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ON THE TOPOLOGICAL ENTROPY OF THE SPECTRA OF LIMIT POINTS OF MULTIERGODIC AVERAGE SEQUENCES

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Abstract

Let (X, d) be a compact metric space and $f : X \to X$, if X^r is the product of r-copies of X, $r \geq 1$ and $\Phi : X^r \to \mathbf{R}$. Then the V-statistics with kernel Φ are defined as

$$V_{\Phi}(n, x) = \frac{1}{n^r} \sum_{0 \leq i_1, \ldots, i_r \leq n-1} \Phi(f^{i_1}(x), \ldots, f^{i_r}(x)).$$

In this article, we are mainly interested in the multifractal decomposition $E_{\Phi, I} = \{x : \mathcal{A}[\Phi](x) = I\}$, where $\mathcal{A}[\Phi](x)$ are limit points of the sequence $n \to V_{\Phi}(n, x)$ and I is a closed interval. We obtain a variational expression for the topological entropy of $E_{\Phi, I}$.

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I. Introduction

The multiple ergodic averages are being considered an interesting field and motive of active research. In his seminal work, Furstenberg [7] studied ergodic averages in a measure-preserving probability space $(X, \mathcal{B}, \mu, f)$ of the form

$$\frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap f^{n} A \cap \cdots \cap f^{kn} A), \tag{1}$$

where $A \in \mathcal{B}$ and $j \in \mathbf{N}$. The main result of that work states that if $\mu(A) > 0$, then $\liminf_{N \to \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap f^{n} A \cap \cdots \cap f^{jn} A) > 0$, which is a dynamical version of the Szemerédi theorem in combinatorial number theory. From this, were studied multiergodic averages not only to make an interplay between Number Theory and Dynamical Systems, but also to be applied in other fields.

In particular, in the area of multifractal analysis problems on convergence of multiple ergodic averages motivate Fan et al. [5] to study the multifractal spectra of V-statistics. Let us consider a topological dynamical system (X, f) with X a compact metric space and f a continuous map. Let $X^{r} = X \times \cdots \times X$ be the product of r-copies of X with $r \geq 1$, if $\Phi : X^{r} \to \mathbf{R}$ is a continuous map, then let

$$V_{\Phi}(n, x) = \frac{1}{n^{r}} \sum_{1 \leq i_1, \ldots, i_r \leq n} \Phi(f^{i_1}(x), \ldots, f^{i_r}(x)). \tag{2}$$

These averages are called the *V-statistics of order r with kernel* Φ. The multifractal decomposition for the spectra of V-statistics is

$$E_{\Phi}(\alpha) = \{x : \lim_{n \to \infty} V_{\Phi}(n, x) = \alpha\}.$$

In [5], the authors established the following variational principle:

$$h_{top}(E_{\Phi}(\alpha)) = \sup\left\{h_{\mu}(f) : \int \Phi d\mu^{\otimes r} = \alpha\right\},$$

where h_{top} is the topological entropy for non-compacts nor invariant sets and $h_{\mu}(f)$ is the measure-theoretic entropy of μ. Here $\mu^{\otimes r}$ means $\mu \times \cdots \times \mu$, r-times. The condition imposed on the dynamics is specification. This principle is a generalization of that established by Takens and Verbitski for $r = 1$ [9].

In a previous article [8], we have analyzed the *irregular part*, or *historic set*, of the spectrum of V-statistics. This is the set of points x for which $\lim_{n\to\infty} V_{\Phi}(n, x)$ does not exist. We proved that the irregular part of the spectrum of multiple ergodic averages, or V-statistics, has the same topological entropy as the whole space X. The condition imposed on the dynamics was specification, under this condition, the system becomes saturated.

