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Global smooth geodesic foliations of the hyperbolic space

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Abstract We consider foliations of the whole three dimensional hyperbolic space $\mathbb{H}^3$ by oriented geodesics. Let $\mathcal{L}$ be the space of all the oriented geodesics of $\mathbb{H}^3$, which is a four dimensional manifold carrying two canonical pseudo-Riemannian metrics of signature $(2, 2)$. We characterize, in terms of these geometries of $\mathcal{L}$, the subsets $\mathcal{M}$ in $\mathcal{L}$ that determine foliations of $\mathbb{H}^3$. We describe in a similar way some distinguished types of geodesic foliations of $\mathbb{H}^3$, regarding to which extent they are in some sense trivial in some directions: On the one hand, foliations whose leaves do not lie in a totally geodesic surface, not even at the infinitesimal level. On the other hand, those for which the forward and backward Gauss maps $\varphi^{\pm} : \mathcal{M} \to \mathbb{H}^3 (\infty)$ are local diffeomorphisms. Besides, we prove that for this kind of foliations, $\varphi^{\pm}$ are global diffeomorphisms onto their images. The subject of this article is within the framework of foliations by congruent submanifolds, and follows the spirit of the paper by Gluck and Warner where they understand the infinite dimensional manifold of all the great circle foliations of the three sphere.

Keywords Geodesic foliation · Hyperbolic space · Space of oriented lines

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1 Introduction

1.1 Geodesic foliations

A smooth geodesic foliation of a Riemannian manifold N is given by a smooth unit vector field V on N all of whose integral curves, the leaves, are geodesics. Throughout the paper, smooth means of class C^∞.

The standard examples of geodesic foliations of $\mathbb{R}^3$ are given by foliating the space by parallel planes which are in turn foliated by parallel lines, with smoothly varying directions. One can construct other examples by writing $\mathbb{R}^3$ smoothly as the disjoint union of the z-axis and one-sheet hyperboloids of revolution around that axis, and considering on each one, coherently, one of the two ways of ruling it (the striction circles of the hyperboloids do not need to be at the same height). Applying a linear isomorphism one obtains new examples.

Notice that the foliations of the hyperbolic three space by totally geodesic surfaces, as well as the foliations of the hyperbolic plane by geodesics, are by far not as rigid as in the Euclidean case [4,6]. So, the hyperbolic analogues of the standard examples of Euclidean geodesic foliations are richer. Recently, Nuchi studied the fiberwise homogeneous geodesic foliations of the three dimensional space forms [11].

Global smooth geodesic foliations of the three dimensional Euclidean space were characterized in [17] in terms of the geometry of the space of oriented lines. Now, we deal with the analogous problem in the hyperbolic context. The general basic theory for the Euclidean case is still useful, but some crucial definitions and arguments in the proofs must be adapted to the hyperbolic setting.

Let $\mathbb{H}^3$ be the three dimensional hyperbolic space of constant sectional curvature -1. Let $\mathcal{L}_0$ and $\mathcal{L}_-$ be the spaces of oriented geodesics of $\mathbb{R}^3$ and $\mathbb{H}^3$, respectively, which are manifolds of dimension four admitting canonical neutral pseudo-Riemannian metrics: $\mathcal{L}_0$ admits one (associated with the cross product) [9,15], and $\mathcal{L}_-$ admits two of them, $g_\times$ and g_K, coming from the cross product and the Killing form on $\mathrm{Iso}\left(\mathbb{H}^3\right)$, respectively [7,16]. See the precise definitions below in the preliminaries. Distinguished geometries on spaces of oriented geodesics are also studied in [1] and [2].

While the geodesic foliations of $\mathbb{R}^3$ are described in terms of the canonical neutral metric on $\mathcal{L}_0$, the characterization of the geodesic foliations of $\mathbb{H}^3$ involves both $g_\times$ and g_K (see Theorem 3.2, one of the main results). This situation appears also in other problems in hyperbolic geometry; for instance, A. Honda needed both canonical neutral metrics on $\mathcal{L}_-$ in the study of the isometric immersions of the hyperbolic plane into $\mathbb{H}^3$ [10].

We will have two types of distinguished foliations, whose characterizations are given in Theorems 4.3 and 4.6. We call a geodesic foliation *nondegenerate* if the leaves do not lie in a totally geodesic surface, not even at the infinitesimal level. More precisely, if the only eigenvectors of ∇V are in $\mathbb{R}V$, where V is the unit vector field that determines the foliation.

We have also another notion, which turns out to be weaker: a *semi-nondegenerate* foliation does not resemble, in any direction, a trivial foliation whose leaves are all orthogonal to a fixed horosphere. In the upper half space model of $\mathbb{H}^3$, these are foliations congruent to the those with vertical geodesics, with both orientations. See Definition 4.1. Both concepts generalize the Euclidean notion of nondegeneracy (see Corollary 4 in [17]). For a higher dimensional (local) analogue, see the foliations of $\mathbb{R}^n$ by pairwise skew p-planes in [12].

We want to emphasize that the statements of the results are similar to those of [17], but the technical meaning of the definitions involved, for instance, (almost) semidefinite

submanifolds and (semi-)nondegenerate foliations are quite different in the Euclidean and the hyperbolic cases.

1.2 Foliations by congruent submanifolds

The general setting of this article is the study of foliations of a smooth manifold N by congruent submanifolds: Suppose that the Lie group G acts on N, let M be a closed submanifold of N, and let $\mathcal{C}$ be the set of all submanifolds of N congruent to M via G. Let H be the subset of all points in G that preserve M. Then H is a closed Lie subgroup of G since M is closed in N and we can identify $\mathcal{C} \cong G/H$. Sometimes $\mathcal{C}$ admits distinguished G-invariant geometries, which are useful to describe the foliations of N by submanifolds congruent to M. More precisely, the problem is the following:

Describe geometrically which subsets $\mathcal{M}$ of $\mathcal{C}$ determine foliations of N.

The paradigm is the paper [8], where foliations of S^3 by great circles are characterized in this way. See also [14] (a partial generalization of [8]) and [17], with the global foliations of $\mathbb{R}^3$ by oriented lines, which includes a pseudo-Riemannian reformulation of the principal result of [8]. M. Czarnecki and R. Langevin are currently working, in this context, on the classification of codimension two totally geodesic foliations of the complex hyperbolic space.

