Linear Algebra and its Applications 478 (2015) 274–293

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

An integral formula for multiple summing norms of operators [☆]

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A R T I C L E I N F O

Article history:
Received 19 September 2014
Accepted 24 March 2015
Available online xxxx
Submitted by P. Semrl

MSC:
15A69
15A60
47B10
47H60
46G25

Keywords:
Absolutely summing operators
Multilinear operators

A B S T R A C T

We prove that the multiple summing norm of multilinear operators defined on some n-dimensional real or complex vector spaces with the p-norm may be written as an integral with respect to stable measures. As an application we show inclusion and coincidence results for multiple summing mappings. We also present some contraction properties and compute or estimate the limit orders of this class of operators.

☆ This work was partially supported by CONICET PIP 0624, ANPCyT PICT 11-1456, ANPCyT PICT 11-0738, UBACyT 1-746 and UBACyT 20020130300052BA.

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http://dx.doi.org/10.1016/j.laa.2015.03.031
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Multiple summing operators
Stable measures

0. Introduction

The rotation invariance of the Gaussian measure on $\mathbb{K}^N$, which we will denote by μ_2^N, allows us to show the Gauss–Khintchine equality. It asserts that if $c_{2,q}$ denotes the q-th moment of the one dimensional Gaussian measure, and ℓ_2^N denotes $\mathbb{K}^N$ with the Euclidean norm, then for any $\alpha \in \mathbb{K}^N$, $1 \leq q < \infty$,

$$c_{2,q}\|\alpha\|_{\ell_2^N} = \Big(\int\limits_{\mathbb{K}^N} |\langle \alpha, z\rangle|^q d\mu_2^N(z)\Big)^{1/q}. \tag{1}$$

We may interpret this formula as follows: the norm of a linear functional α on ℓ_2^N is a multiple of the L^q-norm of the linear functional with respect to the Gaussian measure on ℓ_2^N. One may ask if there is a formula like (1) for linear functionals on some other space, or even for linear or multilinear operators. For linear functionals, an answer is provided by the s-stable Lévy measure (see for example [6, 24.4]): for $s < 2$ there exists a measure on $\mathbb{K}^N$, called the *s-stable Lévy measure* and denoted by μ_s^N, which satisfies that for any $0 < q < s$, $\alpha \in \mathbb{K}^N$,

$$c_{s,q}\|\alpha\|_{\ell_s^N} = \Big(\int\limits_{\mathbb{K}^N} |\langle \alpha, z\rangle|^q d\mu_s^N(z)\Big)^{1/q}, \tag{2}$$

where

$$c_{s,q} = \Big(\int\limits_{\mathbb{K}} |z|^q d\mu_s^1(z)\Big)^{1/q}.$$

The question for linear operators is more subtle because there are many norms which are natural to consider on $\mathcal{L}(\ell_2^N)$. The first result in this direction is due to Gordon [9] (see also [6, 11.10]), who showed that the formula holds for the identity operator on ℓ_2^N, considering the absolutely p-summing norm of $id_{\ell_2^N}$, that is

$$\pi_p(id_{\ell_2^N}) = c_{2,q}\Big(\int\limits_{\mathbb{K}^N} \|z\|_{\ell_2^N}^q \, d\mu_2^N(z)\Big)^{1/q}.$$

Pietsch [16] extended this formula for arbitrary linear operators from $\ell_{s'}^N \to \ell_s^N$, $s \geq 2$ and used it to compute some limit orders (see also [17, 22.4.11]).

To generalize the formula to the multilinear setting there is again a new issue, as there are many natural candidates of classes of multilinear operators that extend the ideal of absolutely p-summing linear operators (for instance the articles [12,14] are devoted to their comparison). Among those candidates, the ideal of multiple summing multilinear operators is considered by many authors the most important of these extensions and is also the one most studied. Some of the reasons are its connections with the Bohnenblust–Hille inequality [15], or the results on the unconditional structure of the space of multiple summing operators [7]. Multiple summing operators were introduced by Bombal, Pérez-García and Villanueva [2] and independently by Matos [10]. In this note we show that multiple summing operators constitute the correct framework for a multilinear generalization of formula (1). For this we present integral formulas for the exact value of the multiple summing norm of multilinear forms and operators defined on ℓ_p^N for some values of p. Moreover, we prove that for some other finite dimensional Banach spaces these formulas hold up to some constant independent of the dimension.

One particularity of the class of multiple summing operators on Banach spaces is that, unlike the linear situation, there is no general inclusion result. In [3,13,19] the authors investigate this problem and prove several results showing that on some Banach spaces inclusion results hold, but on other spaces not. The integral formula for the multiple summing norm, together with Khintchine/Kahane type inequalities will allow us to show some new coincidence and inclusion results for multiple summing operators.

Another application of these formulas deals with unconditionality in tensor products. Defant and Pérez-García showed in [7] that the tensor norm associated to the ideal of multiple 1-summing multilinear forms preserves unconditionality on $\mathcal{L}_r$ spaces. As a consequence of our formulas, we give a simple proof of this fact for ℓ_r with $r \geq 2$. Moreover, we show that vector-valued multiple 1-summing operators also satisfy a kind of unconditionality property in the appropriate range of Banach spaces. Finally, we compute limit orders for the ideal of multiple summing operators.

Our main results are stated in Theorems 1.1 and 1.2, which give an exact formula for the multiple summing norm, and Proposition 1.3, which gives integral formulas for estimating these norms in a wider range of spaces.

1. Main results and their applications

Let $E_1, \dots, E_m, F$ be real or complex Banach spaces. Recall that an m-linear operator $T \in \mathcal{L}(E_1, \dots, E_m; F)$ is *multiple p-summing* if there exists $C > 0$ such that for all finite sequences of vectors $(x^1_{j_1})_{j_1=1}^{J_1} \subset E_1, \dots, (x^m_{j_m})_{j_m=1}^{J_m} \subset E_m$

$$\left(\sum_{j_1,\dots,j_m} \|T(x^1_{j_1}, \dots, x^m_{j_m})\|_F^p \right)^{\frac{1}{p}} \leq C w_p((x^1_{j_1})_{j_1}) \dots w_p((x^m_{j_m})_{j_m}),$$

where

$$w_p((y_j)_j) = \sup \left\{ \left(\sum_j |\gamma(y_j)|^p \right)^{1/p} \, : \, \gamma \in B_{E'} \right\}.$$

The infimum of all those constants C is the multiple p-summing norm of T and is denoted by $\pi_p(T)$. The space of multiple p-summing multilinear operators is denoted by $\Pi_p(E_1, \ldots, E_m; F)$. When $E_1 = \cdots = E_m = E$, the spaces of continuous and multiple p-summing multilinear are denoted by $\mathcal{L}(^m E; F)$ and $\Pi_p(^m E; F)$ respectively.

The following theorems are our main results. Their proofs will be given in Section 2.