In this work, we shall consider the following multifractal decomposition spectra:

$$E_{\Phi, I} = \{x : \mathcal{A}[\Phi](x) = I\}, \tag{3}$$

where $\mathcal{A}[\Phi](x)$ is the set of limit points of the sequence $n \to V_{\Phi}(n, x)$ and I is a closed interval. The study of these spectra may give another point of view in the study of the dimension of the irregular set. In [1], Barreira et al. proved, for the single case $r = 1$ and for the special case of full shifts, that the spectra of limit points are residual, i.e., contain a dense G_{δ} set. In this article, following [10], a G_{δ} set is constructed. The conditions imposed on the dynamical system will be the *Almost Product Property* (*APP*), which is weaker than specification and the *uniform separation property*. The respective definitions will be given in the next section. The result to be proved is:

Theorem. *Let* (X, f) *be a dynamical system with the Almost Product Property* (*APP*) *and the uniform separation property*. *Let* $\Phi \in C(X^r)$, *for* I *a closed real interval*. *Then holds*

$$h_{top}(E_{\Phi, I}) = \inf\left\{h_\mu(f) : \int \Phi d\mu^{\otimes r} \in I\right\}.$$

This theorem is a generalization of the main result of [10] to *V*-statistics.

II. Preliminary Definitions

Firstly, let us recall the Bowen definition of topological entropy of sets. Let $f : X \to X$ with X a compact metric space, for $n \geq 1$ the dynamical metric, or Bowen metric, be $d_n(x, y) = \max\{d(f^i(x), f^i(y)) : i = 0, 1, \ldots, n-1\}$. We denote by $B_{n,\varepsilon}(x)$ the ball of centre x and radius ε in the metric d_n. Let $Z \subset X$ and let $\mathcal{C}(n, \varepsilon, Z)$ be the collection of finite or countable coverings of the set Z by balls $B_{m,\varepsilon}(x)$ with $m \geq n$. Let

$$M(Z, s, n, \varepsilon) = \inf_{\mathcal{B} \in \mathcal{C}(n, \varepsilon, Z)} \sum_{B_{m,\varepsilon}(x) \in \mathcal{B}} \exp(-sm)$$

and set

$$M(Z, s, \varepsilon) = \lim_{n \to \infty} M(Z, s, n, \varepsilon).$$

There is a unique number $\bar{s}$ such that $M(Z, s, \varepsilon)$ jumps from $+\infty$ to 0. Let

$$H(Z, \varepsilon) = \bar{s} = \sup\{s : M(Z, s, \varepsilon) = +\infty\} = \inf\{s : M(Z, s, \varepsilon) = 0\}$$

and

$$h_{top}(Z) = \lim_{\varepsilon \to 0} H(Z, \varepsilon). \tag{4}$$

The number $h_{top}(Z)$ is the *topological entropy* of Z.

The dynamical ball for $f : X \to X$ is

$$B_{n,\varepsilon}(x) = \{y : \max\{d(f^i(x), f^i(y)) : i = 0, 1, ..., n-1\} < \varepsilon\}.$$

Let $g : \mathbf{N} \to \mathbf{N}$ be a non-decreasing, non-bounded function such that

$$\frac{g(n)}{n} < 1 \quad \text{and} \quad \frac{g(n)}{n} \to 0 \text{ as } n \to \infty.$$

The dynamic ball for f and g is defined as:

$$B_{n,\varepsilon}(g, x) = \{y : \text{there is a } \Lambda_n \subset \{0, 1, ..., n-1\} \text{ with}$$

$$card(\{0, 1, ..., n-1\} - \Lambda_n) \leq g(n)$$

$$\text{and } \max\{d(f^i(x), f^i(y)) : i \in \Lambda_n\} < \varepsilon\}.$$

Definition. A map $f : X \to X$ has the *specification property* if: for any $\varepsilon > 0$, there is an integer $M(\varepsilon)$ such that for any collection of intervals $I_j = [a_j, b_j] \subset \mathbb{Z}^+$, $j = 0, ..., k-1$ such that $a_j - b_{j-1} \geq M(\varepsilon)$, $j = 1, ..., k-1$ and for any $x_0, ..., x_{k-1} \in X$, there is an $x \in X$ such that

$$d(f^{a_j+n}(x), f^n(x_j)) < \varepsilon, \text{ for } 0 \leq n \leq b_j - a_j, \quad j = 0, 1, 2, ..., k-1.$$

