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Locally conformally Kähler structures on unimodular Lie groups

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Abstract We study left-invariant locally conformally Kähler structures on Lie groups, or equivalently, on Lie algebras. We give some properties of these structures in general, and then we consider the special cases when its complex structure is bi-invariant or abelian. In the former case, we show that no such Lie algebra is unimodular, while in the latter, we prove that if the Lie algebra is unimodular, then it is isomorphic to the product of $\mathbb{R}$ and a Heisenberg Lie algebra.

1 Introduction

Let (M, J, g) be a $2n$-dimensional Hermitian manifold and let ω be its fundamental 2-form, that is, $\omega(X, Y) = g(JX, Y)$ for any X, Y vector fields on M. The manifold (M, J, g) is called *locally conformally Kähler* (or LCK, for short) if g can be rescaled locally, in a neighborhood of any point in M, so as to be Kähler, or equivalently, if there exists a closed 1-form θ such that

$$d\omega = \theta \wedge \omega.$$

This 1-form θ is called the *Lee form*. This notion was introduced by Libermann [23] in 1954, but the geometry of these manifolds was not developed until the 70's, with the work of I.

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Vaisman. These manifolds are a natural generalization of the class of Kähler manifolds, and they have been much studied by many authors (see for instance [11,31,36]). According to [16], a locally conformally Kähler manifold is in the class $\mathscr{W}_4$ of the Gray-Hervella classification of almost Hermitian manifolds. An important class of LCK metrics is given by those whose Lee form is parallel with respect to the Levi-Civita connection. These LCK structures are called *Vaisman*, and their existence imposes topological and cohomological restrictions on the underlying Hermitian manifold (see for instance [36]).

We will consider locally conformally Kähler structures on solvmanifolds, that is, compact quotients $\Gamma \backslash G$ where G is a simply connected solvable Lie group and Γ is a lattice in G, which are induced by left-invariant locally conformally Kähler structures on the Lie group. According to [25], such a Lie group is necessarily unimodular. These structures have been the subject of study in several recent papers. For instance, it was shown in [30] that if a non-toral nilmanifold (i.e., a solvmanifold as above with G nilpotent) admits an invariant locally conformally Kähler structure, then it is a quotient of $\mathbb{R} \times H_{2n+1}$, where H_{2n+1} is the $(2n + 1)$-dimensional Heisenberg Lie group. In [31] it was proved that any invariant locally conformally Kähler structure on a solvmanifold such that $\omega = -\theta \wedge J\theta + d(J\theta)$ is in fact Vaisman. According to [28] this condition is related to the existence of a potential for the LCK metric. In [21] it is proved the non-existence of Vaisman metrics on solvmanifolds satisfying certain cohomological conditions.

In this article we study LCK structures on unimodular Lie algebras with two special kinds of complex structures. First, we take into account *bi-invariant* complex structures on Lie algebras, i.e., an endomorphism J of a Lie algebra $\mathfrak{g}$ that satisfies

$$J^2 = -Id, \quad J[X, Y] = [X, JY] \text{ for all } X, Y \in \mathfrak{g}.$$

This condition holds if and only if both left- and right-translations on the corresponding simply connected (real) Lie group are holomorhic, or equivalently, this Lie group is in fact a complex Lie group.

The other special kind of complex structures that we consider is given by the so called *abelian* complex structures. Recall that an abelian complex structure on a Lie algebra $\mathfrak{g}$ is an endomorphism J of $\mathfrak{g}$ that satisfies

$$J^2 = -Id, \quad [JX, JY] = [X, Y] \text{ for all } X, Y \in \mathfrak{g},$$

or equivalently, the i-eigenspace of J in $\mathfrak{g}^{\mathbb{C}}$ is an abelian subalgebra of $\mathfrak{g}^{\mathbb{C}}$. There are well known obstructions for the existence of abelian complex structures. For instance, if the Lie algebra $\mathfrak{g}$ admits such a structure, then $\mathfrak{g}$ has abelian commutator (i.e., $\mathfrak{g}$ is two-step solvable), and the center of $\mathfrak{g}$ is J-invariant, among other properties (see Lemma 2.2). These structures are very important in several areas of geometry and they have been studied by many authors recently (see for instance [6,10,24]).

The outline of this article is as follows. In Sect. 2 we review some known results about LCK manifolds and left-invariant complex structures on Lie groups. In Sect. 3 we determine some properties of Lie algebras endowed with an LCK or Vaisman structure. Next, in Sect. 4 we prove that there exists no unimodular Lie algebra $\mathfrak{g}$ equipped with an LCK structure $(J, \langle \cdot, \cdot \rangle)$ where J is bi-invariant (Theorem 4.1). Finally, in Sect. 5 we prove that if $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$ is LCK with an abelian complex structure J and $\mathfrak{g}$ is unimodular then $\mathfrak{g} \simeq \mathbb{R} \times \mathfrak{h}_{2n+1}$, where $\mathfrak{h}_{2n+1}$ is the $(2n + 1)$-dimensional Heisenberg Lie algebra. Moreover, there is only one, up to equivalence, monoparametric family $(J_0, \langle \cdot, \cdot \rangle_\lambda)$, $\lambda > 0$, of LCK structures on this Lie algebra, where the metrics $\langle \cdot, \cdot \rangle_\lambda$ are pairwise non-isometric.

2 Preliminaries

2.1 Locally conformally Kähler manifolds

A Hermitian metric on an almost complex manifold (M, J) is a Riemannian metric g such that $g(X, Y) = g(JX, JY)$ for any vector fields X, Y on M. In this case (M, J, g) is called an almost Hermitian manifold. When the almost complex structure J is integrable (i.e., (M, J) is a complex manifold), then (M, J, g) is called a Hermitian manifold.

Given an almost Hermitian manifold (M, J, g), the *fundamental 2-form* is defined by $\omega(X, Y) = g(JX, Y)$ for any vector fields X, Y on M.

A *Kähler metric* on a complex manifold (M, J) is a Hermitian metric g whose fundamental 2-form ω is closed, that is, $d\omega = 0$. Then M is called a Kähler manifold.

Kähler manifolds are by far the most important Hermitian manifolds. Nevertheless, this condition might be very restrictive in some cases, and therefore, weaker conditions are studied. One way to do so is to consider Hermitian manifolds whose metric is locally conformal to a Kähler metric.

The Hermitian manifold (M, J, g) is *locally conformally Kähler* (LCK) if there exists an open covering $\{U_i\}_{i \in I}$ of M and a family $\{f_i\}_{i \in I}$ of local C^∞-functions, $f_i : U_i \to \mathbb{R}$, such that each local metric

$$g_i = \exp(-f_i) \, g|_{U_i} \tag{1}$$

is Kähler. Also (M, J, g) is *globally conformally Kähler* (GCK) if there exists a global C^∞-function, $f : M \to \mathbb{R}$, such that the metric $\exp(-f)g$ is Kähler.

We recall an important characterization of LCK manifolds, which can be proved by differentiating (1).

Theorem 2.1 [22] *The Hermitian manifold (M, J, g) is LCK if and only if there exists a closed* 1-*form θ globally defined on M such that*

$$d\omega = \theta \wedge \omega. \tag{2}$$

Moreover, (M, J, g) is globally conformally Kähler if and only the 1-*form θ in* (2) *is exact.*

Remark 2.1 (i) A simply connected LCK manifold is GCK, in particular the universal cover of an LCK manifold is GCK.

(ii) An LCK manifold (M, J, g) is Kähler if and only if $\theta = 0$. Indeed, $\theta \wedge \omega = 0$ and ω non-degenerate imply $\theta = 0$. We consider LCK manifolds (M, J, g) such that (M, J, g) is not Kähler, therefore from now on we assume that $\theta \neq 0$.

(iii) For dimensional reasons, if (M, J, g) is a Hermitian manifold with dim $M \geq 6$ such that (2) holds for some 1-form θ, then θ is automatically closed, therefore M is LCK.

The 1-form θ of the previous theorem is called the *Lee form* and it was introduced by H. C. Lee in [22]. The Lee form is uniquely determined by the following formula:

$$\theta = -\frac{1}{n-1}(\delta\omega) \circ J, \tag{3}$$

where ω is the fundamental 2-form, δ is the codifferential and $2n$ is the real dimension of M. In general this formula is used to define the Lee form of any almost Hermitian manifold, which in this more general setting is called the torsion form (see [12]).

Example 2.1 The *Hopf manifolds* are examples of locally conformally Kähler manifolds which are not GCK. Let $\lambda \in \mathbb{C}$, $|\lambda| \neq 1$ and $\triangle_\lambda$ be the cyclic group generated by transformations $z \mapsto \lambda z$ of $\mathbb{C}^n - \{0\}$. The quotient space $CH^n_\lambda = (\mathbb{C}^n - \{0\})/\triangle_\lambda$ is a complex manifold and it is called *Hopf's complex manifold*. It can be seen that CH^n_λ is diffeomorphic to $S^1 \times S^{2n-1}$. Particularly CH^n_λ is compact and its first Betti number is $b_1(CH^n_\lambda) = 1$. Since all odd Betti numbers of a compact Kähler manifold are even, it follows that CH^n_λ cannot admit a Kähler metric.

We consider now the Hermitian metric on $\mathbb{C}^n - \{0\}$

$$h = \sum \frac{dz_j \otimes d\overline{z}_j}{|z|^2},$$

and canonical complex structure J. This metric is invariant by $\triangle_\lambda$, then it induces a Hermitian metric on CH^n_λ which is called the *Boothby metric* (it was discovered by Boothby for $n = 2$ in [7]). This Hermitian structure on CH^n_λ is in fact LCK.

