Comment

Comment on 'Freeman Dyson was right about analyticity and perturbation series'

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Abstract

In this comment we analyze the results derived recently by Bowen *et al* (2014 *Phys. Scr.* **89** 015301). We first show that the eigenvalues of the family of the anharmonic oscillators studied by those authors already exhibit a branch point but it is irrelevant for perturbation theory. Second, we show that one of the authors' main contributions, namely that the variational method yields the correct dependence of the eigenvalues on the coupling constant, has been known since long ago. Third, the authors' statement that their Bohr–Sommerfeld eigenvalues are exact is proved wrong by comparison with accurate numerical results.

Keywords: anharmonic oscillators, bound states, Bohr–Sommerfeld, variational method

1. Introduction

In a recent paper Bowen *et al* [1] discussed the eigenvalues of the family of anharmonic oscillators with potential $V(x) = \frac{g}{(2k)!}x^{2k}$, $k = 1, 2, \ldots$ by means of two approximate methods: the Bohr–Sommerfeld and canonical quantization procedures. They showed that the eigenvalues exhibit a branch-point singularity at $g = 0$ and concluded that there cannot be a convergent power series expansion in the coupling constant. According to the authors this fact severely restricts the analytic continuation of the coupling constant g in the complex plane. The authors also claimed to have obtained exact solutions for the eigenvalues of such a family of monomial potentials.

According to the authors this paper is a follow-up of a previous study [2]. What they forgot to say is that the results of that earlier paper were proved to be incorrect by Bender and Boettcer [3] and Amore and Fernández [4].

The purpose of this comment is to show that the results derived by Bowen *et al* [1] are well known and that some of the conclusions are incorrect or at least misleading. In section 2 we derive the analytic behaviour of the eigenvalues with respect to the coupling constant by means of the Symanzik's scaling relation and discuss the application of perturbation theory to those oscillators. In section 3 we discuss the application of the scaling variational method to those oscillators and show that it yields the correct dependence on the coupling constant. In section 4 the straightforward comparison of accurate numerical eigenvalues with those provided by the Bohr–Sommerfeld formula illustrates the well-known fact that the latter are not exact. Finally, in section 5 we summarize the main results of this comment and draw conclusions.

2. Symanzik's scaling relation

The family of anharmonic oscillators considered by Bowen *et al* [1] is given by

$$H = \frac{p^2}{2m} + \frac{g}{(2k)!}x^{2k}, \ \ k = 1, 2, \ldots, \tag{1}$$

where the coordinate x and momentum p operators satisfy the canonical commutation relation $[x, p] = i\hbar$. One can easily obtain the exact dependence of the eigenvalues $E_n(g)$ on the coupling constant g by means of the canonical transformation

$$U^{\dagger}xU = \alpha x, \ \ U^{\dagger}pU = \alpha^{-1}p, \tag{2}$$

where U is a unitary operator and $U^{\dagger} = U^{-1}$ its adjoint.

The Hamiltonian operator is transformed according to

$$U^{\dagger}HU = \alpha^{-2}\left[\frac{p^2}{2\,m} + \frac{g\alpha^{2k+2}}{(2k)!}x^{2k}\right]. \qquad (3)$$

If we take into account that $U^{\dagger}HU$ and H are isospectral and choose $\alpha = g^{-1/(2k+2)}$ then we conclude that

$$E_n(g) = g^{1/(k+1)}E_n(1), \qquad (4)$$

which is known as the Symanzik's scaling relation [5] (and references therein). Any reasonable approximation to the eigenvalues of the Hamiltonian (1) should at least exhibit this behaviour; for this reason Bowen et al [1] were able to derive it by means of the Bohr–Sommerfeld and canonical quantization procedures.

Equation (4) tells us that each eigenvalue of the Hamiltonian (1) exhibits a branch-point of order k at $g = 0$. In view of this result the authors state that: 'There are several profound consequences of this behavior. Firstly, it is not possible to expand the energy of any level in a convergent power series in g about zero.' It is well known that one cannot obtain a perturbation series about $g = 0$ because as $g \to 0$ the point spectrum of the Hamiltonian (1) becomes a continuous spectrum and, therefore, the application of Rayleigh–Schrödinger perturbation theory (or some of its variants) is nonsensical. In other words: one cannot obtain the discrete spectrum of the Hamiltonian (1) from the continuous spectrum of the unperturbed or reference Hamiltonian $H_0 = \frac{p^2}{2\,m}$ by means of straightforward perturbation theory. Therefore, one of the main conclusions drawn by Bowen et al is at least irrelevant. A rigorous discussion of the analytic properties of the eigenvalues of a closely related family of anharmonic oscillators (for which perturbation theory makes sense) was provided by Simon[5] some time ago (see also the references therein).