Definition. A map $f : X \to X$ satisfies the *g-Almost Property Product* (*APP*) with g a function as above, if there exists a map $m : \mathbf{R}^+ \to \mathbf{N}$ such that for any points $x_1, x_2, ..., x_k \in X$, for any $\varepsilon_1 > 0, \varepsilon_2 > 0, ..., \varepsilon_k > 0$ and for any numbers $n_i \geq m(\varepsilon_i)$, $i = 1, 2, ..., k$ holds

$$\bigcap_{j=1}^{k} f^{n_j-1}(B_{n_j,\varepsilon_j}(g, x_j)) \neq \emptyset.$$

The specification property implies APP [10].

Definition. Two points x, y are (n, ε)-*separated* if $d(f^j(x), f^j(y)) > \varepsilon$ holds for some $j = 0, 1, ..., n$. A set $E \subset X$ is (n, ε)-*separated* if all points of E are (n, ε)-separated. A pair of points x, y are (δ, n, ε)*-separated* if $card\{j = 0, 1, ..., n-1 : d(f^j(x), f^j(z)) > \varepsilon\} \geq \delta n.$ A set $E \subset X$ is (δ, n, ε)*-separated* if all points of E are (δ, n, ε)-separated.

By $\mathcal{M}(X)$ we denote the space of measures in X and by $\mathcal{M}_{inv}(X, f)$ the space of f-invariant measures on X. The space $\mathcal{M}(X)$ can be endowed with a metric D compatible with the metric in X, in the sense that $D(\delta_x, \delta_y) = d(x, y)$, where δ is the point mass measure. More precisely, the metric considered in $\mathcal{M}(X)$ will be

$$D(\mu, \nu) = \sum_{n=1}^{\infty} \frac{\left| \int \varphi_n d\mu - \int \varphi_n d\nu \right|}{2^n \| \varphi_n \|_\infty},$$

where $\{\varphi_n\}$ is a dense set in $C(X)$. We denote by $B_R(\mu)$ the ball of center μ and radius R in the above metric. The topology induced by this metric is the weak $*$-topology and if X is compact, then $\mathcal{M}(X)$ is compact in the weak topology. The weak convergence is the convergence in the metric which induces the weak topology. By $\mathcal{F}_\mu$ is denoted the filter of neighborhoods of μ in the weak $*$-topology.

The *empirical measures* on X associated to the dynamical system (X, f) are defined as

$$\mathcal{E}_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}.$$

We denote the weak limits of the sequence $\{\mathcal{E}_n(x)\}$ by $V(x)$. Since X is compact, $V(x) \neq \emptyset$. If μ is a measure on X, then a point $x \in X$ is μ-generic if $V(x) = \{\mu\}$, by $G(\mu)$ is denoted the set of μ-generic points. A system is called *saturated* if

$$h_{top}(G(\mu)) = h_\mu(f). \tag{5}$$

Bowen [4] proved the inequality

$$h_{top}(G(\mu)) \leq h_\mu(f),$$

while in [6], the opposite inequality for systems satisfying the property of specification was proved.

Let $F \subset \mathcal{M}(X)$ and

$$X_{n, F} := \{x : \mathcal{E}_n(x) \in F\}.$$

By $R_{n, \varepsilon, F}$ is denoted the maximal cardinality of (n, ε)-separated sets contained in $X_{n, F}$ and by $R_{\delta, n, \varepsilon, F}$ the maximal cardinality of (δ, n, ε)-separated sets contained in $X_{n, F}$.

