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Invariant solutions to the conformal Killing–Yano equation on Lie groups

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ABSTRACT

We search for invariant solutions of the conformal Killing–Yano equation on Lie groups equipped with left invariant Riemannian metrics, focusing on 2-forms. We show that when the Lie group is compact equipped with a bi-invariant metric or 2-step nilpotent, the only invariant solutions occur on the 3-dimensional sphere or on a Heisenberg group. We classify the 3-dimensional Lie groups with left invariant metrics carrying invariant conformal Killing–Yano 2-forms.

1. Introduction

The concept of conformal Killing–Yano p-forms (also known in the literature as twistor forms or conformal Killing forms) on Riemannian manifolds was introduced by Tachibana in [1] for $p = 2$ and later by Kashiwada in [2] for general p. Applications of these forms to theoretical physics were found related to quadratic first integrals of geodesic equations, symmetries of field equations, conserved quantities and separation of variables, among others (see, for instance, [3–7]). More recently, since the work of Moroianu, Semmelmann [8,9], a renewed interest in the subject arose among differential geometers (see, for instance, [10–16]).

Next we give the basic definitions and recall some well known properties of conformal Killing–Yano p-forms.

A p-form ω on an n-dimensional Riemannian manifold (M, g) is called *conformal Killing–Yano* (CKY for short) if it satisfies the following equation:

$$\nabla_X\omega = \frac{1}{p+1}\iota_X d\omega - \frac{1}{n-p+1}X^* \wedge d^*\omega, \quad X \in \mathfrak{X}(M) \tag{1}$$

where ∇ is the Levi-Civita connection, X^* is the 1-form dual to X and $d^* = (-1)^{n(p+1)+1} * d *$ is the co-differential. If, moreover, ω is co-closed, that is $d^*\omega = 0$, then it is called *Killing–Yano* (see [17]).

For $p = 1$, a 1-form ω is conformal Killing–Yano if and only if its dual vector field U is conformal, that is, $\mathcal{L}_U g = \varphi g$ for some $\varphi \in C^\infty(M)$. A 1-form ω is Killing–Yano if and only if its dual vector field is Killing.

We list next some properties:

- If ω is a CKY p-form on M, then $*\omega$ is a CKY $(n-p)$-form, where $*$ is the Hodge-star operator. In particular, $*$ interchanges closed and co-closed CKY forms (see [18,9]).

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- Conformal invariance: If ω is a CKY p-form on (M,g) and $\tilde{g} := e^{2f}g$ is a conformally equivalent metric, then the form $\tilde{\omega} := e^{(p+1)f}\omega$ is a CKY p-form on $(M,\tilde{g})$ (see [6]).
- The space of CKY p-forms has finite dimension $\leq \binom{n+2}{p+1}$ with equality attained on the standard n-sphere (see [9]).

It was shown in [15] that every Killing–Yano p-form on a compact quaternionic Kähler manifold is automatically parallel ($p \geq 2$). In [10] the authors prove that a compact simply connected symmetric space carries a non-parallel Killing–Yano p-form if and only if it is isometric to a Riemannian product $S^k \times N$, where S^k is a round sphere and $k > p$. The case of conformal Killing–Yano p-forms on a compact Riemannian product was considered in [16], proving that such a form is a sum of forms of the following types: parallel forms, pull-back of Killing forms on the factors, and their Hodge duals.

A family of examples of manifolds carrying CKY 2-forms is given by spheres, nearly Kähler manifolds and Sasakian manifolds. In this article we will search for other examples, not necessarily compact. We restrict ourselves to the case of Lie groups with a left invariant metric. In Section 2 we consider Eq. (1) for $p = 2$ in the left invariant setting and we give an equivalent condition at the Lie algebra level (Proposition 2.5). We show that strong restrictions to the existence of left invariant KY and CKY 2-forms appear when we consider a compact Lie group with a bi-invariant metric or a 2-step nilpotent Lie group with any left invariant metric. Indeed, in Section 3 we prove that the only compact Lie group with a bi-invariant metric that admits non-coclosed CKY 2-forms is $SU(2)$, and in Section 4 we prove that the only 2-step nilpotent Lie groups carrying non-coclosed CKY 2-forms are the Heisenberg groups. In Section 5 we classify the 3-dimensional Lie groups with left invariant metrics carrying CKY 2-forms, obtaining families of globally defined metrics admitting CKY 2-forms on each of $\mathbb{R}^3$, $S^1 \times \mathbb{R}^2$ and S^3, and give the coordinate expression of the metrics. Moreover, we determine which of them give rise to Sasakian structures.

2. CKY tensors on Lie groups

Let G be an n-dimensional Lie group and let $\mathfrak{g}$ be the associated Lie algebra of all left invariant vector fields on G. If T_eG is the tangent space of G at e, the identity of G, the correspondence $X \to X_e := x$ from $\mathfrak{g} \to T_eG$ is a linear isomorphism. This isomorphism allows to define a Lie algebra structure on the tangent space T_eG setting, for $x, y \in T_eG$, $[x, y] = [X, Y]_e$ where X, Y are the left invariant vector fields defined by x, y, respectively.

A left invariant metric on G is a Riemannian metric such that L_a, the left multiplication by a, is an isometry for every $a \in G$. Every inner product on T_eG gives rise, by left translations, to a left invariant metric. Thus each n-dimensional Lie group possesses a $\frac{1}{2}n(n+1)$-dimensional family of left invariant metrics.

A Lie group equipped with a left invariant metric is a homogeneous Riemannian manifold where many geometric invariants can be computed at the Lie algebra level. In particular, the Levi-Civita connection ∇ associated to a left invariant metric g, when applied to left invariant vector fields, is given by:

$$2\langle \nabla_x y, z\rangle = \langle [x, y], z\rangle - \langle [y, z], x\rangle + \langle [z, x], y\rangle, \quad x, y, z \in \mathfrak{g}, \tag{2}$$

where $\langle\ ,\ \rangle$ is the inner product induced by g on $\mathfrak{g}$. Note that $\nabla g = 0$ implies that ∇_x is a skew-symmetric endomorphism of $\mathfrak{g}$ for any $x \in \mathfrak{g}$.

