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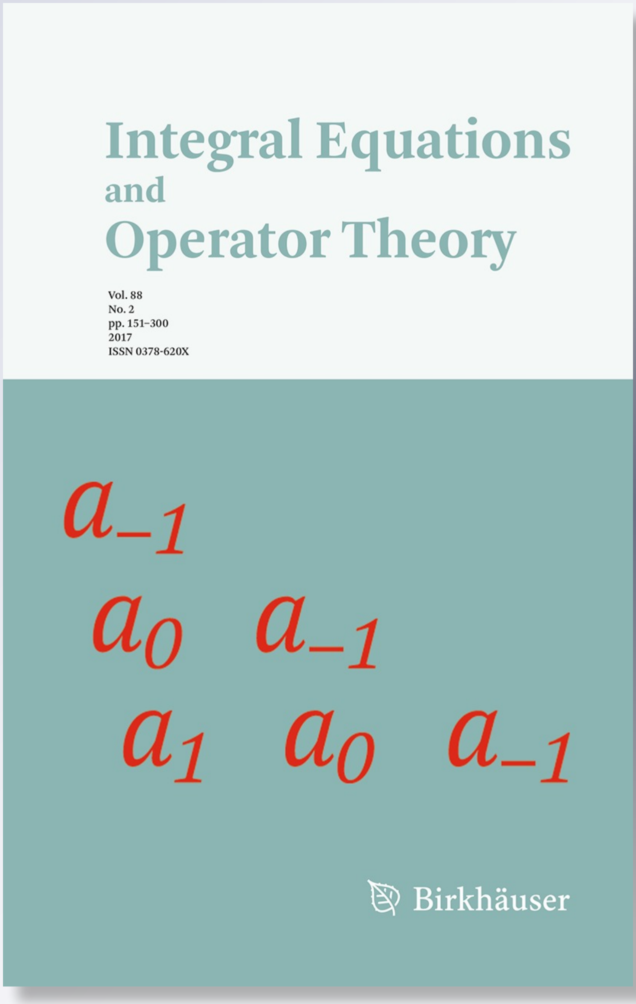

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Products of Idempotent Operators

M. Laura Arias, Gustavo Corach and Alejandra Maestripieri

Abstract. The goal of this article is to study the set of all products EF with E, F idempotent operators defined on a Hilbert space. We present characterizations of this set in terms of operator ranges, Hilbert space decompositions and generalized inverses.

Mathematics Subject Classification. Primary 47A05; Secondary 47A68.

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1. Introduction

Let $\mathcal{H}$ be a Hilbert space. Denote by $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, $\mathcal{Q} = \{E \in L(\mathcal{H}) : E^2 = E)\}$ (idempotents) and $\mathcal{P} = \{P \in \mathcal{Q} : P = P^*\}$ (orthogonal projections). The purpose of this paper is to study the set $\mathcal{QQ}$, which consists of all products EF, where $E, F \in \mathcal{Q}$. The study has been guided, in some sense, by the results of [9], concerning the set $\mathcal{PP} \subseteq \mathcal{QQ}$ of all products PQ, where $P, Q \in \mathcal{P}$. Of course, the (unbounded) set $\mathcal{QQ}$ is much bigger than the (bounded) set $\mathcal{PP}$. We mention a few examples of subsets of operators contained in $\mathcal{QQ}$: nilpotent operators of order 2, normal operators T such that the kernel $N(T)$ and the closure $\overline{R(T)}$ of the range have the same dimension; more generally, every T such that $N(T) \cap N(T^*)$ and $\overline{R(T)} \cap \overline{R(T^*)}$ have the same dimension; and even more generally, every T such that $\overline{R(T)}$ and $N(T)$ have a common complement. This last class is related to a theorem of Lauzon and Treil, who in [19] found a complete characterization of all pairs of closed subspaces $\mathcal{S}, \mathcal{T}$ of $\mathcal{H}$ such that there exists another closed subspace $\mathcal{M}$ with the property $\mathcal{S} \dot{+} \mathcal{M} = \mathcal{T} \dot{+} \mathcal{M} = \mathcal{H}$ (hereafter, $\dot{+}$ denotes a direct sum). Together with some characterizations of $\mathcal{QQ}$ which we describe below, we consider for every $T \in \mathcal{QQ}$ the set of all decompositions of T, i.e., $\{(E, F) \in \mathcal{Q} \times \mathcal{Q} : T = EF\}$. Recall that this has been done for $T \in \mathcal{PP}$ [9], where it is proven that T belongs to $\mathcal{PP}$ if and only if $T = P_{\overline{R(T)}} P_{N(T)^\perp}$ (from here on, if $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ then

