Journal of Process Control 54 (2017) 65–80

Contents lists available at ScienceDirect

Journal of Process Control

journal homepage: www.elsevier.com/locate/jprocont

A robust gradient-based MPC for integrating real time optimizer (RTO) with control

Agustina D'Jorge [a], Antonio Ferramosca [b], Alejandro H. González [a]

[a] Institute of Technological Development for the Chemical Industry (INTEC), CONICET – Universidad Nacional del Litoral (UNL), Güemes 3450, 3000 Santa Fe, Argentina

[b] CONICET – Universidad Tecnolgica Nacional (UTN), Facultad Regional de Reconquista, Calle 27 de Abril, 1000 (3560), Reconquista, Santa Fe, Argentina

ARTICLE INFO

Article history:
Received 8 August 2016
Received in revised form 6 January 2017
Accepted 28 February 2017

Keywords:
Model predictive control
Economic optimization
Robust control

ABSTRACT

A gradient-based model predictive control (MPC) strategy was recently proposed to reduce the computational burden derived from the explicit inclusion of an economic real time optimization (RTO). The main idea is to compute a suboptimal solution, which is the convex combination of a feasible solution and a solution of an approximated (linearized) problem. The main benefits of this strategy are that convergence is still guaranteed and good economic performances are obtained, according to several simulation scenarios. The formulation, however, is developed only for the nominal case, which significantly reduces its applicability. In this work, an extension of the gradient-based MPC to explicitly account for disturbances is made. The resulting robust formulation considers a nominal prediction model, but restricted constraints (in order to account for the effect of additive disturbances). The nominal economic performance is preserved and robust stability is ensured. An illustrative example shows the benefits of the proposal.

1. Introduction

Model predictive control (MPC) is one of the most successful advanced control techniques in the process industries, capable to provide (theoretical) stability, robustness, constraint satisfaction and tractable computation for linear and nonlinear systems [24].

In the last decade researchers were focused not only on the dynamic performance, but also on improving the economic performance of MPC: i.e., the performance of the controller according to a process economic criterion that could not coincide with that of dynamic objectives. The analysis starts with the study of the hierarchical control structure, typically used in process industries [11]: at the top of this structure an economic planner determines the whole plant production (level, quality, etc.); then, the outputs of this layer are sent to a real time optimizer (RTO), which is devoted to compute the stationary setpoints according to economic criteria. This optimizer, which is usually based on a complex nonlinear stationary model of the plant and so has a sampling time different from other layers, computes the setpoints to be sent to the MPC control level, which computes the control actions necessary for the plant to reach those setpoints, taking into account a simplified dynamic model of the plant and the variable constraints.

A main well-known drawback of this hierarchical control structure is, however, that the communication between the economic/stationary and dynamic layers may be inconsistent. The differences between models and time scales produce problems that go from unreachability of the setpoints (in a pure stationary context) to poor transient and stationary economic performances. A proper strategy to unify this objectives becomes then a highly desired challenge from an operating point of view.

Two main strategies to reduced inconsistency are the so-called **two-layer** and **one-layer** structures. In the first case, an extra optimization level – the steady state target optimizer, SSTO – is added in between the RTO and the MPC to decide the best admissible target for the MPC, according to a local approximation of the RTO cost function, and using the same simplified model used in the MPC layer. In this framework, another approach is represented by the dynamic real time optimizer (D-RTO) [6,26,10], which solves a dynamic economic optimization and delivers target trajectories (instead of target steady state) to the MPC layer.

E-mail addresses: agustinadj@gmail.com (A. D'Jorge), ferramosca@santafe-conicet.gov.ar (A. Ferramosca), alejgon@santafe-conicet.gov.ar (A.H. González).

http://dx.doi.org/10.1016/j.jprocont.2017.02.015

The **one-layer** strategy, on the other hand, consists in a unified MPC cost function that simultaneously considers the economic objectives and the dynamic objectives, as for instance in [27,1] or [9,5,17]. The main problem of this strategy is that the economic objectives are usually represented by a complex nonlinear function that turn the one-layer optimization cost also nonlinear and difficult to be solved.

In the **one-layer** strategies context, several solutions were also proposed to overcome the computational-burden problem of such an economic MPC. The more significant one is the use of an approximated economic cost function, instead of the nonlinear original one. A first approach is presented in [8], where the gradient of the economic objective function is included in the controller cost function in order to obtain a computational low-cost strategy. This solution allows one to solve the resulting control/optimization problem as a single QP problem, and the results are promising from both, theoretic and practical points of view. This idea has then been extended in [2], in order to obtain a stable formulation.

The novelty of the latter strategy is that instead of applying to the system the optimal solution of an approximated problem, it applies a suboptimal solution, in the sense that the applied control action is given by the convex combination of an arbitrary (and easy-to-obtain) feasible solution of the original problem and the optimal solution of an approximated/simplified problem. As it is shown in [2], this sub-optimal MPC strategy ensures recursive feasibility and convergence to the optimal steady state in the economic sense, with a significantly reduced computational cost.

The aim of this work is to extend the benefits of the aforementioned formulation – which was developed for the nominal case only – to the robust case. To do that, a bounded additive disturbance scheme is used, and taking into account the ideas presented in [7] (for regulation) and [15] (for set-interval tracking) a robust one-layer MPC is proposed which considers a nominal prediction model, but restricted constraints (in order to account for the effect of the additive disturbance). This robust extension is made with the non trivial objective to preserve the nominal economic performance and to ensure robust stability. The so obtained control formulation is tested by simulating several economic scenarios on a four-tank system.

The paper is organized as follows. In Section 2 the control problem is stated. Section 3 presents the proposed robust controller and its main properties. In Section 4, the application of the robust MPC presented to the four tanks systems shows the properties of the proposed controller. Finally, in Section 5 some conclusions are drawn.

Notation: A positive definite symmetric matrix T is denoted as $T>0$ and $T>P$ denotes that $T-P>0$. For a given symmetric matrix $P>0$, $\|x\|_P$ denotes the weighted Euclidean norm of x, i.e. $\|x\|_P = \sqrt{x'Px}$. Consider $a \in IR^{n_a}$ and $b \in IR^{n_b}$, the vector made from stacking both vectors is defined as $(a, b) \triangleq [a', b']' IR^{n_a+n_b}$; for a set $\Gamma \subset IR^{n_a+n_b}$, the projection of Γ onto a is defined as $Proj_a(\Gamma) = \{a \in IR^{n_a} : \exists b \in IR^{n_b}, (a, b) \in \Gamma\}$. A vector **t** in bold denotes a finite sequence of vectors, that is, a vector defined as $\{t(0), t(1), \ldots, t(N)\}$, where N is deduced from the context. The norm of a signal **t** is defined as $\|t\|_\infty = \sup_{k\geq 0}(t(k))$. A matrix $\mathbf{0}_{n,m} \in IR^{n\times m}$ denotes a matrix of zeros and $I_n \in IR^{n\times n}$ denotes the identity matrix. Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset IR^n$ and $\mathcal{V} \subset IR^n$, the Minkowski sum is defined by $\mathcal{U} \oplus \mathcal{V} \triangleq \{u + v : u \in \mathcal{U}, v \in \mathcal{V}\}$, the Pontryagin set difference is: $\mathcal{U} \ominus \mathcal{V} \triangleq \{u : u \oplus \mathcal{V} \subseteq \mathcal{U}\}$; given a matrix $M \in IR^{p\times n}$, the set $M\mathcal{U} \subset IR^p$ is defined as $M\mathcal{U} \triangleq \{Mu : u \in \mathcal{U}\}$; for a given λ, $\lambda\mathcal{U} \triangleq (\lambda I_n)\mathcal{U}$.

2. Problem statement

Consider a system described by a linear time-invariant discrete time model

$$x^+ = Ax + Bu + w \tag{1}$$

where $x \in IR^n$ is the system state, $u \in IR^m$ is the current control vector, x^+ is the successor state and $w \in IR^n$ is an unknown but bounded state disturbance. In what follows, $x(k)$, $u(k)$ and $w(k)$ denote the state, the manipulable variable and the disturbance respectively, at sampling time k.

The system is subject to constraints on state and input:

$$(x(k), u(k)) \in \mathcal{Z} \tag{2}$$

for all $k \geq 0$, where $\mathcal{Z} = \mathcal{X} \times \mathcal{U}$ is a compact convex polyhedron containing the origin in its interior.

Define also the plant nominal model, given by (1) neglecting the disturbance input w:

$$\bar{x}^+ = A\bar{x} + Bu \tag{3}$$

The solution of this system for a given sequence of control inputs $\mathbf{u} = \left\{u(0), \ldots, u(j-1)\right\}$ and the initial state x is denoted as $\bar{x}(j) = \phi(j; \bar{x}, \mathbf{u})$, $j \in \mathbb{I}_{\geq 1}$, where $\bar{x} = \phi(0; \bar{x}, \mathbf{u})$.

The plant model is assumed to fulfill the following assumption:

Assumption 1. (i) The pair (A, B) is controllable and the state is measured at each sampling time. (ii) The uncertainty vector w is bounded and lies in a compact convex polyhedron, $\mathcal{W}$, containing the origin in its interior: $\mathcal{W} = \{w \in \mathcal{R}^n : A_w w \leq b_w, \}$.

2.1. The robustness approach

In this work, it will be used the robust approach presented in [15,7]. This approach is characterized by the use of a nominal model for predictions and by restricting the constraint sets $\mathcal{X}$ and $\mathcal{U}$ at any step of the prediction horizon. The controller is based on a pre-stabilization of the plant using a state feedback control gain K, such that $A_K = A + BK$ has all its eigenvalues in the interior of the unit circle.

The nominal prediction model is then given by:

$$\begin{aligned} \bar{x}(k+1) &= A_K\bar{x}(k) + Bc(k) \\ u(k) &= K\bar{x}(k) + c(k) \end{aligned} \tag{4}$$

where the control variable $c(k)$ represents the difference between the control input $u(k)$ and the nominal feedback $K\bar{x}(k)$.

In order to introduce the restricted constraints, the following definition is needed.

Definition 1. The reachable set in j steps, $\mathcal{R}_j$, is given by

$$\mathcal{R}_j = \oplus_{i=0}^{j-1} A_K^i \mathcal{W} \tag{5}$$

This is the set of states of the nominal closed-loop systems which are reachable in j steps from the origin, under the disturbance input w. This set satisfies the following properties [7]:

(i) It is given by the recursion $\mathcal{R}_j \oplus A_K^j \mathcal{W} = \mathcal{R}_{j+1}$ with $\mathcal{R}_1 = \mathcal{W}$.
(ii) $A_K^j \mathcal{R}_j \oplus \mathcal{W} = \mathcal{R}_{j+1} = \mathcal{R}_j \oplus A_K^j \mathcal{W}$.
(iii) $\mathcal{R}_j \subseteq \mathcal{R}_{j+1}$
(iv) The sequence of sets $\mathcal{R}_j$ has a limit $\mathcal{R}_\infty$ as $j \to \infty$, and $\mathcal{R}_\infty$ is a robust positive invariant set.
(v) $\mathcal{R}_\infty$ is the minimal robust positively invariant (RPI) set ([19]).

Definition 2. [19] A set Ω is called a robust positively invariant (RPI) set for the uncertain system $x(k+1) = A_K x(k) + w(k)$ with $w(k) \in \mathcal{W}$ if $A_K \Omega \oplus \mathcal{W} \subseteq \Omega$.