Definition. The map $f : X \to X$ has the *uniform separation property* if the following condition is satisfied: for any $\gamma > 0,$ there are numbers $\bar{\delta} > 0,$ $\bar{\varepsilon} > 0$ such that for any ergodic measure μ and for any $F \in \mathcal{F}_\mu,$ there is a natural $\overline{N} = \overline{N}(F, \mu, \gamma)$ such that for any $n \geq \overline{N},$

$$R_{\bar{\delta}, n, \bar{\varepsilon}, F} \geq \exp(n(h_\mu(f) - \gamma)).$$

III. Proof of the Theorem

Let us denote

$$\mathcal{M}_{\Phi, I} = \left\{\mu \in \mathcal{M}_{inv}(X) : \int \Phi d\mu^{\otimes r} \in I\right\}.$$

We begin by proving the inequality

$$h_{top}(E_{\Phi, I}) \geq \inf\{h_\mu(f) : \mu \in \mathcal{M}_{\Phi, I}\}.$$

Following [10], it can be constructed a set $E \subset E_{\Phi, I}$ and such that $h_{top}(E_{\Phi, I}) \geq \bar{h}. = \inf\{h_\mu(f) : \mu \in \mathcal{M}_{\Phi, I}\} - \gamma,$ for $\gamma > 0.$ To construct this set, is used uniform separation Almost Property Product (APP). Let us recall that $R_{n^r, \varepsilon, F^r}$ denote the maximal cardinality of (n^r, ε)-separated sets contained in X_{n, F^r}. By [10], uniform separation property implies that there are numbers $\overline{\delta} > 0$, $\overline{\varepsilon} > 0$ such that for any ergodic measure μ and for any $F^r \in \mathcal{F}_{\mu^{\otimes r}}$, there is a number $\overline{N} = \overline{N}(F^r, \mu, \gamma)$ such that for any $n \geq \overline{N}$,

$$R_{\overline{\delta}, n, \overline{\varepsilon}, F^r} \geq \exp(n^r(h_\mu(f) - \gamma)).$$

Let us choose sequences $\{n_k\}$, $\{R_k\}$, $\{\varepsilon_k\}$ with $R_k \searrow 0$ and $\varepsilon_k \searrow 0$. By the APP, it may be assumed that

$$\overline{\delta} n_k > 2g(n_k) + 1 \text{ and } \frac{g(n_k)}{n_k} < \varepsilon_k,$$

where g is the function of APP definition.

From the above facts, by [10], for a given sequence $\{\rho_1, \rho_2, ..., \rho_k\} \subset \mathcal{M}(X)$ and for $\overline{\varepsilon} > \varepsilon_1$, there exist a $\overline{\delta} > 0$ and a $(\overline{\delta}, n_k, \overline{\varepsilon})$-separated set $\Gamma_k \subset \{x : \mathcal{E}_{n_k}(x) \in B_{R_k}(\rho_k)\}$ such that

$$x \in \Gamma_k, \quad z \in B_{n_k, \varepsilon_k}(g, x) \Rightarrow \mathcal{E}_{n_k}(z) \in B_{R_k + \varepsilon_k}(\rho_k).$$

Let us choose now a strictly increasing sequence $\{N_k\}$ such that

$$n_{k+1} \leq R_k \sum_{j=1}^{k} n_j N_j$$

and

$$\sum_{j=1}^{k-1} n_j N_j \leq R_k \sum_{j=1}^{k} n_j N_j.$$

We consider stretched sequences $\{n'_j\}$, $\{\varepsilon'_j\}$, $\{\Gamma'_j\}$ such that if $j = N_1 + \cdots + N_{k-1} + q$ with $1 \leq q \leq N_k$, then $n'_j = n_k$, $\varepsilon'_j = \varepsilon_k$ and $\Gamma'_j = \Gamma_k$.

Finally, we can define

$$E_k := \bigcap_{j=1}^{k} \left(\bigcup_{x_j \in \Gamma'_j} f^{-M_{j-1}}(B_{n'_j, \varepsilon'_j}(x_j)) \right) \tag{6}$$

with $M_j = n'_1 + n'_2 + \cdots + n'_j$ and

$$E := \bigcap_{k \geq 1} E_k. \tag{7}$$

Any element of E can be labelled by a sequence $x_1, x_2, \ldots$ with $x_j \in \Gamma'_j$. According to Pfister and Sullivan [10], the following holds: let $x_j, y_j \in \Gamma'_j$, $x_j \neq y_j$, if $x \in B_{n_j, \varepsilon_j}(g, x_j)$, $y \in B_{n_j, \varepsilon_j}(g, y_j)$. Then $\max\{d(f^k(x), f^k(y)) : k = 0, \ldots, n_j - 1\} > 2\varepsilon$ with $\varepsilon > \varepsilon_1/4$.