A left invariant p-form ω on G is a p-form such that $L_a^*\omega = \omega$ for all $a \in G$. We will consider left invariant p-forms ω on (G, g) satisfying (1). Since $\nabla\omega$, $d\omega$ and $d^*\omega$ are left invariant as well, we will study $\omega \in \Lambda^p\mathfrak{g}^*$ satisfying (1) for $x \in \mathfrak{g}$. We will call such a form a conformal Killing–Yano p-form on $\mathfrak{g}$.

For $p = 1$, we have the following result.

Proposition 2.1. *Let $\langle\cdot,\cdot\rangle$ be an inner product on a Lie algebra $\mathfrak{g}$. Then any CKY 1-form on $\mathfrak{g}$ is KY. Equivalently, if g is a left invariant metric on the Lie group G then any left invariant conformal vector field on G is a Killing vector field.*

Proof. For an arbitrary Riemannian manifold (M^n, g), a vector field ξ is conformal if $\mathcal{L}_\xi g = \varphi\, g$ with $\varphi \in C^\infty(M)$ given by $\varphi = -\frac{2}{n}d^*\eta$, where η is the 1-form dual to ξ. As a consequence, if g is a left invariant metric on G and ξ is left invariant, then φ is constant on G. Therefore, ξ satisfies the equation $\mathcal{L}_\xi\, g = c\, g$, $c \in \mathbb{R}$. Evaluating in $y, z \in \mathfrak{g}$ we obtain

$$-\langle [\xi, y], z\rangle - \langle y, [\xi, z]\rangle = c\langle y, z\rangle. \tag{3}$$

Setting $y = z = \xi$ in the above equation and assuming $\xi \neq 0$, we get $c = 0$, hence, ξ is a Killing vector field on (G, g). □

Corollary 2.2. *Let G be an n-dimensional Lie group with a left invariant metric g. Then any left invariant CKY $(n-1)$-form on (G, g) is closed.*

Proof. Let ω be a left invariant CKY $(n-1)$-form on (G, g), then $\eta := *\omega$ is a CKY 1-form, hence Proposition 2.1 implies that η is KY. Therefore, $d^*\eta = 0$, that is, $*d*\eta = 0$, which implies $d\omega = 0$, as claimed. □

From now on, we will consider left invariant CKY 2-forms on (G, g), where g is a left invariant metric. With respect to g, any left invariant 2-form ω on G gives rise to a skew-symmetric endomorphism T of the tangent bundle of G defined by $\omega(X, Y) = g(TX, Y)$, for any X, Y vector fields on G, which is also left invariant. We will still denote by T the corresponding endomorphism of $\mathfrak{g}$. When ω is a left invariant CKY 2-form on (G, g), the associated endomorphism T of $\mathfrak{g}$ will be called a conformal Killing–Yano tensor on $\mathfrak{g}$. In the special case when ω is a left invariant KY 2-form on (G, g), the associated endomorphism T will be called a Killing–Yano tensor.

According to [9, Proposition 2.7], a left invariant 2-form ω on (G, g) is CKY if and only if there exists a 1-form θ on G such that

$$(\nabla_X\omega)(Y, Z) + (\nabla_Y\omega)(X, Z) = 2g(X, Y)\theta(Z) - g(X, Z)\theta(Y) - g(Y, Z)\theta(X), \tag{4}$$

for any vector fields X, Y, Z on G. Furthermore, the 1-form θ is given by

$$\theta = -\frac{1}{n-1}d^*\omega, \tag{5}$$

where $n = \dim G$. It follows from (5) that θ is left invariant, hence $\theta \in \mathfrak{g}^*$. The following lemma gives an explicit formula for this 1-form.

Lemma 2.3. *If T is a conformal Killing–Yano tensor on the n-dimensional Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, with associated 1-form θ defined by (4), then θ is given by*

$$\theta(z) = -\frac{1}{n-1}\left(\frac{1}{2}\langle v_T, z\rangle - \operatorname{tr}\operatorname{ad}_{Tz}\right),$$

with $v_T = \sum_{i=1}^n [Te_i, e_i]$, for any orthonormal basis $\{e_i\}_{i=1,\dots,n}$ of $\mathfrak{g}$.

Proof. We recall that for any 2-form ω on a Riemannian manifold M, $d^*\omega$ can be computed as follows: if $\{U_i : i = 1, \dots, n\}$ is a local orthonormal basis of vector fields, then

$$d^*\omega(X) = -\sum_{i=1}^n(\nabla_{U_i}\omega)(U_i, X) \tag{6}$$

for any vector field X on M [19].

The lemma follows by applying (6) to the CKY 2-form ω on $\mathfrak{g}$ defined by T, using (2) for the computation of the Levi-Civita connection of $\langle\cdot,\cdot\rangle$. Indeed, let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\mathfrak{g}$ and $z \in \mathfrak{g}$, then

$$\begin{aligned}
d^*\omega(z) &= -\sum_{i=1}^n(\nabla_{e_i}\omega)(e_i, z)\\
&= \sum_{i=1}^n\left(\langle T\nabla_{e_i}e_i, z\rangle + \langle Te_i, \nabla_{e_i}z\rangle\right)\\
&= -\operatorname{tr}\operatorname{ad}_{Tz} + \frac{1}{2}\sum_{i=1}^n\left(\langle [e_i, z], Te_i\rangle - \langle [z, Te_i], e_i\rangle + \langle [Te_i, e_i], z\rangle\right)\\
&= -\operatorname{tr}\operatorname{ad}_{Tz} + \frac{1}{2}\langle v_T, z\rangle
\end{aligned}$$

since $\sum_{i=1}^n\left(\langle [e_i, z], Te_i\rangle - \langle [z, Te_i], e_i\rangle\right) = \operatorname{tr}(T\operatorname{ad}_z) - \operatorname{tr}(\operatorname{ad}_z T) = 0$. $\square$

Corollary 2.4. *Let T be a CKY tensor on the Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. Then T is KY if and only if*

$$\operatorname{tr}\operatorname{ad}_{Tz} = \frac{1}{2}\langle v_T, z\rangle, \quad z \in \mathfrak{g},$$

with v_T as in Lemma 2.3. In particular, if $\mathfrak{g}$ is unimodular, T is KY if and only if $v_T = 0$.