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$P_{\mathcal{S}}$ denotes the orthogonal projection onto $\mathcal{S}$). This result, which is due to Crimmins (see [22, Theorem 8] for a proof) provides a standard factorization of every $T \in \mathcal{PP}$, which also has some optimal properties among every other $(P, Q) \in \mathcal{P} \times \mathcal{P}$ such that $T = PQ$. It turns out that the situation for $\mathcal{QQ}$ is much more subtle: even if $T \in \mathcal{QQ}$ there exists $(E, F) \in \mathcal{Q} \times \mathcal{Q}$ such that $T = EF$, $R(E) = \overline{R(T)}$ and $N(F) = N(T)$, it happens that, in general, this pair is not unique. Several other properties of operators in $\mathcal{PP}$ do not hold in $\mathcal{QQ}$, in general. Thus, if $T \in \mathcal{PP}$ it holds that $\overline{R(T)} \cap N(T) = \{0\}$, $\overline{R(T)} + N(T)$ is dense in $\mathcal{H}$ and $\overline{R(T)} + N(T) = \mathcal{H}$ if and only if $R(T)$ is closed (see [9]). They all fail, in general, in $\mathcal{QQ}$. These properties even fail, in general, in the smaller set $\mathcal{PQ}$.

We collect here some references on previous results on $\mathcal{PP}, \mathcal{PQ}$ and $\mathcal{QQ}$. There is an excellent survey by Wu [23] about factorizations of type $\mathcal{A}^n$ and $\mathcal{AB}$, where $n \geq 2$ and $\mathcal{A}, \mathcal{B}$ are fixed classes of operators on $\mathcal{H}$ as normal, Hermitian, positive, involutions, partial isometries, orthogonal projections, idempotents, and so on. We mention here a theorem of Ballantine [6]: if T is a a square matrix then $T \in \mathcal{Q}^k$ if and only if $\dim R(T - I) \leq k \dim N(T)$. If $\mathcal{H}$ has infinite dimension, Dawlings [10] proved that $T \in \mathcal{Q}^k$ for some $k \geq 1$ if and only if $T = I$ or $\dim N(T) = \dim N(T^*) = \infty$ or $0 < \dim N(T) = \dim N(T^*)$ and $\dim R(I - T^*) < \infty$. Kuo and Wu [18] proved that , if $\dim \mathcal{H}$ is finite then $T \in \mathcal{P}^k$ for some k if and only if T is unitarily equivalent to a matrix of the form $\begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$ where S is singular and $||S|| < 1$. For $k = 2$, T. Crimmins proved that $T \in \mathcal{PP}$ if and only if $TT^*T = T^2$, and in such case $T = P_{\overline{R(T)}} P_{N(T)^\perp}$, as remarked above; the proof of Crimmins' result appeared in the paper by Radjavi and Williams [22], which contains many factorization results. More recent references include [9], which contains several results on $\mathcal{PP}$, [4] where there is a study of $\mathcal{PL}^+$ where $\mathcal{L}^+$ stands for the set of semi-definite positive operators on $\mathcal{H}$ and [1] with a discussion on several examples of factorizations including $\mathcal{Q}$, partial isometries, unitaries, and so on.