Let us consider any measure ρ_i, $i = 1, 2, \ldots, k$, belonging to the set

$$K = \{\rho : \rho \in \mathcal{M}_{\Phi, I}, \text{ for any } \Phi \in C(X^r)\}.$$

Let us extend the set measures $\{\rho_1, \rho_2, \ldots, \rho_k\}$ to a set $\{\rho_1, \rho_2, \ldots, \rho_k, \ldots\}$ such that $\overline{\{\rho_j : j > n\}} = K$ (the bar means closure) and such that $D(\rho_j, \rho_{j+1}) \to 0$ as $j \to \infty$. By [10], for any $x \in E$, the sequences $\{\mathcal{E}_{M_k}(x)\}$ and $\{\rho_{M_k}\}$ with $M_k = n'_1 + n'_2 + \cdots + n_k$ have the limit points. Therefore, the sequences $\{V_{\tilde{\Phi}}(M_k, x)\}$ and $\left\{\int \tilde{\Phi} d\rho_{M_k}^{\otimes r}\right\}$ have the same limit points and so $\mathcal{A}[\tilde{\Phi}](x) \in I$. Since $\| \Phi - \tilde{\Phi} \|_\infty < \varepsilon$, for any $\varepsilon > 0$, we have that $\mathcal{A}[\Phi](x) \in I$. Thus, $E \subset E_{\Phi, I}$.

Let us show that $h_{top}(E) \geq \overline{h}$. Let $s < \overline{h}$, the set E is closed and so it is compact, let us consider a finite covering $\mathcal{U}$ by balls $B_{m, \varepsilon}(x)$ having non-empty intersection with G. Now

$$M(E,\ s,\ N,\ \varepsilon) = \inf_{\mathcal{U}\in\mathcal{C}(n,\varepsilon,E)} \sum_{B_{m,\varepsilon}(x)\in\mathcal{U}} \exp(-sm).$$

For any finite covering $\mathcal{U}$ of E, we can construct a covering $\mathcal{U}_0$ in the following way: each ball $B_{m,\varepsilon}(x)$ is replaced by a ball $B_{M_{rr},\varepsilon}(x)$ with $M_r \le m \le M_{r+1}$. Thus,

$$M(E,\ s,\ N,\ \varepsilon) = \inf_{\mathcal{U}\in\mathcal{C}(n,\varepsilon,E)} \sum_{B_{m,\varepsilon}(x)\in\mathcal{U}} \exp(-sm)$$

$$\ge \inf_{\mathcal{U}\in\mathcal{C}(N,\varepsilon,E)} \sum_{B_{M_r,\varepsilon}\in\mathcal{U}_0} \exp(-sM_{r+1}).$$

Now we can consider a covering $\mathcal{U}_0$ in which $m = \max\{r :$ there is a ball $B_{M_r,\varepsilon}(x) \in \mathcal{U}_0\}$. We set

$$W_k := \prod_{i=1}^{k} \Gamma_i, \quad \overline{W_m} = \bigcup_{k=1}^{m} W_k.$$

Let $x_j,\ y_j \in \Gamma'_j,\ x_j \ne y_j$ as we pointed out earlier, if $x \in B_{N'_j,\varepsilon'_j}(x_j)$, $y \in B_{N'_j,\varepsilon'_j}(y_j)$, then $d(f^l(x),\ f^l(y)) > 2\varepsilon$, for any $l = 0, ..., N_j - 1$ and with $\varepsilon > \varepsilon_1/4$. Now, for any $x \in B_{M_r,\varepsilon}(z) \cap G$, there is a uniquely determined $z = z(x) \in W_r$. A word $\overline{w} \in W_j$ with $j = 1, 2, ..., k$, is called a *prefix* of a word $w \in W_k$ if the first *j*-letters of $\overline{w}$ agree with the first *j*-letters of *w*. The number of times that each $w \in W_k$ is a prefix of a word in W_m is