The example above is more than LCK: its Lee form θ is in fact parallel with respect to the Levi-Civita connection. LCK manifolds with this property are very special and thus deserve their own definition: Given an LCK manifold (M, J, g), the metric g on M is called *Vaisman* if the Lee form θ is parallel with respect to the Levi-Civita connection of (M, g). A Vaisman manifold is an LCK manifold with a Vaisman metric. This notion is due to I. Vaisman, who used the terminology "generalized Hopf manifolds" (see [36,37]).

It is known that Vaisman manifolds have several special properties which do not necessarily hold in LCK manifolds. For example, the first Betti number of a Vaisman manifold is odd [20,37], while the Oeljeklaus–Toma manifolds are examples of LCK manifolds with even first Betti number [26]. Moreover, Vaisman manifolds always have an LCK potential [27] and have rank 1 [13].

2.2 Complex structures on Lie algebras

A left-invariant almost complex structure J on a Lie group G is a $(1, 1)$-tensor such that $J_g : T_g G \to T_g G$ is an endomorphism with $J_g^2 = -\text{Id}$ for all $g \in G$ and left-translations are holomorphic. As usual, the almost complex structure J on G is called integrable if

$$[JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0,$$

for any X, Y vector fields on G. In this case, J is called a left-invariant complex structure and (G, J) is a complex manifold.

A left-invariant (almost) complex structure is determined by its value on the identity of G, and therefore it is possible to define an (almost) complex structure on the Lie algebra $\mathfrak{g}$ of G. Namely, a *complex structure J* on a Lie algebra $\mathfrak{g}$ is an endomorphism $J : \mathfrak{g} \to \mathfrak{g}$ satisfying $J^2 = -\text{Id}$ and

$$[JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0,$$

for any $X, Y \in \mathfrak{g}$.

In this article we will be interested in two special kinds of almost complex structures on Lie algebras, namely bi-invariant complex structures and abelian complex structures. An almost complex structure J on $\mathfrak{g}$ is called *bi-invariant* if

$$J[X, Y] = [X, JY], \quad \text{for all } X, Y \in \mathfrak{g},$$

and it is called *abelian* if

$$[JX, JY] = [X, Y], \quad \text{for all } X, Y \in \mathfrak{g}.$$

Remark 2.2 Note that in both cases the almost complex structure is automatically integrable. Also, a complex structure on $\mathfrak{g}$ cannot be bi-invariant and abelian at the same time, unless $\mathfrak{g}$ is an abelian Lie algebra.

Remark 2.3 In general, right-translations are not holomorphic on a Lie group G with a left-invariant complex structure J. This holds only when G is a complex Lie group with the holomorphic structure given by J, or equivalently, J is bi-invariant.

Next, we include some properties about abelian complex structures in the following lemma (see [3,5,29] for their proofs).

Lemma 2.2 *Let $\mathfrak{g}$ be a Lie algebra with $\mathfrak{z}(\mathfrak{g})$ its center and $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ its commutator ideal. If J is an abelian complex structure on $\mathfrak{g}$, then*

1. $J\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$.
2. $\mathfrak{g}' \cap J\mathfrak{g}' \subset \mathfrak{z}(\mathfrak{g}' + J\mathfrak{g}')$.
3. *The codimension of $\mathfrak{g}'$ is at least 2, unless $\mathfrak{g}$ is isomorphic to $\mathfrak{aff}(\mathbb{R})$ (the only 2-dimensional non-abelian Lie algebra).*
4. *$\mathfrak{g}'$ is abelian, therefore $\mathfrak{g}$ is 2-step solvable.*

A rich family of Lie algebras with (abelian) complex structures is obtained by considering a finite dimensional real associative algebra $\mathscr{A}$ and $\mathfrak{aff}(\mathscr{A})$, namely, the vector space $\mathscr{A} \oplus \mathscr{A}$ equipped with the Lie bracket given by

$$[(a, b), (a', b')] = (aa' - a'a, ab' - a'b), \quad a, b, a', b' \in \mathscr{A}.$$

If J is the endomorphism of $\mathfrak{aff}(\mathscr{A})$ defined by

$$J(a, b) = (b, -a), \quad a, b \in \mathscr{A},$$

then it is easy to see that J is a complex structure on $\mathfrak{aff}(\mathscr{A})$. This complex structure is called *standard*. Moreover, when $\mathscr{A}$ is commutative, J is abelian. We prove next a result about $\mathfrak{aff}(\mathscr{A})$ that will be used later. We recall that a Lie algebra is called *unimodular* if the adjoint representation is trace-free, i.e., $\text{tr}(\text{ad}_X) = 0$ for any x in the Lie algebra. Moreover, if a Lie group G is unimodular (i.e., its left-invariant Haar measure is also right-invariant), then its Lie algebra $\mathfrak{g}$ is unimodular, and the converse holds when G is connected (see [25]).

Lemma 2.3 *If $\mathscr{A}$ is an associative commutative algebra and $\mathfrak{aff}(\mathscr{A})$ is unimodular, then $\mathscr{A}$ is nilpotent. Therefore $\mathfrak{aff}(\mathscr{A})$ is a nilpotent Lie algebra.*

Proof Suppose that $\mathscr{A}$ is not nilpotent, then there exists $0 \neq e \in \mathscr{A}$ such that $e^2 = e$. We consider $(e, 0) \in \mathfrak{aff}(\mathscr{A})$ and we compute $\text{ad}_{(e,0)}(x, y) = (0, ey) = (0, l_e(y))$ where l_e is the left-multiplication by e. Therefore the matrix of $\text{ad}_{(e,0)}$ is of the form

$$\text{ad}_{(e,0)} = \begin{pmatrix} 0 & 0 \\ 0 & l_e \end{pmatrix} \tag{4}$$

Since $l_e^2 = l_e$ and $l_e \neq 0$, there exists a basis of $\mathscr{A}$ such that

$$l_e = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \tag{5}$$

and therefore $\text{tr}(\text{ad}_{(e,0)}) \neq 0$. $\square$

Remark 2.4 With a similar proof, one can show that if $\mathscr{A}$ is an associative algebra with identity, then $\mathfrak{aff}(\mathscr{A})$ is not unimodular.

3 Left-invariant LCK metrics on Lie groups

One way to produce examples of manifolds equipped with LCK structures is by considering Lie groups carrying left-invariant LCK structures. However, when the Lie group is simply connected, the LCK is in fact *globally* conformally Kähler, according to Remark 2.1(i). In order to obtain LCK structures which are not GCK, we may consider quotients by lattices of the group (i.e., co-compact discrete subgroups), and the LCK structure descends to the quotient. The existence of a lattice imposes restrictions on the Lie group, since according to Milnor [25] such a group has to be unimodular. This is our motivation to study left-invariant LCK structures on unimodular Lie groups.

Let G be a Lie group with a left-invariant complex structure J and a left-invariant metric g. If (G, J, g) satisfies the LCK condition (2), then (J, g) is called a *left-invariant LCK structure* on the Lie group G. That is, there exists a closed 1-form θ on G such that $d\omega = \theta \wedge \omega$. We will see next that the Lee form θ is left-invariant. Therefore θ is determined by its value in the identity.

Proposition 3.1 *Let G be a Lie group with a left-invariant LCK structure (J, g), with θ the associated Lee form. Then θ is left-invariant.*

Proof Recall that if α is any left-invariant form on G, then $d\alpha$, $*\alpha$ and $\delta\alpha = \pm * \circ d \circ *\alpha$ are also left-invariant. Since J is left-invariant, the claim follows from (3). □

This fact allows us to define LCK structures on Lie algebras.

Let $\mathfrak{g}$ a Lie algebra, J a complex structure and $\langle \cdot, \cdot \rangle$ a Hermitian inner product on $\mathfrak{g}$, with ω its fundamental 2-form. The triple $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$ is called *locally conformally Kähler* (LCK) if there exists $\theta \in \mathfrak{g}^*$, with $d\theta = 0$, such that

$$d\omega = \theta \wedge \omega. \tag{6}$$

A Lie algebra $\mathfrak{g}$ with a Hermitian structure $(J, \langle \cdot, \cdot \rangle)$ is *Vaisman* if $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$ is LCK and the Lee form is parallel (see Lemma 3.3 below).

Example 3.1 Let $\mathfrak{g} = \mathbb{R} \times \mathfrak{h}_{2n+1}$, where $\mathfrak{h}_{2n+1}$ is the $(2n + 1)$-dimensional Heisenberg Lie algebra. There is a basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z_1, Z_2\}$ of $\mathfrak{g}$ with Lie brackets given by $[X_i, Y_i] = Z_1$ for $i = 1, \ldots, n$ and Z_2 in the center. We define an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ such that the basis above is orthonormal. Let J_0 be an almost complex structure given by:

$$J_0 X_i = Y_i, \quad J_0 Z_1 = -Z_2 \quad \text{for } i = 1, \ldots, n.$$

It is easily seen that J_0 is a complex structure on $\mathfrak{g}$. Let $\{x^i, y^i, z^1, z^2\}$ be the 1-forms dual to $\{X_i, Y_i, Z_1, Z_2\}$ respectively. Then the fundamental form is:

$$\omega = \sum_{i=1}^{n} (x^i \wedge y^i) - z^1 \wedge z^2.$$

Thus,

$$d\omega = z^2 \wedge \omega,$$

and therefore $(\mathfrak{g}, J_0, \langle \cdot, \cdot \rangle)$ is LCK. It can be seen that the Lee form $\theta = z^2$ is parallel, hence the LCK structure is Vaisman. This example appeared in [8].