Based on this, the sets of **restricted constraints** are defined by:

$$\begin{aligned} \mathcal{X}_j &\triangleq \mathcal{X} \ominus \mathcal{R}_j \\ \mathcal{U}_j &\triangleq \mathcal{U} \ominus K\mathcal{R}_j \end{aligned} \tag{6}$$

These sets are non-empty if the following assumption holds:

Assumption 2. The sets $\mathcal{X}_j$ and $\mathcal{U}_j$ exist if and only if $\mathcal{R}_\infty \subset \mathcal{X}$ and $K\mathcal{R}_\infty \subset \mathcal{U}$

It is also important to note that the sets $\mathcal{X}_j$ and $\mathcal{U}_j$ are polyhedral sets and their computation is made off-line, so it has no practical effects on the MPC problem.

Remark 1. The control gain K has an important role in the proposed robust approach, since it determines the dynamic of the closed-loop system in presence of disturbances and hence, it has to ensure that Assumption 2 holds. In [22] it is proposed an LMI-based method for the calculation of the control gain K which ensures not only that Assumption 2 holds, but also that the set $\mathcal{R}_\infty$ is minimized.

2.2. Equilibria characterization and optimal point

If we consider the joint variable (x, u), the state and input equilibrium subspace, associated to the nominal model (3), is given by $\mathcal{N}([A - I_n \quad B])$, where $\mathcal{N}$ is the null space of a matrix. That is

$$[A - I_n \quad B] \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \mathbf{0}_{n,1}$$

Defining $\overline{\mathcal{Z}} \triangleq \mathcal{X}_N \times \mathcal{U}_N$, the set of admissible equilibriums – for the nominal system, is given by

$$\mathcal{Z}_s \triangleq \{(x, u) \in \rho\overline{\mathcal{Z}} \mid x = Ax + Bu\}$$

where $\rho \in (0, 1)$ is a given parameter added to avoid those steady states and inputs that provide active constraints.

Taking into account the latter equilibrium characterization, the optimal operation point that stabilizes the plant is

Definition 3. The **optimal steady state and input**, (x_s^{eco}, u_s^{eco}), satisfies

$$(x_s^{eco}, u_s^{eco}) = arg \quad \min_{(x,u) \in \mathcal{Z}_s} f_{eco}(x, u, p). \tag{7}$$

where $f_{eco}(x, u, p)$ defines an economic cost function and p is a parameter that takes into account prices, costs or production goals.

Assumption 3. The economic cost function $f_{eco}(x, u, p)$ is strictly convex in (x, u) and twice differentiable.

In addition, according to most real cases, it is assumed that $f_{eco}(x, u, p)$ is nonlinear and its evaluation takes a significant computation time.

3. The one layer robust economic MPC strategy

In this section the proposed controller formulation is presented. The controller follows the ideas presented in [2], which considers an additional cost term (called the offset cost function) as the economic objective and computes the control action sub-optimally; i.e., by means of a convex combination of an arbitrary (and easy-to-obtain) feasible solution of the original problem and the optimal solution of an approximated/simplified problem. Furthermore, here we take into account the robustness approach presented in Section 2.1, to extend the controller to the robust case. The main objective is to preserve the properties of the former formulation (simplicity, feasibility, convergence) when a bounded disturbance is explicitly considered.

The proposed cost function for the general MPC optimization problem is given by:

$$V_N(x, p; \mathbf{u}, x_s, u_s) = V_N^{dyn}(x; \mathbf{u}, x_s, u_s) + V_{eco}(x_s, u_s, p) \tag{8}$$

where $V_N^{dyn}(x; \mathbf{u}, x_s, u_s) = \sum_{j=0}^{N-1} \|\overline{x}(j) - x_s\|_Q^2 + \|u(j) - u_s\|_R^2 + \|\overline{x}(N) - x_s\|_P^2$, for appropriate matrices Q, R, and P and $V_{eco}(x_s, u_s, p)$=$f_{eco}(x_s, u_s, p)$.

Cost (8) is composed by two terms, based on nominal predictions. The first one is a pure dynamic term (since the pair (x_s, u_s) defines an artificial target only forced to be in $\mathcal{Z}_s$) while the second one is a pure stationary term (since it only penalizes the artificial target – which is an admissible equilibrium – according to the economic objectives). This is an extension of the so called **MPC for tracking** [15], which incorporates the artificial target (x_s, u_s) for feasibility reasons.

Considering the prestabilization gain K, we can write $V_N(x, p; \mathbf{u}, x_s, u_s) = V_N(x, p; \mathbf{c}, x_s, u_s)$, where each element of $\mathbf{c}$, $c(j;x)$, fulfill $u(j;x) = K(\bar{x}(j) - x_s) + u_s + c(j;x)$. This way, for any current state x, the optimization problem $P_N(x, p)$ to be solved is given by:

Problem $P_N(x, p)$**.**

$$\begin{aligned}
\min_{\mathbf{c}, x_s, u_s} \quad & V_N(x, p; \mathbf{c}, x_s, u_s) \\
s.t. \quad & \bar{x}(0) = x, \\
& \bar{x}(j+1) = A\bar{x}(j) + Bu(j), \quad j \in \mathbb{I}_{[0:N-1]} \\
& u(j) = K(\bar{x}(j) - x_s) + u_s + c(j), \quad j \in \mathbb{I}_{[0:N-1]} \\
& \bar{x}(j) \in \mathcal{X}_j, \quad j \in \mathbb{I}_{[0:N-1]} \\
& u(j) \in \mathcal{U}_j, \quad j \in \mathbb{I}_{[0:N-1]} \\
& (\bar{x}(N), x_s, u_s) \in \Omega_t,
\end{aligned}$$

In this optimization problem, x and p are the parameters, while the input sequence $\mathbf{c} = \{c(0), \ldots, c(N-1)\}$ and the artificial target variables x_s and u_s are the optimization variables. The last constraint is a terminal inequality constraint added for stability reasons. The set Ω_t is a robust invariant set for tracking [16].

Definition 4. [Robust invariant set for tracking]

Define the extended state $x_a = (x, x_s, u_s)$, and

$$A_a = \begin{bmatrix} A + BK & -BK & B \\ 0 & I_n & 0 \\ 0 & 0 & I_m \end{bmatrix}$$

Define also

$$X_a^i = \{(x, x_s, u_s) : x \in \mathcal{X}_i, K(x - x_s) + u_s \in \mathcal{U}_i, (x_s, u_s) \in \mathcal{Z}_s\}$$

and

$$\Sigma_t = \{x_a : A_a^i x_a \in X_a^i, \quad \text{for } i \geq 0\}$$

Then

$$\Omega_t = \Sigma_t \ominus (\mathcal{R}_N \times \{0\} \times \{0\})$$

The optimal solution to Problem $P_N(x, p)$ and the optimal value of the cost function $V_N(x, p; \mathbf{c}, x_s, u_s)$ are denoted respectively as $\mathbf{v}^0(x, p) = \{\mathbf{c}^0(x, p), x_s^0(x, p), u_s^0(x, p)\}$ and $V_N^0(x, p)$. The optimal control law is given by $\kappa_N^0(x, p) = u^0(0; x, p) = K(x - x_s^0(x, p)) + u_s^0(x, p) + c^0(0; x, p)$, where $c^0(0;x, p)$ is the first element of the solution sequence $\mathbf{c}^0(x, p)$ (in the receding horizon fashion).

Since the set of constraints of Problem $P_N(x, p)$ does not depend on p or (x_s^{eco}, u_s^{eco}), its feasible region does not depend on p or (x_s^{eco}, u_s^{eco}) neither. Then, the feasible set of this controller is a polyhedral region $\Upsilon_N \subseteq IR^n$ given by the set of initial states that can be driven into $\Omega_t^x = Proj_x(\Omega_t)$ in N steps fulfilling the constraints, for all admissible disturbances.

Remark 2. The concept of MPC for tracking [21,13] plays an important role in this class of economic controller. In fact, if the economic criterion changes – for instance due to changes in the prices, expected demand, etc. – the economically optimal admissible steady state and input (x_s^{eco}, u_s^{eco}) where the controller steers the systems may change, and the feasibility of the controller may be lost.

The aim is then to ensure that under any change of the optimal operation point, the closed-loop system maintains the feasibility, and so the convergence to the new point. More precisely, in standard MPCs, the change of (x_s^{eco}, u_s^{eco}) can result in loss of feasibility of the optimization problem for the following reasons: (i) the terminal region computed for a certain equilibrium point may not be admissible for the new setpoint, and (ii) the terminal region at the new setpoint could be unreachable in N steps. In this case, a re-calculation of an appropriate value of the prediction horizon is necessary to ensure feasibility, and this would require an on-line re-design of the controller for each setpoint, which clearly implies a high computational cost.

Remark 3. Note that, following the same arguments as in [7,23,15], if the control gain K is chosen as the gain of the LQR, the dynamic cost $V_N^{dyn}(x; \mathbf{u}, x_s, u_s)$ can be rewritten as

$$V_N^{dyn}(x; \mathbf{c}, x_s, u_s) = \|\bar{x}(0) - x_s\|_P^2 + \sum_{j=0}^{N-1} \|c(j)\|_\Psi^2$$

where $\Psi = R + B'PB$ and P is the unique solution of the Riccati equation

$$P = A'_K P A_K + Q + K'RK$$

3.1. The easy-to-obtain suboptimal solution

Given that the economic cost is generally based on a complex nonlinear model, the main problem to solve $P_N(x, p)$ is the high computational burden. In this context, the suboptimal strategy proposed in [2] seems to be a convenient way to overcome the problem. The idea is to implement a control action that is not a solution of an optimization problem, but a convex combination of two solutions: the first one being a mere **feasible solution** of the original problem $P_N(x, p)$, and the second one being the optimal solution of a linearized version of the original problem $P_N(x, p)$. This latter solution will be referred, for simplicity, as a the **approximated optimal solution**.

Now, for a given time instant k, define the **feasible solution** to problem $P_N(x, p)$, $\hat{\mathbf{c}} = \left\{\hat{c}(0), \ldots, \hat{c}(N-2), 0\right\}$, $\hat{x}_s$, $\hat{u}_s$, as the *shifted solution* of the same problem at time $k-1$. Associated to this solution is the feasible state sequence $\hat{\mathbf{x}} = \left\{\hat{x}(0), \ldots, \hat{x}(N)\right\}$, where $\hat{x}(N) \in \Omega_t^x$ by the terminal constraint.

The cost function corresponding to solution $(\hat{\mathbf{x}}, \hat{\mathbf{c}}, \hat{x}_s, \hat{u}_s)$ (called **feasible cost**) is given by:

$$V_N(x, p; \hat{\mathbf{c}}, \hat{x}_s, \hat{u}_s) = V_N^{dyn}(\hat{x}; \hat{\mathbf{c}}, \hat{x}_s, \hat{u}_s) + V_{eco}(\hat{x}_s, \hat{u}_s, p) \tag{9}$$

Let us define now the **approximated optimal solution** to the original problem $P_N(x, p)$, by solving the following approximated problem:

Problem $P_N^{app}(x, p)$

$$\begin{array}{ll} \min\limits_{\mathbf{c}, x_s, u_s} & V_N^{app}(x, p; \mathbf{c}, x_s, u_s) \\ s.t. & \mathbf{c}, x_s, u_s \in \mathcal{C}_N(x) \end{array}$$

where $\mathcal{C}_N(x)$ defines the set of values $\mathbf{c}$, x_s, u_s that fulfill the constrain in Problem $P_N(x, p)$. The approximated cost is given by

$$V_N^{app}(x, p; \mathbf{c}, x_s, u_s) = V_N^{dyn}(x; \mathbf{c}, x_s, u_s) + V_{eco}(\hat{x}_s, \hat{u}_s, p) + \nabla V'_{eco}(\hat{x}_s, \hat{u}_s, p) \begin{bmatrix} x_s - \hat{x}_s \\ u_s - \hat{u}_s \end{bmatrix}$$

and $\nabla V_{eco}(\hat{x}_s, \hat{u}_s, p)$ represents the gradient of V_{eco} w.r.t. (x, u), evaluated at the point $(\hat{x}_s, \hat{u}_s)$.