Theorem 1.1. *Let ϕ be a multilinear form in $\mathcal{L}(^m \ell_r^N; \mathbb{K})$, $p < r' < 2$ or $r = 2$. Then*

$$\pi_p(\phi) = \frac{1}{c_{r',p}^m} \Big(\int\limits_{\mathbb{K}^N} \cdots \int\limits_{\mathbb{K}^N} |\phi(z^{(1)}, \ldots, z^{(m)})|^p d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)}) \Big)^{1/p}.$$

Before we state our second theorem, let us recall some necessary definitions and facts. For $1 \leq q \leq \infty$ and $1 \leq \lambda < \infty$ a normed space X is called an $\mathcal{L}_{q,\lambda}^g$*-space*, if for each finite dimensional subspace $M \subset X$ and $\varepsilon > 0$ there are $R \in \mathcal{L}(M, \ell_q^m)$ and $S \in \mathcal{L}(\ell_q^m, X)$ for some $m \in \mathbb{N}$ factoring the inclusion map $I_M^X : M \to X$ such that $\|S\|\|R\| \leq \lambda + \varepsilon$:

$$\begin{tikzcd} M \arrow[rr, hook, "I_M^X"] \arrow[dr, "R"'] & & X. \\ & \ell_q^m \arrow[ur, "S"'] & \end{tikzcd} \tag{3}$$

X is called an $\mathcal{L}_q^g$*-space* if it is an $\mathcal{L}_{q,\lambda}^g$-space for some $\lambda \geq 1$. Loosely speaking, $\mathcal{L}_q^g$-spaces share many properties of ℓ_q, since they *locally look like* ℓ_q^m. The spaces $L_q(\mu)$ are $\mathcal{L}_{q,1}^g$-spaces. For more information and properties of $\mathcal{L}_q^g$-spaces see [6, Section 23].

Theorem 1.2. *Let T be a multilinear map in $\mathcal{L}(^m \ell_r^N; X)$, where X is an $\mathcal{L}_{q,1}^g$-space and suppose r, q and $p > 0$ satisfy one of the following conditions*

a) $r = q = 2$;
b) $r = 2$ *and either* $p < q < 2$ *or* $p = q$;
c) $p < r' < 2$ *and either* $p < q \leq 2$ *or* $p = q$.

Then

$$\pi_p(T) = \frac{1}{c_{r',p}^m} \Big(\int\limits_{\mathbb{K}^N} \cdots \int\limits_{\mathbb{K}^N} \|T(z^{(1)}, \ldots, z^{(m)})\|_X^p \, d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)}) \Big)^{1/p}.$$

It is clear that Theorem 1.1 follows from Theorem 1.2, but in fact, the proof of Theorem 1.2 uses the scalar result, which is much simpler and is interesting on its own. We remark that the formula also holds for any multilinear map in $\mathcal{L}(\ell_{r_1}^N, \ldots, \ell_{r_m}^N; X)$, where X is an $\mathcal{L}_{q,1}^g$-space and $r_1, \ldots, r_m$, q and p satisfy conditions analogous to those of Theorem 1.2. Moreover, the formula turns into an equivalence between the π_p norm and the integral if we take general $\mathcal{L}_q^g$-spaces.

On the other hand, if we put ℓ_r in the domain, since multiple summing operators form a maximal ideal, the formula holds with a limit over N in the right hand side (here we consider $\mathbb{K}^N$ as a subset of ℓ_r).

There are situations not covered by the previous theorem where we have an equivalence or, at least, an inequality between the π_p and the $L_p(\mu_s)$ norms.

Proposition 1.3. *Let $T \in \mathcal{L}(^m\ell_r^N; X)$.*

(i) Suppose either $r = 2$ and $p, q < 2$; or $r = 2$ and $q \le p$; or $p < r' < 2$ and $q \le 2$. If X is an $\mathcal{L}_q^g$-space, then we have

$$\pi_p(T) \asymp \left(\int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} \|T(z^{(1)}, \ldots, z^{(m)})\|_X^p \, d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)}) \right)^{1/p},$$

that is, the multiple p-summing and the $L_{r'}(\mathbb{K}^N \times \cdots \times \mathbb{K}^N, \mu_{r'}^N \times \cdots \times \mu_{r'}^N)$ norm are equivalent in $\mathcal{L}(^m\ell_r^N; X)$, with constants which are independent of N.

(ii) If $r = 2$ or $p < r' < 2$ then we have, for any Banach space X,

$$\pi_p(T) \succeq \left(\int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} \|T(z^{(1)}, \ldots, z^{(m)})\|_X^p \, d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)}) \right)^{1/p}.$$

Now we describe some applications of these results. The most direct one is an asymptotically correct relationship between the multiple summing norm of a multilinear operator and the usual (supremum) norm. Cobos, Kühn and Peetre [5] compared the Hilbert–Schmidt norm, π_2, with the usual norm of multilinear forms. They showed that if T is any m-linear form in $\mathcal{L}(^m\ell_2^N, \mathbb{K})$ then

$$\pi_2(T) \le N^{\frac{m-1}{2}} \|T\|.$$

Moreover, the asymptotic bound is optimal in the sense that there exist constants c_m and m-linear forms T on ℓ_2^N with $\|T\| = 1$ and $\pi_2(T) \ge c_m N^{\frac{m-1}{2}}$. It is easy to see from this that the correct exponent for the asymptotic bound for the Hilbert-valued case is $\frac{m}{2}$. The same holds for the multiple p-summing norm for any p because all those norms are equivalent to the Hilbert–Schmidt norm in $\mathcal{L}(^m\ell_2; \ell_2)$, see [10,13]. We see now that the same optimal exponent holds for multiple p-summing operators with values on $\mathcal{L}_q^g$-spaces.

First, note that passing to polar coordinates we have, in the complex case (the real case follows similarly)

$$\int\limits_{\mathbb{K}^N} \cdots \int\limits_{\mathbb{K}^N} \|T(z^{(1)}, \ldots, z^{(m)})\|_X^p \, d\mu_2^N(z^{(1)}) \ldots d\mu_2^N(z^{(m)})$$
$$= \frac{1}{\Gamma(N)^m} \int\limits_{(S^{2N-1})^m} \|T(\omega^{(1)}, \ldots, \omega^{(m)})\|_X^p \, d\sigma_{2N-1}(\omega^{(1)}) \ldots d\sigma_{2N-1}(\omega^{(m)})$$
$$\times \Big(\int\limits_0^\infty 2\rho^{2N+p-1} e^{-\rho^2} d\rho\Big)^m$$
$$\leq \|T\|^p \Big(\frac{\Gamma(N+p/2)}{\Gamma(N)}\Big)^m,$$

where S^{2N-1} denotes the unit sphere in $\mathbb{R}^{2N}$ and σ_{2N-1} the normalized Lebesgue measure defined on it.

As a consequence of Proposition 1.3, we obtain

$$\pi_p(T) \preceq \left(\frac{\Gamma(N+p/2)}{\Gamma(N)}\right)^{m/p} \|T\| \preceq N^{\frac{m}{2}} \|T\| \tag{4}$$

for X an $\mathcal{L}_{q,\lambda}^g$-space and $p \geq q$ or $p, q < 2$.

Let us see that for $p, q \leq 2$, the exponents are optimal. Since for any $T \in \mathcal{L}(^m\ell_2^N; \ell_q)$ we have

$$\Big(\sum_{j_1,\ldots,j_m=1}^N \|T(e_{j_1}, \ldots, e_{j_m})\|_{\ell_q}^p\Big)^{\frac{1}{p}} \leq \pi_p(T) N^{\frac{m}{p}-\frac{m}{2}} \preceq N^{\frac{m}{p}} \|T\|,$$

it suffices to show that the inequality

$$\Big(\sum_{j_1,\ldots,j_m=1}^N \|T(e_{j_1}, \ldots, e_{j_m})\|_{\ell_q}^p\Big)^{\frac{1}{p}} \preceq N^{\frac{m}{p}} \|T\| \tag{5}$$

is optimal. By [1, Theorem 4], there exist symmetric multilinear operators $\tilde{T}_N \in \mathcal{L}(^m\ell_2^N, \ell_2^N) = \mathcal{L}(^{m+1}\ell_2^N)$, such that, $\tilde{T}_N = \sum\limits_{j_1,\ldots,j_{m+1}=1}^N \varepsilon_{j_1,\ldots,j_{m+1}} e_{j_1} \otimes \cdots \otimes e_{j_{m+1}}$, with $\varepsilon_{j_1,\ldots,j_{m+1}} = \pm 1$ and $\|\tilde{T}_N\| \asymp \sqrt{N}$.