2 Preliminaries

A smooth geodesic foliation of $\mathbb{H}^3$ is given by a smooth unit vector field V on $\mathbb{H}^3$ all of whose integral curves, the leaves, are geodesics. The set $\mathcal{M}$ of all the leaves admits a canonical differentiable structure. For the sake of completeness, we include its existence as a proposition.

Proposition 2.1 *The set $\mathcal{M}$ of all the leaves of a geodesic foliation of $\mathbb{H}^3$ admits a unique differentiable structure such that the canonical projection $P : \mathbb{H}^3 \to \mathcal{M}$ is a smooth submersion.*

Proof Let V be the smooth unit vector field on $\mathbb{H}^3$ associated with the geodesic foliation and consider the smooth distribution given by $\mathcal{D} = \mathbb{R}V$. By Theorem VIII in [13], it suffices to prove that $\mathcal{D}$ is regular, that is, that for each $p \in \mathbb{H}^3$ there is a cubical coordinate system $(U, (x_1, x_2, x_3))$ centered at p such that $\left\{ (\partial/\partial x_3)|_q \right\}$ is a basis of $\mathcal{D}_q$ for all $q \in U$ and each leaf of $\mathcal{D}$ intersects U in at most one 1-dimensional slice $(x_1, x_2) = $ const.

Let us see that for each $p \in \mathbb{H}^3$ we have such a coordinate system. Let $u_1, u_2 \in T_p\mathbb{H}^3$ such that $\{u_1, u_2, V(p)\}$ is an orthonormal basis of $T_p\mathbb{H}^3$ and let $F : \mathbb{R}^2 \to \mathbb{H}^3$ be the totally geodesic submanifold given by $F(x, y) = \mathrm{Exp}_p(xu_1 + yu_2)$ (here Exp is the geodesic exponential map). We consider the smooth map

$$\alpha : \mathbb{R}^3 \to \mathbb{H}^3, \qquad \alpha(x, y, t) = \vartheta_t (F(x, y)),$$

where ϑ_t is the flow of V. Since $d\alpha_0$ is an isomorphism, there exist $\varepsilon > 0$ and an open neighborhood $U \subset \mathbb{H}^3$ of p such that $\alpha : (-\varepsilon, \varepsilon)^3 \to U$ is a diffeomorphism. Hence, $\left(U, \alpha^{-1} = (x_1, x_2, x_3)\right)$ is a cubical coordinate system centered at p such that $(\partial/\partial x_3)|_q = V(q)$ for all $q \in U$. The 1-dimensional slices are clearly integral submanifolds of $\mathcal{D}$ and no leaf of $\mathcal{D}$ intersects two different slices of U, since geodesics in $\mathbb{H}^3$ transverse to a totally geodesic surface intersect it at most at one point. $\square$

The space $\mathcal{L}$ of all complete oriented geodesics of $\mathbb{H}^3$ (up to orientation preserving reparametrizations) admits a unique differentiable structure such that the canonical projection $\Pi : T^1\mathbb{H}^3 \to \mathcal{L}$ is a differentiable submersion (by [13], as above, with the spray as the vector field giving the foliation). We may think of $c \in \mathcal{L}$ as the equivalence class of unit speed geodesics $\gamma : \mathbb{R} \to \mathbb{H}^3$ with image c such that $\{\dot{\gamma}(t)\}$ is a positive basis of $T_{\gamma(t)}c$ for all t. If $\ell \in \mathcal{L}$, then by abuse of notation we sometimes write $z \in \ell$, meaning that z is in the underlying line.

Fixing a point $o \in \mathbb{H}^3$, let

$$H : T(T_o^1\mathbb{H}^3) \to \mathcal{L} \tag{1}$$

be the map defined as follows: Let $u \in T_o^1\mathbb{H}^3$ and $v \in T_o\mathbb{H}^3$ with $u \perp v$, then $H(u, v)$ is the oriented geodesic with initial point $\mathrm{Exp}_o(v)$ and initial velocity the parallel transport of u along the geodesic $t \mapsto \mathrm{Exp}_o(tv)$ at $t = 1$. Proposition 4.14 of [3] asserts that H is a diffeomorphism.

Let γ be a complete unit speed geodesic of $\mathbb{H}^3$ and let $\mathcal{J}_\gamma$ be the space of all Jacobi vector fields along γ which are orthogonal to $\dot{\gamma}$. There exists a well-defined canonical isomorphism

$$T_\gamma : \mathcal{J}_\gamma \to T_{[\gamma]}\mathcal{L}, \qquad T_\gamma(J) = \left.\frac{d}{dt}\right|_0 [\gamma_t], \tag{2}$$

where γ_t is any variation of γ by unit speed geodesics associated with J (see [16]).

Given a tangent vector X to a pseudo-Riemannian manifold, we denote $\|X\| = \langle X, X \rangle$, the square norm associated with the metric $\langle . , . \rangle$, and $|X| = \sqrt{|\langle X, X \rangle|}$. Also, given $v \in T\mathbb{H}^3$, we denote by γ_v the unique geodesic in $\mathbb{H}^3$ with initial velocity v.

Now, we recall the definition of the two canonical pseudo-Riemannian metrics $g_\times$ and g_K on $\mathcal{L}$ given in [16, Theorem 1]. In terms of the isomorphism (2) the square norms of these metrics may be written as follows [16, page 362]: For $J \in \mathcal{J}_\gamma$,

$$\|T_\gamma(J)\|_\times = \langle \dot{\gamma} \times J, J' \rangle,$$
$$\|T_\gamma(J)\|_K = |J|^2 - |J'|^2. \tag{3}$$

The cross product $\times$ is induced by a fixed orientation of $\mathbb{H}^3$ and J' denotes the covariant derivative of J along γ (the right hand side in the expressions are constant functions, so they are well defined).