A left invariant 2-form ω on G will satisfy Eq. (4) if the associated left invariant endomorphism T of the tangent bundle of G satisfies the condition in the following proposition.

Proposition 2.5. *If $\mathfrak{g}$ is a Lie algebra with $\dim\mathfrak{g} = n$, then a skew-symmetric endomorphism T is CKY if and only if $\alpha_T = \beta_T$, where*

$$\alpha_T(x, y, z) = \langle [Tx, y] - [x, Ty], z\rangle + \langle -T[y, z] + [Ty, z] + 2[y, Tz], x\rangle + \langle -T[x, z] + [Tx, z] + 2[x, Tz], y\rangle,$$

and

$$\beta_T(x, y, z) = -2\left(2\langle x, y\rangle\theta(z) - \langle y, z\rangle\theta(x) - \langle z, x\rangle\theta(y)\right),$$

with θ as in Lemma 2.3. Moreover, the following identities hold for any $x, y, z \in \mathfrak{g}$:

$$\begin{aligned}
&\alpha_T(x, y, z) = \alpha_T(y, x, z), \qquad \beta_T(x, y, z) = \beta_T(y, x, z),\\
&\alpha_T(x, y, z) + \alpha_T(y, z, x) + \alpha_T(z, x, y) = \beta_T(x, y, z) + \beta_T(y, z, x) + \beta_T(z, x, y) = 0,
\end{aligned}$$

and $\alpha_T \equiv 0$ implies $\beta_T \equiv 0$.

Proof. By expanding the left-hand side of (4) using (2), we obtain

$$(\nabla_x\omega)(y,z) + (\nabla_y\omega)(x,z) = -\frac{1}{2}\alpha_T(x,y,z),$$

for any $x, y, z \in \mathfrak{g}$, and the proposition follows. □

Remark 1. According to [20] a skew-symmetric endomorphism T of $\mathfrak{g}$ is a Killing–Yano tensor if and only if $\alpha_T \equiv 0$.

3. Compact Lie groups with a bi-invariant metric

Compact Lie groups do carry bi-invariant metrics, that is, left invariant metrics which are also right invariant (see [21]). It is well known that the sphere S^3 with its standard metric has CKY 2-forms (see [9]). In the following result, we prove that if a compact Lie group G equipped with a fixed bi-invariant metric admits a left invariant CKY tensor which is not KY, then G is 3-dimensional. Moreover, its Lie algebra is isomorphic to $\mathfrak{su}(2)$.

Theorem 3.1. *Let G be a compact n-dimensional Lie group equipped with a bi-invariant metric g. If there exists a left invariant conformal Killing–Yano tensor T which is not Killing–Yano, then $n = 3$ and the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{su}(2)$.*

Proof. Since G is compact it is unimodular and according to Corollary 2.4 a left invariant conformal Killing–Yano tensor T which is not Killing–Yano has $v_T \neq 0$. Using that ad_x is skew-symmetric for any $x \in \mathfrak{g}$, it is easily verified that the expression for α_T in Proposition 2.5 can be simplified to

$$\alpha_T(x,y,z) = \langle [Tx,y] - [x,Ty], z\rangle, \quad x,y,z \in \mathfrak{g}. \tag{7}$$

We will prove first that G is semisimple. Let $\mathfrak{z}$ denote the center of $\mathfrak{g}$, and let us assume that $\mathfrak{z} \neq \{0\}$. Clearly, $\mathfrak{z}$ is orthogonal to the commutator ideal $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$. For $y \in \mathfrak{z}$, $\|y\| = 1$, we compute

$$\alpha_T(v_T,y,y) = 0, \qquad \beta_T(v_T,y,y) = -\frac{\|v_T\|^2}{n-1},$$

since $v_T \in \mathfrak{g}'$. As $\alpha_T = \beta_T$ (Proposition 2.5), we obtain $v_T = 0$, a contradiction. Therefore, $\mathfrak{z} = \{0\}$.

Next, we will prove that $Tv_T = 0$. Since T is skew-symmetric, this is equivalent to proving that $v_T \in (\mathrm{Im}\,T)^\perp$. There exists an orthonormal basis $\{x_1, y_1, \dots, x_r, y_r, z_1, \dots, z_s\}$ such that

$$Tx_j = a_j y_j, \qquad Ty_j = -a_j x_j, \qquad Tz_k = 0,$$

for some $a_j \in \mathbb{R}$, $1 \le j \le r$, $1 \le k \le s$. Let us compute

$$\alpha_T(x_j,x_j,y_j) = 2\langle [Tx_j,x_j], y_j\rangle = 2a_j\langle [y_j,x_j], y_j\rangle = 0,$$

while

$$\beta_T(x_j,x_j,y_j) = \frac{2}{n-1}\langle v_T, y_j\rangle.$$

Therefore, $\langle v_T, y_j\rangle = 0$ for $1 \le j \le r$. In a similar way, using $\alpha_T(y_j,y_j,x_j) = \beta_T(y_j,y_j,x_j)$ we obtain that $\langle v_T, x_j\rangle = 0$ for $1 \le j \le r$. Since $\mathrm{Im}\,T = \mathrm{span}\{x_j, y_j : j = 1, \dots, n\}$, we have that $v_T \in (\mathrm{Im}\,T)^\perp = \ker T$, and therefore, $Tv_T = 0$.

Let us suppose now that there exists $x \in \ker T$ with $\langle v_T, x\rangle = 0$. We compute

$$\alpha_T(x,x,v_T) = 0, \qquad \beta_T(x,x,v_T) = \frac{2}{n-1}\|x\|^2\|v_T\|^2.$$

Thus, $x = 0$ and consequently $\ker T = \mathrm{span}\{v_T\}$.