Let $T_N = i_{2q} \circ \tilde{T}_N$, where $i_{2q} : \ell_2^N \to \ell_q^N$ is the inclusion. Then, $\|T_N\| \preceq N^{\frac{1}{q}}$ and

$$\Big(\sum_{j_1,\ldots,j_m=1}^N \|T_N(e_{j_1}, \ldots, e_{j_m})\|_{\ell_q}^p\Big)^{\frac{1}{p}} N^{\frac{1}{q}+\frac{m}{p}} \succeq N^{\frac{m}{p}} \|T_N\|.$$

This implies that inequality (5) is optimal and, hence, so is (4).

1.1. Inclusion theorems

The well-known inclusion theorem for absolutely summing linear operators states that for any Banach spaces E, F we have

$$\Pi_s(E, F) \subset \Pi_t(E, F), \qquad \text{when } s \leq t.$$

Although multiple summing mappings share several properties of linear summing operators, there is no general inclusion theorem in the multilinear case (see [15]). It is therefore interesting to investigate in which situations we do have inclusion type theorems. The following theorem summarizes some of the most important known results on this topic.

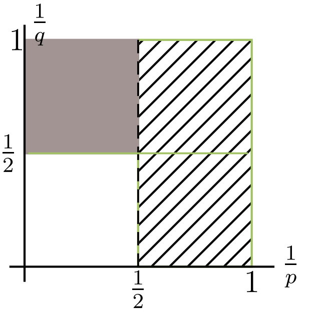
Theorem 1.4. *(See [3,13,19].) (i) If E has cotype $r \geq 2$ then*

$$\Pi_s(^m E, F) = \Pi_1(^m E, F), \qquad \textit{for } 1 \leq s < r^*.$$

(ii) If F has cotype 2 then

$$\Pi_s(^m E, F) \subset \Pi_2(^m E, F), \qquad \textit{for } 2 \leq s < \infty.$$

The following picture illustrates the above theorem in the particular case where $E = \ell_2$ and $F = \ell_q$,



In the ruled area we have $\Pi_{p_1}(^m\ell_2; \ell_q) = \Pi_{p_2}(^m\ell_2; \ell_q)$ and in the shaded area we have the reverse inclusion $\Pi_{p_1}(^m\ell_2; \ell_q) \subset \Pi_2(^m\ell_2; \ell_q)$ for $p_1 \geq 2$.

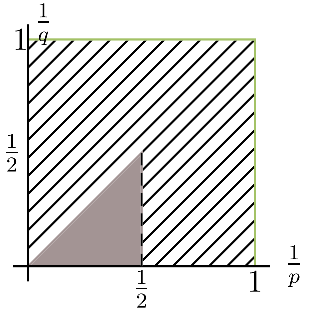
As a consequence of our integral formula, we obtain the following improvement to the previous result, which will be proved in Section 2.

Proposition 1.5. *Let Y be an $\mathcal{L}_2^g$-space and X an $\mathcal{L}_q^g$-space.*

If $p \geq q$, then $\Pi_p(^mY; X) = \Pi_q(^mY; X)$.

If $p \leq q$, then $\Pi_p(^mY; X) \subset \Pi_q(^mY; X)$.

With the information given by the above proposition, we have the following new picture.



In the ruled area we have $\Pi_{p_1}({}^m\ell_2;\ell_q) = \Pi_{p_2}({}^m\ell_2;\ell_q)$ and in the shaded area we have the (direct) inclusion $\Pi_p({}^m\ell_2;\ell_q) \subset \Pi_q({}^m\ell_2;\ell_q)$ for $p \le q$.

1.2. A contraction result and unconditionality

Let us begin with this contraction result for the p-summing norm of multilinear operators.

Theorem 1.6. *Suppose X is an $\mathcal{L}_q^g$-space and let r, q and $p > 0$ satisfy one of the conditions in Proposition 1.3(i). Then, there is a constant K (depending on r, q and p), such that for any finite matrix $(x_{i_1,\dots,i_m})_{i_1,\dots,i_m} \subset X$ and any choice of scalars $\alpha_{i_1,\dots,i_m}$ we have,*

$$\pi_p\left(\sum_{i_1,\dots,i_m} \alpha_{i_1,\dots,i_m}\ e'_{i_1} \otimes \cdots \otimes e'_{i_m}\ x_{i_1,\dots,i_m}\right)$$
$$\le K\|(\alpha_{i_1,\dots,i_m})\|_\infty\ \pi_p\left(\sum_{i_1,\dots,i_m} e'_{i_1} \otimes \cdots \otimes e'_{i_m}\ x_{i_1,\dots,i_m}\right),$$

where the π_p norms are taken in $\Pi_p({}^m\ell_r;X)$.

Proof. If we show the inequality for $\alpha_{i_1,\dots,i_m} = \pm 1$, standard procedures lead to the desired inequality for general scalars, eventually with different constants (see, for example, Section 1.6 in [8]). We set

$$T = \sum_{i_1,\dots,i_m} e'_{i_1} \otimes \cdots \otimes e'_{i_m}\ x_{i_1,\dots,i_m} \quad \text{and} \quad T_\alpha = \sum_{i_1,\dots,i_m} \alpha_{i_1,\dots,i_m} e'_{i_1} \otimes \cdots \otimes e'_{i_m}\ x_{i_1,\dots,i_m}$$

and let $(r_k)_k$ be the sequence of Rademacher functions on $[0,1]$. For any choice of $t_1 \dots, t_m \in [0,1]$, we have

$$\int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} \|T_\alpha(r_1(t_1)z^{(1)}, \ldots, r_n(t_m)z^{(m)})\|_X^p d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)})$$
$$= \int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} \|T_\alpha(z^{(1)}, \ldots, z^{(m)})\|_X^p d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)}).$$

We integrate on $t_j \in [0,1]$, $j = 1, \ldots, m$ and use Fubini's theorem to obtain

$$\begin{aligned}
&\int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} \|T_\alpha(z^{(1)}, \ldots, z^{(m)})\|_X^p d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)}) \\
&= \int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} \int_0^1 \cdots \int_0^1 \left\|T_\alpha(r_1(t_1)z^{(1)}, \ldots, r_n(t_m)z^{(m)})\right\|_X^p dt_1 \ldots dt_m d\mu_{r'}^N(z^{(1)}) \ldots \\
&\quad d\mu_{r'}^N(z^{(m)}) \\
&= \int_{[0;1]^m} \int_{(\mathbb{K}^{\mathbb{N}})^m} \Big\| \sum_{i_1,\ldots,i_m} r_{i_1}(t) \ldots r_{i_m}(t) \alpha_{i_1,\ldots,i_m} z_{i_1}^{(1)} \cdots z_{i_m}^{(m)} x_{i_1,\ldots,i_m} \Big\|_X^p dt_1 \ldots \\
&\quad dt_m d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)}).
\end{aligned} \tag{6}$$