We briefly describe the contents of the paper. In Sect. 2 we collect some characterizations of $\mathcal{QQ}$. By using a slight extension of the well known majorization theorem of R. G. Douglas (see below), we prove that, for $T \in L(\mathcal{H})$ it holds that $T \in \mathcal{QQ}$ if and only if there exists $E \in \mathcal{Q}$ such that $R(E) = \overline{R(T)}$ and $R(T - T^2) \subseteq R(T(I - E))$. Also, $T \in \mathcal{QQ}$ if and only if there exists $E \in \mathcal{Q}$ such that $N(T) + N(E - T) = \mathcal{H}$. This last result is based on a result by Antezana et al. [2, Proposition 4.13] about the existence of idempotent solutions of an operator equation of the type $A = XB$. It is proven that also $\mathcal{PQ}$ and $\mathcal{PP}$ admit similar characterizations. As mentioned before, in [19], Lauzon and Treil parametrized the set $\mathcal{X}$ of all pairs of closed subspaces of $\mathcal{H}$ which admit a common direct complement (for different approaches to this result, see also the papers by Giol [15] and Drivaliaris and Yannakakis [12]). We prove here that every $T \in L(\mathcal{H})$ such that $(\overline{R(T)}, N(T)) \in \mathcal{X}$ belongs to $\mathcal{QQ}$. We also prove that two closed subspaces $\mathcal{S}, \mathcal{T}$ of $\mathcal{H}$ belong to $\mathcal{X}$ if and only if there exists $T \in \mathcal{PQ}$ such that $R(T) = \mathcal{T}^\perp$ and $N(T) = \mathcal{S}$. As a consequence we get that a normal operator T such $N(T)$ and $\overline{R(T)}$ have the same dimension belongs to $\mathcal{QQ}$ and, more generally, that every $T \in L(\mathcal{H})$ such that $\overline{R(T)} \cap \overline{R(T^*)}$ and $N(T) \cap N(T^*)$ have the same dimension belongs to $\mathcal{QQ}$. Section 3 is concerned with the sets $(\mathcal{QQ})_T$ and $[\mathcal{QQ}]_T$ for $T \in \mathcal{QQ}$, namely:

$$(\mathcal{QQ})_T := \{(E, F) \in \mathcal{Q} \times \mathcal{Q} : T = EF\},$$

and

$$[\mathcal{QQ}]_T := \{(E, F) \in (\mathcal{QQ})_T : R(E) = \overline{R(T)} \text{ and } N(F) = N(T)\}.$$