It is known that $\mathfrak{g}$ is the Lie algebra of the Lie group $\mathbb{R} \times H_{2n+1}$, where H_{2n+1}, known as the Heisenberg group, is the group of all matrices with real coefficients which have the following form:

$$P = \begin{pmatrix} 1 & A & c \\ 0 & I_n & B^t \\ 0 & 0 & 1 \end{pmatrix}, \quad c \in \mathbb{R}, \ I_n = Id_{n \times n}.$$

where $A = (a_1, \ldots, a_n) \in \mathbb{R}^n$, $B = (b_1, \ldots, b_n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Let $\Gamma \subset H_{2n+1}$ be the subgroup of all matrices with integer coefficients. Then $\Gamma \backslash H_{2n+1}$ is compact and the nilmanifold $N = S^1 \times \Gamma \backslash H_{2n+1}$ admits an LCK structure which is Vaisman. The nilmanifold N is known as the Kodaira-Thurston manifold, and it was the first example of a compact symplectic manifold that does not admit Kähler structures [32].

Remark 3.1 The complex structure J_0 defined in the previous example is abelian. Moreover, it was proved in [5] that if J is a complex structure on a Lie algebra $\mathfrak{g}$ with dim $\mathfrak{g}' = 1$, then J is abelian.

Example 3.2 In [9] the following example of an LCK solvmanifold was given. Let $\mathfrak{g}$ be the 4-dimensional solvable Lie algebra given by

$$\mathfrak{g} = \text{span}\{A, X, Y, Z\}$$
$$[A, X] = X, \quad [A, Y] = -Y, \quad [X, Y] = Z.$$

Let $\{\alpha, x, y, z\}$ be the dual basis of $\{A, X, Y, Z\}$. We can check by direct computation that

$$d\alpha = 0, \quad dx = -\alpha \wedge x, \quad dy = \alpha \wedge y, \quad dz = -x \wedge y.$$

Let $\langle \cdot, \cdot \rangle$ be a inner product on $\mathfrak{g}$ such that $\{A, X, Y, Z\}$ is an orthonormal basis. If we define J by

$$JA = Y, \quad JZ = X,$$

then $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$ is Hermitian with the fundamental 2-form ω given by

$$\omega = \alpha \wedge y + z \wedge x.$$

Therefore we obtain

$$d\omega = -\alpha \wedge \omega.$$

Hence it is LCK with Lee form $\theta = -\alpha$. This structure is not Vaisman, since α is not parallel (see also Lemma 3.3 below). It was proved in [9] that the associated simply connected solvable Lie group G admits a lattice Γ and therefore the solvmanifold $\Gamma \backslash G$ admits a LCK structure. It is proved in [19] that this LCK solvmanifold is holomorphically homothetic to the Inoue surface Sol_1^4 / Γ equipped with the locally conformal Kähler structure constructed by Tricerri in [33].

Remark 3.2 All 4-dimensional simply connected unimodular Lie groups with left-invariant LCK structures were classified in [18]. In particular, the reductive Lie groups $U(2)$ and $GL(2, \mathbb{R})$ admit such structures. Recently, in [1] it was proved that if a reductive Lie group admits a left-invariant LCK structure then it is locally isomorphic to one of these

4-dimensional groups. Moreover, if a compact Lie group admits a left-invariant LCK structure, then it is locally isomorphic to $U(2)$ and the LCK structure is in fact Vaisman ([1, Theorems 4.6 and 4.15]).

Now we study some properties about Lie algebras equipped with an LCK or Vaisman structure. Recall that if $\alpha \in \mathfrak{g}^*$ and $\eta \in \bigwedge^2 \mathfrak{g}^*$, then their exterior derivatives $d\alpha \in \bigwedge^2 \mathfrak{g}^*$ and $d\eta \in \bigwedge^3 \mathfrak{g}^*$ are given by

$$d\alpha(X, Y) = -\alpha([X, Y]), \qquad d\eta(X, Y, Z) = -\eta([X, Y], Z) - \eta([Y, Z], X) - \eta([Z, X], Y),$$

for any $X, Y, Z \in \mathfrak{g}$.

Let $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$ be LCK and suppose that $\mathfrak{g}$ is not Kähler, that is, $d\omega = \theta \wedge \omega$ where θ is closed and $\theta \neq 0$. Then the codimension of $\ker \theta$ is 1 and then we can choose

$$A \in (\ker \theta)^{\perp} \quad \text{such that} \quad \theta(A) = 1, \tag{7}$$

and therefore $\mathfrak{g}$ can be decomposed orthogonally as

$$\mathfrak{g} = \operatorname{span}\{A\} \oplus \ker \theta, \quad \text{with} \quad \mathfrak{g}' \subset \ker \theta. \tag{8}$$

Note that since $\theta \neq 0$, $\mathfrak{g}$ cannot be a semisimple Lie algebra. Since J is skew-symmetric we obtain $\langle JA, A \rangle = 0$ and therefore $JA \in \ker \theta$. If W is the orthogonal complement of $\operatorname{span}\{JA\}$ in $\ker \theta$, we have

$$\mathfrak{g} = \operatorname{span}\{A, JA\} \oplus^{\perp} W, \tag{9}$$

and W is invariant by J.

According to (8), any $X \in \mathfrak{g}$ can be written as $X = tA + Y$, with $t \in \mathbb{R}$ and $Y \in \ker \theta$. From the definition of A we obtain $t = \theta(X)$. On the other hand, computing $\langle X, A \rangle$ we get $\langle X, A \rangle = t|A|^2$, and consequently the Lee form can be written explicitly as

$$\theta(X) = \frac{\langle X, A \rangle}{|A|^2} \quad \text{for all } X \in \mathfrak{g}. \tag{10}$$

Thus $|\theta(X)|$ represents the length of the orthogonal projection of X with respect to A.

Lemma 3.2 *If $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$ is LCK then $J \circ \operatorname{ad}_{JA}$ is a symmetric endomorphism of $\mathfrak{g}$.*

Proof For any $X, Y \in \mathfrak{g}$ we compute

$$\begin{aligned} d\omega(JA, X, Y) &= -\omega([JA, X], Y) - \omega([X, Y], JA) - \omega([Y, JA], X) \\ &= -\langle J[JA, X], Y \rangle - \langle J[X, Y], JA \rangle - \langle J[Y, JA], X \rangle \\ &= -\langle J[JA, X], Y \rangle - \langle J[Y, JA], X \rangle. \end{aligned}$$

On the other hand, using (10), we obtain

$$\begin{aligned} \theta \wedge \omega(JA, X, Y) &= \theta(X)\omega(Y, JA) + \theta(Y)\omega(JA, X) \\ &= \frac{\langle A, X \rangle}{|A|^2}\langle Y, A \rangle - \frac{\langle A, Y \rangle}{|A|^2}\langle A, X \rangle \\ &= 0. \end{aligned}$$

It follows from (6) that $\langle J[JA, X], Y \rangle = \langle J[JA, Y], X \rangle$ for all $X, Y \in \mathfrak{g}$, hence $J \circ \operatorname{ad}_{JA}$ is symmetric. $\qquad\square$

Now we consider a Vaisman structure $(J, \langle \cdot, \cdot \rangle)$ on $\mathfrak{g}$, with θ its parallel Lee form. In this context the Vaisman condition can be characterized as follows.

Lemma 3.3 *Let* $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$ *be LCK and let* $A \in \mathfrak{g}$ *be as in* (7). *Then* $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$ *is Vaisman if and only if* ad_A *is a skew-symmetric endomorphism of* $\mathfrak{g}$.

Proof Recall first that the Levi-Civita connection ∇ of a left-invariant Riemannian metric on a Lie group is itself left-invariant, that is, $\nabla_X Y$ is a left-invariant vector field whenever X, Y are left-invariant. Similarly, ∇T is a left-invariant tensor if T is a left-invariant tensor. In particular, if η is a left-invariant 1-form, we have that $(\nabla_X \eta)(Y) = -\eta(\nabla_X Y)$ for X, Y left-invariant vector fields.

Let us now compute $\nabla \theta$, where ∇ is the Levi-Civita connection on $\mathfrak{g}$ associated to $\langle \cdot, \cdot \rangle$ and θ is the Lee form. Given $X, Y \in \mathfrak{g}$ we have that

$$(\nabla_X \theta)(Y) = -\theta(\nabla_X Y) = -\frac{\langle \nabla_X Y, A \rangle}{|A|^2} = -\frac{1}{2|A|^2}\{\langle [X, Y], A \rangle - \langle [Y, A], X \rangle + \langle [A, X], Y \rangle\},$$

where we have used the Koszul formula for the Levi-Civita connection in the left-invariant setting. Since A

$$(\nabla_X \theta)(Y) = -\frac{1}{2|A|^2}\{\langle [A, Y], X \rangle + \langle [A, X], Y \rangle\}$$

Therefore $(\nabla_X \theta)(Y) = 0$ if and only if $\langle [A, Y], X \rangle = -\langle [A, X], Y \rangle$ for all $X, Y \in \mathfrak{g}$. □

Remark 3.3 Let H be a Lie group equipped with a left-invariant metric h. It is well known that an endomorphism ad_X of its Lie algebra $\mathfrak{h}$ is skew-symmetric with respect to $h_e = \langle \cdot, \cdot \rangle$ if and only if the left-invariant vector field on H determined by X is Killing.