Since X has nontrivial cotype and local unconditional structure, we can apply a multilinear version of Pisier's deep result [18, Proposition 2.1] (which follows the same lines as the bilinear result) to show that, for any $z^{(1)}, \ldots, z^{(m)} \in \mathbb{K}^N$, we have

$$\int_0^1 \cdots \int_0^1 \Big\| \sum_{i_1,\ldots,i_m} r_{i_1}(t) \ldots r_{i_m}(t)\ \alpha_{i_1,\ldots,i_m}\ z_{i_1}^{(1)} \cdots z_{i_m}^{(m)}\ x_{i_1,\ldots,i_m} \Big\|_X^2 dt_1 \ldots dt_m$$
$$\leq K_X \int_0^1 \cdots \int_0^1 \Big\| \sum_{i_1,\ldots,i_m} r_{i_1}(t) \ldots r_{i_m}(t)\ z_{i_1}^{(1)} \cdots z_{i_m}^{(m)}\ x_{i_1,\ldots,i_m} \Big\|_X^2 dt_1 \ldots dt_m$$

Using a multilinear Kahane inequality (which may be proved by induction on m), the same holds, with a different constant, if we consider the power p in the integrals. This means that we can take the $\alpha_{i_1,\ldots,i_k}$ from (6), at the price of a constant K. Now, we can go all the way back as before to obtain

$$\int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} \|T_\alpha(z^{(1)}, \ldots, z^{(m)})\|_X^p d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)})$$
$$\leq K \int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} \|T(z^{(1)}, \ldots, z^{(m)})\|_X^p d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)}).$$

The integral formula in Proposition 1.3 gives the result. $\square$

Note that, in the scalar-valued case, the previous theorem asserts that the monomials form an unconditional basic sequence in $\Pi_1({}^m\ell_r)$ for $r \geq 2$. This is a particular case of the result of Defant and Pérez-García in [7]. It should be noted that the analogous scalar-valued result is much easier to prove: after introducing the Rademacher functions as in the previous proof, we just have to use a multilinear Khintchine inequality and the integral formula from Theorem 1.1 to obtain the result (Pisier's result is, of course, not needed in this case).

1.3. Limit orders

As a consequence of the integral formula for the p-summing norm, we are able to compute limit orders of multiple summing operators (see definitions below). Limit orders of the ideal of scalar-valued multiple 1-summing forms were computed in [7] for the bilinear case. In the multlinear case, they were computed in [4] for ℓ_r with $1 \leq r \leq 2$ and in [11] for ℓ_r with $r \geq 2$. This latter case can be easily obtained from our integral formula for the multiple summing norm. We will actually use the integral formula to compute some limit orders for the vector-valued case, the mentioned scalar case being very similar.

A subclass $\mathfrak{A}$ of the class $\mathcal{L}$ of all m-linear continuous mappings between Banach spaces is called an *ideal of m-linear mappings* if

(1) For all Banach spaces $E_1, \dots, E_m, F$, the component set $\mathfrak{A}(E_1, \dots, E_m; F) := \mathfrak{A} \cap (E_1, \dots, E_m; F)$ is a linear subspace of $(E_1, \dots, E_m; F)$.
(2) If $T_j \in (E_j; G_j)$, $\phi \in \mathfrak{A}(G_1, \dots, G_m; G)$ and $S \in \mathcal{L}(G, F)$, then $S \circ \phi \circ (T_1, \dots, T_m)$ belongs to $\mathfrak{A}(E_1, \dots, E_m; F)$.
(3) The application $\mathbb{K}^m \ni (\lambda_1, \dots, \lambda_m) \mapsto \lambda_1 \cdot \ldots \cdot \lambda_m \in \mathbb{K}$ is in $\mathfrak{A}(\mathbb{K}, \dots, \mathbb{K}; \mathbb{K})$.

A *normed ideal of m-linear operators* $(\mathfrak{A}, \|\cdot\|_{\mathfrak{A}})$ is an ideal $\mathfrak{A}$ of m-linear operators together with an ideal norm $\|\cdot\|_{\mathfrak{A}}$, that is,

(1) $\|\cdot\|_{\mathfrak{A}}$ restricted to each component is a norm.
(2) If $T_j \in (E_j; G_j)$, $\phi \in \mathfrak{A}(G_1, \dots, G_m; G)$ and $S \in \mathcal{L}(G, F)$, then $\|S \circ \phi \circ (T_1, \dots, T_m)\|_{\mathfrak{A}} \leq \|S\| \|\phi\|_{\mathfrak{A}} \|T_1\| \cdot \dots \cdot \|T_m\|$.
(3) $\|\mathbb{K}^m \ni (\lambda_1, \dots, \lambda_m) \mapsto \lambda_1 \cdot \ldots \cdot \lambda_m \in \mathbb{K}\|_{\mathfrak{A}} = 1$.

Given a normed ideal of m-linear operators $(\mathfrak{A}, \|\cdot\|_{\mathfrak{A}})$, the *limit order* $\lambda_m(\mathfrak{A}, r, q)$ is defined as the infimum of all $\lambda \geq 0$ such that there is a constant $C > 0$ satisfying

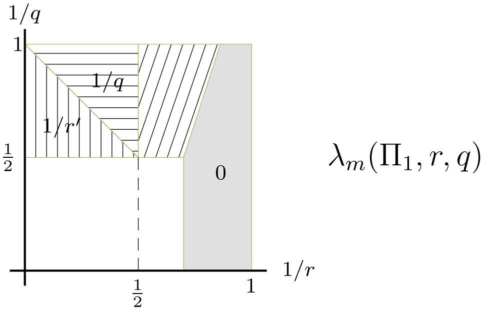
$$\|\Phi_N\|_{\mathfrak{A}} \leq C N^{\lambda},$$

for every $N \geq 1$, where $\Phi_N : \ell_r^N \times \dots \times \ell_r^N \to \ell_q^N$ is the m-linear operator, $\Phi_N(x^1, \dots, x^m) = \sum_{j=1}^N x_j^1 \dots x_j^m e_j$.

Proposition 1.7.

$$\lambda_m(\Pi_1, r, q) = \begin{cases} \frac{1}{q} & \text{if } q \le r' \le 2 \\ \frac{1}{r'} & \text{if } r' \le q \le 2 \\ \frac{1}{q} + \frac{m}{2} - \frac{m}{r} & \text{if } \frac{2mq}{2+mq} < r \le 2 \text{ and } q \le 2 \\ 0 & \text{if } 1 \le r \le \frac{2mq}{2+mq} \end{cases}$$

These values can be represented by the following picture:



The proof will be splitted in several lemmas.

Lemma 1.8. *Let $p \le q \le r' \le 2$ then $\lambda_m(\Pi_p, r, q) = \frac{1}{q}$.*

Proof. Let $p \le q < r' \le 2$. Then by Theorem 1.2,

$$\begin{aligned} c^m_{r',p}\pi_p(\Phi_N) &= \Big(\int\limits_{\mathbb{K}^N} \dots \int\limits_{\mathbb{K}^N} \Big(\sum_j |z_j^{(1)} \dots z_j^{(m)}|^q\Big)^{p/q} d\mu^N_{r'}(z^{(1)}) \dots d\mu^N_{r'}(z^{(m)})\Big)^{1/p} \\ &\le \Big(\int\limits_{\mathbb{K}^N} \dots \int\limits_{\mathbb{K}^N} \sum_j |z_j^{(1)} \dots z_j^{(m)}|^q d\mu^N_{r'}(z^{(1)}) \dots d\mu^N_{r'}(z^{(m)})\Big)^{1/q} \\ &= c^m_{r',q} N^{1/q}. \end{aligned}$$