Notice that, with the obvious notations, $[\mathcal{PP}]_T = \{(P_{\overline{R(T)}}, P_{N(T)^\perp})\}$ and, by [9], $(\mathcal{PP})_T = \{(P_{\mathcal{M}_1}, P_{\mathcal{M}_2}) : \exists \text{ closed subspaces } \mathcal{N}_i \text{ of } \mathcal{M}_i \text{s.t. } \mathcal{M}_1 = \overline{R(T)} \oplus \mathcal{N}_1, \mathcal{M}_2 = N(T)^\perp \oplus \mathcal{N}_2,\ \mathcal{N}_1 \perp \mathcal{N}_2 \text{ and } \mathcal{N}_1 \oplus \mathcal{N}_2 \subseteq \overline{R(T)^\perp \cap N(T)}\}$. For $(E, F) \in (\mathcal{QQ})_T$ it holds that $(E, F) \in [\mathcal{QQ}]_T$ if and only if $\overline{N(E) \dot{+} R(F)} = \mathcal{H}$ (see Lemma 3.3) and this property, together with the use of the closed (unbounded) projection $H_{F,E}$ with $R(H_{F,E}) = R(F)$ and $N(H_{F,E}) = N(E)$, leads to the following new characterization of $\mathcal{QQ}$, (Theorem 3.7): if $T \in L(\mathcal{H})$ then $T \in \mathcal{QQ}$ if and only if there exists a closed projection H such that $THT = T$ and $T^*H^*T^* = T^*$, i.e., H (resp. H^*) is an unbounded inner inverse of T (resp. T^*). In particular, if $R(T)$ is closed, $T \in \mathcal{QQ}$ if and only if $T^\dagger \in \mathcal{PQP}$, where $T^\dagger$ denotes the Moore–Penrose inverse of T. Moreover, for $T \in \mathcal{QQ}$ it holds $\{H_{F,E} : (E, F) \in [\mathcal{QQ}]_T\} = \{H \in \tilde{\mathcal{Q}} : H \in T[1, 2] \text{ and } H^* \in T^*[1]\}$, where $T[1] = \{X : TXT = T\}$, $T[1, 2] = \{X \in T[1] : XTX = X\}$ and $\tilde{\mathcal{Q}}$ is the set of all (not necessarily bounded) closed projections in $\mathcal{H}$. Finally, Sect. 4 deals with splitting properties of $R(T)$ and $N(T)$ for $T \in \mathcal{QQ}$. As we have mentioned before, most of the properties regarding splitting that hold in $\mathcal{PP}$ fail, in general, in $\mathcal{QQ}$. However, we get some results in similar directions. We only mention here a few of them: for $T \in \mathcal{QQ}$ it holds $\overline{R(T)} \cap N(T) = \{0\}$ if and only if $E + F - I$ is injective for some (and then all) $(E, F) \in [\mathcal{QQ}]_T$; $R(T) + N(T)$ is dense if and only if $R(E + F - I)$ is dense; and $R(T) \dot{+} N(T) = \mathcal{H}$ if and only if $E + F - I$ is invertible. The paper finishes with a complementary result to Ballantine's characterization of $\mathcal{QQ}$, for $\mathcal{H}$ finite dimensional, mentioned above. More precisely, we prove that if $T \in L(\mathcal{H})$ with $\dim \mathcal{H} < \infty$ then $T \in \mathcal{QQ}$ if there exists $X \in L(\mathcal{H})$ such that $XTX = X^2$ and $\dim N(X) \leq \dim N(T)$.

2. The Set $\mathcal{QQ}$

Our goal in this section is to describe the set $\mathcal{QQ} := \{EF : E, F \in \mathcal{Q}\}$, where $\mathcal{Q} := \{E \in L(\mathcal{H}) : E^2 = E\}$. Observe that there are neither injective nor dense-range operators in $\mathcal{QQ}$, except for the identity operator.

In [9] it is proven that, if $\mathcal{P} := \{E \in \mathcal{Q} : E^* = E\}$ then for $T \in \mathcal{PP}$ the pair $(P_{\overline{R(T)}}, P_{N(T)^\perp})$ has optimal properties in the set $\{(P, Q) \in \mathcal{P} \times \mathcal{P} : T = PQ\}$, namely, for all $P, Q \in \mathcal{P}$ such that $T = PQ$ it holds that

- $R(P_{\overline{R(T)}}) \subseteq R(P)$, $N(P_{N(T)^\perp}) \subseteq N(Q)$.
- $||(P_{\overline{R(T)}} - P_{N(T)^\perp})x|| \leq ||(P - Q)x||$ for all $x \in \mathcal{H}$.

We show now that the situation in $\mathcal{QQ}$ is completely different, in the sense that there is no such distinguished factorization of a $T \in \mathcal{QQ}$ and it does not look evident how to define an optimal factorization of T. The next result is a key tool in what follows.

Lemma 2.1. *Let $T \in \mathcal{QQ}$. Then, there exist $E, F \in \mathcal{Q}$ such that $T = EF$, $R(E) = \overline{R(T)}$ and $N(F) = N(T)$.*