4 LCK structures with bi-invariant complex structure

In this section we consider a Lie algebra equipped with an LCK structure such that its complex structure is bi-invariant. Equivalently, we are considering left-invariant LCK metrics on complex Lie groups. The aim of this section is to prove that in each even (real) dimension, there is only one Lie algebra admitting such metrics.

First, we exhibit some examples. Beginning with $\mathbb{R}^{2n}$ equipped with a complex structure J_0 and a Hermitian inner product $\langle \cdot, \cdot \rangle$, consider the Lie algebra $\mathbb{R}^2 \ltimes \mathbb{R}^{2n}$ (with $\mathbb{R}^2 = \mathrm{span}\{u, v\}$), where the Lie brackets are given by $[u, v] = 0$, $[u, X] = X$ and $[v, X] = J_0 X$ for all $X \in \mathbb{R}^{2n}$. Extending J_0 by $J_0 u = v$, it is easy to check that J_0 is a bi-invariant complex structure on $\mathbb{R}^2 \ltimes \mathbb{R}^{2n}$. Extend also $\langle \cdot, \cdot \rangle$ to an inner product $\langle \cdot, \cdot \rangle_\lambda$ on $\mathbb{R}^2 \ltimes \mathbb{R}^{2n}$ by setting $\langle \mathbb{R}^2, \mathbb{R}^{2n} \rangle = 0$, $\langle u, v \rangle = 0$ and $|u| = |v| = \lambda$ for some $\lambda > 0$. It is easy to see that $(J_0, \langle \cdot, \cdot \rangle_\lambda)$ is an LCK structure, but it is not Vaisman (see Lemma 3.3). Furthermore, the metrics $\langle \cdot, \cdot \rangle_\lambda$ are pairwise non-isometric, since the scalar curvature of $\langle \cdot, \cdot \rangle_\lambda$ is $-\frac{2n(n+1)}{\lambda^2}$.

Theorem 4.1 *Let* $(J, \langle \cdot, \cdot \rangle)$ *be an LCK structure on the Lie algebra* $\mathfrak{g}$ *with a bi-invariant complex structure* J. *Then* $\mathfrak{g} \simeq \mathbb{R}^2 \ltimes \mathbb{R}^{2n}$ *as above and* $(J, \langle \cdot, \cdot \rangle)$ *is equivalent to a Hermitian structure* $(J_0, \langle \cdot, \cdot \rangle_\lambda)$ *as above for some* $\lambda > 0$.

In order to prove this theorem, we recall the following well known result concerning the existence of Kähler metrics on complex Lie groups [15].

Lemma 4.2 *If* $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$ *is Kähler with* J *bi-invariant, then* $\mathfrak{g}$ *is abelian.*

Proof (of Theorem 4.1) Recall from (8) the orthogonal decomposition $\mathfrak{g} = \mathrm{span}\{A\} \oplus \ker \theta$.

Lemma 4.3 *The endomorphism* $\mathrm{ad}_A : \mathfrak{g} \to \mathfrak{g}$ *is symmetric.*

Proof Since J is bi-invariant we have that $J \circ \mathrm{ad}_{JA} = -\mathrm{ad}_A$, and it follows from Lemma 3.2 that ad_A is symmetric. $\qquad\square$

Next, from (9) we have

$$\mathfrak{g} = \mathrm{span}\{A, JA\} \oplus^{\perp} W,$$

where $\mathfrak{g}' \subset \ker\theta = \mathrm{span}\{JA\} \oplus W$ and W is J-invariant. The fact that J is bi-invariant implies that also $\mathfrak{g}'$ is J-invariant, and therefore $\mathfrak{g}' \subset W$. In fact, $\mathfrak{g}' = W$, since for $X \in W$ we have $d\omega(A, X, JX) = -2\langle[A, X], X\rangle$ and $\theta \wedge \omega(A, X, JX) = |X|^2$, therefore

$$-2\langle[A, X], X\rangle = |X|^2,$$

this implies that $\mathfrak{g}' = W$. Then we obtain

$$\mathfrak{g} = \mathrm{span}\{A, JA\} \oplus^{\perp} \mathfrak{g}',$$

with $J\mathfrak{g}' = \mathfrak{g}'$.

On the other hand $(\mathfrak{g}', J|_{\mathfrak{g}'}, \langle\cdot, \cdot\rangle)$ is Kähler, since the fundamental form of $\mathfrak{g}'$ is the restriction of ω to $\mathfrak{g}' \times \mathfrak{g}'$, and $d\omega = 0$ on $\mathfrak{g}'$. From Lemma 4.2 we get that $\mathfrak{g}'$ is abelian, that is, $\mathfrak{g}' = \mathbb{R}^{2n}$.

Since ad_A is symmetric and $d\omega(A, X, JY) = \theta \wedge \omega(A, X, JY)$, we have that

$$2\langle[A, X], Y\rangle = -\langle X, Y\rangle, \quad X, Y \in \mathfrak{g}'.$$

Therefore $[A, X] = -\frac{1}{2}X$ for all $X \in \mathfrak{g}'$. Setting $B = JA$, we obtain $[A, B] = 0$, $\mathrm{ad}_A|_{\mathbb{R}^{2n}} = -\frac{1}{2}\mathrm{Id}$ and $\mathrm{ad}_B|_{\mathbb{R}^{2n}} = J\mathrm{ad}_A = -\frac{1}{2}J$. Setting $u = -2A$ and $v = -2B$, we obtain $\mathfrak{g} \simeq \mathbb{R}^2 \ltimes \mathbb{R}^{2n}$ with $(J, \langle\cdot, \cdot\rangle)$ equivalent to $(J, \langle\cdot, \cdot\rangle_\lambda)$ for $\lambda = |2A|$. $\qquad\square$

Corollary 4.4 *There exists no unimodular Lie algebra* $\mathfrak{g}$ *with an LCK structure* $(J, \langle\cdot, \cdot\rangle)$ *such that* J *is a bi-invariant complex structure.*

Proof With the notation from the proof of Theorem 4.1, it follows that $\mathrm{tr}(\mathrm{ad}_A) = -\frac{1}{n} \neq 0$, and therefore $\mathfrak{g}$ is not unimodular. $\qquad\square$

Remark 4.1 (i) It follows from Theorem 4.1 and Lemma 3.3 that an LCK structure with a bi-invariant complex structure is never Vaisman.

(ii) The Lie algebra $\mathbb{R}^2 \ltimes \mathbb{R}^{2n}$ with the Lie brackets as above is in fact an almost abelian complex Lie algebra $\mathbb{C} \ltimes \mathbb{C}^n$ such that $[Z, U] = U$ for all $U \in \mathbb{C}^n$, where $\mathbb{C}$ is generated by Z.

Let M be a compact complex parallelizable manifold. According to [38], M may be written as a quotient $\Gamma\backslash G$, where G is a simply connected complex Lie group and Γ is a discrete subgroup. Note that according to [25], G must be unimodular. Let $\pi : G \to M$ denote the holomorphic projection.

Corollary 4.5 *With notation as above, M does not admit any LCK metric g compatible with its holomorphic structure such that π^*g is a left-invariant metric on G.*

Proof Let us assume such a metric exists. Since π^*g is left-invariant, G admits a left-invariant LCK metric with bi-invariant complex structure, and this determines an LCK structure on $\mathfrak{g}$, the Lie algebra of G. But $\mathfrak{g}$ is unimodular (as G is unimodular), and this contradicts Corollary 4.4. $\qquad\square$

Remark 4.2 In [17], it is proved more generally that a compact complex parallelizable manifold does not admit any LCK metric compatible with its holomorphic structure.

The LCK structure $(J_0, \langle \cdot, \cdot \rangle_\lambda)$ on the Lie algebra $\mathbb{R}^2 \ltimes \mathbb{R}^{2n}$ induces a left-invariant LCK structure on the corresponding simply connected Lie group, which is diffeomorphic to $\mathbb{R}^{2n+2}$. Let $(x_1, x_2, u_1, v_1, \ldots, u_n, v_n)$ be the canonical global coordinates, thus the left-invariant metric g_λ associated to $\langle \cdot, \cdot \rangle_\lambda$ is given by

$$g_\lambda = \lambda^2(dx_1^2 + dx_2^2) + e^{-2x_1} \sum_k (du_k^2 + dv_k^2),$$

where $\lambda = |2A|$. The fundamental form is

$$\omega_\lambda = \lambda^2(dx_1 \wedge dx_2) + e^{-2x_1} \sum_k (du_k \wedge dv_k).$$

The conformal metric $h_\lambda = e^{2x_1} g_\lambda$ is Kähler.

Remark 4.3 Note that in the 4-dimensional case the metric g_λ for $\lambda = 1$ is locally conformally hyper-Kähler. Indeed, according to [4], the metric h_1 is hyper-Kähler with respect to a hypercomplex structure I_1, I_2, I_3. Moreover, the bi-invariant complex structure J_0 is not in the 2-sphere of complex structures generated by I_j, $j = 1, 2, 3$.

