.6). This motivates the question of deciding if there is an intrinsic characterization of the Hopf algebras that can arise as lower composition factors (see Question 5.7). Our proof of Theorem 5.9 follows the lines of the classical proof of the Jordan-Hölder theorem in group theory. In particular, we prove analogues of the Zassenhaus' butterfly lemma (Theorem 3.10) and the Schreier's refinement theorem (Theorem 5.8) for finite dimensional Hopf algebras.

We study some properties of lower and upper composition factors and their relation with the composition factors. Unlike for the case of the length, the lower and upper lengths are not additive with respect to exact sequences and they are not invariant under duality in general. Nevertheless, we show that if the lower (respectively, upper) composition factors are simple Hopf algebras, then they coincide, up to permutations, with the composition factors (see Proposition 5.4). We discuss some families of examples that include group algebras and their duals, abelian extensions and Frobenius-Lusztig kernels.

We point out that neither the composition factors nor the upper or lower composition factors of a finite dimensional Hopf algebra H are categorical invariants of H. In other words, they are not invariant under twisting deformations of H. In fact, it was shown in [6] that there exists a (semisimple) Hopf algebra such that H is simple as a Hopf algebra and H is twist equivalent to the group algebra of a solvable group G. In particular, the categories of finite dimensional representations of H and G are equivalent fusion categories. Thus not even the length nor the lower or upper lengths are invariant under twisting deformations. We discuss series of normal (right) coideal subalgebras, instead of normal Hopf subalgebras, in Subsection 5.2. These kind of series are of a categorical nature but, as it turns out, they fail to enjoy a Jordan-Hölder theorem.

The paper is organized as follows. In Section 2 we recall some definitions and facts related to normality and exact sequences. In Section 3 we prove the isomorphism theorems and discuss some consequences, including the butterfly lemma. Theorem 1.2 is proved in Section 4; several properties and examples of composition series and its related invariants are also studied in this section. In Section 5 we study lower and upper composition series and give a proof of Theorem 5.9.

Unless explicitly mentioned, k will denote an arbitrary field. The notation Hom, $\otimes$, etc., will mean, respectively, Hom_k, $\otimes_k$, etc.

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2. Preliminaries

Let H be a Hopf algebra over k. In this section we recall some facts about normal Hopf subalgebras, normal Hopf algebra maps, and exact sequences of Hopf algebras. Our references for these topics are [3], [16], [19].

2.1. **Normal Hopf subalgebras.** The left and right adjoint actions of H on itself are defined, respectively, in the form:

$$h.a = h_{(1)}a\mathcal{S}(h_{(2)}), \qquad a.h = \mathcal{S}(h_{(1)})ah_{(2)}, \qquad a, h \in H.$$

Let K, L be subspaces of H. We shall say that L *right normalizes* K (respectively, *left normalizes* K) if K is stable under the right (respectively, left) adjoint action of L. A subspace $K \subseteq H$ is called *left normal* (respectively, right normal) if it is stable under the left adjoint action of H (respectively, under the right adjoint action of H). K is called *normal* if it is stable under both adjoint actions.

Suppose K is a Hopf subalgebra of H. If H is left faithfully flat over K, then K is right normal if and only if $K^+H \subseteq HK^+$; if H is right faithfully flat over K, then K is left normal if only if $HK^+ \subseteq K^+H$ [19, Theorem 4.4 (a)].

Remark 2.1. (i) Suppose H has a bijective antipode. If $K \subseteq H$ is a Hopf subalgebra, then H is left faithfully flat over K if and only if it is right faithfully flat over K. In this case the antipode of K is also bijective [19, Proposition 3.3 (b)].

(ii) Suppose that A, B are Hopf subalgebras of H with bijective antipodes (which is always the case if H is finite dimensional). Then A right normalizes B if and only if it left normalizes B. If this is the case, then $AB = BA$. In particular, if the antipode of H is bijective and $K \subseteq H$ is a Hopf subalgebra with bijective antipode, then K is left normal in H if and only if it is right normal in H. When H is faithfully flat over K, these conditions are also equivalent to $HK^+ = K^+H$.

2.2. **Normal Hopf algebra maps.** The left and right adjoint coactions of H on itself are defined, respectively, as $h \mapsto h_{(1)}\mathcal{S}(h_{(3)}) \otimes h_{(2)}$, and $h \mapsto h_{(2)} \otimes h_{(1)}\mathcal{S}(h_{(3)})$, $h \in H$.

Let $\pi : H \to \overline{H}$ be a Hopf algebra map. The map π will be called *left normal* (respectively, *right normal*) if the kernel I of π is a subcomodule for the left (respectively, right) adjoint coaction of H, and it will be called *normal* if it is both left and right normal.

Let ${}^{\operatorname{co}\pi}H$ and $H^{\operatorname{co}\pi}$ be the subalgebras of H defined, respectively, by

$${}^{\operatorname{co}\pi}H = \{h \in H : (\pi \otimes \operatorname{id})\Delta(h) = 1 \otimes h\}, \quad H^{\operatorname{co}\pi} = \{h \in H : (\operatorname{id} \otimes \pi)\Delta(h) = h \otimes 1\}.$$

The subalgebra ${}^{\operatorname{co}\pi}H$ is always a right normal right coideal subalgebra of H. Similarly, $H^{\operatorname{co}\pi}$ is a left normal left coideal subalgebra of H. When the Hopf algebra map π is clear from the context, we shall also use the notation ${}^{\operatorname{co}\overline{H}}H$ and $H^{\operatorname{co}\overline{H}}$ to indicate the subalgebras ${}^{\operatorname{co}\pi}H$ and $H^{\operatorname{co}\pi}$, respectively.

If H is left faithfully coflat over $\overline{H}$, then π is right normal if and only if $H^{\operatorname{co}\pi} \subseteq {}^{\operatorname{co}\pi}H$, and if H is right faithfully coflat over $\overline{H}$, then π is left normal if and only if ${}^{\operatorname{co}\pi}H \subseteq H^{\operatorname{co}\pi}$. If $\pi : H \to \overline{H}$ is a left normal Hopf algebra map, then ${}^{\operatorname{co}\pi}H$ is a right normal Hopf subalgebra of H. See [19, Section 4].

Remark 2.2. (i) Suppose that the antipode of H is bijective. Then H is left faithfully coflat over $\overline{H}$ if and only if it is right faithfully coflat. In this case, π is left normal if and only if it is right normal, if and only if $H^{\operatorname{co}\pi} = {}^{\operatorname{co}\pi}H$. Moreover, the antipode of $\overline{H}$ is also bijective [19, Proposition 3.1 and Corollary 4.5 (b)].

(ii) Suppose H has a bijective antipode. Let K be a normal Hopf subalgebra of H. It follows from [19, Theorem 3.2] that H is faithfully flat over K if and only if it is faithfully coflat over H/HK^+. If this is the case, then the antipodes of K and H/HK^+ are also bijective.

Remark 2.3. (i) When H is a commutative Hopf algebra, H is faithfully flat over any Hopf subalgebra [17]. If the coradical of H is cocommutative, then H is (left and right) faithfully coflat over any quotient H-module coalgebra and (left and right) faithfully flat over any Hopf subalgebra [8]. If H is finite dimensional, the Nichols-Zoeller theorem [12] implies that H is free over any Hopf subalgebra. Observe that a Hopf algebra H in any of the above classes has a bijective antipode.

(ii) Every Hopf algebra H is left and right faithfully flat (in fact free) over its finite dimensional normal Hopf subalgebras and left and right faithfully coflat over its finite dimensional normal quotient Hopf algebras [16, 2.1].

2.3. **Exact sequences of Hopf algebras.** An *exact sequence of Hopf algebras* is a sequence of Hopf algebra maps

$$k \longrightarrow H' \xrightarrow{i} H \xrightarrow{\pi} H'' \longrightarrow k, \tag{2.1}$$

satisfying the following conditions:

(a) i is injective and π is surjective , (b) $\ker \pi = Hi(H')^+$, (c) $i(H') = {}^{\operatorname{co}\pi}H$.