As it can be seen, this optimal solution of an approximated problem tries to approach the optimal solution of $P_N(x, p)$ by means of a simplified version of it. This solution is sub-optimal (in the transient regime) with respect to the optimal solution of $P_N(x, p)$, and hence its direct application into the MPC scheme does not guarantee convergence of the closed-loop system to the optimal solution to the original problem $P_N(x, p)$.

Let us denote the **approximated optimal solution** (the optimal solution to problem $P_N^{app}(x, p)$) as $\mathbf{c}^* = \left\{c^*(0), \ldots, c^*(N-1)\right\}$, x_s^*, u_s^*, and the corresponding sate sequence as $\mathbf{x}^* = \left\{x^*(0), \ldots, x^*(N)\right\}$. Then, the cost function corresponding to $(\mathbf{x}^*, \mathbf{c}^*, x_s^*, u_s^*)$ reads:

$$V_N(x, p; \mathbf{c}^*, x_s^*, u_s^*) = V_N^{dyn}(x^*; \mathbf{c}^*, x_s^*, u_s^*) + V_{eco}(x_s^*, u_s^*, p)$$

The idea now is to construct a convex combination of the **feasible solution** and the **approximated optimal solution**,

$$\begin{aligned} \mathbf{c}(\lambda) &= (1-\lambda)\hat{\mathbf{c}} + \lambda\mathbf{c}^* \\ \mathbf{x}(\lambda) &= (1-\lambda)\hat{\mathbf{x}} + \lambda\mathbf{x}^* \\ u_s(\lambda) &= (1-\lambda)\hat{u}_s + \lambda u_s^* \\ x_s(\lambda) &= (1-\lambda)\hat{x}_s + \lambda x_s^*, \quad \text{with} \quad \lambda \in [0, 1], \end{aligned}$$

to obtain the so called **suboptimal solution**, which is feasible and produces a decreasing MPC cost, as it was shown in following theorem:

Theorem 1. [2, Theorem 1] *Let us consider Problem $P_N(x, p)$, with $x \neq x_s^{eco}$, and the aforementioned suboptimal solutions $\mathbf{c}(\lambda)$, $x_s(\lambda)$, $u_s(\lambda)$. Consider also that $(\hat{x}_s, \hat{u}_s) \neq (x_s^{eco}, u_s^{eco})$. Then there exists a $\tilde{\lambda} \in (0, 1]$ such that, for all $0 \le \lambda \le \tilde{\lambda}$*

$$V_N(x, p; \mathbf{c}(\lambda), x_s(\lambda), u_s(\lambda)) < V_N(x, p; \hat{\mathbf{c}}, \hat{x}_s, \hat{u}_s). \tag{10}$$

The proof of theorem can be seen in [2]. Note that, in such a work, a method to calculate $\tilde{\lambda}$ based on the Hessian of the economic cost is proposed. Also, $\tilde{\lambda}$ can be computed heuristically in such a way that condition (10) is fulfilled.

According to the latter result, we can define more precisely the **suboptimal solution** as the one that produces a (positive) decrement in the cost function. That is, $\mathbf{c}^{so}, x_s^{so}, u_s^{so} \triangleq \mathbf{c}(\tilde{\lambda}), x_s(\tilde{\lambda}), u_s(\tilde{\lambda})$, and associated to this solution are the suboptimal state sequence, $\mathbf{x}^{so} \triangleq \mathbf{x}(\tilde{\lambda})$, and the cost function $V_N^{so}(x, p) = V_N(x, p; \mathbf{c}^{so}, x_s^{so}, u_s^{so})$.

Remark 4. The way to implement this suboptimal solution sequentially (i.e., at every time k the MPC control action is implemented) is as follows. In the first place, sets $\mathcal{X}_j$ and $\mathcal{U}_j$ (necessary to define the feasible space of Problem $P_N(x, p)$) must be computed offline. Then

1. Compute the **feasible solution** $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ to problem $P_N(x, p)$, using the shifted solution applied to the system at the sample time $k-1$. If the current time is $k=0$, compute the **feasible solution** $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ by solving the reduced problem $P_N^{dyn}(x)$:

   Problem $P_N^{dyn}(x)$

$$\begin{array}{ll} \min\limits_{\mathbf{c}, x_s, u_s} & V_N^{dyn}(x; \mathbf{c}, x_s, u_s) \\ s.t. & \mathbf{c}, x_s, u_s \in \mathcal{C}_N(x) \end{array}$$

2. Compute the gradient of the economic cost function $V_{eco}(x, u, p)$ w.r.t. (x, u), $\nabla V_{eco}(x, u, p)$.
3. Compute the value of the parameter $\tilde{\lambda}$ that defines the **suboptimal solution**. To obtain such a suboptimal solution, the **approximated optimal solution** to problem $P_N(x, p)$, $(\mathbf{x}^*, \mathbf{u}^*)$, must be computed first, by minimizing the approximated problem $P_N^{app}(x, p)$.
4. From the **suboptimal solution** $\mathbf{c}^{so}, x_s^{so}, u_s^{so} = \mathbf{c}(\tilde{\lambda}), x_s(\tilde{\lambda}), u_s(\tilde{\lambda})$, take the first input of the sequence $\mathbf{c}^{so}$ to implement the implicit MPC control law, $\kappa_N(x, p) \triangleq u^{so}(0; x) = K(x - x_s^{so}) + u_s^{so} + c^{so}(0; x)$.

3.2. Stability and convergence of the proposed controller

In this section some new results are presented regarding the convergence, the economic optimality and the stability of the proposed algorithm. Consider the following assumption on the controller parameters:

Assumption 4.

1. Let $R \in \mathbb{R}^{m\times m}$ be a positive definite matrix and $Q \in \mathbb{R}^{n\times n}$ a positive semi-definite matrix such that the pair $(Q^{1/2}, A)$ is observable.
2. Let $K \in \mathbb{R}^{m\times n}$ be the stabilizing control gain of the LQR controller.
3. Let $P \in \mathbb{R}^{n\times n}$ be a positive definite matrix, solution of the Riccati equation:

$$P = A'_K P A_K + Q + K'RK$$

4. Let the set Ω_t be a robust invariant set for tracking, as described in Definition 4.

In the following theorem, stability and constraints satisfaction of the controlled system are stated.

Theorem 2 (Stability). *Consider that assumptions (1)–(4) hold and consider a given parameter p for the economic cost $V_{eco}(x, u, p) = f_{eco}(x, u, p)$. The system controlled by the proposed MPC controller $\kappa_N(x, p)$ is such that for any initial condition $x(0) \in \Upsilon_N$, the optimization problem $P_N(x, p)$ is recursively feasible and steers the disturbed system (1) to $x_s^{eco} \oplus \mathcal{R}_\infty$.*

Proof. *Recursive feasibility* Recursive feasibility is proved by means of Lemmas 1–4, in Appendix B. *Robust asymptotic stability* In order to prove robust stability, let us define $J(x) = V_N^{so}(x, p) - V_{eco}(x_s^{eco}, u_s^{eco}, p)$. This function is well defined in the feasible set Υ_N. Define also $e(x) = x - x_s^{so}$. Notice that, since the cost $V_N(x, p;\mathbf{v})$ is positive definite, then there exists a $\mathcal{K}$-function, α_1, such that $J(x) \geq \alpha_1(\|e(x)\|)$, for all $x \in \Upsilon_N$.

From Lemma 5, it follows that

$$\alpha_1(\|e(x)\|) \geq \alpha_1(\alpha_e(\|x - x_s^{eco}\|)) = \alpha_J(\|x - x_s^{eco}\|)$$

where α_e and α_J are $\mathcal{K}$-functions. Then, we can conclude that:

(i) $J(x) \geq \alpha_J(\|x - x_s^{eco}\|)$, for all $x \in \Upsilon_N$, since $J(x) \geq \alpha_1(\|e(x)\|)$ and $\alpha_1(\|e(x)\|) \geq \alpha_J(\|x - x_s^{eco}\|)$.
(ii) $J(x) \leq \alpha_2(\|x - x_s^{eco}\|)$, for all $x \in \Upsilon_N$, where α_2 is a $\mathcal{K}_\infty$ function. Since the stage cost function is quadratic and the model is linear, then the cost function $J(x) = V_N^{so}(x, p) - V_{eco}(x_s^{eco}, u_s^{eco}, p)$ is a locally bounded continuous function and furthermore $J(x_s^{eco}) = 0$. So, it there exists a $\mathcal{K}_\infty$ function $\alpha_2(\cdot)$ such that $J(x) \leq \alpha_2(\|x - x_s^{eco}\|)$, for all $x \in \Upsilon_N$ (see [24, Postface, Propositions 1–2]).
(iii) $J(x^+) - J(x) \leq -\alpha_3(\|x - x_s^{eco}\|) + \sigma(\|w\|)$ for all $x \in \mathcal{X}_N$, where α_3 and σ are $\mathcal{K}$ functions. From Appendix A, $J(x^+) - J(x) \leq -\alpha(\|e(x)\|) + \sigma(\|w\|)$, with α and σ being $\mathcal{K}$ functions. Since $\alpha(\|e(x)\|) \geq \alpha_J(\|x - x_s^{eco}\|)$, then $J(x^+) - J(x) \leq -\alpha_3(\|x - x_s^{eco}\|) + \sigma(\|w\|)$, with α_3 (and σ) being $\mathcal{K}$ functions. This implies that, for $k \to \infty$, if $w = 0$, the $x(k) \to x_s^{eco}$; if $w \neq 0$, then $x(k) \to x_s^{eco} \oplus \mathcal{R}_\infty$.

Then, based on these facts, and resorting to ISS arguments [20, Theorem 5], it can be proved that the controlled system is robustly asymptotically stable; that is, there exist a $\mathcal{KL}$-function ϑ and a $\mathcal{K}$-function δ such that

$$\|\bar{x}(k) - x_s^{eco}\| \leq \vartheta(\|\bar{x}(0) - x_s^{eco}\|, k) + \delta(\|w\|)$$

for all $\bar{x}(0) \in \Upsilon_N$ and all disturbances $w(k) \in \mathcal{W}$. □

4. Application to the four-tank system

In this section some simulations results will be presented, to evaluate the proposed control strategy. First, a brief description of the system is shown. Then, the results of dynamic simulations are presented.

4.1. System description

In order to demonstrate the benefits and the properties of the proposed controller, we consider the four-tank plant [18]. This system is a multivariable laboratory plant of interconnected tanks with nonlinear dynamics and subject to state and inputs constraints (Fig. 1).

The four tanks are filled from a storage located at the bottom of the plant. The upper tanks (number 3 and number 4) discharge water into the lower tanks (number 1 and number 2, respectively). The inlet flows from the pumps are crossed, in such a way that the inlet flow from pump a (q_a) enters tank 1 and 4, while the inlet flow from pump b (q_b) enters tank 2 and 3.