Now, for any $x, y \in \mathrm{Im}\,T$ we have that $\alpha_T(x,x,y) = 2\langle [Tx,x], y\rangle$, while $\beta_T(x,x,y) = 0$, and then $[Tx,x] \in \ker T$, so that there exists $\lambda : \mathrm{Im}\,T \to \mathbb{R}$ such that $[Tx,x] = \lambda(x)v_T$, $x \in \mathrm{Im}\,T$.

Now, for any $z \in \mathrm{Im}\,T$, we compute

$$\alpha_T(z,z,v_T) = 2\langle [Tz,z], v_T\rangle = 2\lambda(z)\|v_T\|^2,$$

and

$$\beta_T(z,z,v_T) = \frac{2}{n-1}\|z\|^2\|v_T\|^2,$$

so that $\lambda(z) = \frac{\|z\|^2}{n-1}$ and therefore $[Tz,z] = \frac{\|z\|^2}{n-1}v_T$.

Using this, we can prove that $\mathrm{ad}_{v_T} : \mathrm{Im}\,T \to \mathrm{Im}\,T$ is an isomorphism. Indeed, for any $x \in \mathrm{Im}\,T$, we have $\langle [v_T,x], v_T\rangle = 0$, so that $[v_T,x] \in \mathrm{Im}\,T$. If $[v_T,x] = 0$, then

$$0 = \langle [v_T,x], Tx\rangle = \langle v_T, [x,Tx]\rangle = -\frac{\|x\|^2}{n-1}\|v_T\|^2,$$

hence $x = 0$, proving the assertion.

It follows that any maximal abelian subalgebra containing v_T has dimension 1, and this implies that G has rank 1, therefore $\mathfrak{g}$ is isomorphic to $\mathfrak{su}(2)$. □

The following proposition was stated in [20]. We give below the proof, for the sake of completeness.

Proposition 3.2. *Let G be a Lie group with a bi-invariant Riemannian metric. If T is a left invariant skew-symmetric endomorphism of TG, then T is a Killing–Yano tensor if and only if $T|_{\mathfrak{g}'} = 0$.*

Proof. Using that ad_x is skew-symmetric for any $x \in \mathfrak{g}$, it is easily verified that the expression for α_T in Proposition 2.5 can be simplified to

$$\alpha_T(x, y, z) = \langle [Tx, y] - [x, Ty], z\rangle, \quad x, y, z \in \mathfrak{g}. \tag{8}$$

Therefore, T is Killing–Yano if and only if $[Tx, y] = [x, Ty]$ for all $x, y \in \mathfrak{g}$, that is to say, $\mathrm{ad}_{Tx} = \mathrm{ad}_x T$ for all $x \in \mathfrak{g}$. Since the metric is bi-invariant, we obtain easily that $\mathrm{ad}_x T = -T\,\mathrm{ad}_x$ for any $x \in \mathfrak{g}$.

Now we compute

$$\begin{aligned}[x, T[y, z]] &= -[x, [y, Tz]] \\ &= [y, [Tz, x]] + [Tz, [x, y]] \\ &= T([y, [z, x]] - [z, [x, y]]),\end{aligned}$$

but on the other hand

$$\begin{aligned}[x, T[y, z]] &= -T[x, [y, z]] \\ &= T([y, [z, x]] + [z, [x, y]]),\end{aligned}$$

and it follows that $T([z, [x, y]]) = 0$ for any $x, y, z \in \mathfrak{g}$. Since $\mathfrak{g}'$ is compact semisimple, it follows that $T|_{\mathfrak{g}'} = 0$.

The converse is easily verified. □

The following corollary is probably well-known:

Corollary 3.3. *Let G be a Lie group with a bi-invariant Riemannian metric g. If J is a left invariant almost complex structure on G such that (G, J, g) is nearly Kähler, then G is abelian.*

4. Two-step nilpotent case

We will consider next the case of a 2-step nilpotent Lie algebra $\mathfrak{g}$ equipped with an inner product $\langle\cdot,\cdot\rangle$. We recall that a Lie algebra $\mathfrak{g}$ is called 2-step nilpotent if $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] = 0$, or equivalently, the commutator ideal $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is contained in the center $\mathfrak{z}$ of $\mathfrak{g}$. Such a Lie algebra is unimodular and its associated simply connected Lie group is diffeomorphic to a Euclidean space $\mathbb{R}^n$.

If $\mathfrak{g}$ is a 2-step nilpotent Lie algebra equipped with an inner product $\langle\cdot,\cdot\rangle$, then there is an orthogonal decomposition $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $\mathfrak{v}$ its orthogonal complement. We recall from [22] that the Lie bracket on $\mathfrak{g}$ is completely determined by the linear operator $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ defined by

$$\langle j_z x, y\rangle = \langle z, [x, y]\rangle \quad \text{for } z \in \mathfrak{z},\ x, y \in \mathfrak{v}. \tag{9}$$

We will denote by $\mathfrak{h}_{2m+1}$ the $(2m + 1)$-dimensional Heisenberg Lie algebra given by $\mathfrak{h}_{2m+1} = \mathrm{span}\{e_1, \dots, e_{2m}, z_0\}$ with Lie bracket $[e_{2i-1}, e_{2i}] = z_0,\ 1 \le i \le m$. Clearly, $\mathfrak{h}_{2m+1}$ is 2-step nilpotent. It is well known that any nilpotent Lie algebra with 1-dimensional commutator ideal is isomorphic to $\mathfrak{h}_{2m+1} \times \mathbb{R}^{2k+1}$.