Note that either (b) or (c) imply that $\pi i = \epsilon 1$. If H is faithfully flat over H', then (a) and (b) imply (c). Dually, if H is faithfully coflat over H'', then (a) and (c) imply (b).

Let $K \subseteq H$ be a normal Hopf subalgebra. Then $HK^+ = K^+H$ is a Hopf ideal of H and the canonical map $H \to H/HK^+$ is a Hopf algebra map. Hence, if H is faithfully flat over K, then there is an exact sequence of Hopf algebras $k \longrightarrow K \longrightarrow H \longrightarrow H/HK^+ \longrightarrow k$. Similarly, if $\pi : H \to \overline{H}$ is a normal quotient Hopf algebra, then ${}^{\operatorname{co}\pi}H = H^{\operatorname{co}\pi}$ is a Hopf subalgebra and if H is faithfully coflat over $\overline{H}$, there is an exact sequence of Hopf algebras $k \longrightarrow {}^{\operatorname{co}\pi}H \longrightarrow H \longrightarrow \overline{H} \longrightarrow k$.

Assume that H is finite dimensional. Then any exact sequence (2.1) is cleft. In particular, $H \cong H' \otimes H''$ as left H'-modules and right H''-comodules [15], and therefore $\dim H = \dim H' \dim H''$.

Observe that H is simple if and only if it admits no proper normal quotient Hopf algebra. Furthermore, a sequence of Hopf algebra maps $k \longrightarrow H' \xrightarrow{i} H \xrightarrow{\pi} H'' \longrightarrow k$ is an exact sequence if and only if $k \longrightarrow (H'')^* \xrightarrow{\pi^*} H^* \xrightarrow{i^*} (H')^* \longrightarrow k$ is an exact sequence. Therefore H is simple if and only if H^* is simple.

Remark 2.4. Let (2.1) be an exact sequence of finite dimensional Hopf algebras. By cleftness, H is isomorphic to a bicrossed product $H'\#H''$. This implies that H is semisimple if and only if H' and H'' are semisimple. Indeed, if H is semisimple then every quotient Hopf algebra and every Hopf subalgebra of H are semisimple; for the converse, we use that H is isomorphic as an algebra to a crossed product and [5, Theorem 2.6]. Dualizing the exact sequence (2.1) we also get that H is cosemisimple if and only if H' and H'' are cosemisimple.

3. Isomorphism theorems for Hopf algebras

In this section we prove analogues of the Noether isomorphism theorems. We refer the reader to [9], [19], for a detailed exposition on the quotient theory of Hopf algebras. Our proofs rely on the following result of M. Takeuchi:

Theorem 3.1. ([18, Theorems 1 and 2].) *Let H be a Hopf algebra. Let K be a right coideal subalgebra of H and let $\pi : H \to \overline{H}$ be a quotient left H-module coalgebra. Then the following hold:*

(i) If H is left faithfully flat over K, then H is left faithfully coflat over H/HK^+ and $K = {}^{\mathrm{co}\, H/HK^+}H$.

(ii) If H is right faithfully coflat over $\overline{H}$, then H is right faithfully flat over ${}^{\mathrm{co}\,\pi}H$ and $\ker \pi = H({}^{\mathrm{co}\,\pi}H)^+$. □

3.1. **First isomorphism theorem.** In view of [19, Proposition 1.4 (a)], if $K \subseteq H$ is a right normal right coideal subalgebra, then HK^+ is a Hopf ideal of H. Therefore the quotient H/HK^+ is a Hopf algebra and the canonical map $\pi_K : H \to H/HK^+$ is a Hopf algebra map. Theorem 3.1 can be regarded as an analogue of the first isomorphism theorem of group theory:

Theorem 3.2. (First isomorphism theorem for Hopf algebras.) *Let H be a Hopf algebra and let $\pi : H \to \overline{H}$ be a surjective Hopf algebra map. Suppose that H is right $\overline{H}$-faithfully coflat. Then π induces an isomorphism of Hopf algebras $H/HK^+ \cong \overline{H}$, where $K = {}^{\mathrm{co}\,\pi}H$.* □

As a consequence of Theorem 3.1 we also obtain:

Corollary 3.3. *Let $K \subseteq H$ be a right normal right coideal subalgebra of H and let $\pi : H \to \overline{H}$ be a Hopf algebra map. Assume that H is left faithfully flat over K and right faithfully coflat over $\overline{H}$. Then the following assertions are equivalent:*

(i) $K \subseteq {}^{\mathrm{co}\,\pi}H$.

(ii) There is a unique Hopf algebra map $\overline{\pi} : H/HK^+ \to \overline{H}$ such that $\pi = \overline{\pi}\pi_K$.

Proof. Assume (i). By Theorem 3.1 (ii), we have that $\ker \pi = H({}^{\mathrm{co}\,\pi}H)^+$. Therefore $HK^+ \subseteq \ker \pi$ and there is a unique algebra map $\overline{\pi} : H/HK^+ \to \overline{H}$ that makes the diagram in (ii) commute. Since π is a Hopf algebra map, then so is $\overline{\pi}$. Hence (ii) holds. Conversely, assume (ii). By Theorem 3.1 (i), $K = {}^{\mathrm{co}\, H/HK^+}H$. The relation $\pi = \overline{\pi}\pi_K$ implies $K = {}^{\mathrm{co}\, H/HK^+}H \subseteq {}^{\mathrm{co}\,\pi}H$. Thus (i) holds. □

3.2. **Second isomorphism theorem.** A version of the next theorem was shown, in a finite dimensional context, in [2, (3.3.8)].

Theorem 3.4. (Second isomorphism theorem for Hopf algebras.) *Let H be a Hopf algebra and let A, B be Hopf subalgebras of H such that A right normalizes B. Then the following hold:*

(i) The product AB is a Hopf subalgebra of H.

(ii) $A(A \cap B)^+$ is a Hopf ideal of A and AB^+ is a Hopf ideal of AB.

(iii) Suppose that AB is left faithfully flat over B and A is right faithfully coflat over AB/AB^+. Then the inclusion $A \to AB$ induces canonically an isomorphism of Hopf algebras $A/A(A \cap B)^+ \cong AB/AB^+$.

Proof. (i). It is clear that AB is a subcoalgebra of H. Since A right normalizes B, then, for all $b \in B$, $a \in A$, we have $ba = a_{(1)}\mathcal{S}(a_{(2)})ba_{(3)} \in AB$. Thus $BA \subseteq AB$. Therefore AB is a subalgebra, hence a subbialgebra, of H. In addition, $\mathcal{S}(AB) = \mathcal{S}(B)\mathcal{S}(A) \subseteq BA \subseteq AB$. Then AB is a Hopf subalgebra.

(ii). Since A right normalizes B, then $A \cap B$ is a right normal Hopf subalgebra of A and B is a right normal Hopf subalgebra of AB. Hence $A(A \cap B)^+$ is a Hopf ideal of A and $AB^+ = ABB^+$ is a Hopf ideal of AB [19, Proposition 1.4 (a)].

(iii). It follows from (ii) that $A/A(A \cap B)^+$ is a quotient Hopf algebra of A and AB/AB^+ is a quotient Hopf algebra of AB. The composition of the inclusion $A \to AB$ with the canonical projection $\pi' : AB \to AB/AB^+$ induces a Hopf algebra map $\pi' : A \to AB/AB^+$. Note that π' is surjective; indeed, for all $a \in A$, $b \in B$, we have that $\pi'(ab) = \pi'(a)\pi'(b) = \pi'(a)\epsilon(b) \in \pi'(A)$. Hence $AB/AB^+ = \pi'(AB) = \pi'(A)$.