This way, the water levels in the four tanks (h_1, h_2, h_3, h_4) are the state of the system, while the inlet flows (q_a, q_b) are the manipulated variables.

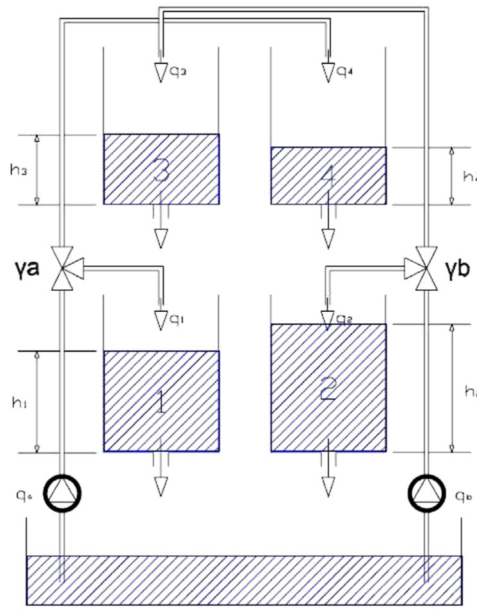

Fig. 1. The quadruple tank process.

A state-space continuous-time nonlinear model of the quadruple-tank process system is given by [18]:

$$\frac{dh_1}{dt} = -\frac{a_1}{S}\sqrt{2gh_1} + \frac{a_3}{S}\sqrt{2gh_3} + \frac{\gamma_a}{S}\frac{q_a}{3600} \tag{11a}$$

$$\frac{dh_2}{dt} = -\frac{a_2}{S}\sqrt{2gh_2} + \frac{a_4}{S}\sqrt{2gh_4} + \frac{\gamma_b}{S}\frac{q_b}{3600} \tag{11b}$$

$$\frac{dh_3}{dt} = -\frac{a_3}{S}\sqrt{2gh_3} + \frac{(1-\gamma_b)}{S}\frac{q_b}{3600} \tag{11c}$$

$$\frac{dh_4}{dt} = -\frac{a_4}{S}\sqrt{2gh_4} + \frac{(1-\gamma_a)}{S}\frac{q_a}{3600} \tag{11d}$$

where, the parameters of the plant are:

- S: Cross-section of the tanks [m^2].
- a_i: Discharge constant of the tank i [m^2].
- h_i: Water level of the tank i (state of the system) [m].
- q_a, q_b: Flow produced by the pumps a and b [m^3/h].
- g: The acceleration of gravity [m/s^2].
- γ_a, γ_b: Ratio of the three-way valves.

The value of these parameters are shown in Table 1. These values have been estimated on the experimental plant located at the Control Laboratory of the University of Seville (Spain). For a detailed description of this experimental plant, please refer to [4].

Table 1
Parameters of the four-tank plant.

	Value	Unit	Description
a_1	1.310e−4	m^2	Discharge constant of tank 1
a_2	1.507e−4	m^2	Discharge constant of tank 2
a_3	9.267e−5	m^2	Discharge constant of tank 3
a_4	8.816e−5	m^2	Discharge constant of tank 4
S	0.06	m^2	Cross-section of all tanks
γ_a	0.3		Parameter of the 3-ways valve
γ_b	0.4		Parameter of the 3-ways valve

Linearizing the model at an operating point given by $h^o = (0.6487, 0.6639, 0.6498, 0.6592)$, $q^o = (1.63, 2)$, and defining $x_i = h_i - h_i^o$, $u_j = q_j - q_j^o$, where $i = 1, \ldots, 4$ and $j = a, b$, we have:

$$\frac{dx}{dt} = \begin{bmatrix} \frac{-1}{\tau_1} & 0 & \frac{1}{\tau_3} & 0 \\ 0 & \frac{-1}{\tau_2} & 0 & \frac{1}{\tau_4} \\ 0 & 0 & \frac{-1}{\tau_3} & 0 \\ 0 & 0 & 0 & \frac{-1}{\tau_4} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma_a}{\chi} & 0 \\ 0 & \frac{\gamma_b}{\chi} \\ 0 & \frac{(1-\gamma_b)}{\chi} \\ \frac{(1-\gamma_a)}{\chi} & 0 \end{bmatrix} u + w$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x$$

where $\chi = 3600 * S$, and $\tau_i = \frac{S}{a_i}\sqrt{\frac{2h_i^o}{g}} \geq 0$, $i = 1, \ldots, 4$, are the time constants of each tank.

The system has been discretized using the zero-order hold method with a sample time of $T_s = 5[s]$.

The set $\mathcal{W}$ of possible disturbances realization is given by $\mathcal{W} = \{w \in \mathcal{R}^4 : \|w\|_\infty \leq 5 \times 10^{-3}\}$, and it was selected to account for plant-model mismatches in the usual operating points [3].

The system must fulfill the following constraints: $\mathcal{X} = \{x \in \mathbb{R}^4 : 0.2 \leq x_{1,2} \leq 1.36; 0.2 \leq x_{3,4} \leq 1.30\}$, and $\mathcal{U} = \{u \in \mathbb{R}^2 : [0, 0] \leq u \leq [3.26, 4]\}$. In what follows, set $\mathcal{Y}$ will denote the set of admissible outputs, given by $\mathcal{Y} = Proj_y(\mathcal{X})$. Matrix K has been chosen as the linear quadratic regulator (LQR) gain, for $Q = I_4$ and $R = 0.0005 I_2$, and it is given by:

$$K = \begin{bmatrix} -8.3586 & -6.3160 & 3.9788 & -28.7655 \\ -7.9006 & -12.4498 & -27.9745 & 3.1314 \end{bmatrix}$$

The prediction horizon has been chosen as $N = 5$.

The economic objective is to minimize the plant energetic consumption [12], by minimizing the voltage of the two pumps, and at the same time to maximize the volume of water in the tanks 1 and 2. Then the economic cost function are given by:

$$f_{eco}(y, u, p) = (q_a^2 + p(1)q_b^2) + p(2)\frac{V_{\min}}{S(h_1 + h_2)}$$

where $V_{min} = 0.012$ [m^3] is the minimum volume of water to be accumulated in the tanks. $y = (h_1, h_2)$, $u = (q_a, q_b)$, and $p = (p(1), p(2))$ are the prices on the cost function. Note that this function is strictly convex in (x, u) and twice differentiable.

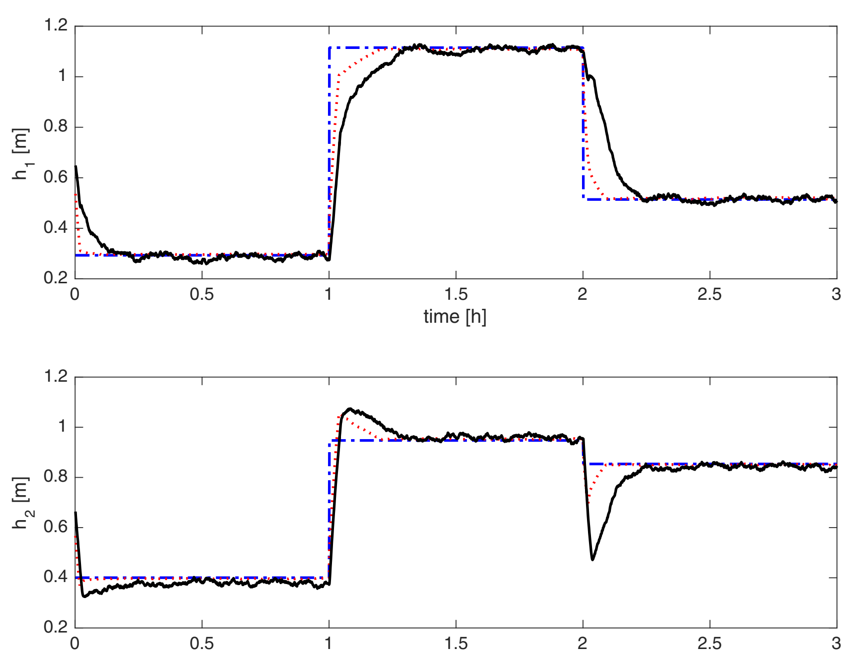

Fig. 2. Time evolution of the outputs h_1 and h_2: system output in black solid line, artificial reference in red dotted line, economic optimum in blue dash-dotted line. (For interpretation of the references to color in this legend, the reader is referred to the web version of the article.)

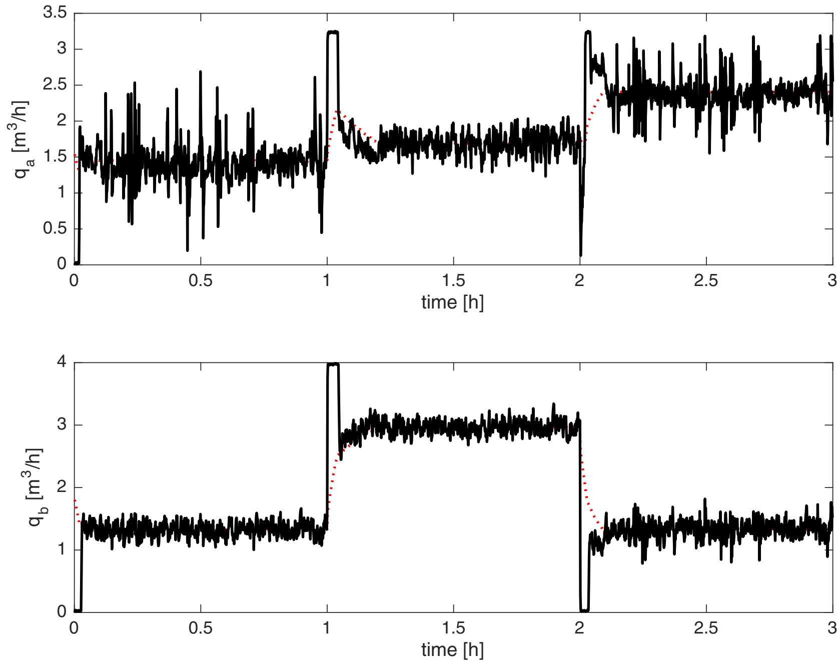

Fig. 3. Time evolution of the control inputs q_a and q_b: system output in black solid line, artificial reference in red dotted line. (For interpretation of the references to color in this legend, the reader is referred to the web version of the article.)

4.2. Dynamic simulations

In order to observe the dynamic performance of the system, it is considered a starting point that is the linearization point of the nominal model h_i^o.