Theorem 4.1. *A 2-step nilpotent Lie algebra $\mathfrak{g}$ admitting an inner product with a CKY tensor T which is not KY is isomorphic to $\mathfrak{h}_{2m+1}$ and if ξ is a unit generator of the center of $\mathfrak{h}_{2m+1}$, then $T\xi = 0$ and $T|_{\mathfrak{v}} = \lambda j_\xi^{-1}$, for some $\lambda \neq 0$, where $\mathfrak{v} = \xi^\perp$.*

Proof. Let $\mathfrak{g}$ be an n-dimensional 2-step nilpotent Lie algebra with center $\mathfrak{z}$ and let $\mathfrak{v}$ denote the orthogonal complement of $\mathfrak{z}$. Since $\mathfrak{g}$ is 2-step nilpotent it is unimodular and by Corollary 2.4 $v_T \neq 0$ for any CKY tensor T which is not KY. Note that $v_T \in \mathfrak{z}$.

We will prove now that $\dim \mathfrak{z} = 1$. Otherwise, if $\dim \mathfrak{z} \ge 2$, let $x \in \mathfrak{z}$, with $\|x\|^2 = 1$, $\langle x, v_T\rangle = 0$, and then $\alpha_T(x, x, v_T) = 0$, whereas

$$\beta_T(x, x, v_T) = \frac{2}{n-1}\|x\|^2\|v_T\|^2.$$

Since $\alpha_T = \beta_T$, we obtain that $v_T = 0$, a contradiction.

Since $\dim \mathfrak{z} = 1$ and $\mathfrak{g}$ is 2-step nilpotent, it follows that $\mathfrak{g}$ is isomorphic to the Heisenberg Lie algebra $\mathfrak{h}_{2m+1}$. Next, we will characterize the CKY tensor T.

Let ξ denote a unit generator of $\mathfrak{z}$, and $v_T = c\xi$ for some $c \in \mathbb{R}$, $c \neq 0$. We will use the condition $\alpha_T = \beta_T$ in several cases, according to the decomposition $\mathfrak{g} = \mathbb{R}\xi \oplus \mathfrak{v}$.

(i) For any $z \in \mathfrak{v}$, we have $\alpha_T(\xi, \xi, z) = \beta_T(\xi, \xi, z)$, and this implies $\langle [T\xi, z], \xi\rangle = 0$. Since $\dim \mathfrak{z} = 1$, it follows that $T\xi = 0$.

(ii) For any $x, y \in \mathfrak{v}$, we have that $\alpha_T(x, y, \xi) = \beta_T(x, y, \xi)$ if and only if

$$\langle [Tx, y] - [x, Ty], \xi\rangle = \frac{2c}{n-1}\langle x, y\rangle;$$

(iii) For any $x, y \in \mathfrak{v}$, we have that $\alpha_T(\xi, x, y) = \beta_T(\xi, x, y)$ if and only if

$$\langle [Tx, y] + 2[x, Ty], \xi\rangle = -\frac{c}{n-1}\langle x, y\rangle.$$

From (ii) and (iii) we obtain

$$[Tx, y] = \frac{c}{n-1}\langle x, y\rangle\xi, \quad x, y \in \mathfrak{v}. \tag{10}$$

Moreover, it can be shown that Eq. (10) implies both (ii) and (iii). From the definition of the map $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ above, it is straightforward that (10) is equivalent to

$$j_\xi T = \frac{c}{n-1}\,\mathrm{Id},$$

and the proposition follows. □

Remark 2. It was proved in [20] that there are no non-trivial solutions to the KY equation on $\mathfrak{h}_{2m+1}$.

5. Dimension 3

In this section we determine all three-dimensional metric Lie algebras which carry non-zero CKY 2-forms. We will show, in Theorem 5.1, that there are 6 isomorphism classes of these Lie algebras. The non-abelian ones are given by the Lie brackets below with respect to a basis $\{f_1, f_2, f_3\}$, where the remaining Lie brackets are zero:

$$\begin{array}{rlll} \mathfrak{e}(2): & [f_3, f_1] = f_2, & [f_3, f_2] = -f_1, & \\ \mathfrak{su}(2): & [f_1, f_2] = f_3, & [f_2, f_3] = f_1, & [f_3, f_1] = f_2, \\ \mathfrak{sl}(2, \mathbb{R}): & [f_1, f_2] = -f_3, & [f_2, f_3] = f_1, & [f_3, f_1] = f_2, \\ \mathfrak{h}_3: & [f_1, f_2] = f_3, & & \\ \mathfrak{aff}(\mathbb{R}): & [f_1, f_2] = f_2. & & \end{array} \tag{11}$$

Note that $\mathfrak{e}(2)$ is the Lie algebra of the isometry group of the Euclidean plane, $\mathfrak{h}_3$ is the Heisenberg Lie algebra and $\mathfrak{aff}(\mathbb{R})$ is the Lie algebra of the group $\mathrm{Aff}(\mathbb{R})$ of affine motions of the real line.

We recall that in order to determine all CKY 2-forms, it is equivalent to find all non-zero CKY 1-forms, since the $*$ operator interchanges CKY 2-forms and CKY 1-forms [9]. This, in turn, is equivalent to the study of Killing vector fields, in view of Proposition 2.1, and this is what we will do next.

5.1. Killing vector fields on 3-dimensional metric Lie algebras

Let ξ be a non-zero Killing vector field on $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. There are two cases: $\xi \notin \mathfrak{z}$ or $\xi \in \mathfrak{z}$. If $\xi \notin \mathfrak{z}$ there exists an orthonormal basis e_1, e_2, e_3 with $e_3 = \frac{\xi}{\|\xi\|}$ and $a, b \in \mathbb{R}$, $a \neq 0$ such that the Lie brackets are given by

$$[e_3, e_1] = ae_2, \qquad [e_2, e_3] = ae_1, \qquad [e_1, e_2] = be_3. \tag{12}$$

The last equation follows by applying the Jacobi identity and using that $a \neq 0$. We point out that the parameters (a, b) and $(-a, -b)$ give rise to isometrically isomorphic Lie algebras and therefore we may assume $a > 0$.