Since AB is left faithfully flat over B, Theorem 3.1 (i) implies that AB is left faithfully coflat over AB/AB^+ and ${}^{\operatorname{co}\pi'}(AB) = {}^{\operatorname{co} AB/AB^+}(AB) = B$.

By assumption, A is right faithfully coflat over AB/AB^+. Then Theorem 3.1 (ii) implies that A is right faithfully flat over ${}^{\operatorname{co}\pi'}A$ and the kernel of $\pi' : A \to AB/AB^+$ coincides with $A({}^{\operatorname{co}\pi'}A)^+$. Note that ${}^{\operatorname{co}\pi'}A = A \cap {}^{\operatorname{co}\pi'}(AB) = A \cap B$. From this we deduce that the kernel of $\pi' : A \to AB/AB^+$ is $A(A \cap B)^+$. Hence π' induces an isomorphism of Hopf algebras $A/A(A \cap B)^+ \to AB/AB^+$, as claimed. □

Corollary 3.5. *Let A, B be finite dimensional Hopf subalgebras of H such that A right normalizes B. Then AB is a finite dimensional Hopf subalgebra of H and we have* $\dim AB = \dfrac{\dim A \dim B}{\dim A \cap B}$.

Proof. By Theorem 3.4 (iii), there is an isomorphism of Hopf algebras $A/A(A \cap B)^+ \cong AB/AB^+$ (c.f. Remark 2.3). This implies the corollary, since $\dim A/A(A \cap B)^+ = \dfrac{\dim A}{\dim A \cap B}$ and $\dim AB/AB^+ = \dfrac{\dim AB}{\dim B}$. □

3.3. **Third isomorphism theorem.** A version of the following theorem was established, under certain finiteness conditions, in [2, (3.3.7)]. Recall that if $K \subseteq H$, then $\pi_K : H \to H/HK^+$ denotes the canonical map.

Theorem 3.6. (Third isomorphism theorem for Hopf algebras.) *Let A be a right normal Hopf subalgebra of H and let B be a right normal right coideal subalgebra of H such that $B \subseteq A$. Then $\pi_B(A)$ is a right normal Hopf subalgebra of H/HB^+ and the following hold:*

(i) π_A induces an isomorphism of Hopf algebras $\dfrac{H/HB^+}{(H/HB^+)\pi_B(A)^+} \cong H/HA^+$.

(ii) Assume that H is left faithfully flat over B and A is right faithfully coflat over $\pi_B(A)$. Then π_B induces an isomorphism of Hopf algebras $A/AB^+ \cong \pi_B(A)$.

Proof. Since π_B is a surjective Hopf algebra map, then $\pi_B(A)$ is a normal Hopf subalgebra of H/HB^+. Since $HB^+ \subseteq HA^+ = \ker \pi_A$, there exists a unique surjective Hopf algebra map $\overline{\pi_A} : H/HB^+ \to H/HA^+$ such that $\pi_A = \overline{\pi_A}\pi_B$. Hence $\ker \overline{\pi_A} = \pi_B(\ker \pi_A) = \pi_B(HA^+) = (H/HB^+)\pi_B(A)^+$. Therefore (i) holds.

Theorem 3.1 (i) implies that ${}^{\operatorname{co}\pi_B}H = B$, since H is left faithfully flat over B. Hence ${}^{\operatorname{co}\pi_B}A = A \cap {}^{\operatorname{co}\pi_B}H = B$. Therefore, since A is right faithfully coflat over $\pi_B(A)$, Theorem 3.2 implies that π_B induces an isomorphism of Hopf algebras $A/AB^+ \cong \pi_B(A)$. This proves part (ii) and finishes the proof of the theorem. □

Corollary 3.7. *Let B be a normal Hopf subalgebra of H such that H is left faithfully flat over B. Assume that the quotient Hopf algebra H/HB^+ is simple. Then for every normal Hopf subalgebra A such that $B \subseteq A$ and H is left faithfully flat over A, we have $A = B$ or $A = H$.*

Proof. Let $\pi_B : H \to H/HB^+$ denote the canonical map. By Theorem 3.1 (i), ${}^{\mathrm{co}\,\pi_B}H = B$, since H is left faithfully flat over B. Since $\pi_B(A)$ is a normal Hopf subalgebra of H/HB^+ and H/HB^+ is a simple Hopf algebra by assumption, then $\pi_B(A) = k$ or $\pi_B(A) = H/HB^+$. If $\pi_B(A) = k$, then $\pi_B|_A = \epsilon_A$. Therefore $A \subseteq {}^{\mathrm{co}\,\pi_B}H = B$ and $A = B$ in this case. Suppose, on the other hand, that $\pi_B(A) = H/HB^+$. From Theorem 3.6, we obtain that $\ker \overline{\pi_A} = (H/HB^+)\pi_B(A)^+ = (H/HB^+)^+$. Hence $H/HA^+ \cong (H/HB^+)/(H/HB^+) \cong k$. Since H is left faithfully flat over A, Theorem 3.1 (i) implies that $A = H$. □

Example 3.8. Let $\mathfrak{g}$ be a finite dimensional semisimple complex Lie algebra. Suppose ℓ is an odd integer, relatively prime to 3 if $\mathfrak{g}$ has a component of type G_2, and let ε be a primitive ℓ-th root of unity in $\mathbb{C}$. Let also $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ over the cyclotomic field $k = \mathbb{Q}(\varepsilon)$ and let $U_\varepsilon(\mathfrak{g})$ be the quantum enveloping algebra introduced by Lusztig [7]. Then $U_\varepsilon(\mathfrak{g})$ is a pointed Hopf algebra over k with bijective antipode.

Consider the Frobenius homomorphism $\mathrm{Fr} : U_\varepsilon(\mathfrak{g}) \to U(\mathfrak{g})$ [7, 8.10, 8.16]. The Frobenius-Lusztig kernel of $U_\varepsilon(\mathfrak{g})$ is the finite dimensional Hopf subalgebra $\mathfrak{u} = U_\varepsilon(\mathfrak{g})^{\mathrm{co}\,\mathrm{Fr}}$ of $U_\varepsilon(\mathfrak{g})$. Fr induces a cleft exact sequence of Hopf algebras $k \longrightarrow \mathfrak{u} \longrightarrow U_\varepsilon(\mathfrak{g}) \xrightarrow{\mathrm{Fr}} U(\mathfrak{g}) \longrightarrow k$; see [2, Lemma 3.4.1 and Proposition 3.4.4]. In particular, there is an isomorphism of Hopf algebras $U_\varepsilon(\mathfrak{g})/U_\varepsilon(\mathfrak{g})\mathfrak{u}^+ \cong U(\mathfrak{g})$.

Suppose $\mathfrak{g}$ is a simple Lie algebra. Then $U(\mathfrak{g})$ is a simple Hopf algebra (this can be seen as a consequence of the Cartier-Kostant-Milnor-Moore theorem). Corollary 3.7 implies that $\mathfrak{u} \subseteq U(\mathfrak{g})$ is a maximal faithfully flat normal inclusion of Hopf algebras.

3.4. **Summary.** The following theorem summarizes the contents of the previous subsections in the finite dimensional case.

Theorem 3.9. *Let H be a finite dimensional Hopf algebra. The following hold:*

(i) Let A, B be Hopf subalgebras of H such that A normalizes B. Then B is a normal Hopf subalgebra of AB, $A \cap B$ is a normal Hopf subalgebra of A and there is an exact sequence of Hopf algebras $k \longrightarrow A \cap B \longrightarrow A \longrightarrow AB/AB^+ \longrightarrow k$.

(ii) Let A be a normal Hopf subalgebra and B a right normal right coideal subalgebra of H such that $B \subseteq A$. Then there exists a unique Hopf algebra map $\pi : H/HB^+ \to H/HA^+$ such that $\pi\pi_B = \pi_A$ and the map π fits into an exact sequence of Hopf algebras $k \longrightarrow A/AB^+ \longrightarrow H/HB^+ \xrightarrow{\pi} H/HA^+ \longrightarrow k$.