$$cardW_m/cardW_k,$$

thus if *W* is a subset of $\overline{W_m}$, then

$$\sum_{k=1}^{m} \frac{card(W \cap W_k)}{card(W_k)} \ge card(W_m).$$

If each word in W_m has a prefix contained in a $W \subset \overline{W_m}$, then

$$\sum_{k=1}^{m} \frac{card(W \cap W_k)}{card(W_k)} \geq 1$$

and since $\mathcal{U}_0$ is a covering each point of W_m has a prefix associated to a ball in $\mathcal{U}_0$. By this and because $cardW_k \geq \exp(\bar{h}M_r)$, we obtain

$$\sum_{B_{M_r, \varepsilon} \in \mathcal{U}_0} \exp(-sM_r) \geq 1.$$

Thus, if r is taken such that $k \geq r$, then $sM_{k+1} \leq \bar{h}M_k$, for $N \geq M_r$, $\mathcal{U} \in \mathcal{G}(N, \varepsilon, e)$. Therefore,

$$\sum_{B_{m, \varepsilon}(x) \in \mathcal{U}} \exp(-sm) \geq 1$$

and so

$$M(e, s, N, \varepsilon) \geq 1.$$

By this, $h_{top}(E) \geq \bar{h}$.

To prove the other inequality, we set

$$\widetilde{E}_{\Phi, I} = \{x : \mathcal{A}[\Phi](x) \text{ has at least a limit point in } I\}$$

and we shall firstly prove that

$$h_{top}(\widetilde{E}_{\Phi, I}) \leq \sup\left\{h_\mu(f) : \int \Phi d\mu^{\otimes r} \in I\right\}.$$

Let us consider the r-empirical measures

$$\mathcal{E}_n^r(x) = \frac{1}{n^r} \sum_{1 \leq i_1, \dots, i_r \leq n} \delta_{(f^{i_1}(x), \dots, f^{i_r}(x))}.$$

Let $F \subset \mathcal{M}(X)$ and $F^r = F \times \cdots \times F \subset \mathcal{M}(X^r)$ be the product of r-copies of F. We set

$$X_{n, F^r} := \{x : \mathcal{E}_n^r(x) \in F^r\}.$$

Let $R_{n^r, \varepsilon, F^r}$ be the maximal cardinality of (n^r, ε)-separated sets contained in X_{n^r, F^r}. In similar way than in [10] can be proved that for any $\mu \in \mathcal{M}(X)$ and for any $\varepsilon > 0$, there is a set $F = F(\mu, \varepsilon) \in \mathcal{F}_\mu$, $\mu^{\otimes r} \in F^r$, there is a natural $N = N(F)$ such that

$$R_{n^r, \varepsilon, F^r} \leq \exp[n^r(h_\mu(f) + \delta)],$$

for any $\delta > 0$. Let $t = \sup\{h_\mu(f) : \mu \in \mathcal{M}_{\Phi, I}\}$ and let $\bar{t}$ be such that $\bar{t} - t = 2\delta > 0$. Then we have

$$M(X_{n^r, F^r}, \bar{t}, n^r, \varepsilon) \leq \sum_{\substack{B_{m, \varepsilon}(x) \in \mathcal{B} \\ n^r \geq m}} \exp(-sm)$$

$$\leq R_{n^r, \varepsilon, F^r} \exp(-\bar{t}n^r) \leq \exp(-\delta n^r).$$

Since the map $\mu \mapsto \int \Phi d\mu^{\otimes r}$ is continuous, $\mathcal{M}_{\Phi, I}$ is a compact subset of $\mathcal{M}(X^r)$, for any $\Phi \in C(X^r)$. Thus, it can be covered by sets of the form $F_j = F(\mu_j, \varepsilon) \in \mathcal{F}_{\mu_j}$ with $\mu_j \in K$, $j = 1, 2, ..., k_\varepsilon$. If $x \in \widetilde{E}_{\Phi, I}$, then the sequence $\{\mathcal{E}_n^r(x)\}$ has a limit point in $\mathcal{M}_{\Phi, I}$, so that there are a $j_0 \in \{1, 2, ..., k_\varepsilon\}$ and an M with $n^r \geq M$ such that $x \in X_{M, F^r(\mu_{j_0}, \varepsilon)}$. Thus,

$$M(\widetilde{E}_{\Phi, I}, \bar{t}, M, \varepsilon) \leq k_\varepsilon \sum_{n^r \geq m} \exp(-\delta n^r).$$