On the other hand, three economic costs have been considered, based on the following values of prices: $p_1 = (5, 10)$ and $p_2 = (0.5, 100)$ and $p_3 = (5, 100)$. For these cases the economically optimal steady conditions and optimal costs are, respectively:

$$(y_{s1}^{eco}, u_{s1}^{eco}) = (0.2988, 0.3986, 1.452, 1.333), \quad f_{eco}(y_{s1}^{eco}, u_{s1}^{eco}, p_1) = 13.6553,$$
$$(y_{s2}^{eco}, u_{s2}^{eco}) = (1.1090, 0.9473, 1.671, 2.974), \quad f_{eco}(y_{s2}^{eco}, u_{s2}^{eco}, p_2) = 16.9311,$$
$$(y_{s3}^{eco}, u_{s3}^{eco}) = (0.5198, 0.8519, 2.408, 1.333), \quad f_{eco}(y_{s3}^{eco}, u_{s3}^{eco}, p_3) = 29.1083.$$

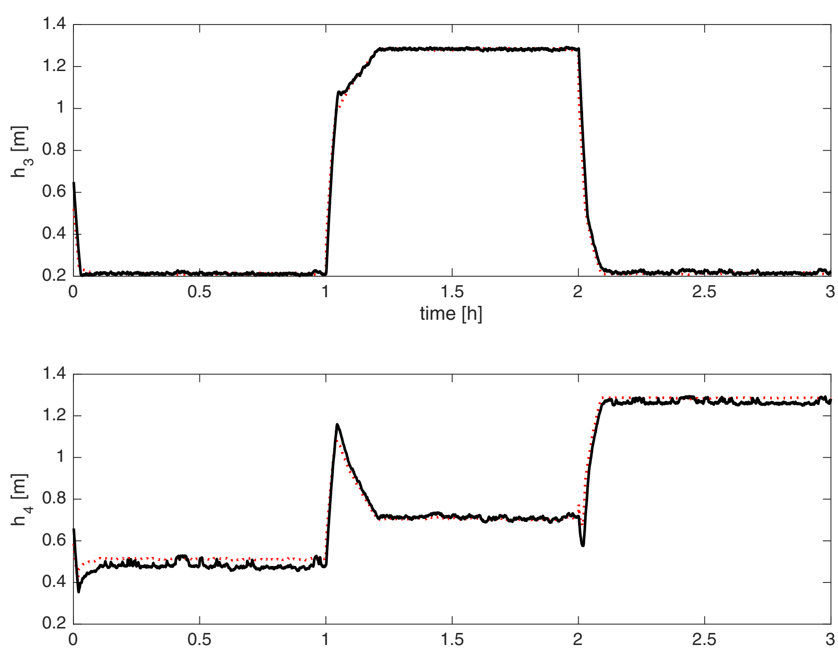

Fig. 4. Time evolution of state h_3 *and* h_4: system output in black solid line, artificial reference in red dotted line. (For interpretation of the references to color in this legend, the reader is referred to the web version of the article.)

The results of the simulation are presented in Figs. 2–7.

In particular, in Fig. 2 the evolution of the control outputs h_1 and h_2 is shown. The economic setpoint, the artificial references and the real output are depicted, respectively, in blue dashdotted, red dashed and black solid lines.

Figs. 3 and 4 show the evolution of the control inputs q_a and q_b and the evolution of h_3 and h_4, respectively.

Fig. 5 shows the economic cost for the different values of p_1, p_2, p_3. We can observe that the economic cost function robustly converges to the optimal value of f_{eco} for all changes of p.

To simulate the disturbance, we considered a truncated Gaussian distribution. The output space evolution of this simulation is shown in Fig. 6. Note how the closed-loop system converges and is maintained inside the blue shaded sets. Those sets represent the projections onto the output space of the minimal RPIs centered in the economic equilibria, that is $y_{si}^{eco} \oplus C\mathcal{R}_\infty$.

On the other hand, Fig. 7 shows the output space evolution for a simulation where the additive disturbances are taken as constant and periodic, as shown in Fig. 8. Such disturbances were generated using the Matlab function *square*(T), which generates a square wave with period 2π for the elements of time vector T.

It can be observed that in both cases $y \to y_s^{eco} \oplus C\mathcal{R}_\infty$ as $\bar{y} \to y_s^{eco}$, with the difference that in the first case the system remains near y_s^{eco} while in the second case, due to the type of disturbance, it is near the limit of the set $y_s^{eco} \oplus C\mathcal{R}_\infty$.

It is important to note that:

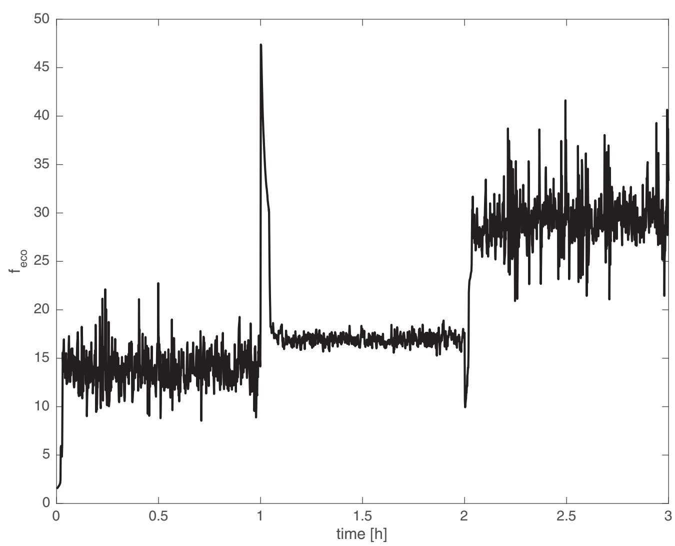

Fig. 5. Evolution of the economic cost f_{eco}, for different values of p.

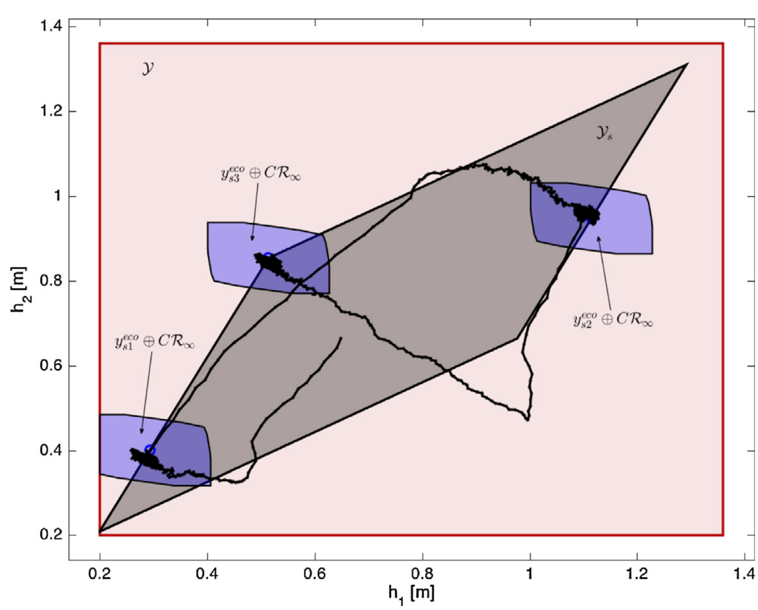

Fig. 6. Output space evolution of the closed-loop system in case of a truncated Gaussian distribution. (For interpretation of the references to color in text, the reader is referred to the web version of the article.)

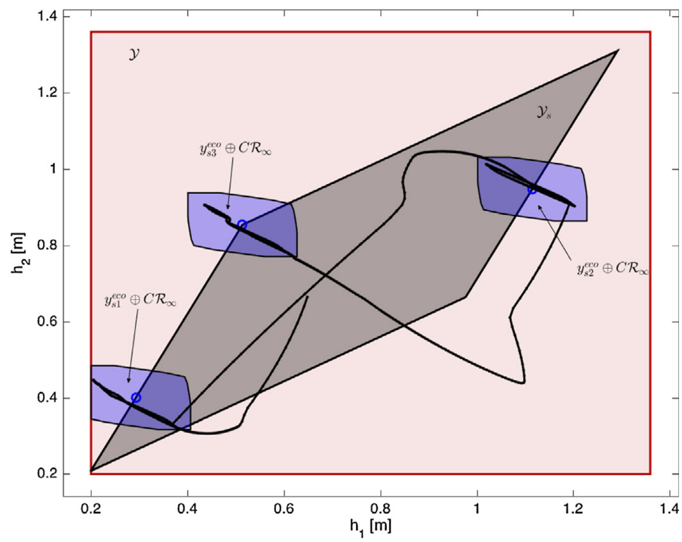

Fig. 7. Output space evolution of the closed-loop system in case of periodic and constant disturbances.

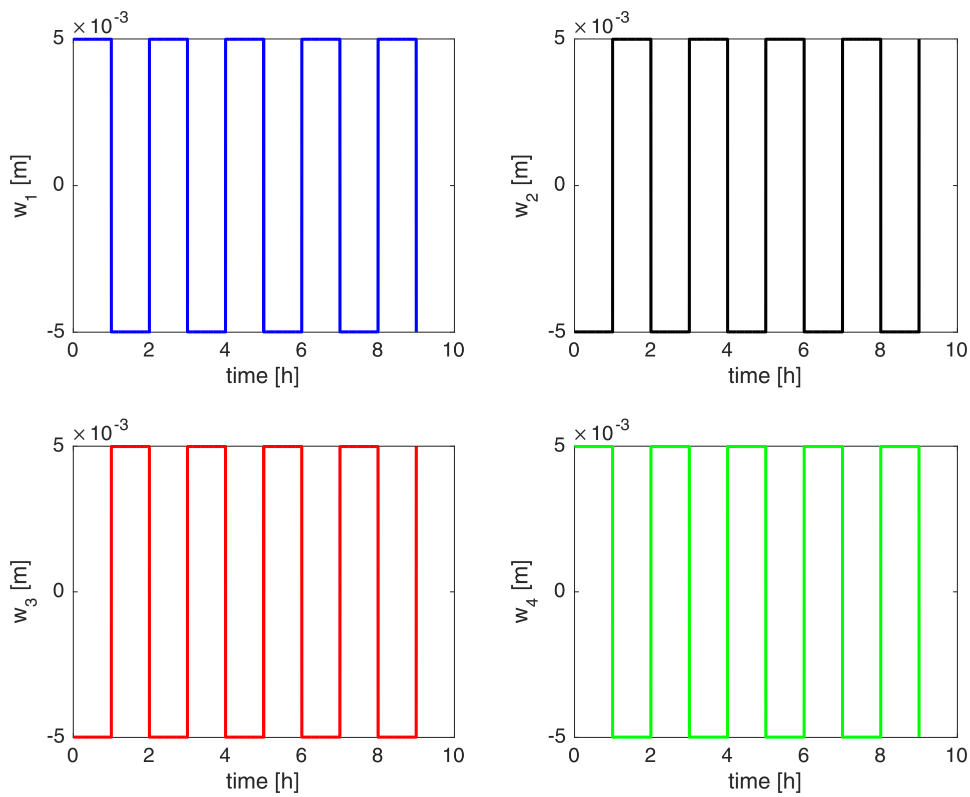

Fig. 8. Time evolution of the periodic disturbances.

- the state evolution never leaves $y_s^{eco} \oplus \mathcal{CR}_\infty$ once it is inside this set. Only when a change occurs in the parameter p, the controller brings the system outside the set, to go to the new operating point.
- the controller brings the system as close as possible to the economically optimal operating point. The deviation is due to the presence of w.
- the artificial reference allows the controller to maintain the feasibility when the economic setpoint changes. This fact can be observed in Fig. 2: when the economic cost changes, and hence the setpoint (blue dash-dotted line) changes, the artificial reference (in red dotted line) provides an admissible and reachable reference, which can be followed by the controller, avoiding any loss of feasibility.

Remark 5. Note that the online computational demand of the proposed robust controller is not greater than the one of a nominal one, since every polytopic set involved in the online MPC optimization problem is computed offline.

5. Conclusions

In this work the design of a robust formulation for the economic gradient-based MPC presented in [2], was proposed, for the case of linear systems with additive and bounded disturbances.

The proposed controller is able to bring the system to an economic optimal operation, maintaining the closed-loop system in a region of the state space around the economically optimal operation point, given by the minimal Robust Positively Invariant Set. Recursive feasibility and robust stability for any realization of the bounded disturbance was proved.

