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On the recognition of neighborhood inclusion posets

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Abstract

Let G be a simple graph. When we order the different closed neighborhoods of G by inclusion, the resulting poset is called the neighborhood inclusion poset. In this paper, we show that the problem of determining whether a poset is a neighborhood inclusion poset is NP-complete. We also apply this result to prove the NP-completeness of another problem about clique trees of chordal graphs and compatible trees of dually chordal graphs.

Keywords: Poset, neighborhood inclusion, chordal graph, clique tree, dually chordal graph, compatible tree

1 Introduction

A *poset* P is a pair $(X, \leq)$ where X is a set and $\leq$ is a partial order (a reflexive, antisymmetric and transitive relation) defined on X. Every poset can be modelled through the inclusion of sets. In fact, if we make every $x \in X$ correspond to the set $D_x = \{y \in X : \ y \leq x\}$, then it holds that $u \leq v$ if and only if $D_u \subseteq D_v$, for every pair u, v of elements of X.

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Many special posets have been studied, like for example the posets that can be modelled through the inclusion of intervals of the real line [6,7]. In this paper, we will consider the inclusion order of the neighborhoods of a graph.

Let G be a graph. The closed neighborhood $N[v]$ of a vertex v of G consists of v and of all the vertices adjacent to it. Denote the set of all the different closed neighborhoods of G by $\mathcal{N}_G$. The *neighborhood inclusion poset* of G is defined to be the pair $(\mathcal{N}_G, \subseteq)$ consisting of the different closed neighborhoods of G ordered by inclusion. We say that a poset P is a *neighborhood inclusion* poset if there exists a graph G such that $(\mathcal{N}_G, \subseteq)$ is isomorphic to P.

Fig. 1. A graph and the Hasse diagram of its neighborhood inclusion poset.

The structure and recognition of neighborhood inclusion orders in graphs has already been considered previously, but restricted to some particular classes of graphs [1]. In this paper, we consider the problem of determining whether a given poset is the neighborhood inclusion poset of some graph and we show that it is NP-complete by establishing a connection with the Set Basis problem.

This proof was motivated by another problem about chordal and dually chordal graphs, i.e., clique graphs of some chordal graph. Both classes can be characterized by the existence of tree representations. A *clique tree* of a graph G is a tree T whose vertex set is the set $\mathcal{C}(G)$ of maximal cliques of G and such that, for every $v \in V(G)$, the set $\mathcal{C}_v$ of maximal cliques of G that contain v induces a subtree in T. A graph is chordal if and only if it has a clique tree [8]. A *compatible tree* of a graph G is a tree T that has the same vertices as G and such that every maximal clique and every closed neighborhood of G induces a subtree in G. A graph is dually chordal if and only if it has a compatible tree [2].

Given a family $\mathcal{T}$ of trees on a vertex set V, the problem of determining whether $\mathcal{T}$ is the family of all clique trees of some chordal graph G can be solved in polynomial time [4]. We also know that every family of compatible trees of a dually chordal graph is also the family of clique trees of some chordal graph [5]. However, it is shown in [3] that not every family of clique trees of a chordal graph is the family of compatible trees of some dually chordal graph, leaving it an open problem to determine the complexity of recognizing a family of compatible trees of a dually chordal graph.

We show in the last part of the paper that the problem of determining whether the family of clique trees of a given chordal graph is also the family of compatible trees of a dually chordal graph is NP-complete. For that purpose, we use the techniques of [5] to reformulate each instance of the problem as an instance of the Neighborhood Inclusion Poset Recognition problem.

2 The Neighborhood Inclusion Poset Recognition problem is NP-complete

As mentioned in the Introduction, in order to prove that the Neighborhood Inclusion Poset Recognition problem is NP-complete, we relate it to the Set Basis problem, which was proved to be NP-complete by Stockmeyer in 1975 [9]. An instance of the Set Basis Problem consists of a collection $\mathcal{C}$ of subsets of a finite set S and a positive integer $k \leq |\mathcal{C}|$, the question being whether there exists a collection $\mathcal{B}$ of subsets of S with $|\mathcal{B}| = k$ such that, for every $C \in \mathcal{C}$, there is a subcollection of $\mathcal{B}$ whose union is exactly C. Let us call a collection like $\mathcal{B}$ a *union basis*. If we replace unions by intersections, we get a new NP-complete problem whose formulation is easier to relate to the Neighborhood Inclusion Poset Recognition problem:

`INTERSECTION BASIS PROBLEM`
`INSTANCE:` a collection $\mathcal{C}$ of subsets of a finite set S and an integer $k \leq |\mathcal{C}|$.
`QUESTION:` is there a collection $\mathcal{B}$ of subsets of S with $|\mathcal{B}| = k$ such that, for every $C \in \mathcal{C}$, there is a subcollection of $\mathcal{B}$ whose intersection is exactly C?