Thus, $\lambda_m(\Pi_p, r, q) \le \frac{1}{q}$. On the other hand,

$$\begin{aligned} c^m_{r',p}\pi_p(\Phi_N) &= \Big(\int\limits_{\mathbb{K}^N} \dots \int\limits_{\mathbb{K}^N} \Big(\sum_j |z_j^{(1)} \dots z_j^{(m)}|^q\Big)^{p/q} d\mu^N_{r'}(z^{(1)}) \dots d\mu^N_{r'}(z^{(m)})\Big)^{1/p} \\ &\ge \Big(\int\limits_{\mathbb{K}^N} \dots \int\limits_{\mathbb{K}^N} N^{p/q-1} \sum_j |z_j^{(1)} \dots z_j^{(m)}|^p d\mu^N_{r'}(z^{(1)}) \dots d\mu^N_{r'}(z^{(m)})\Big)^{1/p} \\ &= c^m_{r',p} N^{1/q}. \end{aligned}$$

Hence, $\lambda_m(\Pi_p, r, q) \ge \frac{1}{q}$ and the proof is done. $\square$

Lemma 1.9. *Let $p < r' \leq q < 2$. Then $\lambda_m(\Pi_p, r, q) = \frac{1}{r'}$.*

Proof. Let $1 < s < r' \leq q < 2$. Then, by Theorem 1.2,

$$\begin{aligned}
c_{r',1}^m \pi_1(\Phi_N) &= \int\limits_{\mathbb{K}^N} \dots \int\limits_{\mathbb{K}^N} \Big(\sum_j |z_j^{(1)} \dots z_j^{(m)}|^q\Big)^{1/q} d\mu_{r'}^N(z^{(1)}) \dots d\mu_{r'}^N(z^{(m)}) \\
&\leq \int\limits_{\mathbb{K}^N} \dots \int\limits_{\mathbb{K}^N} \Big(\sum_j |z_j^{(1)} \dots z_j^{(m)}|^s\Big)^{1/s} d\mu_{r'}^N(z^{(1)}) \dots d\mu_{r'}^N(z^{(m)}) \\
&\leq \Big(\int\limits_{\mathbb{K}^N} \dots \int\limits_{\mathbb{K}^N} \sum_j |z_j^{(1)} \dots z_j^{(m)}|^s d\mu_{r'}^N(z^{(1)}) \dots d\mu_{r'}^N(z^{(m)})\Big)^{1/s} \\
&= c_{r',s}^m N^{1/s}.
\end{aligned}$$

Since this is true for every $s < r'$, $\lambda_m(\Pi_1, r, q) \leq \frac{1}{r'}$.

On the other hand, let $\Psi_N : \ell_r^N \times \dots \ell_r^N \times \ell_{q'}^N \to \mathbb{C}$, the $(m+1)$-linear form induced by Φ_N. By [15, Proposition 2.2] or [10, Proposition 2.5], $\pi_1(\Phi_N) \geq \pi_1(\Psi_N)$. Thus, by Theorem 1.1 taking into account the comments after Theorem 1.2, we have

$$\begin{aligned}
&c_{r',1}^m c_{q,1} \pi_1(\Psi_N) \\
&= \int\limits_{\mathbb{K}^N} \dots \int\limits_{\mathbb{K}^N} |\Psi_N(z^{(1)}, \dots, z^{(m+1)})| d\mu_{r'}^N(z^{(1)}) \dots d\mu_{r'}^N(z^{(m)}) d\mu_q^N(z^{(m+1)}) \\
&= \int\limits_{\mathbb{K}^N} \dots \int\limits_{\mathbb{K}^N} \Big|\sum_j z_j^{(1)} \dots z_j^{(m+1)}\Big| d\mu_{r'}^N(z^{(1)}) \dots d\mu_{r'}^N(z^{(m)}) d\mu_q^N(z^{(m+1)}) \\
&= c_{r',1} \int\limits_{\mathbb{K}^N} \dots \int\limits_{\mathbb{K}^N} \Big(\sum_j |z_j^{(2)} \dots z_j^{(m+1)}|^{r'}\Big)^{1/r'} d\mu_{r'}^N(z^{(2)}) \dots d\mu_{r'}^N(z^{(m)}) d\mu_q^N(z^{(m+1)}) \\
&\geq c_{r',1} N^{\frac{1}{r'}-1} \int\limits_{\mathbb{K}^N} \dots \int\limits_{\mathbb{K}^N} \sum_j |z_j^{(2)} \dots z_j^{(m+1)}| d\mu_{r'}^N(z^{(2)}) \dots d\mu_{r'}^N(z^{(m)}) d\mu_q^N(z^{(m+1)}) \\
&= c_{r',1}^m c_{q,1} N^{\frac{1}{r'}-1} N \;=\; c_{r',1}^m c_{q,1} N^{\frac{1}{r'}}
\end{aligned}$$

Therefore $\lambda_m(\Pi_1, r, q) = \frac{1}{r'}$.

This proves our assertions for $p = 1$. By [3, Theorem 4.7], $\Pi_p(\ell_r, \ell_q) = \Pi_1(\ell_r, \ell_q)$ for every $1 \leq p \leq 2$, and the lemma follows. □

Since ℓ_r has cotype 2 for $1 \leq r \leq 2$, given any m-linear form $T \in \mathcal{L}(\ell_r^N, \dots, \ell_r^N; \mathbb{C})$, we know from [7, Lemma 4.5] that

$$\pi_1(T) \asymp \sup \pi_1(T \circ (D_{\sigma_1}, \dots, D_{\sigma_m})), \tag{7}$$

where the supremum is taken over the set of norm one diagonal operators $D_{\sigma_j} : \ell_2^N \to \ell_r^N$. The vector-valued version of this result follows the same lines, so (7) holds for any m-linear map $T \in \mathcal{L}(\ell_r^N, \ldots, \ell_r^N; Y)$, for every Banach space Y.

Lemma 1.10. *Let $1 \leq p, q, r \leq 2$. Then*

(*i*) $\lambda_m(\Pi_p, r, q) = 0$ *for* $1 \leq r \leq \frac{2mq}{2+mq}$.
(*ii*) $\lambda_m(\Pi_p, r, q) = \frac{1}{q} + \frac{m}{2} - \frac{m}{r}$ *for* $\frac{2mq}{2+mq} < r \leq 2$.

Proof. Let $\frac{1}{t} = \frac{1}{r} - \frac{1}{2}$, then for any diagonal operator we have $\|D_\sigma\|_{\mathcal{L}(\ell_2^N;\ell_r^N)} = \|\sigma\|_{\ell_t^N}$. Since $\Phi_N \circ (D_{\sigma_1}, \ldots, D_{\sigma_m}) \in \mathcal{L}({}^m\ell_2^N; \ell_q^N)$, by Theorem 1.2 we have

$$\begin{aligned}&\pi_1(\Phi_N \circ (D_{\sigma_1}, \ldots, D_{\sigma_m}))\\ &\quad\asymp \int\limits_{\mathbb{K}^N} \cdots \int\limits_{\mathbb{K}^N} \Big(\sum_{j=1}^N |\sigma_1(j) z_j^{(1)} \ldots \sigma_m(j) z_j^{(m)}|^q\Big)^{1/q} d\mu_2^N(z^{(1)}) \ldots d\mu_2^N(z^{(m)}).\end{aligned}$$

(*i*) The assumption $1 \leq r \leq \frac{2mq}{2+mq}$ implies $t \leq mq$. Then

$$\Big(\sum_{j=1}^N |\sigma_1(j) z_j^{(1)} \ldots \sigma_m(j) z_j^{(m)})|^q\Big)^{1/q} \leq \|\sigma_1\|_{\ell_t^N} \ldots \|\sigma_m\|_{\ell_t^N} \sup_j |z_j^{(1)} \ldots z_j^{(m)}|.$$

Consequently, for any $s \geq 1$, we have

$$\begin{aligned}\pi_1(\Phi_N \circ (D_{\sigma_1}, \ldots, D_{\sigma_m})) &\preceq \int\limits_{\mathbb{K}^N} \cdots \int\limits_{\mathbb{K}^N} \sup_j |z_j^{(1)} \ldots z_j^{(m)}| d\mu_2^N(z^{(1)}) \ldots d\mu_2^N(z^{(m)})\\ &\leq \Big(\int\limits_{\mathbb{K}^N} \cdots \int\limits_{\mathbb{K}^N} \sum_{j=1}^N |z_j^{(1)} \ldots z_j^{(m)}|^s d\mu_2^N(z^{(1)}) \ldots d\mu_2^N(z^{(m)})\Big)^{1/s}\\ &= c_{2,s}^m N^{\frac{1}{s}},\end{aligned}$$

which implies that $\lambda_m(\Pi_1, r, q) = 0$.