Proof. Let $T = E'F'$ with $E', F' \in \mathcal{Q}$. Trivially, $\overline{R(T)} \subseteq R(E')$ and $N(F') \subseteq N(T)$. Define $E = P_{\overline{R(T)}}E'$ and $F = F'P_{N(T)^\perp}$. Clearly, $T = EF$. Let us see that E, F satisfy the conditions of the lemma. First, $E^2 = P_{\overline{R(T)}}E'P_{\overline{R(T)}}E' = P_{\overline{R(T)}}E' = E$ since $\overline{R(T)} \subseteq R(E')$. Moreover, $R(E) \subseteq \overline{R(T)}$, and given $x \in \overline{R(T)}$ then $x = P_{\overline{R(T)}}E'x = Ex$, i.e., $R(E) = \overline{R(T)}$. On the other hand, $F^2 = F'P_{N(T)^\perp}F'P_{N(T)^\perp} = F'P_{N(T)^\perp} = F$, because $N(F') \subseteq N(T) = N(P_{N(T)^\perp})$. In addition, $N(T) \subseteq N(F)$ and given $x \in N(F)$ then $P_{N(T)^\perp}x \in N(F') \subseteq N(T) \cap N(T)^\perp = \{0\}$, i.e., $x \in N(T)$ and so $N(T) = N(F)$ as desired. □

It should be noticed that, for a general $T \in \mathcal{QQ}$, a factorization $T = EF$, with $E, F \in \mathcal{Q}$ and $R(E) = \overline{R(T)}$, $N(F) = N(T)$ is not unique. For example, consider $T = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$, $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $F = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & 0 \\ \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E' = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $F' = \begin{pmatrix} \frac{3}{2} & \frac{-3}{2} & 2 \\ \frac{1}{2} & \frac{-1}{2} & 2 \\ 0 & 0 & 1 \end{pmatrix}$. Therefore, a simple computation shows that $T = EF = E'F'$; $E, F, E', F' \in \mathcal{Q}$ and $R(E) = R(E') = \overline{R(T)}$, $N(F) = N(F') = N(T)$.

Given $T \in \mathcal{QQ}$, the preceding lemma motivates the next definitions:

$$(\mathcal{QQ})_T := \{(E, F) \in \mathcal{Q} \times \mathcal{Q} : T = EF\},$$

and

$$[\mathcal{QQ}]_T := \{(E, F) \in (\mathcal{QQ})_T : R(E) = \overline{R(T)} \text{ and } N(F) = N(T)\}.$$

We will frequently use the fact that $(E, F) \in [\mathcal{QQ}]_T$ if and only if $(F^*, E^*) \in [\mathcal{QQ}]_{T^*}$.

By the proof of Lemma 2.1, $(P_{\overline{R(T)}}E, FP_{N(T)^\perp}) \in [\mathcal{QQ}]_T$ if $(E, F) \in (\mathcal{QQ})_T$. Observe that this defines a retraction map:

$$\phi : (\mathcal{QQ})_T \to [\mathcal{QQ}]_T. \tag{2.1}$$

With the obvious notations, $[\mathcal{PP}]_T = \{(P_{\overline{R(T)}}, P_{N(T)^\perp})\}$. In particular, it says that there exists a natural cross section of the product map $\pi : \mathcal{P} \times \mathcal{P} :\to \mathcal{PP}$, namely, $s : \mathcal{PP} \to \mathcal{P} \times \mathcal{P}$, $s(T) = (P_{\overline{R(T)}}, P_{N(T)^\perp})$. Unfortunately, this section is not continuous and it is not useful to obtain topological facts on $\mathcal{PP}$. In any case, there is not such section for the map $\mathcal{Q} \times \mathcal{Q} :\to \mathcal{QQ}$; in fact, as it was mentioned above, there is no distinguished factorization of $T \in \mathcal{QQ}$.

In order to prove our first characterization of $\mathcal{QQ}$, we introduce the well known Douglas' theorem on factorization of operators [13]. Here, we present a simple generalization of this result whose proof is similar to Douglas original proof, see [3].