Proof. A finite dimensional Hopf algebra is free over any Hopf subalgebra and over any right coideal subalgebra [12], [14]. The statement follows from Theorems 3.1, 3.4 and 3.6. □

3.5. **Zassenhaus' butterfly lemma.** We assume in this subsection that H is finite dimensional. As an application of Theorem 3.6 we obtain the following analogue of Zassenhaus' butterfly lemma of group theory:

Theorem 3.10. *Let A, B be Hopf subalgebras of H and let A', B' be normal Hopf subalgebras of A and B, respectively. Then the following hold:*

(i) $A'(A \cap B')$ is a normal Hopf subalgebra of $A'(A \cap B)$.

(ii) $B'(A' \cap B)$ is a normal Hopf subalgebra of $B'(A \cap B)$.

(iii) $A'(A \cap B') \cap A \cap B = (A' \cap B)(A \cap B') = B'(A' \cap B) \cap A \cap B$.

(iv) There is an isomorphism of Hopf algebras

$$\frac{A'(A \cap B)}{A'(A \cap B)(A'(A \cap B'))^+} \cong \frac{B'(A \cap B)}{B'(A \cap B)(B'(A' \cap B))^+}.$$

Proof. Note that, since A' is normal in A, then $A'(A \cap B')$ and $A'(A \cap B)$ are indeed Hopf subalgebras of H. Similarly, $B'(A' \cap B)$ and $B'(A \cap B)$ are Hopf subalgebras of H, because B' is normal in B. Part (i) follows from the fact that $A'(A\cap B')$ is stable under both adjoint actions of A' (since $A'\subseteq A'(A\cap B')$) and also under both adjoint actions of $A\cap B$, because A' is normal in A and B' is normal in B. Part (ii) reduces to (i), interchanging the rôles of A and B.

We next prove (iii). It is clear that $(A'\cap B)(A\cap B')\subseteq A'(A\cap B')\cap A\cap B$. By Corollary 3.5, we have

$$\dim (A'\cap B)(A\cap B') = \frac{\dim A'\cap B \dim A\cap B'}{\dim A'\cap B'}. \tag{3.1}$$

On the other side, using again Corollary 3.5, we compute

$$\begin{aligned}\dim A'(A\cap B')\cap A\cap B &= \frac{\dim A'(A\cap B')\dim A\cap B}{\dim A'(A\cap B')(A\cap B)} = \frac{\dim A'(A\cap B')\dim A\cap B}{\dim A'(A\cap B)}\\ &= \frac{\dim A' \dim A\cap B' \dim A\cap B}{\dim A'\cap B' \dim A'(A\cap B)} = \frac{\dim A\cap B' \dim A'\cap B}{\dim A'\cap B'}.\end{aligned}$$

Comparing this formula with (3.1), we find that $(A'\cap B)(A\cap B')$ and $A'(A\cap B')\cap A\cap B$ have the same (finite) dimension, and therefore they are equal. By symmetry, this also shows that $(A'\cap B)(A\cap B') = B'(A'\cap B)\cap A\cap B$. Hence we get (iii).

To prove part (iv) we argue as follows. From Theorem 3.4, we obtain

$$\begin{aligned}\frac{A'(A\cap B)}{A'(A\cap B)(A'(A\cap B'))^+} &= \frac{A'(A\cap B')(A\cap B)}{A'(A\cap B)(A'(A\cap B'))^+}\\ &\cong \frac{A\cap B}{(A\cap B)(A'(A\cap B')\cap A\cap B)^+},\end{aligned}$$

and

$$\begin{aligned}\frac{B'(A\cap B)}{B'(A\cap B)(B'(A'\cap B))^+} &= \frac{B'(A'\cap B)(A\cap B)}{B'(A\cap B)(B'(A'\cap B))^+}\\ &\cong \frac{A\cap B}{(A\cap B)(B'(A'\cap B)\cap A\cap B)^+}.\end{aligned}$$

Then part (iv) follows from (iii). This finishes the proof of the theorem. □

4. Composition series of finite dimensional Hopf algebras

Let H be a finite dimensional Hopf algebra over k. Recall that a composition series of H is a sequence of finite dimensional simple Hopf algebras $\mathfrak{H}_1,\dots,\mathfrak{H}_n$ defined as $\mathfrak{H}_1 = H$ and $n=1$, if H is simple, and $n=m+l$, $\mathfrak{H}_i=\mathfrak{A}_i$, if $1\leq i\leq m$, $\mathfrak{H}_i = \mathfrak{B}_{i-m}$, if $m<i\leq m+l$, whenever $k\subsetneq A\subsetneq H$ is a normal Hopf subalgebra, and $\mathfrak{A}_1,\dots,\mathfrak{A}_m$, $\mathfrak{B}_1,\dots,\mathfrak{B}_l$, are composition series of A and $B=H/HA^+$, respectively. See Definition 1.1.

Lemma 4.1. *Every finite dimensional Hopf algebra admits a composition series.*

Proof. If H is simple, then $H_1 = H$ is a composition series of H. Otherwise, H contains a normal Hopf subalgebra $k\subsetneq A\subsetneq H$. We have $\dim A, \dim H/HA^+ < \dim H$, because $\dim H = \dim A \dim H/HA^+$. The lemma follows by induction. □

As an application of the results in Section 3 we are now able to prove the Jordan-Hölder Theorem 1.2 for finite dimensional Hopf algebras:

Proof of Theorem 1.2. The proof is by induction on the dimension of H. If H is simple, there is nothing to prove. Assume that H is not simple. Let $k\subsetneq A\subsetneq H$ be a normal Hopf subalgebra, and let $\mathfrak{A}_1,\dots,\mathfrak{A}_m$, $\mathfrak{B}_1,\dots,\mathfrak{B}_l$, be composition series of A and $B=H/HA^+$, respectively.

Suppose that there exists a normal Hopf subalgebra K such that $A\subsetneq K\subsetneq H$. Let $\mathfrak{K}_1,\dots,\mathfrak{K}_r$, $\mathfrak{L}_1,\dots,\mathfrak{L}_s$, be composition series of K and $L=H/HK^+$, respectively. Then $\mathfrak{K}_1,\dots,\mathfrak{K}_r,\mathfrak{L}_1,\dots,\mathfrak{L}_s$ is a composition series of H. Let $\mathfrak{C}_1,\dots,\mathfrak{C}_p$

be a composition series of K/KA^+. Since $\dim K < \dim H$, it follows by induction that $r = m + p$ and the sequence $\mathfrak{K}_1, \dots, \mathfrak{K}_r$ is a permutation of the sequence $\mathfrak{A}_1, \dots, \mathfrak{A}_m, \mathfrak{C}_1, \dots, \mathfrak{C}_p$.

By Theorem 3.6 (iii), there is an exact sequence of Hopf algebras

$$k \longrightarrow K/KA^+ \longrightarrow H/HA^+ \longrightarrow H/HK^+ \longrightarrow k.$$

We also have $\dim H/HA^+ < \dim H$. Hence, by induction, $l = p + s$ and the sequence $\mathfrak{B}_1, \dots, \mathfrak{B}_l$ is a permutation of the sequence $\mathfrak{C}_1, \dots, \mathfrak{C}_p, \mathfrak{L}_1, \dots, \mathfrak{L}_s$. In conclusion, $r + s = m + l$ and the sequence $\mathfrak{K}_1, \dots, \mathfrak{K}_r, \mathfrak{L}_1, \dots, \mathfrak{L}_s$ is a permutation of the sequence $\mathfrak{A}_1, \dots, \mathfrak{A}_m, \mathfrak{B}_1, \dots, \mathfrak{B}_l$.