If $b = 0$ then $de^3 = 0$ and therefore the Lie algebra is isomorphic to $\mathfrak{e}(2)$. Moreover, $d^*e^3 = 0$ since e_3 is Killing and it follows from (1) that e^3, hence also e_3, is parallel. Finally, the corresponding CKY 2-form $*e^3$ is parallel. If $b \neq 0$ then e^3 is a contact form, since $e^3 \wedge de^3 \neq 0$. According to [21], the Lie algebra is isomorphic to $\mathfrak{su}(2)$ when $b > 0$ and it is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ when $b < 0$.

If $\xi \in \mathfrak{z}$, let $e_3 = \frac{\xi}{\|\xi\|}$ and complete to an orthonormal basis e_1, e_2, e_3. The Lie bracket is given by

$$[e_1, e_2] = xe_1 + ye_2 + qe_3, \quad x, y, q \in \mathbb{R}. \tag{13}$$

If $x = y = 0$ and $q \neq 0$ then the Lie algebra is isomorphic to $\mathfrak{h}_3$ and e^3 is a contact form, since $de^3 = -qe^1 \wedge e^2$. Moreover, the parameters q and $-q$ give rise to isometrically isomorphic Lie algebras and therefore we may assume $q > 0$. If $x^2 + y^2 \neq 0$ then by making an orthogonal change if necessary, we may assume that $[e_1, e_2] = pe_2 + qe_3$, $p \neq 0$. These Lie algebras are isomorphic to $\mathfrak{aff}(\mathbb{R}) \times \mathbb{R}$. Note that if $q = 0$ then e^3 is a closed KY 1-form, hence it is parallel.

We analyze next the metrics in each case.

• $\mathfrak{g} \simeq \mathfrak{e}(2)$: This Lie algebra is obtained when $b = 0$. Since e_3 acts on the abelian ideal $\mathrm{span}_{\mathbb{R}}\{e_1, e_2\}$ by a skew-symmetric endomorphism, then the metric is flat (see [21, Theorem 1.5]).

• $\mathfrak{g} \simeq \mathfrak{su}(2)$: This Lie algebra is obtained when $b > 0$. Considering the basis $f_1 = \frac{1}{\sqrt{ab}}e_1,\ f_2 = \frac{1}{\sqrt{ab}}e_2,\ f_3 = \frac{1}{a}e_3$, we obtain the Lie brackets as in (11) and the inner product is

$$g_{a,b} = \begin{pmatrix} \frac{1}{ab} & & \\ & \frac{1}{ab} & \\ & & \frac{1}{a^2} \end{pmatrix} = \frac{1}{ab}\begin{pmatrix} 1 & & \\ & 1 & \\ & & \frac{b}{a} \end{pmatrix}. \tag{14}$$

Therefore any $g_{a,b}$ is homothetic to a Berger metric $g_t = \begin{pmatrix} 1 & & \\ & 1 & \\ & & t \end{pmatrix}$, with $t := \frac{b}{a} > 0$. It is well known that these Berger metrics are pairwise non isometric. For any $t \neq 1$, the only Killing vector field up to a multiple is f_3 and the only CKY 2-forms are $kf^1 \wedge f^2, k \in \mathbb{R}$. When $t = 1$, we obtain the bi-invariant metric on $SU(2)$. Note that in this case, any left invariant vector field is Killing and consequently any left invariant 2-form is CKY.

We note that the scalar curvature of g_t is given by $\rho_t = -\frac{1}{2}t + 2$.

• $\mathfrak{g} \simeq \mathfrak{sl}(2, \mathbb{R})$: This Lie algebra is obtained when $b < 0$. Considering the basis $f_1 = \frac{1}{\sqrt{-ab}}e_1,\ f_2 = \frac{1}{\sqrt{-ab}}e_2,\ f_3 = \frac{1}{a}e_3$, we obtain the Lie brackets as in (11) and the inner product is

$$g_{a,b} = \begin{pmatrix} -\frac{1}{ab} & & \\ & -\frac{1}{ab} & \\ & & \frac{1}{a^2} \end{pmatrix} = -\frac{1}{ab}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -\frac{b}{a} \end{pmatrix}. \tag{15}$$

Therefore any $g_{a,b}$ is homothetic to the metric $g_t = \begin{pmatrix} 1 & & \\ & 1 & \\ & & t \end{pmatrix}$, with $t := -\frac{b}{a} > 0$. It is easy to verify, comparing the eigenvalues of the Ricci operator, that these metrics are pairwise non isometric. For any $t > 0$, the only Killing vector field up to a multiple is f_3 and the only CKY 2-forms are $kf^1 \wedge f^2, k \in \mathbb{R}$.

We note that the scalar curvature of g_t is given by $\rho_t = -\frac{1}{2}t - 2$.

• $\mathfrak{g} \simeq \mathfrak{h}_3$: This Lie algebra is obtained when $x = y = 0$ and $q > 0$.

Considering the basis $f_1 = e_1,\ f_2 = e_2,\ f_3 = qe_3$, we obtain the Lie brackets as in (11) and the inner product is

$$g_q = \begin{pmatrix} 1 & & \\ & 1 & \\ & & q^2 \end{pmatrix}, \quad q > 0. \tag{16}$$

It turns out that g_q is isometric, up to scaling, to g_1 (see [23]) and the only Killing vector field up to a multiple is f_3 and the only CKY 2-forms are $kf^1 \wedge f^2, k \in \mathbb{R}$.

Since any inner product on $\mathfrak{h}_3$ is isometric, up to scaling, to g_1, it follows that any left invariant metric on the Heisenberg Lie group H_3 admits non-zero left invariant CKY tensors.

• $\mathfrak{g} \simeq \mathfrak{aff}(\mathbb{R}) \times \mathbb{R}$: Recall that the Lie bracket is given by

$$[e_1, e_2] = pe_2 + qe_3, \qquad [e_1, e_3] = [e_2, e_3] = 0, \quad p \neq 0,\ q \in \mathbb{R}.$$

Two of these metric Lie algebras are isometrically isomorphic if and only if $p' = \pm p$ and $q' = \pm q$, therefore we may assume $p > 0,\ q \geq 0$.