Let us call a collection like $\mathcal{B}$ an *intersection basis*. The connection between both problems lies in the fact that a collection $\mathcal{C} = \{C_1, C_2, ..., C_n\}$ has union basis $\{B_1, B_2, ..., B_k\}$ if and only if the collection $\widehat{\mathcal{C}} = \{\overline{C_1}, \overline{C_2}, ..., \overline{C_n}\}$ has intersection basis $\{\overline{B_1}, \overline{B_2}, ..., \overline{B_k}\}$.

Let G be a graph and $P = (X, \leq)$ be its neighborhood inclusion poset. If G does not have twin vertices, we can assume that $X = V(G)$, since the correspondence between the vertices and their neighborhoods is one-to-one. Let v be any vertex of G and let $N[v] = \{w_1, w_2, ..., w_n\}$. Then it holds that $\cap_{i=1}^{n} N[w_i] = \{w \in V(G) : \; v \leq w\}$. Hence, the closed neighborhoods of the vertices of G form an intersection basis for the sets $\{w \in V(G) : \; v \leq w\}$, $v \in V(G)$. That being said, we state the main result:

Theorem 2.1 *The Neighborhood Inclusion Poset Recognition Problem is NP-complete.*

Proof sketch. We start by considering an instance of the Intersection Basis problem consisting of a collection $\mathcal{C} = \{C_1, C_2, ..., C_n\}$ of subsets of a set $S = \{s_1, s_2, ..., s_m\}$ and a number $k \leq n$. Let us assume that no C_i is empty, and that $C_i \neq \{x\}$ whenever x is an element that is in every set of $\mathcal{C}$. This assumption is possible because these cases can be easily reduced to other instances of the problem. Let $x_1, x_2, ..., x_n$ be new elements that are not in S and let $S' = S \cup \{x_1, x_2, ..., x_n\}$. For every i between 1 and n, let $C'_i = C_i \cup \{x_j : \ C_j \subseteq C_i\}$ and use these sets to obtain a new collection $\mathcal{C}'$. It is not difficult to verify that the solution to the intersection basis problem is the same when S is replaced by S', $\mathcal{C}$ is replaced by $\mathcal{C}'$ and k stays the same.

Define now the relation $\vdash$ on S' such that two elements v and w satisfy $v \vdash w$ if and only if every set in $\mathcal{C}'$ that contains v also contains w. This relation is reflexive and transitive, but not necessarily antisymmetric. In order to deal with this situation, we partition S' so that two elements v and w are in the same set of the partition if and only if $v \vdash w$ and $w \vdash v$. For every set of the partition, take a single element of it to obtain a subset S'' of S'. It follows that $\vdash$ induces a partial order on S''.

Let $\widetilde{S} = S'' \cup \{u, a, y_1, y_2, ..., y_{k+1}, z_1, z_2, ..., z_{k+2}\}$, where u is in S'' if and only if u is a maximal element of S'' when ordered by $\vdash$ and it is unique. Every other element of the second set of the union is not in S''. We define an order $\leq$ on $\widetilde{S}$ according to the following rules: (1) when restricted to S'', we have $v \leq w$ if and only if $v \vdash w$, (2) $v \leq u$ for every $v \in \widetilde{S}$, (3) $a \leq v$ for every $v \in S''$, (4) $a \leq y_1 \leq y_2 \leq ... \leq y_{k+1}$, (5) $z_{k+2} \leq z_i$ for every $1 \leq i \leq k+1$, and (6) $v \leq y_{k+1}$ for every $v \in S'' \setminus \{u\}$.

In order to complete the proof, it is necessary to demonstrate that the answer to the intersection basis problem is affirmative if and only if $(\widetilde{S}, \leq)$ is a neighborhood inclusion poset. □

3 An application to chordal and dually chordal graphs

In order to establish a connection between the Neighborhood Inclusion Poset Recognition problem and clique trees of chordal graphs and compatible trees of dually chordal graphs, we consider a poset $P = (X, \leq)$ as a starting point.