5 LCK structures with abelian complex structure

In this section we consider a Lie algebra equipped with an LCK structure such that its complex structure is abelian. Our aim is to prove that the only unimodular Lie algebras that admit such metrics are the product of a Heisenberg Lie algebra by $\mathbb{R}$, and the LCK structure is in fact Vaisman. From now on we assume that the Lie algebras we work with are at least 4-dimensional.

Before stating the main result, we consider the following variation of Example 3.1. Recall that $\mathbb{R} \times \mathfrak{h}_{2n+1}$ has a basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z_1, Z_2\}$ such that $[X_i, Y_i] = Z_1$ for $i = 1, \ldots, n$, and that this Lie algebra admits an abelian complex structure J_0 given by $J_0 X_i = Y_i$, $J_0 Z_1 = -Z_2$. For any $\lambda > 0$, consider the inner product $\langle \cdot, \cdot \rangle_\lambda$ such that the basis above is orthogonal, with $|X_i| = |Y_i| = 1$ but $|Z_1| = |Z_2| = \frac{1}{\lambda}$. It is easy to see (just as in Example 3.1) that $(J_0, \langle \cdot, \cdot \rangle_\lambda)$ is an LCK structure, in fact, it is Vaisman. Furthermore, the metrics $\langle \cdot, \cdot \rangle_\lambda$ are pairwise non-isometric, since the scalar curvature of $\langle \cdot, \cdot \rangle_\lambda$ is $-\frac{n\lambda^2}{2}$.

Theorem 5.1 *Let $(J, \langle \cdot, \cdot \rangle)$ be an LCK structure on $\mathfrak{g}$ with abelian complex structure J. If $\dim \mathfrak{g} \geq 4$ and $\mathfrak{g}$ is unimodular then $\mathfrak{g} \simeq \mathbb{R} \times \mathfrak{h}_{2n+1}$, where $\mathfrak{h}_{2n+1}$ is the $(2n+1)$-dimensional Heisenberg Lie algebra, and $(J, \langle \cdot, \cdot \rangle)$ is equivalent to $(J_0, \langle \cdot, \cdot \rangle_\lambda)$ for some $\lambda > 0$.*

We will provide the proof of this theorem in a series of results. Recall from (8) that

$$\mathfrak{g} = \mathrm{span}\{A\} \oplus \ker \theta,$$

where $A \in (\ker \theta)^\perp$ such that $\theta(A) = 1$. This series of results will be divided roughly in two parts. The first part will culminate in Proposition 5.10, where it is proved that both A and JA are central elements of $\mathfrak{g}$. In the second part, we shall determine all the Lie brackets on $\mathfrak{g}$, which will allow us to establish an isomorphism $\mathfrak{g} \simeq \mathbb{R} \times \mathfrak{h}_{2n+1}$.

Lemma 5.2 *The endomorphism $\mathrm{ad}_A : \mathfrak{g} \to \mathfrak{g}$ is symmetric.*

Proof It is an immediate consequence of Lemma 3.2.   □

In particular $\mathrm{ad}_A \mid_{\ker \theta} : \ker \theta \to \ker \theta$ is symmetric, therefore it is diagonalizable and there is an orthogonal decomposition

$$\ker \theta = \sum_{\lambda \in S} \mathfrak{g}_\lambda,$$

where $S \subset \mathbb{R}$ is the spectrum of $\mathrm{ad}_A \mid_{\ker \theta}$ and $\mathfrak{g}_\lambda$ is the eigenspace associated with the eigenvalue λ.

According to Lemma 2.2 (iii) the codimension of $[\mathfrak{g}, \mathfrak{g}]$ is at least 2. Therefore $\mathfrak{g}_0 \neq \{0\}$, that is $0 \in S$. Hence we obtain the orthogonal decomposition

$$\mathfrak{g} = \mathbb{R}A \oplus \mathfrak{g}_0 \oplus \sum_{\lambda \in S^*} \mathfrak{g}_\lambda, \tag{11}$$

where $S^* := S - \{0\}$. Note that the Jacobi identity and the fact that $\mathfrak{g}'$ is abelian imply that:

- $\mathfrak{g}_\lambda$ is an ideal for $\lambda \in S^*$,
- $\mathfrak{g}_0$ is a subalgebra.

Now we consider $\mathfrak{g}_0' = [\mathfrak{g}_0, \mathfrak{g}_0]$ and $(\mathfrak{g}_0')^\perp$, its orthogonal complement in $\mathfrak{g}_0$, that is,

$$\mathfrak{g}_0 = \mathfrak{g}_0' \oplus (\mathfrak{g}_0')^\perp.$$

Note also that $\mathfrak{g}' = \mathfrak{g}_0' \oplus \sum_{\lambda \in S^*} \mathfrak{g}_\lambda$.

For any $X, Y \in \ker \theta$, we compute

$$\begin{aligned}
d\omega(A, X, Y) &= -\omega([A, X], Y) - \omega([X, Y], A) - \omega([Y, A], X) \\
&= -\langle J[A, X], Y \rangle - \langle J[X, Y], A \rangle - \langle J[Y, A], X \rangle \\
&= \langle [A, X], JY \rangle + \langle [X, Y], JA \rangle + \langle [Y, A], JX \rangle \\
\theta \wedge \omega(A, X, Y) &= \theta(A)\omega(X, Y) + \theta(X)\omega(Y, A) + \theta(Y)\omega(A, X) \\
&= \langle JX, Y \rangle.
\end{aligned}$$

Therefore we have

$$\langle [A, X], JY \rangle + \langle [X, Y], JA \rangle + \langle [Y, A], JX \rangle = \langle JX, Y \rangle,$$

for all $X, Y \in \ker \theta$. From this we get three equations that will be important later:

$$\langle [X, Y], JA \rangle = \langle JX, Y \rangle, \text{ for } X, Y \in \mathfrak{g}_0. \tag{12}$$

$$(\lambda + \mu + 1)\langle JX, Y \rangle = 0, \text{ for } X \in \mathfrak{g}_\lambda, Y \in \mathfrak{g}_\mu \text{ and } \lambda, \mu \in S^*. \tag{13}$$

$$\langle [X, Y], JA \rangle = (\mu + 1)\langle JX, Y \rangle, \text{ for } X \in \mathfrak{g}_0, Y \in \mathfrak{g}_\mu, \mu \in S^*. \tag{14}$$

Lemma 5.3 $J(\mathfrak{g}_0') \subset \mathbb{R}A \oplus (\mathfrak{g}_0')^\perp.$

Proof If $X, Y \in \mathfrak{g}_0$ we can write $J[X, Y] = aA + Z_0 + \sum_{\lambda \in S^*} Z_\lambda$ with $a \in \mathbb{R}$, $Z_0 \in \mathfrak{g}_0$ and $Z_\lambda \in \mathfrak{g}_\lambda$. Then $[A, J[X, Y]] = [A, \sum_{\lambda \in S^*} Z_\lambda] = \sum_{\lambda \in S^*} \lambda Z_\lambda$.

On the other hand, from (11), $JA = X_0 + \sum_{\lambda \in S^*} X_\lambda$, with $X_0 \in \mathfrak{g}_0$ y $X_\lambda \in \mathfrak{g}_\lambda$ for $\lambda \in S^*$.

Then

$$[A, J[X, Y]] = -[JA, [X, Y]] = -[X_0 + \sum_{\lambda \in S^*} X_\lambda, [X, Y]] = -[X_0, [X, Y]] \in \mathfrak{g}_0$$

since $\mathfrak{g}_0$ is subalgebra and $\mathfrak{g}'$ is abelian. Therefore $Z_\lambda = 0$ for all $\lambda \in S^*$.

Moreover, it follows from (12) and the fact that $\mathfrak{g}'$ is abelian that $J(\mathfrak{g}'_0)$ and $\mathfrak{g}'_0$ are orthogonal. Then we must have $Z_0 \in (\mathfrak{g}'_0)^\perp$, and this implies the result. $\square$

Now, we define $\Lambda \subset S^*$ in the following way: $\lambda \in \Lambda$ if and only if there is not any $\lambda' \in S^*$ such that $\lambda + \lambda' + 1 = 0$, or equivalently, $\Lambda = \{\lambda \in S^* : -(\lambda + 1) \notin S^*\}$. Note that $\lambda \notin \Lambda$ if and only if $-(\lambda + 1) \notin \Lambda$.

Lemma 5.4 *Let $\lambda \in S^*$. Then,*

(i) *if $\lambda \in \Lambda$ then $J(\mathfrak{g}_\lambda) \subset \mathbb{R}A \oplus (\mathfrak{g}'_0)^\perp$.*
(ii) *if $\lambda \in \Lambda^c$ then $J(\mathfrak{g}_\lambda) \subset \mathbb{R}A \oplus (\mathfrak{g}'_0)^\perp \oplus \mathfrak{g}_{\lambda'}$, where $\lambda + \lambda' + 1 = 0$.*

Proof (i) If $\lambda \in \Lambda$, from (13) we have that $J(\mathfrak{g}_\lambda)$ is orthogonal to $\mathfrak{g}_\mu$ for all $\mu \in S^*$, and therefore $J(\mathfrak{g}_\lambda) \subset \mathbb{R}A \oplus \mathfrak{g}_0$. Moreover, for $X_\lambda \in \mathfrak{g}_\lambda$, $Y \in \mathfrak{g}'_0$, it follows from Lemma 5.3 that $\langle JX_\lambda, Y \rangle = -\langle X_\lambda, JY \rangle = 0$. This proves (i), and in a similar way we prove (ii). $\square$

Let $\mathfrak{h}$ be the orthogonal complement of $\mathfrak{g}' + J\mathfrak{g}'$ in $\mathfrak{g}$. Note that $\mathfrak{h}$ is J-invariant. Thus, we can write

$$\mathfrak{g} = (\mathfrak{g}' + J\mathfrak{g}') \oplus \mathfrak{h}. \tag{15}$$

We will show that this orthogonal complement $\mathfrak{h}$ is non-zero. We begin with an auxiliary result.