Let now $k \subsetneq A' \subsetneq H$ be another normal Hopf subalgebra, and let $\mathfrak{A}'_1, \dots, \mathfrak{A}'_e$, $\mathfrak{B}'_1, \dots, \mathfrak{B}'_d$, be composition series of A' and $B' = H/HA'^+$, respectively. We want to show that $m + l = e + d$ and $\mathfrak{A}_1, \dots, \mathfrak{A}_m$, $\mathfrak{B}_1, \dots, \mathfrak{B}_l$ is a permutation of $\mathfrak{A}'_1, \dots, \mathfrak{A}'_e$, $\mathfrak{B}'_1, \dots, \mathfrak{B}'_d$. We may assume that $A \neq A'$. Furthermore, the first part of the proof implies that we may also assume that A and A' are maximal proper normal Hopf subalgebras. Since AA' is a normal Hopf subalgebra of H containing strictly A and A', then $AA' = H$. In view of Theorem 3.4, $A \cap A'$ is a normal Hopf subalgebra of A and of A' and there are isomorphisms $H/HA^+ \cong A'/A'(A' \cap A)^+$, $H/HA'^+ \cong A/A(A' \cap A)^+$.

Let $\mathfrak{U}_1, \dots, \mathfrak{U}_f$ be a composition series of $A \cap A'$. Since $\dim A, \dim A' < \dim H$, the inductive assumption implies that $e = f + l$, $m = f + d$, $\mathfrak{A}'_1, \dots, \mathfrak{A}'_e$ is a permutation of $\mathfrak{U}_1, \dots, \mathfrak{U}_f, \mathfrak{B}_1, \dots, \mathfrak{B}_l$, and $\mathfrak{A}_1, \dots, \mathfrak{A}_m$ is a permutation of $\mathfrak{U}_1, \dots, \mathfrak{U}_f$, $\mathfrak{B}'_1, \dots, \mathfrak{B}'_d$. Hence $e + d = m + l$ and the sequence $\mathfrak{A}'_1, \dots, \mathfrak{A}'_e$, $\mathfrak{B}'_1, \dots, \mathfrak{B}'_d$ is a permutation of $\mathfrak{A}_1, \dots, \mathfrak{A}_m$, $\mathfrak{B}_1, \dots, \mathfrak{B}_l$, as claimed. This finishes the proof of the theorem. □

Theorem 1.2 permits to introduce the following invariants of a finite dimensional Hopf algebra:

Definition 4.2. Let H be a finite dimensional Hopf algebra and let $\mathfrak{H}_1, \dots, \mathfrak{H}_n$ be a composition series of H. The simple Hopf algebras $\mathfrak{H}_i$, $1 \leq i \leq n$, will be called the *composition factors of* H. The number n will be called the *length of* H.

Corollary 4.3. *Suppose $k \longrightarrow A \longrightarrow H \longrightarrow B \longrightarrow k$ is any exact sequence of Hopf algebras. Then the length of H equals the sum of the lengths of A and B.* □

Corollary 4.4. *Let H be a finite dimensional Hopf algebra and let $\mathfrak{H}_1, \dots, \mathfrak{H}_n$, be the compostion factors of H. Then the composition factors of H^* are $\mathfrak{H}_1^*, \dots, \mathfrak{H}_n^*$. In particular the length of H^* coincides with the length of H.*

Proof. The proof is by induction on the dimension of H, using the fact that a sequence $k \longrightarrow A \longrightarrow H \longrightarrow B \longrightarrow k$ is exact if and only if the sequence $k \longrightarrow B^* \longrightarrow H^* \longrightarrow A^* \longrightarrow k$ is exact. □

Proposition 4.5. *Let H be a finite dimensional Hopf algebra. Then H is semisimple (respectively, cosemisimple) if and only if all its composition factors are semisimple (respectively, cosemisimple).*

Proof. We shall prove the statement concerning semisimplicity. This implies the statement for cosemisimplicity, in view of Corollary 4.4. We may assume that H is not simple. Then there is an exact sequence $k \longrightarrow A \longrightarrow H \longrightarrow B \longrightarrow k$, where $\dim A, \dim B < \dim H$. In particular, H is semisimple if and only if A and B are semisimple (see Remark 2.4). By definition, the composition factors of H are $\mathfrak{A}_1, \dots, \mathfrak{A}_m$, $\mathfrak{B}_1, \dots, \mathfrak{B}_l$, where $\mathfrak{A}_1, \dots, \mathfrak{A}_m$ are the commposition factors of A and $\mathfrak{B}_1, \dots, \mathfrak{B}_l$ are the composition factors of B. The proposition follows by an inductive argument. □

Example 4.6. Let G be a finite group and let $H = kG$ be the group Hopf algebra of G. The normal Hopf subalgebras of H are exactly those group algebras kN, where N is a normal subgroup of G and $kG/kG(kN)^+ \cong k(G/N)$. Thus the composition factors of H are exactly the group algebras of the composition factors of G. In view of Corollary 4.4 the composition factors of the dual group algebra k^G are the dual group algebras of the composition factors of G.

Example 4.7. Let F, Γ be finite groups and let H be an abelian extension of k^Γ by kG, that is, H is a Hopf algebra fitting into an exact sequence $k \longrightarrow k^\Gamma \longrightarrow H \longrightarrow kF \longrightarrow k$ (see Example 5.15). Then the composition factors of H are the group algebras of the composition factors of F and the dual group algebras of the composition factors of Γ.

As an example, consider the Drinfeld double $D(G)$ of a finite group G; $D(G)$ fits into an exact sequence $k \longrightarrow k^G \longrightarrow D(G) \longrightarrow kG \longrightarrow k$. Therefore if $G_1, \dots, G_n$ are the composition factors of G, then the composition factors of $D(G)$ are the Hopf algebras $k^{G_1}, \dots, k^{G_n}, kG_1, \dots, kG_n$. In particular, the length of $D(G)$ is twice the length of G.

5. Upper and lower composition series

Along this section, H will be a finite dimensional Hopf algebra over k. The following definition extends the notion of subnormal series of a group.

Definition 5.1. A *lower subnormal series* of H is a series of Hopf subalgebras

$$k = H_n \subseteq H_{n-1} \subseteq \cdots \subseteq H_1 \subseteq H_0 = H, \tag{5.1}$$

with H_{i+1} normal in H_i, for all i. The *factors* of the series (5.1) are the quotient Hopf algebras $\overline{H}_i = H_i/H_iH_{i+1}^+$, $i = 0, \dots, n-1$.

An *upper subnormal series* of H is a series of surjective Hopf algebra maps

$$H = H_{(0)} \to H_{(1)} \to \cdots \to H_{(n)} = k, \tag{5.2}$$

such that $H_{(i+1)}$ is a normal quotient Hopf algebra of $H_{(i)}$, for all $i = 0, \dots, n-1$. The *factors* of (5.2) are the Hopf algebras $\underline{H}_i = {}^{\operatorname{co} H_{(i+1)}}H_{(i)} \subseteq H_{(i)}$, $i = 0, \dots, n-1$.

Lower and upper subnormal series were introduced in [11, Section 3] under the names *normal* lower and upper series, respectively.

Definition 5.2. A *refinement* of (5.1) is a lower subnormal series

$$k = H'_m \subseteq H'_{m-1} \subseteq \cdots \subseteq H'_1 \subseteq H'_0 = H, \tag{5.3}$$

such that for all $0 < i < n$, there exists $0 < N_i < m$, with $N_1 < N_2 < \cdots < N_m$, and $H_i = H'_{N_i}$. If (5.3) is a refinement of (5.1) and it does not coincide with (5.1), we shall say that it is a *proper refinement*.

Two lower subnormal series $k = H_n \subseteq H_{n-1} \subseteq \cdots \subseteq H_1 \subseteq H_0 = H$ and $k = H'_m \subseteq H'_{m-1} \subseteq \cdots \subseteq H'_1 \subseteq H'_0 = H$ will be called *equivalent* if there exists a bijection $f : \{0, \dots, n\} \to \{0, \dots, m\}$ such that the corresponding factors are isomorphic as Hopf algebras, that is, such that $H_i/H_iH_{i+1}^+ \cong H'_{f(i)}/H'_{f(i)}{H'_{f(i)+1}}^+$.