Theorem 2.2. *Let $A \in L(\mathcal{H},\mathcal{K})$ and $B \in L(\mathcal{F},\mathcal{K})$. Then, there exists $C \in L(\mathcal{F},\mathcal{H})$ such that $AC = B$ if and only if $R(B) \subseteq R(A)$. In such case, if $\mathcal{M}$ is a topological complement of $N(A)$ then there exists a unique solution $X_{\mathcal{M}} \in L(\mathcal{F},\mathcal{H})$ of the equation $AX = B$ such that $R(X_{\mathcal{M}}) \subseteq \mathcal{M}$. The operator $X_{\mathcal{M}}$ will be called the* **reduced solution for** *$\mathcal{M}$ of the equation $AX = B$.*

Theorem 2.3. *Let $T \in L(\mathcal{H})$. The next conditions are equivalent:*

1. *$T \in \mathcal{QQ}$.*
2. *$R(T - T^2) \subseteq R(T(I - E))$ for some $E \in \mathcal{Q}$ with $R(E) = \overline{R(T)}$.*
3. *$R((T - T^2)^*) \subseteq R(((I - F)T)^*)$ for some $F \in \mathcal{Q}$ with $N(F) = N(T)$.*

Proof. $1 \Leftrightarrow 2$. Assume that $T \in \mathcal{QQ}$ and let $(E, F) \in [\mathcal{QQ}]_T$. Then, $T - T^2 = T(I - T) = EF(I - E)F = T(I - E)F$. Therefore, $R(T - T^2) = R(T(I - E)F) \subseteq R(T(I - E))$ where $E \in \mathcal{Q}$ and $R(E) = \overline{R(T)}$.

Conversely, suppose that $R(T - T^2) \subseteq R(T(I - E))$ for some $E \in \mathcal{Q}$ with $R(E) = \overline{R(T)}$. Then, by Theorem 2.2, the operator equation $T - T^2 = T(I - E)X$ has a solution in $L(\mathcal{H})$. Now, as $N(T(I - E)) = N(I - E) \dot{+} R(I - E) \cap N(T) = \overline{R(T)} \dot{+} N(E) \cap N(T)$ and $\mathcal{H} = \overline{R(T)} \dot{+} N(E)$ there exists a closed subspace $\mathcal{S} \subseteq N(E)$ such that $\mathcal{H} = N(T(I - E)) \dot{+} \mathcal{S}$, (for example, $\mathcal{S} = N(E) \ominus N(E) \cap N(T)$). Let X_0 be the reduced solution for $\mathcal{S}$ of $T - T^2 = T(I - E)X$. Notice that $EX_0 = 0$, i.e., $T - T^2 = TX_0$. Moreover, from these two last equalities it can be proven that $T - T^2 = T(I - E)(X_0T + X_0^2)$, i.e., $X_0T + X_0^2$ is a solution of $T - T^2 = T(I - E)X$ with $R(X_0T + X_0^2) \subseteq R(X_0) \subseteq \mathcal{S}$. Hence, by the uniqueness of the reduced solution, $X_0T + X_0^2 = X_0$. Now, define $F := T + X_0$. Hence, $F^2 = (T + X_0)(T + X_0) = T^2 + TX_0 + X_0T + X_0^2 = T + X_0 = F$, i.e., $F \in \mathcal{Q}$ and $T = EF$. Therefore, $T \in \mathcal{QQ}$.

$1 \Leftrightarrow 3$. Taking into account that $T \in \mathcal{QQ}$ if and only if $T^* \in \mathcal{QQ}$, then this equivalence follows by applying $1 \Leftrightarrow 2$ to T^*. □

Remark 2.4. Ballantine [6] found a nice characterization of $\mathcal{QQ}$ for matrices; he proved that $T \in \mathbb{C}^{n\times n}$ belongs to $\mathcal{QQ}$ if and only if $\dim R(T - I) \leq 2 \dim N(T)$. Observe that Theorem 2.3 can be interpreted as an extension of this result for $T \in L(\mathcal{H})$. In fact, $R(T - T^2) \subseteq R(T(I - E))$ if and only if $R(T - I) \subseteq R(I - E) + N(T)$. Hence, in matrices, this last inclusion implies that $\dim R(T - I) \leq \dim R(I - E) + \dim N(T) = 2 \dim N(T)$ since $\dim R(I - E) = \dim N(T)$ for all $E \in \mathcal{Q}$ with $R(E) = \overline{R(T)}$. We shall return on this at the end of the paper.