(*ii*) The assumption $\frac{2mq}{2+mq} \leq r < 2$ implies $t > mq$. Let $\frac{1}{q} = \frac{m}{t} + \frac{1}{s}$. Then

$$\Big(\sum_{j=1}^N |\sigma_1(j) z_j^{(1)} \ldots \sigma_m(j) z_j^{(m)}|^q\Big)^{1/q} \leq \|\sigma_1\|_{\ell_t^N} \ldots \|\sigma_m\|_{\ell_t^N} \Big(\sum_{j=1}^N |z_j^{(1)} \ldots z_j^{(m)})|^s\Big)^{1/s}.$$

Thus we have,

$$\pi_1(\Phi_N \circ (D_{\sigma_1}, \ldots, D_{\sigma_m})) \preceq \int\limits_{\mathbb{K}^N} \cdots \int\limits_{\mathbb{K}^N} \Big(\sum_{j=1}^N |z_j^{(1)} \ldots z_j^{(m)}|^s\Big)^{1/s} d\mu_2^N(z^{(1)}) \ldots d\mu_2^N(z^{(m)})$$

$$\leq \Big(\int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} \sum_{j=1}^{N} |z_j^{(1)} \ldots z_j^{(m)}|^s d\mu_2^N(z^{(1)}) \ldots d\mu_2^N(z^{(m)}) \Big)^{1/s}$$
$$= c_{2,s}^m N^{1/s} \quad \asymp N^{\frac{1}{q} + \frac{m}{2} - \frac{m}{r}}.$$

On the other hand,

$$\pi_1(\Phi_N \circ (D_{\sigma_1}, \ldots, D_{\sigma_m}))$$
$$\succeq N^{-1/q'} \int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} \sum_{j=1}^{N} |\sigma_1(j) z_j^{(1)} \ldots \sigma_m(j) z_j^{(m)}| d\mu_2^N(z^{(1)}) \ldots d\mu_2^N(z^{(m)})$$
$$= N^{-1/q'} c_{2,1}^m \sum_{j=1}^{N} |\sigma_1(j) \ldots \sigma_m(j)|.$$

Taking supremum over $\sigma_k \in B_{\ell_t^N}$, $k = 1, \ldots, m$, and using (7) we get that

$$\pi_1(\Phi_N) \succeq N^{-1/q'} N^{1-m/t} c_{2,1}^m \asymp N^{\frac{1}{q} + \frac{m}{2} - \frac{m}{r}}.$$

This proves our assertions for $p = 1$. Since ℓ_r has cotype 2, by [3, Theorem 4.6], $\Pi_p(\ell_r, \ell_q)$ coincides with $\Pi_1(\ell_r, \ell_q)$ for every $1 \leq p \leq 2$, and the lemma follows. □

2. Proofs of the main results

The proofs will be splitted in a few lemmas. We will also use the following result, which is [13, Proposition 3.1].

Proposition 2.1 *(Pérez-García). Let $T \in \Pi_p^m(X_1, \ldots, X_m; Y)$ and let (Ω_j, μ_j) be measure spaces for each $1 \leq j \leq m$. We have*

$$\Big(\int_{\Omega_1} \cdots \int_{\Omega_m} \|T(f_1(w_1), \ldots, f_m(w_m))\|_Y^p \, d\mu_1(w_1) \ldots d\mu_m(w_m) \Big)^{1/p}$$
$$\leq \pi_p(T) \prod_{j=1}^{m} \sup_{x_j^* \in B_{X_j^*}} \Big(\int_{\Omega_j} |\langle x_j^*, f_j(w_j) \rangle|^p d\mu_j(w_j) \Big)^{1/p},$$

for every $f_j \in L_p(\mu_j, X_j)$.

A simple consequence of this proposition is the following.

Lemma 2.2. *Let T be a multilinear operator in $\mathcal{L}({}^m\ell_r^N;Y)$, and $p < r' < 2$ or $r = 2$. Then*

$$\Big(\int\limits_{\mathbb{K}^N} \cdots \int\limits_{\mathbb{K}^N} \|T(z^{(1)},\dots,z^{(m)})\|_Y^p \, d\mu_{r'}^N(z^{(1)}) \dots d\mu_{r'}^N(z^{(m)})\Big)^{1/p} \le c_{r',p}^m \pi_p(T).$$

Proof. Let $(\Omega_j,\mu_j) = (\mathbb{K}^N,\mu_{r'})$, $f_j \in L_p((\mathbb{K}^N,\mu_{r'}),\mathbb{K}^N)$, $f_j(z) = z$ for all j and $p < r' < 2$ or $r = 2$. By Proposition 2.1 and rotation invariance of stable measures,

$$\begin{aligned}
&\Big(\int\limits_{\mathbb{K}^N} \cdots \int\limits_{\mathbb{K}^N} \|T(z^{(1)},\dots,z^{(m)})\|_Y^p \, d\mu_{r'}^N(z^{(1)}) \dots d\mu_{r'}^N(z^{(m)})\Big)^{1/p} \\
&\quad \le \pi_p(T) \prod_{j=1}^m \sup_{w_j \in B_{\ell_{r'}^N}} \Big(\int\limits_{\mathbb{K}^N} |\langle z^{(j)}, w_j\rangle|^p d\mu_{r'}^N(z^{(j)})\Big)^{1/p} \\
&\quad = \pi_p(T)\Big(\int\limits_{\mathbb{K}^N} |e_1'(z)|^p d\mu_{r'}^N(z)\Big)^{m/p} = \pi_p(T) c_{r',p}^m. \qquad \square
\end{aligned}$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. One inequality is given in the previous lemma. We prove the reverse inequality by induction on m. For $m = 1$, we have $\phi \in (\ell_r^N)' = \ell_{r'}^N$ and then

$$\begin{aligned}
\pi_p(\phi) = \|\phi\|_{\ell_{r'}^N} &= \Big(\sum_{j=1}^N |e_j'(\phi)|^{r'}\Big)^{1/r'} = c_{r',p}^{-1}\Big(\int\limits_{\mathbb{K}^N} \Big|\sum_{j=1}^N e_j'(\phi) z_j\Big|^p d\mu_{r'}^N(z)\Big)^{1/p} \\
&= c_{r',p}^{-1}\Big(\int\limits_{\mathbb{K}^N} |\phi(z)|^p d\mu_{r'}^N(z)\Big)^{1/p}.
\end{aligned}$$

Suppose that for any k-linear form $\psi : \ell_r^N \times \cdots \times \ell_r^N \to \mathbb{K}$, with $k < m$, we have,

$$\begin{aligned}
&\sum_{n_1,\dots,n_k} |\psi(u_{n_1}^{(1)},\dots,u_{n_k}^{(k)})|^p \\
&\quad \le c_{r',p}^{-kp}\Big(\int\limits_{\mathbb{K}^N} \cdots \int\limits_{\mathbb{K}^N} |\psi(z^{(1)},\dots,z^{(k)})|^p d\mu_{r'}^N(z^{(1)}) \dots d\mu_{r'}^N(z^{(k)})\Big),
\end{aligned}$$

for all sequences $(u_{n_j}^{(j)}) \subset \ell_r^N$, with $w_p(u_{n_j}^{(j)}) = 1$, $j = 1,\dots,k$.