A *lower composition series* of H is a *strictly decreasing* lower subnormal series which does not admit a proper refinement. In other words, a lower normal series (5.1) is a composition series if and only if H_i is a maximal normal Hopf subalgebra of H_{i-1}, for all $i = 1, \dots, n$.

Upper composition series can be defined similarly. It is clear that every finite dimensional Hopf algebra H admits lower and upper composition series.

Remark 5.3. It follows from [11, Theorem 3.2] that lower subnormal series of H with factors $\overline{H}_i$, $i = 0, \dots, n-1$, correspond to upper subnormal series of H^* with factors $\underline{H^*}_i \cong (\overline{H}_i)^*$, $i = 0, \dots, n-1$. Similarly, upper subnormal series of H with factors $\underline{H}_i$, $i = 0, \dots, n-1$, correspond to lower subnormal series of H^* with factors $\overline{H^*}_i \cong \underline{H}_i^*$, $i = 0, \dots, n-1$. In particular, upper composition series of H with factors $\underline{H}_i$ correspond to lower composition series of H^* with factors $\underline{H}_i^*$.

Proposition 5.4. *Let H be a finite dimensional Hopf algebra. Then a lower (respectively, upper) subnormal series of H such that all of its factors are simple Hopf algebras is a lower (respectively, upper) composition series. The factors of such series coincide, up to permutations, with the composition factors of H.*

Proof. Suppose (5.1) is a lower subnormal series with simple factors. By Corollary 3.7, (5.1) is a lower composition series. Let $\overline{H}_i$, $i = 0, \dots, n-1$, be the lower composition factors of H. Observe that $\overline{H}_i$, $i \geq 1$, are the lower composition factors of H_1, and we may assume inductively that they coincide with its composition factors. Since H_1 is normal in H and $\overline{H}_0 = H/HH_1^+$ is simple, then the composition factors of H are, up to permutations, $\overline{H}_0$ and $\overline{H}_i$, $i = 1, \dots, n-1$, as claimed. □

Example 5.5. Assume H is a cocommutative Hopf algebra. Then every quotient Hopf algebra of H is normal. Suppose B is a normal Hopf subalgebra and $H/HB^+ \to Q$ is a proper quotient Hopf algebra. Then ${}^{\operatorname{co} Q}H$ is a normal Hopf subalgebra of H such that $B \subseteq {}^{\operatorname{co} Q}H$ and ${}^{\operatorname{co} Q}H \neq H$. This shows that the converse of Corollary 3.7 is true in this case. Hence a lower subnormal series of H is a lower composition series if and only if all its factors are simple Hopf algebras. By Proposition 5.4 the composition factors of H coincide with its lower composition factors. In particular, if H is the group algebra of a finite group G, then the lower composition factors of H are the group algebras of the composition factors of G.

The next example shows that the converse of Proposition 5.4 is not true.

Example 5.6. Let G be a finite group and let k^G be the commutative Hopf algebra of k-valued functions on G. The Hopf subalgebras of k^G are of the form $k^{G/S}$, where S is a normal subgroup of G, and every Hopf subalgebra is normal.

A lower composition series of k^G corresponds to a *principal series* (or *chief series*) of G, that is, a strictly increasing series of subgroups $S_0 = \{e\} \subseteq S_1 \subseteq \cdots \subseteq S_{n-1} \subseteq S_n = G$, where S_i is a normal subgroup of G, $0 \leq i \leq n-1$, and there exists no normal subgroup S of G such that $S_i \subsetneq S \subsetneq S_{i+1}$ (see [13, Section 1.3]).

Recall that a *characteristic subgroup* of a group G is a subgroup stable under the action of the automorphism group of G. The factors of a principal series (called the principal or chief factors of G) are *characteristically simple* groups, that is, they contain no proper characteristic subgroup. It is a known fact that a finite group is characteristically simple if and only if it is isomorphic to a direct product of isomorphic simple groups (see for instance [13, 3.3.15]). In particular, the lower composition factors of k^G are not necessarily simple Hopf algebras.

The fact that a lower composition series can have factors which are not simple motivates the following question:

Question 5.7. *Can a lower (or upper) composition series of a finite dimensional Hopf algebra be recognized by the structure of its factors?*

5.1. **Jordan-Hölder theorem.** In this subsection we focus our discussion on lower composition series. We point out that, by duality, analogous results hold for upper composition series. See Remark 5.3.

Theorem 5.8. (Schreier's refinement theorem.) *Let H be a finite dimensional Hopf algebra. Then any two lower subnormal series of H admit equivalent refinements.*

Proof. Let $(\mathcal{A})$, $(\mathcal{B})$ be two lower subnormal series of H, where

$$(\mathcal{A}): \ k = A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_0 = H, \quad (\mathcal{B}): \ k = B_m \subseteq B_{m-1} \subseteq \cdots \subseteq B_0 = H,$$

$n, m \geq 1$. For every $0 \leq i \leq n-1$, $0 \leq j \leq m$, let $A_{i,j} = A_{i+1}(B_j \cap A_i)$. By Theorem 3.10 (i), $A_{i,j}$ is a normal Hopf subalgebra of $A_{i,j-1}$, for all $j = 1, \dots, m$. In addition, $A_{i,0} = A_i$, for all $i = 0, \dots, n-1$, and A_{i+1} is a normal Hopf subalgebra of $A_{i,j}$, for all $i = 0, \dots, n-1$, $j = 0, \dots, m$. Hence we obtain a refinement of $(\mathcal{A})$:

$$\begin{aligned} k &\subseteq A_{n-1,m-1} \subseteq A_{n-1,m-2} \subseteq \cdots \subseteq A_{n-1,0} = A_{n-1} \subseteq A_{n-2,m-1} \subseteq \cdots \subseteq A_{n-2,0} \\ &= A_{n-2} \subseteq \cdots \subseteq A_{1,m-1} \subseteq \cdots \subseteq A_{1,0} = A_1 \subseteq A_{0,m-1} \subseteq \cdots \subseteq A_{0,0} = H. \end{aligned}$$

Similarly, let $B_{j,i} = B_{j+1}(A_i \cap B_j)$, for all $0 \leq j \leq m-1$, $0 \leq i \leq n$. Then $B_{j,i+1}$ is a normal Hopf subalgebra of $B_{j,i}$, and we obtain a refinement of $(\mathcal{B})$:

$$\begin{aligned} k &\subseteq B_{n-1,m-1} \subseteq B_{n-1,m-2} \subseteq \cdots \subseteq B_{n-1,0} = B_{n-1} \subseteq B_{n-2,m-1} \subseteq \cdots \subseteq B_{n-2,0} \\ &= B_{n-2} \subseteq \cdots \subseteq B_{1,m-1} \subseteq \cdots \subseteq B_{1,0} = B_1 \subseteq B_{0,m-1} \subseteq \cdots \subseteq B_{0,0} = H. \end{aligned}$$

There is a bijection between the set of indices in these new subnormal series, induced by the map $(i,j) \to (j,i)$, $0 \leq i \leq n-1$, $0 \leq j \leq m-1$. Note that $A_{i,0} = A_{i-1,m}$ and $B_{j,0} = B_{j-1,m}$, $i, j \geq 1$. The corresponding factors are, respectively

$$\begin{aligned} \frac{A_{i,j}}{A_{i,j}A_{i,j+1}^+} &= \frac{A_{i+1}(B_j \cap A_i)}{A_{i+1}(B_j \cap A_i)(A_{i+1}(B_{j+1} \cap A_i))^+}, \quad \text{and} \\ \frac{B_{j,i}}{B_{j,i}B_{j,i+1}^+} &= \frac{B_{j+1}(A_i \cap B_j)}{B_{j+1}(A_i \cap B_j)(B_{j+1}(A_{i+1} \cap B_j))^+}. \end{aligned}$$

It follows from Theorem 3.10 (iv) that the corresponding factors are isomorphic. Therefore the series $(\mathcal{A})$ and $(\mathcal{B})$ are equivalent. □

Since a lower composition series of H admits no proper refinement, we obtain:

Theorem 5.9. (Jordan-Hölder theorem for lower subnormal series.) *Any two lower composition series of a finite dimensional Hopf algebra H are equivalent.* □

Theorem 5.9 allows us to introduce the following invariants of a finite dimensional Hopf algebra H.