Let $\mathcal{M}$ be a submanifold of $\mathcal{L}$ and we take $[\gamma] \in \mathcal{M}$. Next we show that any tangent vector in $T_{[\gamma]}\mathcal{M}$ corresponds (via T_γ) to a Jacobi vector field in $\mathcal{J}_\gamma$ associated with a variation of γ by unit speed geodesics whose equivalence classes are in $\mathcal{M}$. In fact, given $X \in T_{[\gamma]}\mathcal{M}$, there exists a smooth curve $c : (-\varepsilon, \varepsilon) \to \mathcal{M}$ with $c(0) = [\gamma]$ and $\dot{c}(0) = X$. By Proposition 3 in [16], there exists a standard presentation of c, that is, a function $\varphi : \mathbb{R} \times (-\varepsilon, \varepsilon) \to \mathbb{H}^3$ such that $s \mapsto \alpha_t(s) := \varphi(s, t)$ is a unit speed geodesic of $\mathbb{H}^3$ satisfying $c(t) = [\alpha_t]$, $\langle \dot{\beta}(t), \dot{\alpha}_t(0) \rangle = 0$ for all $t \in (-\varepsilon, \varepsilon)$, where $\beta(t) = \varphi(0, t)$, and $\varphi(0, 0) = \gamma(0)$. It is easy to see that

$$J(s) = \left.\frac{d}{dt}\right|_0 \alpha_t(s)$$

is a Jacobi field in $\mathcal{J}_\gamma$ and it satisfies

$$T_\gamma(J) = \left.\frac{d}{dt}\right|_0 [\alpha_t] = \left.\frac{d}{dt}\right|_0 c(t) = X.$$

3 Global geodesic foliations of $\mathbb{H}^3$

In this section we characterize, in terms of the canonical neutral metrics on $\mathcal{L}$, the subsets $\mathcal{M}$ in $\mathcal{L}$ that determine foliations of $\mathbb{H}^3$. To this end, it is convenient to give the following definition.

Definition 3.1 A submanifold $\mathcal{M}$ of $\mathcal{L}$ is said to be *almost semidefinite* if $\|X\|_\times = 0$ for $X \in T\mathcal{M}$ only if $\|X\|_K \geq 0$.

Theorem 3.2 *Let $\mathcal{M}$ be a surface contained in $\mathcal{L}$ (the inclusion is a priori not even smooth). Then the following statements are equivalent:*

(a) *The surface $\mathcal{M}$ is the space of leaves of a smooth foliation of $\mathbb{H}^3$ by oriented geodesics, with the canonical differentiable structure.*
(b) *The surface $\mathcal{M}$ is a closed almost semidefinite connected submanifold of $\mathcal{L}$.*

Besides, if $\mathcal{M}$ satisfies (a) *or* (b), *$\mathcal{M}$ is diffeomorphic to $\mathbb{R}^2$.*

Let o be a fixed point in $\mathbb{H}^3$. We recall that

$$T(T_o^1\mathbb{H}^3) = \{(u, v) \in T_o^1\mathbb{H}^3 \times T_o\mathbb{H}^3 : \langle u, v\rangle = 0\} \cong TS^2.$$

Let $f : \mathcal{L} \to \mathbb{H}^3$ be the map that assigns to each oriented unit speed geodesic ℓ of the hyperbolic space its closest point to o. Considering the following diagram,

$$
\begin{array}{ccc}
T\left(T_o^1\mathbb{H}^3\right) & \xrightarrow{H} & \mathcal{L} \\
\pi_2 \downarrow & & \downarrow f \\
T_o\mathbb{H}^3 & \xrightarrow{\mathrm{Exp}_o} & \mathbb{H}^3
\end{array}
$$

we have that f is a smooth map, where H is the diffeomorphism given in (1) and π_2 is the projection onto the second component.

Let $D : \mathcal{L} \to \mathbb{R}$ be the square distance from o. In particular, if $\ell = H(u, v)$, we have that $D(\ell) = |v|^2$ and so D is smooth.

For any unit speed geodesic γ, let $\psi_\gamma : T_{[\gamma]}\mathcal{L} \simeq \mathcal{J}_\gamma \to \dot\gamma(0)^\perp$ be the linear map defined by $\psi_\gamma(J) = J(0)$.

Lemma 3.3 *Let $\mathcal{M}$ be an almost semidefinite closed connected two-dimensional submanifold of $\mathcal{L}$.*

(a) *For any $\ell = [\gamma] \in \mathcal{M}$, $\psi_\gamma\big|_{T_\ell\mathcal{M}}$ is surjective.*
(b) *Any critical point ℓ of $D|_\mathcal{M}$ is a strict local minimum of $D|_\mathcal{M}$ with $D(\ell) = 0$. Moreover, $D(\ell_n) \to \infty$ as $n \to \infty$ for any sequence ℓ_n in $\mathcal{M}$ without cluster points.*

Proof (a) It suffices to show that the map is injective ($T_\ell\mathcal{M}$ and $\dot\gamma(0)^\perp$ have the same dimension). If $\psi_\gamma(J) = 0$, then $J(0) = 0$ and from (3) we have that $\|T_\gamma(J)\|_\times = 0$. Since $\mathcal{M}$ is almost semidefinite, using (3) we obtain that $J'(0) = 0$, thus $J \equiv 0$.

(b) Let $(u, v) \in T(T_o^1\mathbb{H}^3)$ and let $H(u, v) = \ell$. In particular, $\ell = [\gamma_U]$ with $U = \tau_{\gamma_v}\big|_0^1(u)$, where τ_{γ_v} denotes the parallel transport along γ_v. By (a), there exists a Jacobi vector field $J \in \mathcal{J}_{\gamma_U}$, with $T_{\gamma_U}(J) \in T_\ell\mathcal{M}$, such that $J(0) = \dot\gamma_v(1)$. We take a variation of γ_U by unit speed geodesics $\Gamma(s, t) = \gamma_t(s)$ associated with J, with $[\gamma_t] = H(u_t, v_t) \in \mathcal{M}$. We call α the smooth curve in $\mathcal{M}$ given by $\alpha(t) = [\gamma_t]$. We have $v_t = \pi_2 \circ H^{-1} \circ \alpha(t)$, thus v_t is a smooth curve in $T_o\mathbb{H}^3$.