In what follows we give a characterization of $\mathcal{QQ}$ in terms of subspaces. By $Gr(\mathcal{H})$ we denote the set of all closed subspaces of $\mathcal{H}$ and the symbol $E_{\mathcal{S}//\mathcal{T}}$ stands for the operator in $\mathcal{Q}$ with range $\mathcal{S}$ and nullspace $\mathcal{T}$ provided that $\mathcal{S}, \mathcal{T} \in Gr(\mathcal{H})$ and $\mathcal{S} \dot{+} \mathcal{T} = \mathcal{H}$. If $\mathcal{T} = \mathcal{S}^\perp$ then we simply write $P_{\mathcal{S}}$ instead of $E_{\mathcal{S}//\mathcal{S}^\perp}$.

Proposition 2.5. *Let $T \in L(\mathcal{H})$. The next conditions are equivalent:*

1. *$T \in \mathcal{QQ}$.*
2. *There exist $\mathcal{S}, \mathcal{W} \in Gr(\mathcal{H})$ such that $\overline{R(T)} \dot{+} \mathcal{S} = \mathcal{H}$, $\mathcal{W} \dot{+} N(T) = \mathcal{H}$, and $P_{\mathcal{S}^\perp} T P_{\mathcal{W}} \in \mathcal{PP}$.*

Proof. 1 $\Rightarrow$ 2. Let $T = EF$ with $(E, F) \in [\mathcal{QQ}]_T$. Let $\mathcal{S} := N(E)$ and $\mathcal{W} := R(F)$. Hence, $\overline{R(T)} \dot{+} \mathcal{S} = \mathcal{H}$ and $\mathcal{W} \dot{+} N(T) = \mathcal{H}$. Moreover, $P_{\mathcal{S}^\perp} T P_{\mathcal{W}} = P_{\mathcal{S}^\perp} EF P_{\mathcal{W}} = P_{\mathcal{S}^\perp} P_{\mathcal{W}} \in \mathcal{PP}$.

2 $\Rightarrow$ 1. Define $E := Q_{\overline{R(T)}//\mathcal{S}}$ and $F := Q_{\mathcal{W}//N(T)}$ and let $P_1, P_2 \in \mathcal{P}$ such that $P_{\mathcal{S}^\perp} T P_{\mathcal{W}} = P_1 P_2$. There is no loss of generality in assuming that $R(P_1) = \overline{R(P_{\mathcal{S}^\perp} T P_{\mathcal{W}})}$ and $N(P_2) = N(P_{\mathcal{S}^\perp} T P_{\mathcal{W}})$. Thus, $R(P_1) \subseteq \mathcal{S}^\perp$ or, equivalently $N(E) = \mathcal{S} \subseteq N(P_1)$ and $\mathcal{W}^\perp \subseteq N(P_2)$ or, equivalently, $R(P_2) \subseteq \mathcal{W} = R(F)$. Therefore, $P_1 = P_1 E$ and $FP_2 = P_2$. Thus, $(EP_1)^2 = EP_1EP_1 = EP_1$ and $(P_2F)^2 = P_2FP_2F = P_2F$, i.e., $EP_1, P_2F \in \mathcal{Q}$. Now,

$$T = ETF = EP_{\mathcal{S}^\perp} T P_{\mathcal{W}} F = EP_1P_2F \in \mathcal{QQ},$$

and the proof is finished. □

The next result due to Antezana et al. [2, Proposition 4.13] will be useful in order to obtain another characterization of $\mathcal{QQ}$:

Proposition 2.6. *Given $A, B \in L(\mathcal{H}, \mathcal{K})$, the following statements are equivalent:*

1. *$\overline{R(A)} \dot{+} \overline{R(B - A)}$ is closed.*
2. *There exists $E \in \mathcal{Q}$ such that $A = EB$.*