3. The scaling-variational method

Bowen et al [1] obtained an approximate expression for the lowest eigenvalue of the family of anharmonic oscillators (1) by means of the so-called canonical quantization procedure. The result proves to be exact for the harmonic oscillator and in the other cases it exhibits the correct dependence on the coupling constant. In what follows we show that the canonical quantization procedure is merely the well known scaling-variational method [6].

To do so we resort to the canonical transformation (2) and construct a trial function $\varphi_\alpha = U\varphi$, where $\langle\varphi\mid\varphi\rangle = 1$. The expectation value of H is then given by

$$\langle H\rangle(\alpha) = \left\langle\varphi_\alpha\right|H\left|\varphi_\alpha\right\rangle = \langle\varphi\mid U^{\dagger}HU\mid\varphi\rangle$$
$$= \frac{\left\langle p^2\right\rangle(1)}{2\,m\alpha^2} + \langle V\rangle(1)\alpha^{2k}, \qquad (5)$$

so that the variational method $\partial\langle H\rangle(\alpha)/\partial\alpha\left.\right|_{\alpha_{op}} = 0$ yields the

optimum value of α

$$\alpha_{op} = \left[\frac{\left\langle p^2\right\rangle(1)}{2\,km\,\langle V\rangle(1)}\right]^{\frac{1}{2k+2}}, \qquad (6)$$

and the approximate energy

$$\langle H\rangle(\alpha_{op}) = (k+1)\left[\frac{\left\langle p^2\right\rangle(1)^k\,\langle V\rangle(1)}{(2\,km)^k}\right]^{\frac{1}{k+1}}. \qquad (7)$$

Note that the approximate variational energy exhibits the correct g-dependence $\langle H\rangle(\alpha_{op}) \propto g^{1/(k+1)}$ disregarding the chosen function φ. Similar expressions appear in the treatment of more complex problems such as atoms and molecules [6]. That the scaling-variational method yields the correct dependence of the energy on the coupling parameter for homogeneous interactions ($\mathbf{r}.\nabla V = sV$) has been known for some time [7]; however, over and over again unaware researchers rediscover it.

It is worth adding that by a suitable choice of φ the scaling-variational method also yields the correct semi-classical dependence on the quantum number $n = 0, 1, \ldots$ (see next section). In fact, in the case of the Hamiltonian (1) one may obtain $\langle H\rangle(\alpha_{op}) = C_k(n + 1/2)^{2k/(k+1)}g^{1/(k+1)}$ where C_k is independent of both g and n [7].

4. The Bohr–Sommerfeld quantization condition

Bowen et al [1] also calculated the eigenvalues of the Hamiltonian (1) by means of the Bohr–Sommerfeld quantization formula

$$\frac{2}{h}\int_{x_-}^{x_+}\sqrt{2\,m[E - V(x)]}\,dx = \left(n + \frac{1}{2}\right), \qquad (8)$$

where $x_{\pm}$ are the classical turning points given by $E = V(x_{\pm})$. They compared the eigenvalues obtained by this expression with those coming from straightforward diagonalization of the Hamiltonian operator and appeared to conclude that they had in fact obtained the exact eigenvalues of the anharmonic oscillators. They used expressions such as 'This means that there is now a much larger set of exactly soluble Hamiltonians' and 'This means that we now have a whole family of exactly diagonalized Hamiltonians of which the SHO is a member' to refer to their Bohr–Sommerfeld results.

It is well known that the Bohr–Sommerfeld formula (8) is equivalent to the leading term of the WKB expansion. Bender et al [8] have explicitly shown how the accuracy of the approximate WKB eigenvalues for the quartic oscillator ($k = 2$) increases with the addition of correction terms. Besides, the accuracy of the WKB series truncated to any order increases with the quantum number. The Bohr–Sommerfeld formula (8) is by no means exact and its supposed agreement with the numerical results in figure 2 of Bowen et al's paper is merely due to the scale used. Therefore, the statements quoted above from this paper are wrong or at least

unhappily phrased in a misleading way. In order to make this point clearer to the reader in what follows we compare Bohr–Sommerfeld eigenvalues with accurate ones. To do so we first obtain suitable dimensionless eigenvalues that are independent of the physical and coupling constants. If we change the coordinate $x = \alpha q$, where $\alpha = \left[\frac{(2k)\,!\,\hbar^2}{2\,mg} \right]^{\frac{1}{2k+2}}$, then the Hamiltonian becomes

$$H = \left[\frac{\hbar^{2k}g}{(2k)\,!\,(2\,m)^k} \right]^{\frac{1}{k+1}} \left(-\frac{d^2}{dq^2} + q^k \right). \qquad (9)$$