Lemma 5.5 $\mathfrak{g}' \cap J\mathfrak{g}' = \sum_{\lambda \in \Lambda^c} \mathfrak{g}_\lambda \cap J\left(\sum_{\lambda \in \Lambda^c} \mathfrak{g}_\lambda\right).$

Proof Given $Y \in \mathfrak{g}' \cap J\mathfrak{g}'$ then $Y = JZ$ for $Z \in \mathfrak{g}'$. Since $\mathfrak{g}' = \mathfrak{g}'_0 \oplus \sum_{\lambda \in S^*} \mathfrak{g}_\lambda$, we can write

$$Y = Y_0 + \sum_{\lambda \in S^*} Y_\lambda, \quad Z = Z_0 + \sum_{\lambda \in S^*} Z_\lambda$$

with $Y_0, Z_0 \in \mathfrak{g}'_0$ and $Y_\lambda, Z_\lambda \in \mathfrak{g}_\lambda$. Then $JZ = JZ_0 + \sum_{\lambda \in S^*} JZ_\lambda$. From Lemma 5.3 and Lemma 5.4 we obtain $JZ \in \mathbb{R}A \oplus (\mathfrak{g}'_0)^\perp \oplus \sum_{\lambda \in \Lambda^c} \mathfrak{g}_\lambda$. Since $JZ = Y \in \mathfrak{g}'_0 \oplus \sum_{\lambda \in S^*} \mathfrak{g}_\lambda$, we get $Y_0 = 0$ and $Y_\lambda = 0$ for all $\lambda \in \Lambda$, therefore $Y \in \sum_{\lambda \in \Lambda^c} \mathfrak{g}_\lambda$. In the same way, $Z \in \sum_{\lambda \in \Lambda^c} \mathfrak{g}_\lambda$, and therefore $\mathfrak{g}' \cap J\mathfrak{g}' \subset \sum_{\lambda \in \Lambda^c} \mathfrak{g}_\lambda \cap J\left(\sum_{\lambda \in \Lambda^c} \mathfrak{g}_\lambda\right)$. The other inclusion is clear. $\square$

Lemma 5.6 *With notation as above, $\mathfrak{h} \neq 0$.*

Proof If we suppose that $\mathfrak{h} = \{0\}$ we get from (15) that $\mathfrak{g} = \mathfrak{g}' + J\mathfrak{g}'$.

Claim $\mathfrak{g}' \cap J\mathfrak{g}' = \{0\}$.

Indeed, according to Lemma 5.5 and to Lemma 2.2 (ii) we have

$$\mathfrak{g}' \cap J\mathfrak{g}' = \sum_{\lambda \in \Lambda^c} \mathfrak{g}_\lambda \cap J\left(\sum_{\lambda \in \Lambda^c} \mathfrak{g}_\lambda\right) \subset \mathfrak{z}(\mathfrak{g}).$$

Given $Y \in \mathfrak{g}' \cap J\mathfrak{g}'$, it can be written as $Y = \sum_{\lambda \in \Lambda^c} Y_\lambda$. Then $0 = [A, Y] = \sum_{\lambda \in \Lambda^c} \lambda Y_\lambda$, and therefore $Y = 0$. This proves the claim.

As a consequence, we have the direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}' \oplus J\mathfrak{g}'.$$

According to [2, Corollary 3.3], the Lie bracket on $\mathfrak{g}$ induces a structure of commutative associative algebra on $\mathfrak{g}'$ given by $X * Y = [JX, Y]$. Furthermore if $\mathscr{A}$ denotes the commutative associative algebra $(\mathfrak{g}', *)$, then $\mathscr{A}^2 = \mathscr{A}$ and $\mathfrak{g}$ is holomorphically isomorphic to $\mathfrak{aff}(\mathscr{A})$ with its standard complex structure (see Sect. 2.2). Since $\mathfrak{g}$ is unimodular, it follows from Lemma 2.3 that $\mathscr{A}$ is nilpotent. This is a contradiction with the fact that $\mathscr{A}^2 = \mathscr{A}$, hence $\mathfrak{h} \neq \{0\}$. □

Since A is orthogonal to $\mathfrak{g}'$, we have that JA is orthogonal to $J\mathfrak{g}'$. More precisely, we have

Lemma 5.7 $JA \in \mathfrak{g}'$.

Proof Let $\mathfrak{u}$ be the orthogonal complement of $\mathfrak{g}' \cap J\mathfrak{g}'$ in $\mathfrak{g}'$, that is

$$\mathfrak{g}' = \mathfrak{u} \oplus (\mathfrak{g}' \cap J\mathfrak{g}').$$

Since $\mathfrak{g}' \cap J\mathfrak{g}'$ is J-invariant we have

$$J\mathfrak{g}' = J\mathfrak{u} \oplus (\mathfrak{g}' \cap J\mathfrak{g}'),$$

and therefore

$$\mathfrak{g}' + J\mathfrak{g}' = \mathfrak{u} \oplus (\mathfrak{g}' \cap J\mathfrak{g}') \oplus J\mathfrak{u}.$$

As JA is orthogonal to $J\mathfrak{g}'$, it follows from (15) that $JA = U + \beta$ for some $U \in \mathfrak{u}$ and $\beta \in \mathfrak{h}$. Since $\mathfrak{u} \subset \mathfrak{g}'$, the lemma will follow if we prove $\beta = 0$.

Now, for any $X \in \mathfrak{g}$ such that $\langle A, X \rangle = \langle JA, X \rangle = 0$, we compute

$$d\omega(J\beta, X, JX) = \theta \wedge \omega(J\beta, X, JX)$$

$$-\langle [J\beta, X], X \rangle + \langle [X, JX], -\beta] \rangle + \langle [JX, J\beta], JX \rangle = \frac{\langle A, J\beta \rangle}{|A|^2} |X|^2$$

$$\langle [J\beta, X], X \rangle + \langle [J\beta, JX], JX \rangle = \frac{|\beta|^2}{|A|^2} |X|^2. \tag{16}$$

Moreover $\langle [J\beta, A], A \rangle = 0$ and $\langle [J\beta, JA], JA \rangle = 0$, due to Lemma 3.2 and the fact that $\mathfrak{h}$ is orthogonal to $\mathfrak{g}'$. Let $\{X_1, JX_1, \ldots, X_r, JX_r\}$ be an orthonormal basis of W, where W is given in (9). Note that $r \geq 1$ since $\dim \mathfrak{g} \geq 4$. We compute next $\text{tr}(\text{ad}_{J\beta})$, taking into account (16):

$$\text{tr}(\text{ad}_{J\beta}) = \frac{1}{|A|^2} \langle [J\beta, A], A \rangle + \frac{1}{|A|^2} \langle [J\beta, JA], JA \rangle$$

$$+ \sum_{j=1}^r \langle [J\beta, X_j], X_j \rangle + \langle [J\beta, JX_j], JX_j \rangle$$

$$= \frac{|\beta|^2}{|A|^2} \sum_{j=1}^r |X|^2.$$

Since $\mathfrak{g}$ is unimodular, it follows that $\beta = 0$. □

Remark 5.1 It follows from Lemma 5.7 and (15) that if $H \in \mathfrak{h}$, then H is orthogonal to A and JA.

Lemma 5.8 *If $H \in \mathfrak{h}$, then*

(i) $\langle [H, JH], JA \rangle = |H|^2$,
(ii) $|[H, JH]|^2 = \frac{|H|^4}{|A|^2}$.

Proof For $H \in \mathfrak{h}$, we compute first

$$d\omega(A, H, JH) = \theta \wedge \omega(A, H, JH)$$

$$\langle [A, H], J^2 H \rangle + \langle [H, JH], JA \rangle + \langle [JH, A], JH \rangle = |H|^2 - \frac{\langle A, H \rangle^2}{|A|^2} - \frac{\langle JA, H \rangle^2}{|A|^2}$$

$$\langle [H, JH], JA \rangle = |H|^2,$$

since $\mathfrak{h}$ is J-invariant and orthogonal to $\mathfrak{g}'$. This proves (i).

Now we compute

$$d\omega(J[H, JH], H, JH) = \theta \wedge \omega(J[H, JH], H, JH)$$

$$-|[H, JH]|^2 = \frac{\langle A, J[H, JH] \rangle}{|A|^2} |H|^2$$

$$|[H, JH]|^2 = \frac{|H|^4}{|A|^2},$$

where we used (i) for the last equality. This proves (ii).   □

Lemma 5.9 *If $H \in \mathfrak{h}$, then*

(i) $[H, JH] = \frac{|H|^2}{|A|^2} JA$,
(ii) $[H, Y] = 0$ *for all $Y \in \mathfrak{h}$ such that $\langle Y, JH \rangle = 0$,*
(iii) $[H, \mathfrak{g}_0'] = 0$,
(iv) $[H, \mathfrak{g}_\lambda] = 0$ *for all $\lambda \in S^* - \{-\frac{1}{2}\}$.*

Proof (i) Using Lemma 5.8 and the Cauchy–Schwarz inequality we obtain

$$|H|^4 = \langle [H, JH], JA \rangle^2 \leq |[H, JH]|^2 |A|^2 = \frac{|H|^4}{|A|^2} |A|^2 = |H|^4,$$

so that we have equality everywhere and therefore for all $H \in \mathfrak{h}$ there exists $c(H) > 0$ such that

$$[H, JH] = c(H) JA.$$

From Lemma 5.8 (ii) again we get that $|H|^2 = c(H)|A|^2$, and therefore $[H, JH] = \frac{|H|^2}{|A|^2} JA$ for all $H \in \mathfrak{h}$.