Moreover, the proposed controller still enjoys the good properties of the nominal gradient-based economic MPC such as:

- **Simple optimization problem.** The controller implementation requires the solution of just one QP.
- **Simplicity.** There is no need to compute the Hessian of f_{eco}, provided that an heuristic procedure is used to compute $\tilde{\lambda}$.
- **Feasibility.** The controller remains feasible under any change of the economic objective and any disturbance realization, thanks to the use of artificial references and of a relaxed terminal constraint.

Acknowledgment

This work was supported by the Argentinian National Scientific and Technical Research Council, CONICET.

Appendix A. Proof of Theorem 2

Robust cost decreasing

Consider the suboptimal solution to problem $P_N(x(k), p)$ at time k, given by $\mathbf{v}^{so}(x(k), p) = \{\mathbf{c}^{so}(x(k), p), x_s^{so}(x(k), p), u_s^{so}(x(k), p)\}$.

The value of the cost function at time k, due to $\mathbf{v}^{so}(x(k), p)$, is $V_N^{so}(x(k), p)$, where[1]

$$V_N^{so}(x(k), p) = \sum_{j=0}^{N-1} \|\bar{x}^{so}(j;k) - x_s^{so}(k)\|_Q^2 + \|u^{so}(j;k) - u_s^{so}(k)\|_R^2 + \|\bar{x}^{so}(N;k) - x_s^{so}(k)\|_P^2 + V_{eco}(x_s^{so}(k), u_s^{so}(k), p)$$

where

$$\mathbf{u}^{so}(k) = \{u^{so}(0;k), u^{so}(1;k), \ldots, u^{so}(N-1;k)\}$$

$$\mathbf{c}^{so}(k) = \{c^{so}(0;k), c^{so}(1;k), \ldots, c^{so}(N-1;k)\}$$

$$\bar{\mathbf{x}}^{so}(k) = \{\bar{x}^{so}(0;k), \bar{x}^{so}(1;k), \ldots, \quad \bar{x}^{so}(N-1;k), \bar{x}^{so}(N;k)\}$$

with $\bar{x}^{so}(0;k) = x(k)$, $\bar{x}^{so}(j;k) = A\bar{x}^{so}(j-1;k) + Bu^{so}(j-1;k)$, $\bar{x}^{so}(N;k) \in \Omega_t^x = Proj_x(\Omega_t)$ (due to the terminal stabilizing constraint) and $u^{so}(j;k) = K(\bar{x}^{so}(j;k) - x_s^{so}(k)) + u_s^{so}(k) + c^{so}(j;k)$

Define the control sequence

$$\tilde{\mathbf{c}}(x(k+1), p) = \{c^{so}(1;k), \ldots, c^{so}(N-1;k), 0\}$$

and define $\tilde{x}_s(k+1) = x_s^{so}(k)$, and $\tilde{u}_s(k+1) = u_s^{so}(k)$. Define also the following sequence:

$$\begin{aligned} \tilde{\mathbf{u}}(k+1) &= \{\tilde{u}(0;k+1), \tilde{u}(1;k+1), \ldots, \tilde{u}(N-2;k+1), \tilde{u}(N-1;k+1)\} \\ &= \{u^{so}(1;k), u^{so}(2;k), \ldots, u^{so}(N-1;k), \tilde{u}(N-1;k+1)\} \end{aligned}$$

with $\tilde{u}(N-1;k+1) = K(\bar{x}^{so}(N;k) - x_s^{so}(k)) + u_s^{so}(k) + \overbrace{\tilde{c}(N-1;k+1)}^{=0}$. Define also the sequence of predictions starting from the uncertain measured state $x(k+1)$ due to the feasible control sequence $\tilde{\mathbf{u}}(k+1)$

$$\tilde{\mathbf{x}}(k+1) = \{\tilde{x}(0;k+1), \tilde{x}(1;k+1), \ldots, \tilde{x}(N;k+1)\},$$

with $\tilde{x}(0;k+1) = x(k+1)$, $\tilde{x}(j;k+1) = A\tilde{x}(j-1;k+1) + B\tilde{u}(j-1;k+1)$, $j \quad in\mathbb{I}_{[1:N]}$

From Lemmas 1–3, it is derived that $\tilde{\mathbf{v}}(x(k+1), p) = \{\tilde{\mathbf{c}}(k+1), \tilde{x}_s(k+1), \tilde{u}_s(k+1)\}$ is a feasible solution to problem $P_N(x(k+1), p)$. Moreover, notice that, from Lemma 1, $\tilde{x}(j;k+1) = \bar{x}^{so}(j+1;k) + A_K^j w(k)$, for $j \quad in\mathbb{I}_{[1:N]}$, and from Lemma 4 $\tilde{x}(N;k+1) \in \Omega_t^x$.

The value of the cost function, at time $k+1$, due to the feasible solution $\tilde{\mathbf{v}}(x(k+1), p)$, is given by

$$\begin{aligned} V_N(x(k+1), p; \tilde{\mathbf{v}}(x(k+1), p)) &= \sum_{j=0}^{N-1} \|\tilde{x}(j;k+1) - \tilde{x}_s(k+1)\|_Q^2 + \|\tilde{u}(j;k+1) - \tilde{u}_s(k+1)\|_R^2 + \|\tilde{x}(N;k+1) - \tilde{x}_s(k+1)\|_P^2 \\ &\quad + V_{eco}(\tilde{x}_s(k+1), \tilde{u}_s(k+1), p) \\ &= \sum_{j=0}^{N-1} \|\tilde{x}(j;k+1) - x_s^{so}(k)\|_Q^2 + \|\tilde{u}(j;k+1) - u_s^{so}(k)\|_R^2 + \|\tilde{x}(N;k+1) - x_s^{so}(k)\|_P^2 + V_{eco}(x_s^{so}(k), u_s^{so}(k), p) \end{aligned}$$

[1] In what follows, the dependence from (x, p) will be omitted for the sake of clarity, i.e., $c^{so}(j;k)$ will denote $c^{so}(j;x(k), p)$ and $x_s^{so}(k)$ will denote $x_s^{so}(x(k), p)$.

As standard in MPC [24, Chapter 2], let us compare the two costs $V_N(x(k+1), p; \tilde{\mathbf{v}}(k+1))$ and $V_N^{so}(x(k), p)$. Since the measured state $x(k+1)$ is uncertain, let us define the following

$$\begin{aligned}\tilde{\Delta} V_N &= V_N(x(k+1), p; \tilde{\mathbf{v}}(k+1)) - V_N^{so}(x(k), p)\\ &= V_N(x(k+1), p; \tilde{\mathbf{v}}(k+1)) - \overline{V}_N(\overline{x}(k+1), p; \tilde{\mathbf{v}}(k+1)) + \overline{V}_N(\overline{x}(k+1), p; \tilde{\mathbf{v}}(k+1)) - V_N^{so}(x(k), p)\\ &= \tilde{\Delta} V_N^{w} + \tilde{\Delta} V_N^{w=0}\end{aligned}$$

where

$$\begin{aligned}\tilde{\Delta} V_N^{w} &= V_N(x(k+1), p; \tilde{\mathbf{v}}(k+1)) - \overline{V}_N(\overline{x}(k+1), p; \tilde{\mathbf{v}}(k+1))\\ \tilde{\Delta} V_N^{w=0} &= \overline{V}_N(\overline{x}(k+1), p; \tilde{\mathbf{v}}(k+1)) - V_N^{so}(x(k), p)\end{aligned}$$

and $\overline{x}(k+1)$ is the successor state if $w(k)=0$, which means (from Lemma 1) that the successor state is only given by the evolution of the nominal model, that is $\overline{x}(k+1) = \overline{x}^{so}(1; k) = Ax(k) + Bu^{so}(0; k)$.

Then,

$$\begin{aligned}\Delta V_N^{w=0} &= \overline{V}_N(\overline{x}^{so}(1; k), p; \tilde{\mathbf{v}}(k+1)) - V_N^{so}(x(k), p)\\ &= \sum_{j=1}^{N-1} \|\overline{x}^{so}(j; k) - x_s^{so}(k)\|_Q^2 + \|u^{so}(j; k) - u_s^{so}(k)\|_R^2 + \|\overline{x}^{so}(N; k) - x_s^{so}(k)\|_Q^2 + \|u^{so}(N; k) - u_s^{so}(k)\|_R^2 + \|\overline{x}^{so}(N+1; k) - x_s^{so}(k)\|_P^2\\ &\quad + V_{eco}(x_s^{so}(k), u_s^{so}(k), p) - \|\overline{x}^{so}(0; k) - x_s^{so}(k)\|_Q^2 - \|u^{so}(0; k) - u_s^{so}(k)\|_R^2 - \left(\sum_{j=1}^{N-1} \|\overline{x}^{so}(j; k) - x_s^{so}(k)\|_Q^2 + \|u^{so}(j; k) - u_s^{so}(k)\|_R^2\right)\\ &\quad - \|\overline{x}^{so}(N; k) - x_s^{so}(k)\|_P^2 - V_{eco}(x_s^{so}(k), u_s^{so}(k), p)\\ &= -\|x(k) - x_s^{so}(k)\|_Q^2 - \|u^{so}(0; k) - u_s^{so}(k)\|_R^2 + \|\overline{x}^{so}(N; k) - x_s^{so}(k)\|_Q^2 + \|K(\overline{x}^{so}(N; k) - x_s^{so}(k))\|_R^2 + \|A_K(\overline{x}^{so}(N; k)\\ &\quad - x_s^{so}(k))\|_P^2 - \|\overline{x}^{so}(N; k) - x_s^{so}(k)\|_P^2\\ &= -\|x(k) - x_s^{so}(k)\|_Q^2 - \|u^{so}(0; k) - u_s^{so}(k)\|_R^2 + \|\overline{x}^{so}(N; k) - x_s^{so}(k)\|_{(Q+K'RK+A'_K PA_K - P)}^2\\ &= -\|x(k) - x_s^{so}(k)\|_Q^2 - \|u^{so}(0; k) - u_s^{so}(k)\|_R^2.\end{aligned}$$

Let us analyze $\tilde{\Delta} V_N^{w}$. From the definitions given above,

$$\begin{aligned}\tilde{\Delta} V_N(w) &= V_N(x(k+1), p; \tilde{\mathbf{v}}(k+1)) - \bar{V}_N(\bar{x}^{so}(1; k), p; \tilde{\mathbf{v}}(k+1))\\ &= V_N^{dyn}(x(k+1); \tilde{\mathbf{c}}, x_s^{so}(k), u_s^{so}(k)) + V_{eco}(x_s^{so}(k), u_s^{so}(k), p) - V_N^{dyn}(\bar{x}^{so}(1; k); \tilde{\mathbf{c}}, x_s^{so}(k), u_s^{so}(k)) - V_{eco}(x_s^{so}(k), u_s^{so}(k), p)\\ &= \sum_{j=0}^{N-1} \|\tilde{c}(j; k+1)\|_\Psi^2 + \|x(k+1) - x_s^{so}(k)\|_P^2 - \sum_{j=0}^{N-1} \|\tilde{c}(j; k+1)\|_\Psi^2 - \|\bar{x}^{so}(1; k) - x_s^{so}(k)\|_P^2\\ &= \|x(k+1) - x_s^{so}(k)\|_P^2 - \|\bar{x}^{so}(1; k) - x_s^{so}(k)\|_P^2\\ &\leq \left| \|x(k+1) - x_s^{so}(k)\|_P^2 - \|\bar{x}^{so}(1; k) - x_s^{so}(k)\|_P^2 \right|\\ &\leq \sigma(|x(k+1) - \bar{x}^{so}(1; k)|)\\ &= \sigma(|w(k)|)\end{aligned}$$

where σ is a $\mathcal{K}$-function, the third equality comes from Remark 3, the second inequality comes from the fact that $\|\cdot\|_P^2$ is uniformly continuous (since $\|\cdot\|_P^2$ is continuous and $\mathcal{X}$ is compact), and the last equality comes from Lemma 1.