Considering the basis $f_1 = \frac{1}{p}e_1,\ f_2 = e_2 + \frac{q}{p}e_3,\ f_3 = e_3$, we obtain $[f_1, f_2] = f_2$ and the inner product with respect to this basis is given by

$$g_{p,q} = \begin{pmatrix} \frac{1}{p^2} & & \\ & \frac{p^2+q^2}{p^2} & \frac{q}{p} \\ & \frac{q}{p} & 1 \end{pmatrix}, \quad p > 0,\ q \geq 0. \tag{17}$$

It follows from [21] that any inner product on $\mathfrak{aff}(\mathbb{R}) \times \mathbb{R}$ is isometric to $g_{p,q}$ for some $p > 0,\ q \geq 0$, and therefore any left invariant metric on $\mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$ admits non-zero left invariant CKY tensors.

Moreover, it can be shown that $p^2 g_{p,q}$ is isometric to $g_{1,t}$, with $t := \frac{q}{p}$. Therefore each $g_{p,q}$ is isometric up to scaling to

$$g_{1,t} = \begin{pmatrix} 1 & & \\ & 1+t^2 & t \\ & t & 1 \end{pmatrix}, \quad t > 0. \tag{18}$$

For any $t > 0$, the only Killing vector field up to a multiple is f_3 and the only CKY 2-forms are $kf^1 \wedge f^2$, $k \in \mathbb{R}$. When $t = 0$ these CKY 2-forms are parallel, while for $t \neq 0$ they are CKY and non-coclosed.

We point out that the scalar curvature of $g_{1,t}$ is $\rho_{1,t} = -2 - \frac{1}{2}t^2$.

To summarize the discussion above, we state the following result. We will say that a CKY form is *strict* if it is not KY, that is, it is not co-closed.

Theorem 5.1. *The 3-dimensional non-abelian Lie algebras admitting an inner product with non-zero CKY 2-forms are* $\mathfrak{e}(2)$, $\mathfrak{su}(2)$, $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{h}_3$ *and* $\mathfrak{aff}(\mathbb{R}) \times \mathbb{R}$. *Furthermore,*

(1) *Any left invariant metric on* $E(2)$ *admitting CKY 2-forms is flat and any of these 2-forms is parallel.*
(2) *On* $SU(2)$ *and* $SL(2,\mathbb{R})$ *there is a one-parameter family of left invariant metrics, pairwise non-isometric up to scaling, that admit CKY 2-forms and any of these 2-forms is strict.*
(3) *Any left invariant metric on the Lie group* H_3 *admits CKY 2-forms and any of these 2-forms is strict.*
(4) *Any left invariant metric on the Lie group* $\mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$ *admits CKY 2-forms. Up to scaling, each of these metrics is isometric to* $g_{1,t}$ *as in* (18) *for one and only one* $t \geq 0$. *For* $t = 0$ *the CKY 2-forms are all parallel, while for* $t > 0$, *the CKY 2-forms are all strict.*

Corollary 5.2. *A left invariant metric* g *on* $SU(2)$ *admits non trivial left invariant CKY tensors if and only if* g *is homothetic to a Berger metric.*

Remark 3. We point out that, according to Corollary 2.2, all the CKY 2-forms obtained in Theorem 5.1 are closed. Furthermore, when these CKY 2-forms are strict, their Hodge duals are contact forms.

Left invariant metrics on H_3 and $\mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$ from Theorem 5.1(iii) and (iv) define homogeneous Riemannian metrics on $\mathbb{R}^3$. We will show next that a metric on $\mathbb{R}^3$ arising from H_3 cannot be isometric, up to scaling, to a metric arising from $\mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$ by studying the eigenvalues of the Ricci operator.

Proposition 5.3. *A homogeneous Riemannian metric on* $\mathbb{R}^3$ *invariant by the Lie group* H_3 *is not isometric, up to scaling, to a homogeneous Riemannian metric on* $\mathbb{R}^3$ *invariant by the Lie group* $\mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$.

Proof. A homogeneous Riemannian metric on $\mathbb{R}^3$ invariant by the Lie group H_3 (resp. $\mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$) amounts to a left invariant metric on H_3 (resp. $\mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$). Moreover, any left invariant metric on H_3 is isometric, up to scaling, to g_1 as in (16), whereas any left invariant metric on $\mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$ is isometric, up to scaling, to $g_{1,t}$ as in (18).

The eigenvalues of the Ricci operator corresponding to g_1 are $-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}$, while the eigenvalues of the Ricci operator corresponding to $g_{1,t}$ are $-1-\frac{t^2}{2}, -1-\frac{t^2}{2}, \frac{t^2}{2}$. If $g_{1,t}$ were isometric up to scaling to g_1, then we should have $|-1-\frac{t^2}{2}| = \frac{t^2}{2}$, and this is not possible. □

5.2. Coordinate expression of the metrics

In order to obtain the coordinate expression of the left invariant metrics we give a basis of left invariant 1-forms in local coordinates on each Lie group.

• $\mathfrak{su}(2)$: using the Euler angles (ψ, θ, ϕ) as a local coordinate system on $SU(2)$, a basis of left invariant 1-forms is given by

$$\sigma_1 = \sin\theta \sin\psi \, d\phi + \cos\psi \, d\theta, \qquad \sigma_2 = \sin\theta\cos\psi \, d\phi - \sin\psi \, d\theta, \qquad \sigma_3 = d\psi + \cos\theta \, d\phi.$$

There are two cases:

(i) $t > 0,\ t \neq 1$. The left invariant Berger metric g_t can be written as

$$g_t = \sigma_1^2 + \sigma_2^2 + t\sigma_3^2,$$

and any left invariant CKY 2-form on $SU(2)$ with respect to this metric is a constant multiple of $\omega = \sigma_1 \wedge \sigma_2$.

(ii) $t = 1$. The left invariant metric g_1 can be written as

$$g_1 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2,$$

and any left invariant 2-form on $SU(2)$ is CKY with respect to this bi-invariant metric.