Definition 5.10. Let H be a finite dimensional Hopf algebra and let $k = H_n \subseteq H_{n-1} \subseteq \cdots \subseteq H_1 \subseteq H_0 = H$ be a lower composition series of H. The factors of the series will be called the *lower composition factors of* H are the factors of H. The number n will be called the *lower length of* H.

Upper composition factors and upper length can be defined similarly.

Remark 5.11. Examples 5.5 and 5.6 show that the lower lengths of a Hopf algebra and its dual may be different. Compare with Corollary 4.4.

Proposition 5.12. *Let m be the length of H and let n, u be its lower and upper lengths, respectively. Then $n \leq m$ and $u \leq m$.*

Note that all three lengths coincide if all lower (or upper) factors are simple Hopf algebras. See Proposition 5.4.

Proof. We shall prove the inequality for the lower length. This implies the inequality for the upper length in view of Corollary 4.4 and Remark 5.3. The proof is by induction on the dimension of H. If $\dim H = 1$, there is nothing to prove. Suppose that $\dim H > 1$ and let $k = H_n \subseteq \cdots \subseteq H_1 \subseteq H_0 = H$ be a lower composition series of H. Then H_1 is a normal Hopf subalgebra and $\overline{H}_0 = H/H{H_1}^+$ is a composition factor of H. It follows from Corollary 4.3 that the length m of H equals $m_1 + m_1'$, where $m_1' \geq 1$ is the length of $\overline{H}_0$. Observe that $k = H_n \subseteq \cdots \subseteq H_1$ is a lower composition series of H_1, hence the lower length of H_1 equals $n-1$. The inductive assumption implies that $n-1 \leq m_1$. Then $m = m_1 + m_1' \geq n - 1 + m_1' \geq n$. □

Example 5.13. Keep the notation in Example 3.8. Suppose $\mathfrak{g}$ is a simple Lie algebra of rank n. Let $A = (a_{ij})_{i,j}$ be the Cartan matrix of $\mathfrak{g}$ and let $D = (d_1, \dots, d_n)$ be a diagonal matrix with relatively prime entries such that $DA = AD$.

Let $\mathfrak{j} \cong k\mathbb{Z}_2^{(n)}$ be the central Hopf subalgebra of $\mathfrak{u}$ considered in [2, p. 27]. Let us denote $\overline{\mathfrak{u}} = \mathfrak{u}/\mathfrak{u}\mathfrak{j}^+$. Suppose that ℓ and the determinant of the matrix $(d_i a_{ij})_{ij}$ are relatively prime (which is always the case if $\mathfrak{g}$ is not of type A). Then $\overline{\mathfrak{u}}$ is a simple Hopf algebra [2, Appendix]. Then $k \subseteq k\mathbb{Z}_2 \subseteq \cdots \subseteq k\mathbb{Z}_2^{(n-1)} \subseteq \mathfrak{j} \subseteq \mathfrak{u}$ is a lower composition series of $\mathfrak{u}$ with factors $k\mathbb{Z}_2$ (with multiplicity n) and $\overline{\mathfrak{u}}$. Since all lower composition factors are simple Hopf algebras, then these are also the composition factors of $\mathfrak{u}$. Hence, in this case, the length of $\mathfrak{u}$ is $n+1$ and it coincides with its lower and upper lengths.

Remark 5.14. Let (5.1) be a lower subnormal series of H with factors $\overline{H_i}$. Then H_i is isomorphic as a Hopf algebra to a bicrossed product $H_i \cong H_{i+1} \# \overline{H}_i$. Therefore $H = H_0$ can be obtained from $H_{n-1} = \overline{H}_{n-1}$ through an iterated sequence of bicrossed products by the factors of the series: $H \cong ((\dots(\overline{H}_{n-1} \# \overline{H}_{n-2}) \# \dots) \# \overline{H}_0)$. In view of Remark 2.4, we get that H is semisimple (respectively, cosemisimple) if and only if all its lower composition factors are semisimple (respectively, cosemisimple). Compare with Proposition 4.5.

Example 5.15. Let Γ and F be finite groups. Consider an *abelian* exact sequence

$$k \longrightarrow k^\Gamma \longrightarrow H \longrightarrow kF \longrightarrow k. \tag{5.4}$$

It is known that (5.4) gives rise to mutual actions by permutations $\triangleleft : \Gamma \times F \to \Gamma$ and $\triangleright : \Gamma \times F \to F$ that make (F, Γ) into a *matched pair of groups*. Moreover, there exist invertible normalized 2-cocycles $\sigma : F \times F \to k^\Gamma$ and $\tau : \Gamma \times \Gamma \to k^F$, which are compatible in an appropriate sense, such that $H \cong k^{\Gamma\,\tau}\#_\sigma kF$ is a bicrossed product. See [10].

For every $s \in \Gamma$, let $e_g \in k^\Gamma$ be defined by $e_s(t) = \delta_{s,t}$, $t \in \Gamma$. Then $(e_s \# x)_{s\in\Gamma, x\in F}$ is a basis of $k^{\Gamma\,\tau}\#_\sigma kF$ and, in this basis, the multiplication, comultiplication and antipode of H are given, respectively, by the formulas

$$(e_g\#x)(e_h\#y) = \delta_{g\triangleleft x,h}\, \sigma_g(x,y) e_g\#xy, \quad \Delta(e_g\#x) = \sum_{st=g} \tau_x(s,t)\, e_s\#(t\triangleright x)\otimes e_t\#x,$$

$$\mathcal{S}(e_g\#x) = \sigma_{(g\triangleleft x)^{-1}}((g \triangleright x)^{-1}, g \triangleright x)^{-1}\, \tau_x(g^{-1}, g)^{-1}\, e_{(g\triangleleft x)^{-1}}\#(g \triangleright x)^{-1},$$

for all $g, h \in \Gamma$, $x, y \in F$, where $\sigma_g(x,y) = \sigma(x,y)(g)$ and $\tau_x(g,h) = \tau(g,h)(x)$.

Furthermore, the exact sequence (5.4) is equivalent to the sequence $k \longrightarrow k^\Gamma \xrightarrow{i} k^{\Gamma\,\tau}\#_\sigma kF \xrightarrow{\pi} kF \longrightarrow k$, where $i(e_g) = e_g \otimes 1$, $g \in \Gamma$, and $\pi = \epsilon \otimes \mathrm{id}$.

A subnormal series $\{e\} = F_m \subseteq \cdots \subseteq F_0 = F$ of F will be called a Γ-*subnormal series* if every term F_i is a Γ-stable subgroup of F. By a Γ-*composition series* of F we shall understand a Γ-subnormal series which does not admit a proper refinement consisting of Γ-stable subgroups.