Then

$$h_{top}(\widetilde{E}_{\Phi, I}) \leq t = \sup\left\{h_\mu(f) : \int \Phi d\mu^{\otimes r} \in I\right\}.$$

Because $E_{\Phi, I} \subset \widetilde{E}_{\Phi, \left\{\int \Phi d\mu^{\otimes r}\right\}}$, for any $\mu \in \mathcal{M}_{\Phi, I}$, we have

$$h_{top}(E_{\Phi, I}) \leq h_\mu(f), \text{ for any } \mu \in \mathcal{M}_{\Phi, I}$$

and so

$$h_{top}(E_{\Phi, I}) \leq \inf\{h_\mu(f) : \mu \in \mathcal{M}_{\Phi, I}\}. \qquad \square$$

Let

$$\overline{E}_{\Phi, I} = \{x : \mathcal{A}[\Phi](x) \subset I\}.$$

The following proposition is a generalization of the main result in [5], which corresponds to $I = \{\alpha\}$.

Proposition. *For dynamical systems with the specification property holds*

$$h_{top}(\overline{E}_{\Phi, I}) = \sup\left\{h_\mu(f) : \int \Phi d\mu^{\otimes r} \in I\right\}.$$

Proof. The inequality

$$h_{top}(\overline{E}_{\Phi, I}) \leq \sup\left\{h_\mu(f) : \int \Phi d\mu^{\otimes r} \in I\right\}$$

is an immediate consequence of the above theorem. To prove the opposite inequality is used the following result appeared in [5]: for any $\Phi \in C(X^r)$ and for any $\varepsilon > 0$, there is a map $\widetilde{\Phi} : X^r \to \mathbf{R}$ of the form

$$\widetilde{\Phi} = \sum_{j=1}^{n} \varphi_j^{(1)} \otimes \cdots \otimes \varphi_j^{(r)}$$

with $\varphi_j^{(i)} \in C(X)$ and such that $\| \Phi - \widetilde{\Phi} \|_\infty < \varepsilon$.

Let $x \in G(\mu)$ with $\mu \in \mathcal{M}_{\Phi, I}$ so that

$$\lim_{n\to\infty} V_{\tilde{\Phi}}(n, x) = \int \tilde{\Phi} d\mu^{\otimes r}.$$

We have

$$\left| \lim_{n\to\infty} V_{\Phi}(n, x) - \int \Phi d\mu^{\otimes r} \right| \leq \left| \lim_{n\to\infty} V_{\Phi}(n, x) - \lim_{n\to\infty} V_{\tilde{\Phi}}(n, x) \right|$$

$$+ \left| \lim_{n\to\infty} V_{\tilde{\Phi}}(n, x) - \int \tilde{\Phi} d\mu^{\otimes r} \right|$$

$$+ \left| \int \tilde{\Phi} d\mu^{\otimes r} - \int \Phi d\mu^{\otimes r} \right| < 2\varepsilon.$$

Thus, $\lim_{n\to\infty} V_{\Phi}(n, x) = \int \Phi d\mu^{\otimes r} \in I$, since $\mu \in \mathcal{M}_{\Phi, I}$. In this way is proved that $G(\mu) \subset \overline{E}_{\Phi, I}$, from this and the saturatedness, which holds under specification property, is obtained

$$h_{top}(\overline{E}_{\Phi, I}) \geq h_{top}(G(\mu)) \geq h_{\mu}(f),$$

then taken sup over the measures $\mu \in \mathcal{M}_{\Phi, I}$, we get

$$h_{top}(\overline{E}_{\Phi, I}) \geq \sup\{h_{\mu}(f) : \mu \in \mathcal{M}_{\Phi, I}\}. \qquad \square$$

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