Let ϕ be an m-linear form, and $(u_{n_j}^{(j)}) \subset \ell_r^N$, with $w_p(u_{n_j}^{(j)}) = 1$, $j = 1, \ldots, m$. Then

$$\begin{aligned}
&\sum_{n_1,\ldots,n_m} |\phi(u_{n_1}^{(1)}, \ldots, u_{n_m}^{(m)})|^p \\
&= \sum_{n_1} \sum_{n_2,\ldots,n_m} |\phi(u_{n_1}^{(1)}, \ldots, u_{n_m}^{(m)})|^p \\
&\leq c_{r',p}^{-(m-1)p} \sum_{n_1} \Big(\int_{\mathbb{K}^N} \ldots \int_{\mathbb{K}^N} |\phi(u_{n_1}^{(1)}, z^{(2)}, \ldots, z^{(m)})|^p d\mu_{r'}^N(z^{(2)}) \ldots d\mu_{r'}^N(z^{(m)}) \Big) \\
&= c_{r',p}^{-(m-1)p} \Big(\int_{\mathbb{K}^N} \ldots \int_{\mathbb{K}^N} \sum_{n_1} |\phi(u_{n_1}^{(1)}, z^{(2)}, \ldots, z^{(m)})|^p d\mu_{r'}^N(z^{(2)}) \ldots d\mu_{r'}^N(z^{(m)}) \Big) \\
&\leq c_{r',p}^{-(m-1)p} \int_{\mathbb{K}^N} \ldots \int_{\mathbb{K}^N} \Big(c_{r',p}^{-p} \int_{\mathbb{K}^N} |\phi(z^{(1)}, z^{(2)}, \ldots, z^{(m)})|^p d\mu_{r'}^N(z^{(1)}) \Big) d\mu_{r'}^N(z^{(2)}) \ldots \\
&\quad d\mu_{r'}^N(z^{(m)}) \\
&= c_{r',p}^{-mp} \Big(\int_{\mathbb{K}^N} \ldots \int_{\mathbb{K}^N} |\phi(z^{(1)}, \ldots, z^{(m)})|^p d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)}) \Big).
\end{aligned}$$

Therefore,

$$c_{r',p}^m \pi_p(\phi) \leq \Big(\int_{\mathbb{K}^N} \ldots \int_{\mathbb{K}^N} |\phi(z^{(1)}, \ldots, z^{(m)})|^p d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)}) \Big)^{1/p}. \qquad \square$$

Let us continue our way to the proof of Theorem 1.2.

Lemma 2.3. *Let T be an m-linear mapping in $\mathcal{L}(^m E; \ell_q^M)$, $0 < p < q < 2$ or $q = 2$. Then*

$$c_{q,p} \pi_p(T) \leq \Big(\int_{\mathbb{K}^M} \pi_p(z \circ T)^p d\mu_q^M(z) \Big)^{1/p}.$$

In particular, if T is linear,

$$c_{q,p} \pi_p(T) \leq \Big(\int_{\mathbb{K}^M} \|T'(z)\|_{E'}^p \, d\mu_q^M(z) \Big)^{1/p}.$$

Proof. For $(u_{k_j}^j) \subset \ell_2^N$ with $w_p((u_{k_j}^j)) = 1$, $j = 1, \ldots, m$, we have

$$\sum_{k_1,\ldots,k_m} \|T(u_{k_1}^1, \ldots, u_{k_m}^m)\|_{\ell_q^M}^p = \sum_{k_1,\ldots,k_m} \Big(\sum_{j=1}^M |e_j' \circ T(u_{k_1}^1, \ldots, u_{k_m}^m)|^q \Big)^{p/q}$$

$$
\begin{aligned}
&= \sum_{k_1,\dots,k_m} c_{q,p}^{-p}\Big(\int\limits_{\mathbb{K}^M} \Big|\sum_{j=1}^{M} e'_j \circ T(u^1_{k_1},\dots,u^m_{k_m}) z_j\Big|^p d\mu_q^M(z)\Big) \\
&= \sum_{k_1,\dots,k_m} c_{q,p}^{-p}\Big(\int\limits_{\mathbb{K}^M} |z \circ T(u^1_{k_1},\dots,u^m_{k_m})|^p d\mu_q^M(z)\Big) \\
&\le c_{q,p}^{-p}\Big(\int\limits_{\mathbb{K}^M} \pi_p(z\circ T)^p d\mu_q^M(z)\Big).
\end{aligned}
$$

Therefore,

$$
c_{q,p}\pi_p(T) \le \Big(\int\limits_{\mathbb{K}^M} \pi_p(z\circ T)^p d\mu_q^M(z)\Big)^{1/p}.
$$

For $m = 1$, $z \circ T$ is a linear form, and then we have $\pi_p(z\circ T) = \|z\circ T\|_{E'} = \|T'(z)\|_{E'}$. □

By a *Banach sequence space* we mean a Banach space $X \subset \mathbb{K}^{\mathbb{N}}$ of sequences in $\mathbb{K}$ such that $\ell_1 \subset X \subset \ell_\infty$ with norm one inclusions satisfying that if $x \in \mathbb{K}^{\mathbb{N}}$ and $y \in X$ are such that $|x_n| \le |y_n|$ for every $n \in \mathbb{N}$, then x belongs to X and $\|x\|_X \le \|y\|_X$. We will now show that if we consider multilinear mappings whose range are certain Banach sequence spaces, then the norm of the multilinear mapping defined by the integral formula is equivalent to the multiple summing norm.

We will need the following remark, which may be seen as a Khintchine/Kahane type multilinear inequality for the stable measures.

Remark 2.4. If T is an m-linear form on $\mathbb{K}^N$ and $q \le p < s < 2$, or $q \le p$ and $s = 2$, then

$$
\begin{aligned}
&c_{s,p}^{-m}\Big(\int\limits_{\mathbb{K}^N}\dots\int\limits_{\mathbb{K}^N} |T(z^{(1)},\dots,z^{(m)})|^p d\mu_s^N(z^{(1)})\dots d\mu_s^N(z^{(m)})\Big)^{1/p} \\
&\quad\le c_{s,q}^{-m}\Big(\int\limits_{\mathbb{K}^N}\dots\int\limits_{\mathbb{K}^N} |T(z^{(1)},\dots,z^{(m)})|^q d\mu_s^N(z^{(1)})\dots d\mu_s^N(z^{(m)})\Big)^{1/q}.
\end{aligned}
$$

For $m = 1$ it follows from property (2) of Lévy stable measures, and then we just apply induction on m.