(ii) We calculate $[H, Y]$ for $Y \in \mathfrak{h}$ such that $\langle Y, JH \rangle = 0$.

$$d\omega(J[H, Y], H, Y) = \langle [J[H, Y], H], JY \rangle - |[H, Y]|^2 + \langle [Y, J[H, Y]], JH \rangle$$

$$= -|[H, Y]|^2,$$

since $\mathfrak{h}$ is J-invariant and orthogonal to $\mathfrak{g}'$. On the other hand

$$\theta \wedge \omega(J[H, Y], H, Y) = \frac{\langle A, J[H, Y]\rangle}{|A|^2}\langle JH, Y\rangle + \frac{\langle A, H\rangle}{|A|^2}\langle JY, J[H, Y]\rangle$$
$$- \frac{\langle A, Y\rangle}{|A|^2}\langle [H, Y], H\rangle = 0,$$

since A is orthogonal to $\mathfrak{h}$ and $\langle JH, Y\rangle = 0$. Therefore $[H, Y] = 0$.

Finally, both (iii) and (iv) will follow from the next computation. Given $H \in \mathfrak{h}$, $X, Y \in \mathfrak{g}'$ we compute

$$d\omega(H, X, JY) = -\omega([H, X], JY) - \omega([X, JY], H) - \omega([JY, H], X)$$
$$= -\langle J[H, X], JY\rangle - \langle J[X, JY], H\rangle - \langle J[JY, H], X\rangle$$
$$= -\langle [H, X], Y\rangle + \langle [X, JY], JH\rangle + \langle [JY, H], JX\rangle$$
$$= -\langle [H, X], Y\rangle + \langle [JY, H], JX\rangle,$$

since $\mathfrak{h}$ is J-invariant and orthogonal to $\mathfrak{g}'$. On the other hand we have

$$\theta \wedge \omega(H, X, JY) = \theta(H)\omega(X, JY) + \theta(X)\omega(JY, H) + \theta(JY)\omega(H, X)$$
$$= \frac{\langle A, H\rangle}{|A|^2}\langle X, Y\rangle - \frac{\langle A, X\rangle}{|A|^2}\langle Y, H\rangle + \frac{\langle A, JY\rangle}{|A|^2}\langle JH, X\rangle$$
$$= 0,$$

since $\langle H, A\rangle = 0$ and $\langle \mathfrak{h}, \mathfrak{g}'\rangle = 0$. Therefore we get $\langle [H, X], Y\rangle = \langle [JY, H], JX\rangle$. In particular, if we take $Y = [H, X]$ we obtain

$$\|[H, X]\|^2 = \langle [J[H, X], H], JX\rangle. \tag{17}$$

(iii) If $X \in \mathfrak{g}'_0$ it follows from Lemma 5.3 that $JX \in \mathbb{R}A \oplus (\mathfrak{g}'_0)^\perp$. Since $[J[H, X], H] \in \mathfrak{g}'$ we get from (17) that $\|[H, X]\|^2 = 0$.

(iv) If $X_\lambda \in \mathfrak{g}_\lambda$, $\lambda \in \Lambda$, it follows from Lemma 5.4 (i) that $JX \in \mathbb{R}A \oplus (\mathfrak{g}'_0)^\perp$. In the same way as above we get $\|[H, X]\|^2 = 0$.

However, if $X_\lambda \in \mathfrak{g}_\lambda$, $\lambda \in \Lambda^c$ and $\lambda \neq -\frac{1}{2}$ from Lemma 5.4 (ii) we obtain that $JX \in \mathbb{R}A \oplus (\mathfrak{g}'_0)^\perp \oplus \mathfrak{g}_{\lambda'}$, where $\lambda' = -\lambda - 1$ and $\lambda' \neq \lambda$ since $\lambda \neq -\frac{1}{2}$. On the other hand $[J[H, X], H] = -[[H, X], JH] \in \mathfrak{g}_\lambda$ since $\mathfrak{g}_\lambda$ is an ideal. Therefore from (13) and (17) we get $\|[H, X]\|^2 = 0$. $\square$

Proposition 5.10 *With notation as above, we have:*

(i) $S = \{0\}$, *that is*, $A, JA \in \mathfrak{z}(\mathfrak{g})$,
(ii) $\mathfrak{g} = \mathfrak{g}' \oplus J\mathfrak{g}' \oplus \mathfrak{h}$, *an orthogonal sum.*

Proof (i) Let $\lambda \in S^* - \{-\frac{1}{2}\}$ and take $H \in \mathfrak{h}$, $H \neq 0$ and $X_\lambda \in \mathfrak{g}_\lambda$. Lemma 5.9 (i) implies that

$$[[H, JH], JX_\lambda] = \frac{|H|^2}{|A|^2}[JA, JX_\lambda] = \frac{|H|^2}{|A|^2}\lambda X_\lambda,$$

whereas Lemma 5.9 (iv) and the fact that $\mathfrak{h}$ is J-invariant imply that

$$[[H, JH], JX_\lambda] = -[[JH, JX_\lambda], H] - [[JX_\lambda, H], JH] = 0.$$

Then $X_\lambda = 0$ and therefore $S^* - \{-\frac{1}{2}\} = \emptyset$. If $-\frac{1}{2} \in S^*$ then it is the only eigenvalue in S^*, hence $\mathfrak{g}$ is not unimodular, that is a contradiction. As a consequence, $S^* = \emptyset$, that is, $S = \{0\}$, or equivalently, $A \in \mathfrak{z}(\mathfrak{g})$. It follows from Lemma 2.2 that $JA \in \mathfrak{z}(\mathfrak{g})$ too.

(ii) It follows from Lemma 5.5 that $\mathfrak{g}' \cap J\mathfrak{g}' = \{0\}$. Therefore

$$\mathfrak{g} = \mathfrak{g}' \oplus J\mathfrak{g}' \oplus \mathfrak{h}$$

where $\mathfrak{g}' = \mathfrak{g}'_0$. Moreover, this decomposition is orthogonal, because of Lemma 5.3. □

Remark 5.2 If $(\mathfrak{g}, J, \langle \cdot, \cdot \rangle)$ is Vaisman with J abelian, it is much easier to show that $A, JA \in \mathfrak{z}(\mathfrak{g})$. Indeed, from Lemmas 5.2 and 3.3 we have that ad_A is symmetric and skew-symmetric, so $A \in \mathfrak{z}(\mathfrak{g})$. Then J abelian implies that $JA \in \mathfrak{z}(\mathfrak{g})$ too.

Proposition 5.11 *The commutator ideal $\mathfrak{g}'$ is 1-dimensional, generated by JA.*

Proof Let us assume that $\dim \mathfrak{g}' \geq 2$, and let $X \in \mathfrak{g}'$, $|X| \neq 0$, such that $\langle X, JA \rangle = 0$.

Claim $[X, JY] = \frac{|\langle X, Y \rangle|}{|A|^2} JA$ for any $Y \in \mathfrak{g}'$ such that $\langle Y, JA \rangle = 0$.

Indeed, we compute

$$d\omega(J[X, JY], X, JY) = -\langle [J[X, JY], X], Y \rangle - \langle [X, JY], [X, JY] \rangle$$
$$+ \langle [JY, J[X, JY]], JX \rangle.$$

Since $\mathfrak{g}'$ and $J\mathfrak{g}'$ are orthogonal, we get $\langle [JY, J[X, JY]], JX \rangle = 0$. From Jacobi identity and the fact that $\mathfrak{g}'$ is abelian we have that

$$[J[X, JY], X] = -[[X, JY], JX] = [[JY, JX], X] + [[JX, X], JY] = [[JX, X], JY].$$

Therefore $d\omega(J[X, JY], X, JY) = -|[X, JY]|^2 + \langle \mathrm{ad}_{J[JX,X]} Y, Y \rangle$. On the other hand,

$$\begin{aligned}
\theta \wedge \omega(J[X, JY], X, JY) &= \frac{\langle A, J[X, JY] \rangle}{|A|^2} \langle X, Y \rangle - \frac{\langle A, X \rangle}{|A|^2} \langle Y, J[X, JY] \rangle \\
&\quad - \frac{\langle A, JY \rangle}{|A|^2} \langle [X, JY], X \rangle \\
&= \frac{\langle A, J[X, JY] \rangle}{|A|^2} \langle X, Y \rangle \\
&= -\frac{\langle X, Y \rangle^2}{|A|^2},
\end{aligned}$$

where we have used $\langle A, X \rangle = \langle Y, JA \rangle = 0$ in the second equality and (12) in the last equality. Therefore

$$\langle \mathrm{ad}_{J[JX,X]} Y, Y \rangle = |[X, JY]|^2 - \frac{\langle X, Y \rangle^2}{|A|^2}. \tag{18}$$

Using (12), the Cauchy–Schwarz inequality and (18) we get

$$\langle X, Y \rangle^2 = \langle [X, JY], JA \rangle^2 \leq |[X, JY]|^2 |A|^2 = \langle \mathrm{ad}_{J[JX,X]} Y, Y \rangle |A|^2 + \langle X, Y \rangle^2, \tag{19}$$

and this implies

$$\langle \mathrm{ad}_{J[JX,X]} Y, Y \rangle \geq 0.$$

Recalling that $\mathrm{tr}(\mathrm{ad}_{J[JX,X]}) = 0$, since $\mathfrak{g}$ is unimodular, we will show next that the inequality above is in fact an equality.