Proposition 5.16. *Let* $\{e\} = \Gamma_n \subseteq \cdots \subseteq \Gamma_0 = \Gamma$ *be a principal series of* Γ *and let* $\{e\} = F_m \subseteq \cdots \subseteq F_0 = F$ *be a* Γ*-composition series of* F*. Then*

$$k \subseteq k^{\Gamma/\Gamma_1} \subseteq \cdots \subseteq k^\Gamma \subseteq k^{\Gamma\,\tau}\#_\sigma kF_{m-1} \subseteq \cdots \subseteq k^{\Gamma\,\tau}\#_\sigma kF_1 \subseteq H \tag{5.5}$$

is a lower composition series of H*.*

Proof. Since the subgroups F_i are Γ-stable, then (Γ, F_i) is a matched pair by restriction and $k^{\Gamma\,\tau}\#_\sigma kF_i$ can be identified with a Hopf subalgebra of H. Moreover, since F_i is normal in F_{i-1}, then $k^{\Gamma\,\tau}\#_\sigma kF_i$ is a normal Hopf subalgebra of

$k^{\Gamma}{}^{\tau}\#_{\sigma}kF_{i-1}$, for all $i = 1, \dots, m$. It was already observed (Example 5.6) that k^{Γ/Γ_i} is a maximal normal Hopf subalgebra of $k^{\Gamma/\Gamma_{i+1}}$, for all $i = 0, \dots, n-1$. Let $1 \leq i \leq m$, and suppose that L is a normal Hopf subalgebra of $k^{\Gamma}{}^{\tau}\#_{\sigma}kF_{i-1}$ such that $k^{\Gamma}{}^{\tau}\#_{\sigma}kF_i \subsetneq L \subseteq k^{\Gamma}{}^{\tau}\#_{\sigma}kF_{i-1}$. Under the canonical projection $\pi = \epsilon \otimes \mathrm{id} : H \to kF$, we have $\pi(L) \subseteq \pi(k^{\Gamma}{}^{\tau}\#_{\sigma}kF_{i-1}) = kF_{i-1}$. Let $\tilde{F}$ be a subgroup of F_{i-1} such that $\pi(L) = k\tilde{F}$. Then $\dim L = \dim L^{\mathrm{co}\,\pi} \dim \pi(L) = |\Gamma||\tilde{F}|$. Since L is a normal in $k^{\Gamma}{}^{\tau}\#_{\sigma}kF_{i-1}$ and $k^{\Gamma} \subseteq L$, then $\tilde{F}$ is a Γ-stable normal subgroup of F_{i-1}. Notice that $kF_i = \pi(k^{\Gamma}{}^{\tau}\#_{\sigma}kF_i) \subseteq k\tilde{F}$, hence $F_i \subseteq \tilde{F}$ and $F_i \neq \tilde{F}$, because $|\Gamma||\tilde{F}| = \dim L > \dim k^{\Gamma}{}^{\tau}\#_{\sigma}kF_i = |\Gamma||F_i|$. By maximality of the series $(F_i)_i$, we get $\tilde{F} = F_{i-1}$. As before, this implies that $\dim L = \dim k^{\Gamma}{}^{\tau}\#_{\sigma}kF_{i-1}$ and then $L = k^{\Gamma}{}^{\tau}\#_{\sigma}kF_{i-1}$. This shows that $k^{\Gamma}{}^{\tau}\#_{\sigma}kF_i$ is a maximal normal Hopf subalgebra of $k^{\Gamma}{}^{\tau}\#_{\sigma}kF_{i-1}$. Then (5.5) is a lower composition series of H, as claimed. □

Remark 5.17. Recall that the length of a finite dimensional Hopf algebra is additive with respect to exact sequences; see Corollary 4.3. In contrast with this situation, Proposition 5.16 provides nontrivial examples of exact sequences of Hopf algebras $k \longrightarrow H' \longrightarrow H \longrightarrow H'' \longrightarrow k$, such that the lower length of H is different from the sum of the lower lengths of H' and H''. Moreover, a lower composition factor of H needs not be a lower composition factor of H' or of H''.

Example 5.18. Suppose that (5.4) is a *cocentral* exact sequence (equivalently, that the action $\triangleright : \Gamma \times F \to F$ is trivial). Then a Γ-composition series of F is the same as a composition series of F. Hence the lower composition factors of H are the principal factors of Γ and the composition factors of F.

An example of a cocentral abelian extension is given by the Drinfeld double $D(G)$ (see Example 4.7). As a Hopf algebra, $D(G)$ is a bicrossed product $D(G) = k^G\#kG$, with respect to the trivial action $\triangleright : G \times G \to G$ and the adjoint action $\triangleleft : G \times G \to G$, where $\sigma = 1$, $\tau = 1$. Therefore the lower composition factors of $D(G)$ are the dual group algebras of the principal factors of G and the group algebras of the composition factors of G. In particular, the lower length of $D(G)$ equals the sum of the principal length and the length of G. On the other hand, the dual Hopf algebra $D(G)^*$ is a bicrossed product $k^G\#kG$, with respect to the adjoint action $\triangleright : G \times G \to G$ and the trivial action $\triangleleft : G \times G \to G$. In this case, a G-composition series of G is the same as a normal series of G. Hence the upper composition factors of $D(G)$ are the dual group algebras and the group algebras of the principal factors of G; see Remark 5.3. In particular, if G is any finite group whose composition length is different from its principal length, then $D(G)$ provides an example of a Hopf algebra for which the lower length, the upper length and the length are pairwise distinct.

5.2. **On lower composition series of normal right coideal subalgebras.** Let H be a finite dimensional Hopf algebra over k. The underlying algebra of H is an algebra in the category $\mathcal{YD}_H^H$ of Yetter-Drinfeld modules over H with respect to the right adjoint action and the right coaction given by the comultiplication. We shall use the notation $\mathcal{H}$ to indicate this structure. Observe that the subalgebras of $\mathcal{H}$ in $\mathcal{YD}_H^H$ are exactly the right normal right coideal subalgebras of H. By [14] H is free over its coideal subalgebras. Hence, by [19, Theorem 3.2], the maps $K \mapsto H/HK^+$, $\overline{H} \mapsto {}^{\mathrm{co}\,\overline{H}}H$, determine inverse bijections between the following two sets:

(a) Right normal right coideal subalgebras $K \subseteq H$.
(b) Quotient Hopf algebras $H \to \overline{H}$.

Since every tensor subcategory of $\operatorname{Rep} H$ is of the form $\operatorname{Rep} \overline{H}$, for some quotient Hopf algebra $\overline{H}$, it follows that the map $A \mapsto \operatorname{Rep} H/HA^+$ determines an anti-isomorphism between the following lattices:

(a') Yetter-Drinfeld subalgebras A of $\mathcal{H}$.
(b') Tensor subcategories of $\operatorname{Rep} H$.

Let us consider a maximal series of algebras in ${}^H_H\mathcal{YD}$ (*i.e.*, normal right coideal subalgebras of H): $k = A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_1 \subseteq A_0 = \mathcal{H}$. This corresponds to a maximal series of tensor subcategories $\operatorname{vect}_k = \operatorname{Rep} H/HA_0^+ \subseteq \operatorname{Rep} H/HA_1^+ \subseteq \cdots \subseteq \operatorname{Rep} H/HA_n^+ = \operatorname{Rep} H$. The following example shows that these series fail to admit a Jordan-Hölder theorem.

Example 5.19. Let G be a finite group and consider the Hopf algebra $H = k^G$. Every right coideal subalgebra of k^G is of the form $k^{H\backslash G}$, where H is a subgroup of G and $H\backslash G = \{Hg : g \in G\}$ is the space of right cosets of G modulo H. Then a maximal series of right coideal subalgebras corresponds to a maximal series of (not necessarily normal) subgroups $\{e\} = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_n = G$.

Take for instance $G = \mathbb{A}_5$. Then the following are maximal series of subgroups: $\{e\} \subsetneq \langle(123)\rangle \subsetneq \langle(123), (12)(45)\rangle \subsetneq \mathbb{A}_5$, and $\{e\} \subsetneq \langle(12)(34)\rangle \subsetneq \langle(12)(34), (14)(23)\rangle \subsetneq \langle(123), (12)(34)\rangle \subsetneq \mathbb{A}_5$. In the first case, we have $\langle(123), (12)(45)\rangle \cong \mathbb{S}_3$, while in the second case, $\langle(123), (12)(34)\rangle \cong \mathbb{A}_4$ and $\langle(12)(34), (14)(23)\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. We observe in this example that not even the length of a maximal series of normal right coideal subalgebras is a well-defined invariant of H.

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