Recall that a Banach sequence space X is called *q-concave*, $q \ge 1$, if there exists $C > 0$ such that for any $x_1,\dots,x_n \in X$ we have

$$
\Big(\sum_{k=1}^n \|x_k\|_X^q\Big)^{\frac{1}{q}} \le C\Big\|\Big(\sum_{k=1}^n |x_k|^q\Big)^{1/q}\Big\|_X.
$$

Lemma 2.5. *Let X be a q-concave Banach sequence space with constant C and let $T \in \mathcal{L}(^mE; X)$ be an m-linear operator. Denote by T_j the j-coordinate of T (T_j is a scalar m-linear form). Then $\pi_q(T) \le C\|(\pi_q(T_j))_j\|_X$.*

Proof. Just note that for finite sequences $\big(u_{n_k}^{(k)}\big)_{n_k} \subset X$, with $w_q\Big(\big(u_{n_k}^{(k)}\big)_{n_k}\Big) = 1$ we have

$$\Big(\sum_{n_1,\ldots,n_m} \|T(u_{n_1}^{(1)}, \ldots, u_{n_m}^{(m)})\|_X^q\Big)^{1/q} \le C\Big\|\Big(\Big(\sum_{n_1,\ldots,n_m} |T_j(u_{n_1}^{(1)}, \ldots, u_{n_m}^{(m)})|^q\Big)^{1/q}\Big)_j\Big\|_X$$
$$\le C\big\|(\pi_q(T_j))_j\big\|_X. \qquad \square$$

Lemma 2.6. *Let X be a Banach sequence space and let $T \in \mathcal{L}(^m\ell_r^N; X)$ be an m-linear operator, $r \ge 2$. Then if either $p, q < r' < 2$, or $r = 2$, then*

$$\|(\pi_p(T_j))_j\|_X \le c_{r',1}^{-m} \int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} \|T(z^{(1)}, \ldots, z^{(m)})\|_X d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)})$$
$$\le (c_{r',q}/c_{r',1})^m \pi_q(T).$$

Proof. By Theorem 1.1, Remark 2.4 and Lemma 2.2 we have

$$\begin{aligned}
\|(\pi_p(T_j))_j\|_X &= c_{r',p}^{-m}\Bigg\|\Bigg(\Big(\int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} |T_j(z^{(1)}, \ldots, z^{(m)})|^p d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)})\Big)^{1/p}\Bigg)_j\Bigg\|_X \\
&\le c_{r',1}^{-m}\Big\|\Big(\int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} |T_j(z^{(1)}, \ldots, z^{(m)})| d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)})\Big)\Big\|_X \\
&\le c_{r',1}^{-m} \int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} \|T(z^{(1)}, \ldots, z^{(m)})\|_X d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)}) \\
&\le c_{r',1}^{-m}\Big(\int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} \|T(z^{(1)}, \ldots, z^{(m)})\|_X^q d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)})\Big)^{1/q} \\
&\le (c_{r',q}/c_{r',1})^m \pi_q(T). \qquad \square
\end{aligned}$$

As a consequence of Lemma 2.2 we obtain one inequality in the following result. For the other inequality, note that if X is q-concave, then it is also s-concave for any $s \ge q$ and apply the previous two lemmas.

Corollary 2.7. *Let X be a q-concave Banach sequence space and let $T \in \mathcal{L}(^m\ell_r^N; X)$. Then for $r = 2$ and $q \le s$, or $q \le s < r' < 2$, we have*

$$\pi_s(T) \asymp \Big(\int_{\mathbb{K}^N} \cdots \int_{\mathbb{K}^N} \|T(z^{(1)}, \ldots, z^{(m)})\|_X^q d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)})\Big)^{1/q}$$

Standard localization techniques and the previous corollary readily show the coincidence of multiple s-summing and multiple q-summing operators from $\mathcal{L}_r^g$-spaces to q-concave Banach sequence spaces.

Corollary 2.8. *Let X be a q-concave Banach sequence space, and let E be an $\mathcal{L}_r^g$-space. Then*

$$\Pi_s({}^mE; X) = \Pi_q({}^mE; X),$$

for $q \leq s < r' < 2$, or $q \leq s$ and $r = 2$.

Proceeding as above we may prove the following.

Corollary 2.9. *Let X be an $\mathcal{L}_{q,1}^g$-space and let $T \in \mathcal{L}({}^m\ell_r^N; X)$ be an m-linear operator. If $q < r' < 2$ or, $r = 2$, then we have*

$$\pi_q(T) = c_{r',q}^{-m}\Big(\int\limits_{\mathbb{K}^N} \cdots \int\limits_{\mathbb{K}^N} \|T(z^{(1)}, \ldots, z^{(m)})\|_X^q d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)})\Big)^{1/q}.$$

We have almost finished the proofs of the main results.

Proof of Theorem 1.2. It is clearly enough to show the result for operators with range in ℓ_q^M for some $M \in \mathbb{N}$.

One inequality is Lemma 2.2. For the other inequality, if either $r = q = 2$ or; $r = 2$ and $p < q < 2$ or; $p < r' < 2$ and $p < q \leq 2$, a combination of the previous results gives:

$$\begin{aligned}
\pi_p(T) &\leq c_{q,p}^{-1}\Big(\int\limits_{\mathbb{K}^N} \pi_p(z \circ T)^p d\mu_q^N(z)\Big)^{1/p} \\
&\leq c_{r',p}^{-m} c_{q,p}^{-1}\Big(\int\limits_{\mathbb{K}^N}\Big(\int\limits_{\mathbb{K}^N} \cdots \int\limits_{\mathbb{K}^N} |z \circ T(z^{(1)}, \ldots, z^{(m)})|^p d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)})\Big) d\mu_q^N(z)\Big)^{1/p} \\
&\leq c_{r',p}^{-m}\Big(\int\limits_{\mathbb{K}^N} \cdots \int\limits_{\mathbb{K}^N} \Big(c_{q,p}^{-p}\int\limits_{\mathbb{K}^N} |z \circ T(z^{(1)}, \ldots, z^{(m)})|^p d\mu_q^N(z)\Big) d\mu_{r'}^N(z^{(1)}) \ldots \\
&\quad d\mu_{r'}^N(z^{(m)})\Big)^{1/p} \\
&\leq c_{r',p}^{-m}\Big(\int\limits_{\mathbb{K}^N} \cdots \int\limits_{\mathbb{K}^N} \pi_p\big(T(z^{(1)}, \ldots, z^{(m)}); \ell_{q'}^M \to \mathbb{K}\big)^p d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)})\Big)^{1/p} \\
&= c_{r',p}^{-m}\Big(\int\limits_{\mathbb{K}^N} \cdots \int\limits_{\mathbb{K}^N} \|T(z^{(1)}, \ldots, z^{(m)})\|_{\ell_q^M}^p d\mu_{r'}^N(z^{(1)}) \ldots d\mu_{r'}^N(z^{(m)})\Big)^{1/p},
\end{aligned}$$

where by $\pi_p\big(T(z^{(1)}, \dots, z^{(m)}); \ell_{q'}^M \to \mathbb{K}\big)$ we denote the absolutely p-summing norm of the vector $T(z^{(1)}, \dots, z^{(m)})$ thought of as a linear functional on $\ell_{q'}^M$, whose norm is just the ℓ_q^M-norm of the vector.

The cases where $p = q$ follow from Corollary 2.9. □

Proof of Proposition 1.3. (i) For $r = 2$, the equivalence of norms is a consequence of Theorem 1.2 when $p < q \leq 2$ or $p = q$ and of Corollary 2.7 for $q \leq p$.

For $r > 2$, the equivalence of norms is a consequence of Theorem 1.2 when $p < r'$ and $p < q \leq 2$ and of Corollary 2.7 $q \leq p < r'$.

(ii) This statement follows from Lemma 2.2. □

Proof of Proposition 1.5. The first assertion follows from Corollary 2.7 and localization. For the second assertion just combine Lemma 2.5 with Lemma 2.6. □

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