Therefore we can compare the dimensionless eigenvalues

$$\epsilon_n = \left[\frac{(2k)\,!\,(2\,m)^k}{\hbar^{2k}g} \right]^{\frac{1}{k+1}} E_n, \qquad (10)$$

calculated by both methods. We obtained accurate numerical eigenvalues by means of the Riccati–Padé method (RPM) [9, 10].

Table 1 shows the first eigenvalues of the quartic oscillator ($k = 2$) calculated with the RPM (accurate to ten digits) and with the Bohr–Sommerfeld formula. We appreciate that the error (fourth column) decreases steadily but rather slowly as n increases. This fact clearly illustrates the semiclassical nature of the Bohr–Sommerfeld formula. In order to obtain more accurate results by means of the semiclassical method we have to add WKB corrections of higher order [8] as mentioned above.

5. Conclusions

In this comment we want to make clear the following facts:

First, it is true (and well known for some time) that the eigenvalues of the family of the anharmonic oscillators (1) exhibit a branch point of order k. Here we have illustrated how to derive such dependence without resorting to any approximate method. On the other hand, Bowen *et al* conjectured that such a dependence was correct by comparing the results of two approximate methods. However, this knowledge is irrelevant for perturbation theory because its application is hindered by the fact that the unperturbed Hamiltonian $H_0 = H(g = 0)$ does not exhibit a discrete spectrum (for this reason nobody would seriously attempt to carry out such calculation).

Second, the fact that the variational method yields the correct dependence of the eigenvalues on the coupling constant has been known for some time [7]. Here we have also shown that such an approximate method may also yield the semiclassical dependence on the quantum number.

Third, we have illustrated the well-known fact that the Bohr–Sommerfeld formula is approximate while Bowen *et al* appeared to believe that their results were exact. Table 1 clearly proves them wrong.

It is likely that some of the present conclusions may also apply to the future paper that the authors are preparing for submission: 'We have shown elsewhere (to be submitted) that

Table 1. First eigenvalues ϵ_n of the quartic oscillator calculated by means of the Bohr–Sommerfeld formula (BS) and the Riccati–Padé method (RPM). The fourth column is the percent error.

n	BS	RPM	Error%
0	0.8671453268	1.060362090	18.2
1	3.751919924	3.799673030	1.26
2	7.413988256	7.455697938	0.559
3	11.61152535	11.64474551	0.285
4	16.23361470	16.26182602	0.173
5	21.21365337	21.23837292	0.116
6	26.50633552	26.52847118	0.0834
7	32.07846412	32.09859771	0.0627
8	37.90447186	37.92300103	0.0489
9	43.96394838	43.98115810	0.0391
10	50.24015233	50.25625452	0.0320
11	56.71905712	56.73421406	0.0267
12	63.38870794	63.40304699	0.0226
13	70.23877146	70.25239463	0.0194
14	77.26021015	77.27320048	0.0168
15	84.44504013	84.45746627	0.0147
16	91.78614736	91.79806681	0.0130
17	99.27714528	99.28860666	0.0115
18	106.9122625	106.9233074	0.0103
19	114.6862530	114.6969174	0.00930
20	122.5943241	122.6046390	0.00841
21	130.6320760	130.6420687	0.00765
22	138.7954533	138.8051479	0.00698
23	147.0807035	147.0901213	0.00640
24	155.4843424	155.4935023	0.00589
25	164.0031248	164.0120436	0.00544
26	172.6340188	172.6427120	0.00504
27	181.3741850	181.3826662	0.00468
28	190.2209569	190.2292387	0.00435
29	199.1718254	199.1799188	0.00406
30	208.2244232	208.2323390	0.00380
31	217.3765143	217.3842617	0.00356
32	226.6259809	226.6335685	0.00335
33	235.9708145	235.9782504	0.00315

the other members of the family of potentials will show the same agreement.'

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