Then

$$\tilde{\Delta} V_N \leq -\|x(k) - x_s^{so}(k)\|_Q^2 - \|u^{so}(0; k) - u_s^{so}(k)\|_R^2 + \sigma(|w(k)|)$$

By Theorem 1, $V_N^{so}(x(k+1), p) \leq V_N(x(k+1), p; \tilde{\mathbf{v}}(k+1))$. Therefore,

$$\begin{aligned}\Delta V_N^{so} &= V_N^{so}(x(k+1), p) - V_N^{so}(x(k), p)\\ &\leq -\|x(k) - x_s^{so}(k)\|_Q^2 - \|u^{so}(0; k) - u_s^{so}(k)\|_R^2 + \sigma(\|w(k)\|)\\ &\leq -\alpha(\|x(k) - x_s^{so}(k)\|) + \sigma(\|w(k)\|)\end{aligned} \tag{A.1}$$

where α is a $\mathcal{K}$-function.

Appendix B. Lemmata

In what follows, we introduce some lemmas necessary to the proof of Theorem 2.

Let us first define as $\mathbf{v}^{so}(x(k), p)$ the suboptimal solution of problem $P_N(x, p)$ at the time instant k, where

$$\mathbf{v}^{so}(x(k), p) = \{\mathbf{c}^{so}(x(k), p), x_s^{so}(x(k), p), u_s^{so}(x(k), p)\}$$

$$\mathbf{c}^{so}(x(k),p)=\{c^{so}(0;x(k),p),c^{so}(1;x(k),p),\ldots,c^{so}(N-1;x(k),p)\}$$

Define the control sequence

$$\tilde{\mathbf{c}}(x(k+1),p)=\{c^{so}(1;x(k),p),\ldots,c^{so}(N-1;x(k),p),0\}$$

and define $\tilde{x}_s(x(k+1),p)=x_s^{so}(x(k),p)$, and $\tilde{u}_s(x(k+1),p)=u_s^{so}(x(k),p)$. Moreover, define as $\tilde{x}(j;x(k+1),p)$ the jth step prediction, given $x(k+1)$. Hence

$$\tilde{x}(j;x(k+1),p)=\tilde{x}_s(x(k+1),p)+A_K^j(x(k+1)-\tilde{x}_s(x(k+1),p))+\sum_{i=0}^{j-1}A_K^iB\tilde{c}(j-i-1;x(k+1),p)$$

In what follows, the dependence from (x,p) will be omitted for the sake of clarity, namely, $x(j;k)$ will denote $x(j;x(k),p)$.

Lemma 1. [15, Lemma 1] *For all* $j=0,\ldots,N-1$

$$\tilde{x}(j;k+1)-x(j+1;k)=A_K^jw(k)$$

Proof. Since

$$x^{so}(j+1;k)=x_s^{so}(k)+A_K^j(x^{so}(1;k)-x_s^{so}(k))+\sum_{i=0}^{j-1}A_K^iBc^{so}(j-i;k)$$

and

$$\begin{aligned}\tilde{x}(j;k+1)&=\tilde{x}_s(k+1)+A_K^j(x(k+1)-\tilde{x}_s(k+1))+\sum_{i=0}^{j-1}A_K^iB\tilde{c}(j-i-1;k+1)\\&=x_s^{so}(k)+A_K^j(x(k+1)-x_s^{so}(k))+\sum_{i=0}^{j-1}A_K^iBc^{so}(j-i;k)\end{aligned}$$

hence

$$\tilde{x}(j;k+1)-x(j+1;k)=A_K^j[x(k+1)-x^{so}(1;k)]=A_K^jw(k)$$

□

Lemma 2. [15, Lemma 2] *If* $x^{so}(j;k)\in\mathcal{X}_j$, *then* $\tilde{x}(j-1;k+1)\in\mathcal{X}_{j-1}$, *for all* $j=0,\ldots,N$.

Proof. Since $\tilde{x}(j-1;k+1)=x^{so}(j;k)+A_K^{j-1}w(k)$, then

$$\begin{aligned}\tilde{x}(j-1;k+1)\in\mathcal{X}_j\oplus A_K^{j-1}\mathcal{W}&=\mathcal{X}\ominus[\oplus_{i=0}^{j-1}A_K^i\mathcal{W}]\oplus A_K^{j-1}\mathcal{W}\\&\subseteq\mathcal{X}\ominus[\oplus_{i=0}^{j-2}A_K^i\mathcal{W}]\\&\subseteq\overline{\mathcal{X}}_{j-1}\end{aligned}$$

□

Lemma 3. [15, Lemma 3] *If* $u^{so}(j;k)\in\mathcal{U}_j$, *then* $\tilde{u}(j-1;k+1)\in\mathcal{U}_{j-1}$, *for all* $j=1,\ldots,N-1$.

Proof. Taking into account that

$$\begin{aligned}u^{so}(j;k)&=K(x^{so}(j;k)-x_s^{so}(k))+u_s^{so}(k)+c^{so}(j;k)\\\tilde{u}(j-1;k+1)&=K(\tilde{x}(j-1;k+1)-\tilde{x}_s(k+1))+\tilde{u}_s(k+1)+\tilde{c}(j-1;k+1)\\&=K(\tilde{x}(j-1;k+1)-x_s^{so}(k))+u_s^{so}(k)+\tilde{c}(j-1;k+1)\end{aligned}$$

and

$$K(x^{so}(j;k)-x_s^{so}(k))+u_s^{so}(k)+c^{so}(j;k)=K(\tilde{x}(j-1;k+1)-x_s^{so}(k))-KA_K^{j-1}w(k)+u_s^{so}(k)+\tilde{c}(j-1;k+1)$$

hence

$$\tilde{u}(j-1;k+1)=K(\tilde{x}(j-1;k+1)-x_s^{so}(k))+u_s^{so}(k)+\tilde{c}(j-1;k+1)\in\mathcal{U}_j\oplus KA_K^{j-1}\mathcal{W}$$

and

$$\mathcal{U}_j\oplus KA_K^{j-1}\mathcal{W}=\mathcal{U}\ominus K\mathcal{R}_j\oplus KA_K^{j-1}\mathcal{W}=\mathcal{U}\ominus K\mathcal{R}_{j-1}=\mathcal{U}_{j-1}$$

□

Lemma 4. [15, Lemma 4]*[Recursive feasibility of the terminal constraint] For all $k \geq 0$,*

$$(\bar{x}^{so}(N;k), x_s^{so}(k), u_s^{so}(k)) \in \Omega_t$$

Proof. Consider that at time k, $(\bar{x}^{so}(N;k), x_s^{so}(k), u_s^{so}(k)) \in \Omega_t$. Since $\Omega_t = \Sigma_t \ominus (\mathcal{R}_N \times \{0\} \times \{0\})$, hence

$$(\bar{x}^{so}(N-1;k+1), x_s^{so}(k+1), u_s^{so}(k+1)) \in \Sigma_t \ominus (\mathcal{R}_N \times \{0\} \times \{0\}) \oplus (A_K^{N-1}\mathcal{W} \times \{0\} \times \{0\})$$

Then, since $(\bar{x}^{so}(N;k+1), x_s^{so}(k+1), u_s^{so}(k+1)) = A_a(\bar{x}^{so}(N-1;k+1), x_s^{so}(k+1), u_s^{so}(k+1))$, hence

$$(\bar{x}^{so}(N;k+1), x_s^{so}(k+1), u_s^{so}(k+1)) \in A_a(\Sigma_t \ominus (\mathcal{R}_N \times \{0\} \times \{0\}) \oplus (A_K^{N-1}\mathcal{W} \times \{0\} \times \{0\}))$$

Taking into account that

$$\begin{aligned} A_a(\Sigma_t \ominus (\mathcal{R}_N \times \{0\} \times \{0\}) & \oplus (A_K^{N-1}\mathcal{W} \times \{0\} \times \{0\})) \\ &= A_a\Sigma_t \ominus (A_K\mathcal{R}_N \times \{0\} \times \{0\}) \oplus (A_K^{N}\mathcal{W} \times \{0\} \times \{0\}) \\ &= A_a\Sigma_t \ominus (\oplus_{j=1}^{N} A_K^{j}\mathcal{W} \times \{0\} \times \{0\}) \oplus (A_K^{N}\mathcal{W} \times \{0\} \times \{0\}) \\ &= A_a\Sigma_t \ominus (\oplus_{j=1}^{N-1} A_K^{j}\mathcal{W} \times \{0\} \times \{0\}) \ominus (A_K^{N}\mathcal{W} \times \{0\} \times \{0\}) \oplus (A_K^{N}\mathcal{W} \times \{0\} \times \{0\}) \\ &\subseteq A_a\Sigma_t \ominus (\oplus_{j=1}^{N-1} A_K^{j}\mathcal{W} \times \{0\} \times \{0\}) \\ &\subseteq (\Sigma_t \ominus (\mathcal{W} \times \{0\} \times \{0\})) \ominus (\oplus_{j=1}^{N-1} A_K^{j}\mathcal{W} \times \{0\} \times \{0\}) \\ &= \Sigma_t \ominus (\oplus_{j=0}^{N-1} A_K^{j}\mathcal{W} \times \{0\} \times \{0\}) \\ &= \Sigma_t \ominus (\mathcal{R}_N \times \{0\} \times \{0\}) \end{aligned}$$

where the second from last equality comes from $A_a\Sigma_t \oplus (\mathcal{W} \times \{0\} \times \{0\}) \subseteq \Sigma_t \Leftrightarrow A_a\Sigma_t \subseteq \Sigma_t \ominus (\mathcal{W} \times \{0\} \times \{0\})$.

Hence,

$$(\bar{x}^{so}(N;k+1), x_s^{so}(k+1), u_s^{so}(k+1)) \in \Sigma_t \ominus (\mathcal{R}_N \times \{0\} \times \{0\}) = \Omega_t$$

□

Lemma 5. [14, Lemma 6] *Consider that assumptions (1)–(4) hold and $w = 0$. Let x_s^{eco} be the optimal steady state, such that function $V_{eco}(x, u, p)$ is minimized. For all $x \in \Upsilon_N$ and $x_s^{so}(x) \in \mathcal{X}_s = Proj_x(\mathcal{Z}_s)$, define the function $e(x) = x - x_s^{so}(x)$. Then, there exists a $\mathcal{K}$-function α_e such that*

$$\|e(x)\| \geq \alpha_e(\|x - x_s^{eco}\|) \tag{B.1}$$

Proof. Notice that, due to convexity, $e(x)$ is a continuous function [24]. Moreover, let us consider these two cases.

1. $\|e(x)\| = 0$ iff $x = x_s^{eco}$. In fact, (i) if $e(x) = 0$, then $x = x_s^{so}(x)$, and from Lemma 6, this implies that $x_s^{so}(x) = x_s^{eco}$; (ii) if $x = x_s^{eco}$, then by optimality $x_s^{so}(x) = x_s^{eco}$, and then $x = x_s^{so}(x)$. Then, $\|e(x)\| = 0$.
2. $\|e(x)\| > 0$ for all $\|x - x_s^{eco}\| > 0$. In fact, for any $x \neq x_s^{eco}$, $\|e(x)\| \neq 0$ and moreover $\|x - x_s^{eco}\| > 0$. Then, $\|e(x)\| > 0$.