Suppose that $\ell \in \mathcal{M}$ is a critical point of $D|_{\mathcal{M}}$. First we verify that $D(\ell) = 0$. We compute

$$0 = \dot{\alpha}(0)(D) = \frac{d}{dt}\Big|_0 D(\alpha(t)) = \frac{d}{dt}\Big|_0 |v_t|^2 = 2\langle v, v_0' \rangle. \tag{4}$$

Now, let λ_t be the smooth curve in $\mathbb{R}$ such that $\mathrm{Exp}_o(v_t) = \gamma_t(\lambda_t)$ (in particular, $\lambda_0 = 0$) and consider the Jacobi vector field K associated with the geodesic variation $(s,t) \to \Delta(s,t) = \Gamma(s + \lambda_t, t)$, that is,

$$K(s) = \lambda_0' \dot{\gamma}_U(s) + J(s).$$

Since $\mathrm{Exp}_o(v_t) = \Delta(0,t)$, we have

$$(d\,\mathrm{Exp}_o)_v(v_0') = K(0) = \lambda_0' U + J(0). \tag{5}$$

Besides, $(d\,\mathrm{Exp}_o)_v(v) = J(0)$. Then, by (4), the Gauss Lemma and (5), we obtain

$$0 = \langle v, v_0' \rangle = \langle \left(d\,\mathrm{Exp}_o\right)_v(v), \left(d\,\mathrm{Exp}_o\right)_v(v_0') \rangle = |J(0)|^2 = |v|^2 = D(\ell), \tag{6}$$

as desired. Next we see that ℓ is a strict local minimum. Let X be a nonzero vector in $T_\ell \mathcal{M}$ and let $J \in \mathcal{J}_{\gamma_u}$ such that $X = T_{\gamma_u}(J)$. Since J is not an identically zero Jacobi vector field, by (a) we have $J(0) \neq 0$. As above, we take a smooth curve $[\gamma_t] = H(u_t, v_t)$ in $\mathcal{M}$ such that its initial velocity is X and $J(s) = \frac{d}{dt}\big|_0 \gamma_t(s)$. Then,

$$\frac{d^2}{dt^2}\Big|_0 D([\gamma_t]) = 2(\langle v_0, v_0'' \rangle + |v_0'|^2) = 2|v_0'|^2 > 0,$$

since $v_0 = v = 0$ by (6) and $v_0' \neq 0$ by (5).

The last statement is proved in a similar way as in Lemma 5(b) of [17]. □

Proof of Theorem 3.2 (a)$\Rightarrow$(b) Suppose that the foliation is given by a smooth unit vector field V on $\mathbb{H}^3$ and let $P : \mathbb{H}^3 \to \mathcal{M}$ be the smooth submersion induced by V as in Proposition 2.1. Since $\mathcal{M} = P(\mathbb{H}^3)$, we have that $\mathcal{M}$ is connected. The fact that the inclusion $i : \mathcal{M} \to \mathcal{L}$ is a submanifold is proved in the same way as in the Euclidean case (the beginning of (a)$\Rightarrow$(b) in the proof of Theorem 2 in [17]).

Let us see that $\mathcal{M}$ is almost semidefinite. Let $X \in T_{[\gamma]}\mathcal{M}$ with $\|X\|_\times = 0$ and let $J \in \mathcal{J}_\gamma$ with $X = T_\gamma(J)$. We want to see that $\|X\|_K \geq 0$. First, we observe that if γ_t is any variation of γ by geodesics in the foliation, associated with J, we have that $\dot{\gamma}_t(s) = V(\gamma_t(s))$ and so we compute

$$J'(s) = \frac{D}{ds}\frac{d}{dt}\Big|_0 \gamma_t(s) = \frac{D}{dt}\Big|_0 \frac{d}{ds}\gamma_t(s) = \frac{D}{dt}\Big|_0 \dot{\gamma}_t(s) = \nabla_{J(s)}V. \tag{7}$$

By (3), $J(0)$ and $J'(0)$ are linearly dependent. If $J(0) = 0$, then $J'(0) = 0$ by (7), and so $\|X\|_K = 0$. If $J(0) \neq 0$, there exists $a \in \mathbb{R}$ such that $J'(0) = aJ(0)$. So, $J(s) = (a \sinh s + \cosh s)Z(s)$, where Z is a parallel vector field along γ and orthogonal to $\dot{\gamma}$. If $|a| > 1$ there exists $s_o = (\tanh)^{-1}(-1/a)$ such that $J(s_o) = 0$. By (7), we have that $J'(s_o) = 0$. Hence, $J \equiv 0$, which is a contradiction. Therefore, $|a| \leq 1$ and consequently $\|X\|_K = (1 - a^2)|J(0)|^2 \geq 0$, as desired.

Next we show that $\mathcal{M}$ is closed. Let $[\gamma_n] = H(u_n, v_n)$ be a sequence in $\mathcal{M}$ with $\lim_{n \to \infty}[\gamma_n] = [\gamma] \in \mathcal{L}$. Let $(u, v) \in T(T_o^1\mathbb{H}^3)$ such that $[\gamma] = H(u, v)$. Since H is a diffeomorphism we have that $(u_n, v_n) \to (u, v)$. Let $\bar{H} : T(T_o^1\mathbb{H}^3) \to T^1\mathbb{H}^3$ be the smooth map defined by $\bar{H}(u, v) = \tau_{\gamma_v}\big|_0^1(u)$ and recall that $H = \Pi \circ \bar{H}$ holds by definition of H, where $\Pi : T^1\mathbb{H}^3 \to \mathcal{L}$ is the canonical projection. Since $[\gamma_n] \in \mathcal{M}$, $\bar{H}(u_n, v_n) = V(\mathrm{Exp}_o(v_n))$.

So, to prove that $\mathcal{M}$ is closed we have to see that $\bar{H}(u, v) = V(\text{Exp}_o(v))$. Now, the assertion follows from the continuity of $\bar{H}$, Exp_o and V.

(b) $\Rightarrow$ (a) The facts that the union of all geodesics in $\mathcal{M}$ covers the whole space $\mathbb{H}^3$ and that two distinct geodesics in $\mathcal{M}$ do not intersect are proved in a similar way as in Theorem 2 of [17], but using in this case Lemma 3.3 (b). As in that theorem, the hypotheses force the existence of only one critical point (cf. the second paragraph of Remark 3.4) and that $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^2$.