• $\mathfrak{sl}(2,\mathbb{R})$: using the diffeomorphism $SL(2,\mathbb{R})\simeq S^1\times\mathbb{R}^2$ corresponding to the Iwasawa decomposition, we may consider a local coordinate system

$$(\theta,r,s)\longrightarrow\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}e^r & 0\\ 0 & e^{-r}\end{pmatrix}\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}\in SL(2,\mathbb{R}).$$

A basis of left invariant 1-forms is given by

$$\sigma_1=dr-se^{2r}\,d\theta,\qquad \sigma_2=2s\,dr+ds-(s^2e^{2r}+e^{-2r})\,d\theta,\qquad \sigma_3=e^{2r}\,d\theta,$$

and the left invariant metric g_t can be written as

$$g_t=4\sigma_1^2+(\sigma_2+\sigma_3)^2+t(-\sigma_2+\sigma_3)^2.$$

Any left invariant CKY 2-form on $SL(2,\mathbb{R})$ with respect to this metric is a constant multiple of $\omega=\sigma_1\wedge(\sigma_2+\sigma_3)$.

• $\mathfrak{h}_3$: Let $H_3=\left\{\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}:x,y,z\in\mathbb{R}\right\}$ be the simply connected Lie group with Lie algebra $\mathfrak{h}_3$. A basis of left invariant 1-forms on H_3 is

$$\sigma_1=dx,\qquad \sigma_2=dy,\qquad \sigma_3=dz-x\,dy,$$

and the left invariant metric g_1 can be written as

$$g_1=\sigma_1^2+\sigma_2^2+\sigma_3^2.$$

Moreover, any left invariant CKY 2-form on H_3 is a constant multiple of $\omega=\sigma_1\wedge\sigma_2=dx\wedge dy$.

• $\mathfrak{aff}(\mathbb{R})\times\mathbb{R}$: Let $\mathrm{Aff}(\mathbb{R})\times\mathbb{R}=\left\{\begin{pmatrix}e^x & y & 0\\ 0 & 1 & 0\\ 0 & 0 & e^z\end{pmatrix}:x,y,z\in\mathbb{R}\right\}$. Then (x,y,z) define global coordinates on $\mathrm{Aff}(\mathbb{R})\times\mathbb{R}$, and a basis of left invariant 1-forms is given by

$$\sigma_1=dx,\qquad \sigma_2=e^{-x}\,dy,\qquad \sigma_3=dz.$$

The left invariant metric $g_{1,t}$ is

$$g_{1,t}=\sigma_1^2+\sigma_2^2+(t\sigma_2+\sigma_3)^2,$$

and any left invariant CKY 2-form is a constant multiple of $\omega=\sigma_1\wedge\sigma_2=e^{-x}dx\wedge dy$.

5.3. Sasakian structures

Among the CKY tensors found in Theorem 5.1, we determine next which ones correspond to a Sasakian structure (compare [24]). Recall that a Riemannian manifold (M,g) is called Sasakian if there exists a unit length Killing vector field ψ such that for all X vector field on M,

$$\nabla_X d\psi^*=-2X^*\wedge\psi^* \tag{19}$$

where $\psi^*(Y)=g(\psi,Y)$. It was proved in [9] that $d\psi^*$ is a CKY 2-form. Note that, if g is a left invariant metric on a Lie group G and ψ is left invariant, it suffices to verify (19) for left invariant vector fields X in order to prove that (G,g) is Sasakian.

Corollary 5.4. *On H_3 there is a unique left invariant Sasakian metric. On $SU(2)$, $SL(2,\mathbb{R})$ and $\mathrm{Aff}(\mathbb{R})\times\mathbb{R}$ there exists a one-parameter family of pairwise non-isometric left invariant Sasakian metrics.*

Proof. Let $(\mathfrak{g},\langle\cdot,\cdot\rangle)$ be a three dimensional metric Lie algebra and $\psi\in\mathfrak{g}$ a unit Killing vector field. We argue as in Section 5.1.

If $\psi\notin\mathfrak{z}$, there exists an orthonormal basis e_1,e_2,e_3 with $e_3=\psi$ such that the Lie brackets are given as in (12) for some $a,b\in\mathbb{R}$, $a>0$.

If $b=0$ then $\mathfrak{g}\simeq\mathfrak{e}(2)$ and the metric is flat, hence it is not Sasakian.

If $b\neq 0$, using that $\omega=e^1\wedge e^2$ satisfies $de^3=-b\omega$, $d\omega=0$ and $d^*\omega=-be^3$, it is easy to verify that Eq. (19) holds if and only if $b=\pm 2$.

When $b=2$ we obtain a family g_a of left invariant Sasakian metrics on $SU(2)$, for $a>0$, such that the scalar curvature is given by $\rho_a=4a-2$. Observe that in the particular case when $a=b$, that is, the metric is bi-invariant, any unit vector field is Killing and defines a left invariant Sasakian structure on $SU(2)$. Identifying $SU(2)$ with S^3, this metric corresponds to the round metric on the 3-sphere.

When $b=-2$, we obtain a family g_a of left invariant Sasakian metrics on $SL(2,\mathbb{R})$, for $a>0$, such that the scalar curvature is given by $\rho_a=-4a-2$.

If $\psi\in\mathfrak{z}$, there exists an orthonormal basis e_1,e_2,e_3 with $e_3=\psi$ such that the non-zero Lie brackets are given either by

$$[e_1,e_2]=qe_3,\quad q>0,$$

or

$$[e_1, e_2] = pe_2 + qe_3, \quad p > 0, \; q \geq 0.$$

In both cases, we have that $\omega = e^1 \wedge e^2$ satisfies $de^3 = -q\omega$, $d\omega = 0$ and $d^*\omega = -qe^3$ and it can be shown that Eq. (19) holds if and only if $q = 2$. Therefore, in the first case, we obtain one left invariant Sasakian metric on H_3 up to isomorphism, with scalar curvature $\rho = -2$. In the second case there is a curve g_p, $p > 0$, of pairwise non-isometric left invariant Sasakian metrics on $\mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$ with corresponding scalar curvature $\rho_p = -2p^2 - 2$. $\square$

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