Then, since Υ_N is compact, in virtue of [25, Ch. 5, Lemma 6, pag. 148], there exists a $\mathcal{K}$-function α_e such that $\|e(x)\| \geq \alpha_e(\|x - x_s^{eco}\|)$ on Υ_N. □

Lemma 6. [2, Lemma 3] *Consider that assumptions (1)–(4) hold and $w = 0$. Let the optimal solution to Problem $P_N(x, p)$, at time k, be such that $x(k) = x_s^{so}(x(k))$, and $\bar{u}(k) = u_s^{so}(x(k))$. Let (x_s^{eco}, u_s^{eco}) be the optimal, such that function $V_{eco}(x, u, p)$ is minimized. Then $x(k) = x_s^{eco}$, and $u(k) = u_s^{eco}$.*

Proof. Consider that $(x_s^{so}(x(k)), u_s^{so}(x(k)))$ is the solution to $P_N(x, p)$ obtained at time k. Then, recalling that $w = 0$,

$$V_N^{so}(x(k)) = V_{eco}(x_s^{so}(x(k)), u_s^{so}(x(k)), p)$$

In what follows, the time dependence is removed for the sake of clarity.

This Lemma will be proved by contradiction. Assume that the stationary point at time k is not the optimal one, that is $(x_s^{so}(x), u_s^{so}(x)) \neq (x_s^{eco}, u_s^{eco})$. Then, by convexity, there exists a $\beta \in [0, 1]$ such that

$$(\hat{x}_s, \hat{u}_s) = \beta(x_s^{so}(x), u_s^{so}(x)) + (1-\beta)(x_s^{eco}, u_s^{eco})$$

characterizes a stationary point and moreover

$$V_{eco}(\hat{x}_s, \hat{u}_s, p) < V_{eco}(x_s^{so}, u_s^{so}, p) \tag{B.2}$$

That is, since the real system is not at the optimal point (x_s^{eco}, u_s^{eco}), it is more convenient to move towards $(\hat{x}_s, \hat{u}_s)$, than to remain in $(x_s^{so}(x), u_s^{so}(x))$. Define the feasible sequence $\hat{\mathbf{u}} = \{\hat{u}(0), \hat{u}(1), \ldots, \hat{u}(N-1)\}$ that drive the closed-loop system from $(x_s^{so}(x), u_s^{so}(x))$ to $(\tilde{x}_s, \tilde{u}_s)$

in N steps. Notice that, since $x(k)=x_s^{so}(k)$ then $\hat{c}(j)=0$, and $\hat{u}(j)=K(\hat{x}(j)-\hat{x}_s)+\hat{u}_s$, where $\hat{x}(j+1)=A\hat{x}(j)+B\hat{u}(j)$, $\hat{x}(0)=x_s^{so}(x)$. Then, the cost to drive the system from $(x_s^{so}(x), u_s^{so}(x))$ to $(\hat{x}_s, \hat{u}_s)$ is given by

$$\begin{aligned} V_N(x_s^{so}(x), p; \hat{\mathbf{u}}, \hat{x}_s, \hat{u}_s) &= \sum_{j=0}^{N-1} \|\hat{x}(j)-\hat{x}_s\|_Q^2 + \|\hat{u}(j)-\hat{u}_s\|_R^2 + \|\hat{x}(N)-\hat{x}_s\|_P^2 + V_{eco}(\hat{x}_s, \hat{u}_s, p) \\ &= \|x_s^{so}(x)-\hat{x}_s\|_P^2 + V_{eco}(\hat{x}_s, \hat{u}_s, p) \\ &= (1-\beta)^2 \|x_s^{so}(x)-x_s^{eco}\|_P^2 + V_{eco}(\hat{x}_s, \hat{u}_s, p) \end{aligned}$$

Now define $W(\beta)=(1-\beta)^2\|x_s^{so}(x)-x_s^{eco}\|_P^2+V_{eco}(\hat{x}_s, \hat{u}_s, p)$ and notice that for $\beta=1$, $W(1)=V_{eco}(x_s^{so}(x), u_s^{so}(x), p)$. Taking the partial of this function with respect to β, and evaluating it for $\beta=1$ we obtain:

$$\left.\frac{\partial W}{\partial \beta}\right|_{\beta=1} = g^{0'}(x_s^{so}(x), u_s^{so}(x), p)$$

where $g^{0'} \in \partial V_{eco}(x_s^{so}(x), u_s^{so}(x), p)$, defining $\partial V_{eco}(x_s^{so}(x), u_s^{so}(x), p)$ as the subdifferential of $V_{eco}(x_s^{so}(x), u_s^{so}(x), p)$. From convexity and from (B.2),

$$\begin{aligned} \left.\frac{\partial W}{\partial \beta}\right|_{\beta=1} &= g^{0'}(x_s^{so}(x), u_s^{so}(x), p) \\ &\geq V_{eco}(x_s^{so}(x), u_s^{so}(x), p) - V_{eco}(\hat{x}_s, \hat{u}_s, p) > 0 \end{aligned}$$

This means that there exists a value of $\beta \in [0, 1)$ such that $V_N(x_s^{so}(x), p; \hat{\mathbf{u}}, \hat{x}_s, \hat{u}_s)$ is smaller than the value of the cost $V_N(x_s^{so}(x), p; \hat{\mathbf{u}}, \hat{x}_s, \hat{u}_s)$ for $\beta=1$, which is $V_{eco}(x_s^{so}(x), u_s^{so}(x), p)$. This contradicts the optimality of the solution to Problem $P_N(x, p)$ at time k. Then it has to be that $(x_s^{so}(x), u_s^{so}(x))=(x_s^{eco}, u_s^{eco})$. Moreover, we can state that this point is the one that minimizes the offset cost function $V_{eco}(x, u, p)$. So the Lemma is proved. □

References

[1] V. Adetola, M. Guay, Integration of real-time optimization and model predictive control, J. Process Control 20 (2) (2010) 125–133.
[2] T. Alamo, A. Ferramosca, A.H. González, D. Limon, D. Odloak, A gradient-based strategy for the one-layer RTO + MPC controller, J. Process Control 24 (4) (2014) 435–447.
[3] I. Alvarado, Model Predictive Control for Tracking Constrained Linear Systems (Ph.D. thesis), Univ. de Sevilla, 2007.
[4] I. Alvarado, D. Limon, D. Muñoz de la Peña, J.M. Maestre, M.A. Ridao, H. Scheu, W. Marquardt, R.R. Negenborn, B.D. Schutter, F. Valencia, J. Espinosa, A comparative analysis of distributed MPC techniques applied to the HD-MPC four-tank benchmark, J. Process Control 21 (2011) 800–815.
[5] D. Angeli, R. Amrit, J.B. Rawlings, On average performance and stability of economic model predictive control, IEEE Trans. Autom. Control 57 (7) (2012) 1615–1626.
[6] L.T. Biegler, Technology advances for dynamic real-time optimization, in: 10th International Symposium on Process Systems Engineering, Comput. Aided Chem. Eng. 27 (2009) 1–6.
[7] L. Chisci, J.A. Rossiter, G. Zappa, Systems with persistent disturbances: predictive control with restricted constraints, Automatica 37 (2001) 1019–1028.
[8] G. De Souza, D. Odloak, A.C. Zanin, Real time optimization (RTO) with model predictive control (MPC), Comput. Chem. Eng. 34 (12) (2010) 1999–2006.
[9] M. Diehl, R. Amrit, J.B. Rawlings, A Lyapunov function for economic optimizing model predictive control, IEEE Trans. Autom. Control 6 (3) (2011) 703–707.
[10] M. Ellis, P.D. Christofides, Integrating dynamic economic optimization and model predictive control for optimal operation of nonlinear process systems, Control Eng. Pract. 22 (1) (2014) 242–251.
[11] S. Engell, Feedback control for optimal process operation, J. Process Control 17 (3) (2007) 203–219.
[12] A. Ferramosca, A.H. González, D. Limón, D. Odloak, One-layer robust MPC: a multi-model approach, 19th IFAC World Congress. IFAC Proceedings Volumes 47 (3) (2014) 11067–11072.
[13] A. Ferramosca, D. Limon, I. Alvarado, T. Alamo, E.F. Camacho, MPC for tracking with optimal closed-loop performance, Automatica 45 (8) (2009) 1975–1978.
[14] A. Ferramosca, D. Limon, E.F. Camacho, Economic MPC for a changing economic criterion for linear systems, IEEE Trans. Autom. Control 59 (10) (2014) 2657–2667.
[15] A. Ferramosca, D. Limon, A.H. González, I. Alvarado, E.F. Camacho, Robust MPC for tracking zone regions based on nominal predictions, J. Process Control 22 (10) (2012) 1966–1974.
[16] A. Ferramosca, D. Limon, A.H. González, D. Odloak, E.F. Camacho, MPC for tracking zone regions, J. Process Control 20 (4) (2010) 506–516.
[17] A. Ferramosca, J.B. Rawlings, D. Limon, E.F. Camacho, Economic MPC for a changing economic criterion, in: Proceedings of 49th IEEE Conference on Decision and Control, CDC (2010), Atlanta, GE, USA, 2010.
[18] K.H. Johansson, The quadruple-tank process: a multivariable laboratory process with an adjustable zero, IEEE Trans. Control Syst. Technol. 8 (2000) 456–465.
[19] I. Kolmanovsky, E.G. Gilbert, Theory and computation of disturbance invariant sets for discrete-time linear systems, Math. Probl. Eng. Theory Methods Appl. 4 (1998) 317–367.
[20] D. Limon, T. Alamo, D.M. Raimondo, D.M. de la Peña, J.M. Bravo, A. Ferramosca, E.F. Camacho, Input-to-state stability: an unifying framework for robust model predictive control, in: L. Magni, D.M. Raimondo, F. Allgöwer (Eds.), Nonlinear Model Predictive Control – Towards New Challenging Applications, Springer, 2009, pp. 1–26.
[21] D. Limon, I. Alvarado, T. Alamo, E.F. Camacho, MPC for tracking of piece-wise constant references for constrained linear systems, Automatica 44 (9) (2008) 2382–2387.
[22] D. Limon, I. Alvarado, T. Alamo, E.F. Camacho, Robust tube-based MPC for tracking of constrained linear systems with additive disturbances, JPC 20 (2010) 248–260.
[23] G. Pannocchia, E.C. Kerrigan, Offset-Free Receding Horizon Control of Constrained Linear Systems Subject to Time-Varying Set Points and Persistents Unmeasured Disturbances. Tech. Rep. CUED/F-INFENG/TR. 468, Department of Engineering, University of Cambridge, 2003.
[24] J.B. Rawlings, D.Q. Mayne, Model Predictive Control: Theory and Design, 1st ed., Nob-Hill Publishing, 2009.
[25] M. Vidyasagar, Nonlinear Systems Theory, 2nd ed., Prentice-Hall, 1993.
[26] L. Würth, J.B. Rawlings, W. Marquardt, Economic dynamic real-time optimization and nonlinear model predictive control on infinite horizons, IFAC Proc. Vol. 42 (11) (2009) 219–224.
[27] A.C. Zanin, M.T. de Gouva, D. Odloak, Integrating real time optimization into the model predictive controller of the FCC system, Control Eng. Pract. 10 (8) (2002) 819–831.