Next, we define the vector field V which determines the foliation. Given $z \in \mathbb{H}^3$, let $V(z) = \dot{\gamma}(t)$, where $[\gamma]$ is the unique element in $\mathcal{M}$ such that z is in the trajectory of γ and $z = \gamma(t)$. Now, we verify that V is smooth. The arguments differ from those in the Euclidean case only at the end, but we include the details for the sake of completeness. The image of V coincides with $\Pi^{-1}(\mathcal{M})$, and hence it is a smooth submanifold of $T^1\mathbb{H}^3$, since Π is a fiber bundle. We have to check that zero is the only vertical (with respect to $p : T^1\mathbb{H}^3 \to \mathbb{H}^3$) tangent vector η of the image of V. Suppose that $(dp)_{V(z)}(\eta) = 0$ and let $t \mapsto V \circ c(t)$ be a smooth curve in $T^1\mathbb{H}^3$ such that $c(0) = z$ and with initial velocity equal to η. So, we have that $c'(0) = 0$. Let ℓ be the curve in $\mathcal{M}$ defined by $\ell(t) = \Pi(V(c(t)))$ and set $\ell'(0) = X$. Let $\ell(0) = [\gamma]$ with $\gamma(0) = c(0)$ and let $J(s) = \frac{d}{dt}\big|_0 \gamma_{V(c(t))}(s)$. We compute $J(0) = c'(0) = 0$ and we have that $J'(0)$ is orthogonal to $\dot{\gamma}(0)$, since V is a unit vector field. Hence, $X = T_\gamma(J)$ and $\|X\|_\times = 0$ by (3) and so $\|X\|_K \geq 0$ since $\mathcal{M}$ is almost semidefinite. This implies, again by (3), that $J'(0) = 0$. Finally, if we consider the isomorphism

$$(dp_{V(z)}, \mathcal{K}_{V(z)}) : T_{V(z)}T\mathbb{H}^3 \to T_z\mathbb{H}^3 \times T_z\mathbb{H}^3, \tag{8}$$

where $\mathcal{K}_{V(z)}$ is the connection operator, we obtain that η is equal to zero, since $(dp_{V(z)}, \mathcal{K}_{V(z)})(\eta) = (J(0), J'(0)) = (0, 0)$. $\qquad\square$

Remark 3.4 We construct in Proposition 3.5 below an example of a two dimensional submanifold $\mathcal{M}$ of $\mathcal{L}$ satisfying all conditions of part (b) in Theorem 3.2, except to be closed. The geodesics in $\mathcal{M}$ not only fail to foliate the whole $\mathbb{H}^3$, as expected, but they do not even foliate the open set $\mathcal{U}$ in $\mathbb{H}^3$ given by the union of all their trajectories (there exist two geodesics in $\mathcal{M}$ intersecting at a point in $\mathcal{U}$).

The same proposition shows that if the hypothesis that $\mathcal{M}$ is closed in $\mathcal{L}$ is removed in Lemma 3.3, there might exist two different critical points ℓ_1 and ℓ_2 of $D|_{\mathcal{M}}$. One can take $o = f(2, 0)$ and as ℓ_1 and ℓ_2 the geodesics through o with initial velocities $V_\lambda(2, 0)$ and $V_\lambda(2, 2\pi)$.

We begin by defining an immersion f of an open set of the plane into a totally geodesic submanifold S of $\mathbb{H}^3$ covering an annulus in S in a non-injective way. Let U be an open set in the half plane $\{(r, t) \mid r > 0\}$ containing the rectangle $R = [1, 3] \times [-\delta, 2\pi + \delta]$. Fix $o \in \mathbb{H}^3$ and define $f : U \to \mathbb{H}^3$ by

$$f(r, t) = \text{Exp}_o(r \cos t \, u_o + r \sin t \, v_o),$$

where $u_o, v_o \in T_o\mathbb{H}^3$ are unit orthogonal vectors. We consider vector fields u, v, w along f forming an orthonormal basis of $T_{f(r,t)}\mathbb{H}^3$ for each r, t. They are given by

$$u = \frac{\partial f}{\partial r}, \quad v = \frac{1}{\sinh r} \frac{\partial f}{\partial t}, \quad w = u \times v.$$

Now, let $\alpha_\lambda(r, t) = \alpha_0 + \lambda t - \lambda r$ for some $\lambda > 0$ and $\alpha_0 \in (0, \pi/2)$, and let V_λ be the vector field along f defined by $V_\lambda = \cos \alpha_\lambda \, v + \sin \alpha_\lambda \, w$. Let R^o be the interior of R.

Proposition 3.5 *For some $\lambda > 0$, the map*

$$F_\lambda : R^o \to \mathcal{L}, \qquad F_\lambda(r,t) = \left[\gamma_{V_\lambda(r,t)} \right]$$

is an immersion and $g_\times$ induces a Riemannian metric on its image $F_\lambda(R^o) = \mathcal{M}$. Moreover, the trajectories of $\gamma_{V_\lambda(r,0)}$ and $\gamma_{V_\lambda(r,2\pi)}$ intersect at $f(r,0)$, for each $r \in (1,3)$.

Proof We fix $(r,t) \in R^o$ and $0 \neq (x,y) \in T_{(r,t)}R^\circ$. For the sake of simplicity we omit the subindex λ and write α instead of $\alpha_\lambda(r,t)$. Let us see that $\left\| dF_{(r,t)}(x,y) \right\|_\times > 0$. Let J be the Jacobi field associated with the variation $s \mapsto \gamma_{V(r+sx,t+sy)}$. We compute that $J(0) = x\,u + y\sinh r\,v$. Now, since

$$\nabla_u u = 0, \ \nabla_u v = 0, \ \nabla_u w = 0, \ \nabla_v u = (\coth r)\,v, \ \nabla_v v = -(\coth r)\,u \ \text{and} \ \nabla_v w = 0,$$

we obtain that

$$J'(0) = -(y\cosh r\cos\alpha)\,u + \lambda(x-y)(\sin\alpha)\,v - \lambda(x-y)(\cos\alpha)\,w.$$

On the other hand, calling $\gamma = \gamma_{V(r,t)}$, we have that

$$\dot\gamma(0) \times J(0) = V(r,t) \times J(0) = -(y\sinh r\sin\alpha)\,u + (x\sin\alpha)\,v - (x\cos\alpha)\,w.$$

Then, since the expression for $g_\times$ in (3) is valid also if J is not orthogonal to $\dot\gamma$, we have that

$$\left\| dF_{(r,t)}(x,y) \right\|_\times = \langle \dot\gamma(0) \times J(0), J'(0) \rangle = \lambda x^2 - \lambda xy + \tfrac{1}{4}(\sinh 2r\sin 2\alpha)\,y^2.$$