We know that $\mathrm{ad}_{J[JX,X]} A = \mathrm{ad}_{J[JX,X]} JA = 0$ (since $A, JA \in \mathfrak{z}(\mathfrak{g})$), $\mathrm{ad}_{J[JX,X]} JZ = 0$ for all $Z \in \mathfrak{g}'$ (since $\mathfrak{g}'$ is abelian) and $\mathrm{ad}_{J[JX,X]} H = 0$ for all $H \in \mathfrak{h}$ (due to Lemma

5.9). Therefore $\operatorname{tr}(\operatorname{ad}_{J[JX,X]}) = \sum_j \langle \operatorname{ad}_{J[JX,X]} e_j, e_j \rangle$ for any orthonormal basis $\{e_j\}$ of $\mathfrak{g}'$. Choosing $e_1 = \frac{JA}{|A|}$ and $e_2 = \frac{X}{|X|}$, we have

$$0 = \operatorname{tr}(\operatorname{ad}_{J[JX,X]}) = \frac{1}{|X|^2} \langle \operatorname{ad}_{J[JX,X]} X, X \rangle + \sum_{j \geq 3} \langle \operatorname{ad}_{J[JX,X]} e_j, e_j \rangle. \qquad (20)$$

Thus $\langle \operatorname{ad}_{J[JX,X]} Y, Y \rangle = 0$ for any $Y \in \mathfrak{g}'$ with $\langle Y, JA \rangle = 0$, since such Y is a linear combination of $\{e_j \mid j \geq 2\}$.

As a consequence, we obtain from (19) that $[X, JY] = c(X, Y)JA$ for some $c(X, Y) \geq 0$. It follows from (18) that $c(X, Y) = \frac{|\langle X, Y \rangle|}{|A|^2} JA$. Therefore $[X, JY] = \frac{|\langle X, Y \rangle|}{|A|^2} JA$ for all $Y \in \mathfrak{g}'$ with $\langle Y, JA \rangle = 0$.

This proves the claim.

In order to finish the proof of Proposition 5.11, recall the following orthogonal decompositions of $\mathfrak{g}$:

$$\mathfrak{g} = \operatorname{span}\{A\} \oplus \mathfrak{g}_0 = \mathfrak{g}' \oplus J\mathfrak{g}' \oplus \mathfrak{h},$$

with $A \in J\mathfrak{g}'$, $JA \in \mathfrak{g}'$, where $\mathfrak{g}'$ and $J\mathfrak{g}'$ are abelian subalgebras. Since $\mathfrak{g}_0' = \mathfrak{g}'$, it follows from Lemma 5.9 (iii) that $[\mathfrak{h}, \mathfrak{g}'] = [\mathfrak{h}, J\mathfrak{g}'] = 0$ and, moreover, the only non-zero brackets in $\mathfrak{h}$ are $[H, JH] = \frac{|H|^2}{|A|^2} JA$. From this and the claim just proved, it is clear that the only non-zero brackets in $\mathfrak{g}$ are multiples of JA, therefore we arrive at a contradiction with the assumption that $\dim \mathfrak{g}' \geq 2$.

Therefore, $\dim \mathfrak{g}' = 1$, and $\mathfrak{g}'$ is generated by JA. □

Proof (of Theorem 5.1) As a consequence of Lemma 5.9, Proposition 5.10 and Proposition 5.11, the only non-vanishing brackets on $\mathfrak{g}$ are $[X, JX] = \frac{|X|^2}{|A|^2} JA$ for $X \in \mathfrak{g}$ with $\langle X, A \rangle = 0$ and $\langle X, JA \rangle = 0$. Considering an orthonormal basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ of $\mathfrak{h}$, with $JX_i = Y_i$, we have that the only non-vanishing brackets on $\mathfrak{g}$ are $[X_i, Y_i] = \frac{JA}{|A|^2}$. Setting $Z_1 = \frac{JA}{|A|^2}$, $Z_2 = \frac{A}{|A|^2}$, it is clear that $\mathfrak{g}$ is isomorphic to $\mathbb{R} \times \mathfrak{h}_{2n+1}$ and $(J, \langle \cdot, \cdot \rangle)$ is equivalent to $(J_0, \langle \cdot, \cdot \rangle_\lambda)$ for $\lambda = |A|$. □

Remark 5.3 We will give a sketch of an alternative proof of Theorem 5.1. From Proposition 5.10 we have that $A, JA \in \mathfrak{z}(\mathfrak{g})$ then $\mathfrak{g} = \operatorname{span}\{A, JA\} \oplus W$ and $\mathfrak{g}' \subset \operatorname{span}\{JA\} \oplus W$. If h is the projection of the Lie bracket on $\mathfrak{g}$ to W, then it can be seen that $(W, h, \langle \cdot, \cdot \rangle|_{W \times W}, J|_W)$ is a unimodular Kähler Lie algebra with an abelian complex structure. According to [3, Theorem 4.1], it follows that (W, h) is abelian. Then $\mathfrak{g}' = \operatorname{span}\{JA\}$, and moreover, the only non-zero Lie brackets are $[X, JX] = \frac{|X|^2}{|A|^2} JA$ for $X \in W$, obtaining in this way the same result as above.

Remark 5.4 On $\mathbb{R} \times \mathfrak{h}_{2n+1}$ there are $\left[\frac{n}{2}\right] + 1$ equivalence classes of complex structures, all of them abelian (see [2, Proposition 2.2]). It follows from the proof of Theorem 5.1 that if $(J, \langle \cdot, \cdot \rangle)$ is an LCK structure on this Lie algebra, then J is equivalent to the complex structure J_0, so that representatives of only one equivalence class of complex structures may admit LCK metrics (compare [34]).

In terms of solvmanifolds, we can rewrite Theorem 5.1 as follows.

Corollary 5.12 *Let $\Gamma \backslash G$ be a compact solvmanifold with an LCK structure induced from a left invariant LCK structure on G with an abelian complex structure, and G simply connected. Then G is isomorphic to $\mathbb{R} \times H_{2n+1}$, and $\mathbb{R} \times H_{2n+1}$ has a left-invariant LCK structure induced*

by $(J_0, \langle \cdot, \cdot \rangle_\lambda)$ for some $\lambda > 0$. In particular, $\Gamma \backslash G$ is a nilmanifold and the LCK structure is Vaisman.

The LCK structure on the Lie algebra $\mathbb{R} \times \mathfrak{h}_{2n+1}$ induces a left-invariant LCK structure on $\mathbb{R} \times H_{2n+1}$. Let $(t, x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ be a global coordinate system on $\mathbb{R} \times H_{2n+1}$, which is diffeomorphic to $\mathbb{R}^{2n+2}$. The left-invariant metric g_λ associated to $\langle \cdot, \cdot \rangle_\lambda$ is given by

$$g_\lambda = \sum_i \left(dx_i^2 + dy_i^2 \right) + \lambda^{-2} \left(dt^2 + \left(dz - \sum_i x_i dy_i \right)^2 \right),$$

where $\lambda = |A|$, while the fundamental form ω_λ is given by

$$\omega_\lambda = \sum_i dx_i \wedge dy_i + \lambda^{-2} dt \wedge (dz - \sum_i x_i dy_i).$$

This left-invariant LCK structure on $\mathbb{R} \times H_{2n+1}$ descends to an LCK structure on the nilmanifold $\Gamma \backslash (\mathbb{R} \times H_{2n+1}) \cong S^1 \times \tilde{\Gamma} \backslash H_{2n+1}$, where $\Gamma = \mathbb{Z} \times \tilde{\Gamma}$ is a lattice in $\mathbb{R} \times H_{2n+1}$, with $\tilde{\Gamma}$ a lattice in H_{2n+1}.

Remark 5.5 The lattices in H_{2n+1} have been classified in [14]. We fix for each $k \in \mathbb{N}$ the lattice Γ_k in H_{2n+1} given by

$$\Gamma_k = \left\{ \begin{pmatrix} 1 & a_1 & \ldots & a_n & c/k \\ & 1 & & & b_1 \\ & & \ddots & & \vdots \\ & & & 1 & b_n \\ & & & & 1 \end{pmatrix} : a_j, b_j, c \in \mathbb{Z} \right\}.$$

Clearly, $\Gamma_r \subset \Gamma_s$ if and only if r divides s. Thus, $\Gamma_1 \backslash H_{2n+1}$ covers $\Gamma_k \backslash H_{2n+1}$ for all k. Moreover, it can be shown that $\Gamma_k / [\Gamma_k, \Gamma_k]$ is isomorphic to $\mathbb{Z}^{2n} \oplus \mathbb{Z}_k$. Hence, the nilmanifolds

$$M_k = (\mathbb{Z} \times \Gamma_k) \backslash (\mathbb{R} \times H_{2n+1}) = S^1 \times (\Gamma_k \backslash H_{2n+1})$$

are non-homeomorphic for different values of k.

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