We now define a poset P' as follows: if P has a unique maximal element, then we set $P' = P$. Otherwise, we set $P' = (X \cup \{u\}, \preceq)$, where u is an element that is not in X and $\preceq$ is an extension of $\leq$ such that $x \preceq u$ for every $x \in X$. The following lemma is easy to prove.

Lemma 3.1 *P is a neighborhood inclusion poset if and only if P' is a neigh-*

borhood inclusion poset. Moreover, if P' is a neighborhood inclusion poset with more than one element, then $X \backslash \{u\}$ has more than one maximal element.

Recall that, given a family $\mathcal{F}$ of sets, its *intersection graph* $L(\mathcal{F})$ has vertex set $\mathcal{F}$ and, for every two different sets F_1 and F_2 in $\mathcal{F}$, F_1 is adjacent to F_2 in $L(\mathcal{F})$ if and only if $F_1 \cap F_2 \neq \emptyset$.

Now we introduce some notation from [5]. Let G be a chordal graph and $\mathcal{C}(G)$ be the set of its maximal cliques. We denote by $\mathcal{SC}(G)$ the family of subsets D of $\mathcal{C}(G)$ such that the induced subgraph $T[D]$ is a subtree for every clique tree T of G. The *basis* of $\mathcal{SC}(G)$ is the minimum subfamily $\mathcal{B}$ of $\mathcal{SC}(G)$ satisfying that every $D \in \mathcal{SC}(G)$ can be expressed as $D = \bigcup_{i=1}^{n} B_i$, where $\{B_1, B_2, ..., B_n\}$ is a subfamily of $\mathcal{B}$ such that $L(\{B_1, B_2, ..., B_n\})$ is a connected graph. The basis is always unique [5]. The choice of the term basis here is independent from the Set Basis Problem, and was made at a time when the connection between the two was not known.

Let G be dually chordal. We denote by $\mathcal{SDC}(G)$ the family of subsets A of $V(G)$ such that $T[A]$ is a subtree for every compatible tree T of G. The basis of $\mathcal{SDC}(G)$ can be defined like in the previous paragraph. Then we have:

Proposition 3.2 *Let P' be the poset defined above and let u be its unique maximal element. Let P' be such that $X \setminus \{u\}$ has more than one maximal element. For every $x \neq u$, let $U_x = \{y \in X \cup \{u\} : \ x \preceq y\}$. Let G be the intersection graph of $\mathcal{F} = \{U_x : \ x \in X \setminus \{u\}\} \cup \{\{x\} : \ x \in X \setminus \{u\}\}$. Then G is chordal and the basis of $\mathcal{SC}(G)$ is $\{U_x : \ x \in X \setminus \{u\}\}$.*

It is important to note that we are making every $x \in X \cup \{u\}$ correspond to the clique of G consisting of all the sets in $\mathcal{F}$ that contain x.

If a dually chordal graph G' is such that its compatible trees are exactly the clique trees of the graph G in the previous proposition, then we have by definition that $\mathcal{SC}(G) = \mathcal{SDC}(G')$, so their bases are the same. Furthermore:

Proposition 3.3 *Let G' be a dually chordal graph with universal vertex u. For every vertex v different from u, define $B_v = \{w \in V(G') : \ N[v] \subseteq N[w]\}$. Then the basis of $\mathcal{SDC}(G')$ consists of the sets B_v.*

Proof sketch. Since u is a universal vertex, the star centered at u is a compatible tree of G'. It is known [5] that, given a compatible tree T, the elements of the basis can be obtained by finding, for each edge vw of T, the set $\bigcap_{x \in N[v] \cap N[w]} N[x]$. We reach the desired conclusion by combining both results. $\square$

Corollary 3.4 *Let P' and G be like in Proposition 3.2. Then P' is a Neighborhood Inclusion Poset if and only if the family of clique trees of G is also* *the family of compatible trees of a dually chordal graph.*

Proof sketch. If P' is the neighborhood inclusion poset of a graph G', then G' is dually chordal because it has a universal vertex. By Propositions 3.2 and 3.3, the bases of $\mathcal{SC}(G)$ and $\mathcal{SDC}(G')$ are the same, and the clique trees of G are exactly the compatible trees of G'. Conversely, if the clique trees of G are exactly the compatible trees of a graph G' then we use the same propositions to conclude that the neighborhood inclusion poset of G' is equal to P'. □

This reduction allows to conclude the main result of the section.

Theorem 3.5 *The problem of determining whether the family of clique trees of a chordal graph G is also the family of compatible trees of some dually chordal graph is NP-complete.*

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