Thus, for $\mathcal{M}$ to be Riemannian, it is enough that λ makes this bilinear form positive definite for all $(r,t) \in R$. Equivalently, that $h_\lambda(r,t) > 0$ for all $(r,t) \in R$, where for each $\lambda > 0$, $h_\lambda : R \to \mathbb{R}$ is defined by

$$h_\lambda(r,t) = \sinh(2r)\sin(2(\alpha_0 + \lambda t - \lambda r)) - \lambda.$$

Now, h_λ converges pointwise (and also uniformly, since R is compact) to $\sinh(2r)\sin(2\alpha_0)$ as $\lambda \to 0^+$. Since the limit function is positive, for $\lambda > 0$ small enough, $h_\lambda(r,t) > 0$ for all $(r,t) \in R$, as desired. $\square$

4 Global nondegenerate geodesic foliations of $\mathbb{H}^3$

Two unit speed geodesics γ and α of $\mathbb{H}^3$ are said to be *asymptotic* if there exists a positive constant C such that $d(\gamma(s), \sigma(s)) \leq C, \forall s \geq 0$ [5]. Two unit vectors $v, w \in T^1\mathbb{H}^3$ are said to be asymptotic if the corresponding geodesics γ_v and γ_w have this property.

A point at infinity for $\mathbb{H}^3$ is an equivalence class of asymptotic geodesics of $\mathbb{H}^3$. The set of all points at infinity for $\mathbb{H}^3$ is denoted by $\mathbb{H}^3(\infty)$ and has a canonical differentiable structure diffeomorphic to the 2-sphere. The equivalence class represented by a geodesic γ is denoted by $\gamma(\infty)$ and the equivalence class represented by the oppositely oriented geodesic $s \mapsto \gamma(-s)$ is denoted by $\gamma(-\infty)$. Let $\varphi^\pm : \mathcal{L} \to \mathbb{H}^3(\infty)$ be the forward Gauss map (for $+$) and the backward Gauss map (for $-$), defined by $\varphi^\pm([\gamma]) = \gamma(\pm\infty)$, which are smooth.

In the introduction we commented on some distinguished types of geodesic foliations of $\mathbb{H}^3$, regarding to which extent they are in some sense trivial in some directions. That motivates the following precise definitions. Before we recall that by Theorem 3.2, any smooth geodesic foliation of $\mathbb{H}^3$ has an associated submanifold $\mathcal{M}$ of $\mathcal{L}$.

Definition 4.1 We say that a smooth foliation by oriented geodesics of $\mathbb{H}^3$ is *semi-nondegenerate* if the Gauss maps $\varphi^{\pm} : \mathcal{M} \to \mathbb{H}^3(\infty)$ are local diffeomorphisms, where $\mathcal{M} \subset \mathcal{L}$ is the space of leaves. And we say that it is *nondegenerate* if the only eigenvectors of ∇V are in $\mathbb{R}V$, where V is the unit vector field that determines the foliation.

In order to characterize the semi-nondegenerate and nondegenerate global geodesic foliations of the hyperbolic space in terms of the geometry of $\mathcal{L}$, we have the next definition.

Definition 4.2 A submanifold $\mathcal{M}$ of $\mathcal{L}$ is said to be *semidefinite* if $\|X\|_{\times} = 0$ for a nonzero $X \in T\mathcal{M}$ only if $\|X\|_K > 0$.

Theorem 4.3 *Let $\mathcal{M}$ be the submanifold of $\mathcal{L}$ associated with a foliation of $\mathbb{H}^3$ by oriented geodesics. Then*

(a) *the foliation is semi-nondegenerate if and only if $\mathcal{M}$ is semidefinite.*
(b) *the foliation is nondegenerate if and only if $g_{\times}$ induces on $\mathcal{M}$ a definite metric.*

Some definitions and lemmas will be necessary to prove the theorem.

A Jacobi vector field J along a geodesic γ of $\mathbb{H}^3$ is said to be *stable* (*unstable*) if there exists a constant $c > 0$ such that

$$|J(s)| \leq c, \qquad \forall s \geq 0 \quad (\forall s \leq 0).$$

It is well-known that a Jacobi vector field J along a geodesic γ of $\mathbb{H}^3$ and orthogonal to $\dot{\gamma}$ is stable (respectively, unstable) if and only if $J(s) = e^{-s}U(s)$ (respectively, $J(s) = e^s U(s)$) for some parallel vector field U along γ orthogonal to $\dot{\gamma}$.

We recall that given $v \in T^1\mathbb{H}^3$ and any point $p \in \mathbb{H}^3$, there exists a unique unit tangent vector at p that is asymptotic to v (see [5, Proposition 1.7.3]). A smooth vector field W in $\mathbb{H}^3$ is called an asymptotic vector field if $W(p)$ and $W(q)$ are asymptotic for every $p, q \in \mathbb{H}^3$.

Lemma 4.4 *Let c be a smooth curve in $\mathbb{H}^3$. Then an asymptotic vector field W on $\mathbb{H}^3$ satisfies the following differential equation*

$$\nabla_{\dot{c}(t)} W = \langle \dot{c}(t), W_{c(t)} \rangle W_{c(t)} - \dot{c}(t). \tag{9}$$

Proof After decomposing $\dot{c}(t)$ into its components tangent and orthogonal to $W_{c(t)}$, the statement follows directly from the following equations:

$$\nabla_X W = -X, \text{ if } X \perp W \quad \text{and} \quad \nabla_W W = 0.$$

The first one is true (see (1.10.9) in [5]), since it is well known that the shape operator of a horosphere is the identity. The second one holds, since the integral curves of an asymptotic vector field are geodesics. $\square$

Lemma 4.5 *Let γ be a geodesic of $\mathbb{H}^3$ and let $J \in \mathcal{J}_\gamma$ be given by $J(s) = \frac{d}{dt}\big|_0 \gamma_{u_t}(s)$, where $t \mapsto u_t$ is a smooth curve in $T^1\mathbb{H}^3$, with foot points $c(t)$ (in particular, $u_0 = \dot{\gamma}(0) \perp \dot{c}(0)$). If $v_t \in T^1_{\gamma(0)}\mathbb{H}^3$ is the asymptotic vector to u_t for each $t \in \mathbb{R}$, then the Jacobi vector field K along γ associated with v_t satisfies*

$$K'(0) = J(0) + J'(0).$$

Proof For each $s, t \in \mathbb{R}$, let $V(s, t)$ be the unique unit vector at $c(s)$ that is asymptotic to u_t. In particular, $V(0, t) = v_t$ and $V(t, t) = u_t$. We compute

$$\frac{D}{dt}\bigg|_0 u_t = \frac{D}{dt}\bigg|_0 V(t, t) = \frac{D}{dt}\bigg|_0 V(t, 0) + \frac{D}{dt}\bigg|_0 V(0, t) = \frac{D}{dt}\bigg|_0 V(t, 0) + \frac{D}{dt}\bigg|_0 v_t \tag{10}$$

The second equality follows from the well-known corresponding identity in the calculus of several variables (writing V in coordinates). The vector field $V(t, 0)$ is an asymptotic vector field along $t \mapsto c(t)$. So, using that J is orthogonal to $\dot\gamma$ (in particular $\langle \dot{c}(0), V(0, 0)\rangle = 0$) and (9) with $W_{c(t)} = V(t, 0)$, we have that

$$\frac{D}{dt}\bigg|_0 V(t, 0) = -\dot{c}(0). \tag{11}$$

Finally, since $J'(0) = \frac{D}{dt}\big|_0 u_t$ and $K'(0) = \frac{D}{dt}\big|_0 v_t$, by (10) and (11) we obtain $K'(0) = J(0) + J'(0)$, as desired. $\square$

Proof of Theorem 4.3. (a) Suppose that the foliation is semi-nondegenerate. Let $[\gamma] \in \mathcal{M}$ and let $0 \neq X \in T_{[\gamma]}\mathcal{M}$ such that $\|X\|_\times = 0$. We want to see that $\|X\|_K > 0$. By (a) $\Rightarrow$ (b) in Theorem 3.2 we have $\|X\|_K \geq 0$. Suppose now that $\|X\|_K = 0$. Let $J \in \mathcal{J}_\gamma$ be the Jacobi vector field associated with X via the isomorphism T_γ given in (2). Hence, by (3), $\{J(0), J'(0)\}$ is linearly dependent and $|J'(0)|^2 = |J(0)|^2$. Since $X \neq 0$, we have that J is a stable or an unstable vector field. If J is a stable Jacobi vector field, by Proposition 1.10.7 in [5], $J(s) = \frac{d}{dt}\big|_0 \gamma_{u_t}(s)$ for some smooth curve $t \mapsto u_t \in T^1\mathbb{H}^3$ with $u_0 = \dot\gamma(0)$ and u_t asymptotic for all t. Then,

$$(d\varphi^+)_{[\gamma]}X = \frac{d}{dt}\bigg|_0 \varphi^+([\gamma_{u_t}]) = 0,$$

which is a contradiction since φ^+ is a local diffeomorphism. The case J unstable is similar. Therefore, $\mathcal{M}$ is semidefinite.

Conversely, if $\mathcal{M}$ is semidefinite, we have to prove that the foliation is semi-nondegenerate, that is, that the backward and forward Gauss maps $\varphi^\pm$ are local diffeomorphisms. So, let $[\gamma] \in \mathcal{M}$ and $0 \neq X \in T_{[\gamma]}\mathcal{M}$. Let $J \in \mathcal{J}_\gamma$ be the Jacobi vector field associated with X via (2) and consider a smooth curve $t \mapsto u_t \in T^1\mathbb{H}^3$ such that $[\gamma_{u_t}] \in \mathcal{M}$ and $J(s) = \frac{d}{dt}\big|_0 \gamma_{u_t}(s)$. Let us prove that $(d\varphi^+)_{[\gamma]}X \neq 0$ (the proof of the corresponding assertion for φ^- instead of φ^+ is similar). We have to see that the initial velocity of $t \mapsto \gamma_{u_t}(\infty)$ is different from zero. By the definition of the differentiable structure of $\mathbb{H}^3(\infty)$ we have that the map that assigns to each $v \in T^1_{\gamma(0)}\mathbb{H}^3$ the equivalence class of γ_v in $\mathbb{H}^3(\infty)$ is a diffeomorphism. So, we consider the smooth curve $t \mapsto v_t \in T^1_{\gamma(0)}\mathbb{H}^3$ such that $\gamma_{u_t}(\infty) = \gamma_{v_t}(\infty)$ and we show that $\frac{d}{dt}\big|_0 v_t \neq 0$. Let K be the Jacobi vector field along γ given by $K(s) = \frac{d}{dt}\big|_0 \gamma_{v_t}(s)$ with initial conditions $K(0) = 0$ and $K'(0) = \frac{D}{dt}\big|_0 v_t$. By the isomorphism given in (8), it suffices to see that $K'(0) \neq 0$. Now, by Lemma 4.5, $K'(0) = J(0) + J'(0)$. If $\|X\|_\times \neq 0$, the set $\{J(0), J'(0)\}$ is linearly independent and so $K'(0) \neq 0$. Now suppose that $\|X\|_\times = 0$. Since $X \neq 0$ and $\mathcal{M}$ is semidefinite, then $\|X\|_K = |J(0)|^2 - |J'(0)|^2 > 0$. By (3), the set $\{J(0), J'(0)\}$ is linearly dependent, and $J(0) \neq 0$. Hence, $J'(0) = \lambda J(0)$ with $\lambda \in \mathbb{R} - \{\pm 1\}$. Consequently, $K'(0) = (1 + \lambda)J(0) \neq 0$, as desired.

(b) First, suppose that the foliation is nondegenerate, that is, the only eigenvectors of ∇V are in $\mathbb{R}V$. We want to see that $\|X\|_\times \neq 0$ for all $0 \neq X \in T\mathcal{M}$. Suppose that there exists a nonzero vector $X \in T_{[\gamma]}\mathcal{M}$ such that $\|X\|_\times = 0$. Let $J \in \mathcal{J}_\gamma$ the Jacobi vector field associated with X via the isomorphism T_γ defined in (2). By (7), $\nabla_{J(0)}V = J'(0)$ and since $X \neq 0$ we obtain that $J(0) \neq 0$. Now, since

$$0 = \|X\|_\times = \|T_\gamma(J)\|_\times = \langle \dot\gamma(0) \times J(0), J'(0)\rangle,$$

we have that $J(0) \times J'(0)$ is orthogonal to $\dot{\gamma}(0) = V(\gamma(0))$. Or equivalently, $J'(0)$ is a multiple of $J(0)$. Again by (7), $J(0)$ is an eigenvector of ∇V orthogonal to $V(\gamma(0))$ (recall that $J \in \mathcal{J}_\gamma$), which is a contradiction.

Conversely, let $u \in T_p \mathbb{H}^3$ be an eigenvector of ∇V with eigenvalue λ. Let $c : (-\varepsilon, \varepsilon) \to \mathbb{H}^3$ be a smooth curve such that $\dot{c}(0) = u - \langle u, V(p) \rangle V(p) \perp V(p)$. So, the Jacobi vector field given by $J(s) = \left. \dfrac{d}{dt} \right|_0 \gamma_{V(c(t))}(s)$ is in $\mathcal{J}_\gamma$, where $\gamma = \gamma_{V(p)}$. Since $J(0) = \dot{c}(0)$ and $\nabla_{V(p)} V = 0$, we have that $\nabla_{J(0)} V = \lambda u$. As $\nabla_{J(0)} V = J'(0)$ by (7), $T_\gamma(J) \in T\mathcal{M}$ satisfies

$$\| T_\gamma(J) \|_\times = \langle \dot{\gamma}(0) \times J(0), J'(0) \rangle = \langle V(p) \times \dot{c}(0), \lambda u \rangle = \langle V(p) \times u, \lambda u \rangle = 0.$$

Since $g_\times$ induces on $\mathcal{M}$ a definite metric, we have that $T_\gamma(J) = 0$ and so $\dot{c}(0) = J(0) = 0$. Thus, u is a multiple of $V(p)$, as desired. □

The definition of semi-nondegenerate foliation says that geodesics varying smoothly within the foliation do not meet at infinity. The following theorem states that this local condition implies in fact the global property that geodesics in the foliation do not meet at infinity at all. In the proof we have to use coordinates in $\mathbb{H}^3$.

Theorem 4.6 *Let $\mathcal{M}$ be the space of leaves of a semi-nondegenerate smooth foliation of $\mathbb{H}^3$ by oriented geodesics. Then the forward and backward Gauss maps $\varphi^\pm : \mathcal{M} \to \mathbb{H}^3(\infty)$ are one to one. In particular, they are diffeomorphisms onto their images.*

Proof Let $P : \mathbb{H}^3 \to \mathcal{M}$ be the map assigning to each point q in the hyperbolic space the oriented geodesic in the foliation containing q, that is, $P(q) = [\gamma_{V(q)}]$. This is a fiber bundle with typical fiber $\mathbb{R}$. Since $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^2$ by Theorem 3.2, there exists a global section $S : \mathcal{M} \to \mathbb{H}^3$. Let $F : \mathcal{M}' \times \mathbb{R} \to \mathbb{H}^3$ be the diffeomorphism given by $F(q, t) = \gamma_{V(q)}(t)$, where $\mathcal{M}' = S(\mathcal{M}) \subset \mathbb{H}^3$. Let

$$F^\pm : \mathcal{M}' \to \mathbb{H}^3(\infty), \qquad F^\pm(q) = \gamma_{V(q)}(\pm\infty),$$

which satisfies $F^\pm \circ S = \varphi^\pm$. Clearly, it suffices to prove that $F^\pm$ is one to one.

We consider the upper half space model of the hyperbolic space, that is, $\{(x, y, z) \mid z > 0\}$ with the Riemannian metric $ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2)$. Without loss of generality, we may suppose that $[\gamma_o] \in \mathcal{M}$, where $\gamma_o(t) = (0, 0, e^t)$, and that $S([\gamma_o]) = (0, 0, 1)$. In this model, $\mathbb{H}^3(\infty) = (\mathbb{R}^2 \times \{0\}) \cup \{\infty\}$, $\varphi^+[\gamma_o] = \gamma_o(\infty) = \infty$ and $\varphi^-[\gamma_o] = \gamma_o(-\infty) = 0$.

Since $\varphi^\pm$ (or equivalently $F^\pm$) are local diffeomorphisms, there exists a neighborhood A of $(0, 0, 1)$ in $\mathcal{M}'$, and neighborhoods U_+ and U_- of ∞ and 0 in $\mathbb{H}^3(\infty)$, respectively, such that $F^\pm : A \to U_\pm$ is a diffeomorphism. Let $B_+ \subset U_+$ be the complement of a closed disk centered at 0 of radius R in $\mathbb{R}^2 \times \{0\}$. Let $A' \subset A$ and $B_- \subset U_-$ be such that $F^\pm : A' \to B_\pm$ are diffeomorphisms. Taking, if necessary, a larger R, we may suppose that B_- is contained in the disk of radius δ (also centered at 0).

Let us see that $F(A' \times \mathbb{R})$ contains the horoball $\{(x, y, z) \mid 2z \geq R + \delta\}$. If $\partial A'$ is the border of A' in $\mathcal{M}'$, then $F(\partial A' \times \mathbb{R})$ is a cylinder in $\mathbb{H}^3$ separating the space in two connected components, in such a way that $F(A' \times \mathbb{R})$ is the component containing the trajectory of γ_o. The assertion follows from the fact that the cylinder is built up with trajectories of geodesics in $\mathbb{H}^3$ (vertical semicircles with center in $\mathbb{R}^2 \times \{0\}$) whose z-component is smaller than $\frac{1}{2}(R + \delta)$.

Finally, given $[\sigma] \in \mathcal{M}$ such that $\sigma(\infty) = \gamma_o(\infty)$, we want to see that $[\sigma] = [\gamma_o]$. The geodesic σ must have the form $\sigma(t) = (x_o, y_o, z_o e^t)$ for some real numbers x_o, y_o, z_o, with $z_o > 0$. For t large enough, $\sigma(t)$ is in the horoball. In particular, there exists t_1 such

that $\sigma(t_1) = F(q, s)$ for some $(q, s) \in A' \times \mathbb{R}$. Hence $\sigma(t_1) = \gamma_{V(q)}(s)$. Now, since for each point of $\mathbb{H}^3$ passes only one geodesic in $\mathcal{M}$, we have that $[\sigma] = [\gamma_{V(q)}]$. Consequently, $\gamma_{V(q)}(\infty) = \infty$ and so $q = (0, 0, 1)$, since F^+ is one to one on A'. Therefore $[\sigma] = [\gamma_o]$. The injectivity of φ^- is verified in a similar way. □

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