Applying the previous result and recalling that $T \in \mathcal{QQ}$ if and only if $T^* \in \mathcal{QQ}$ we obtain the following:

Proposition 2.7. *Let $T \in L(\mathcal{H})$. The next conditions are equivalent:*

1. *$T \in \mathcal{QQ}$.*
2. *There exists $E \in \mathcal{Q}$ such that $\overline{R(T)} \dot{+} \overline{R(E - T)}$ is closed.*
3. *There exists $E \in \mathcal{Q}$ such that $\mathcal{H} = N(T) + N(E - T)$.*

Following the same lines we get the next characterizations of $\mathcal{PQ}$ and $\mathcal{PP}$.

Proposition 2.8. *Let $T \in L(\mathcal{H})$. The next conditions are equivalent:*

1. *$T \in \mathcal{PQ}$.*
2. *There exists a topological complement $\mathcal{M}$ of $N(T)$ such that $||Tx||^2 = \langle Tx, x\rangle$ for all $x \in \mathcal{M}$.*
3. *$T^*T = T^*E$ for some $E \in \mathcal{Q}$.*
4. *$\overline{R(T^*T)} \dot{+} \overline{R(T - T^*T)}$ is closed.*

Proof. 1 $\Rightarrow$ 2. Let $T = PE$ with $P \in \mathcal{P}$ and $E \in \mathcal{Q}$. Without loss of generality, we can consider $N(E) = N(T)$. Let $\mathcal{M} = R(E)$. Then, if $x \in \mathcal{M}$ we have that $||Tx||^2 = \langle x, T^*Tx\rangle = \langle x, E^*PEx\rangle = \langle x, E^*Px\rangle = \langle PEx, x\rangle = \langle Tx, x\rangle$, as desired.

2 $\Rightarrow$ 3. Assume that $||Tx||^2 = \langle Tx, x\rangle$ for all $x \in \mathcal{M}$, with $\mathcal{M} \dot{+} N(T) = \mathcal{H}$. Define $E := E_{\mathcal{M}//N(T)} \in \mathcal{Q}$. Then, $||TEx||^2 = \langle TEx, Ex\rangle$ for all $x \in$

$\mathcal{H}$. Now, as $N(E) = N(T)$ then $TE = T$ and so $\langle T^*Tx, x\rangle = ||Tx||^2 = \langle Tx, Ex\rangle = \langle E^*Tx, x\rangle$ for all $x \in \mathcal{H}$. Thus, $T^*T = E^*T$, i.e., $T^*T = T^*E$.

3 $\Rightarrow$ 1. Suppose that $T^*T = T^*E$ for some $E \in \mathcal{Q}$. Then, $T^*T = T^*P_{\overline{R(T)}}E$ and so $T = P_{\overline{R(T)}}E$.

3 $\Leftrightarrow$ 4. It follows by Proposition 2.6. □

Proposition 2.9. *Let $T \in L(\mathcal{H})$. The next conditions are equivalent:*

1. *$T \in \mathcal{PP}$.*
2. *$T^*T = T^*P$ for some $P \in \mathcal{P}$.*
3. *$R(T^*T) \perp R(T - T^*T)$.*

Proof. 1 $\Leftrightarrow$ 2. If $T = P_{\overline{R(T)}}P_{N(T)^\perp}$ then $T^*T = T^*P_{N(T)^\perp}$. Conversely, if $T^*T = T^*P$ for some $P \in \mathcal{P}$ then $T^*T = T^*P_{\overline{R(T)}}P$ and so $T, P_{\overline{R(T)}}P$ are both reduced solutions for $N(T^*)^\perp$ of $T^*X = T^*T$. Hence, by the uniqueness of the reduced solution, we get that $T = P_{\overline{R(T)}}P \in \mathcal{PP}$, as desired.

1 $\Leftrightarrow$ 3. If $T = P_1P_2$ with $P_1, P_2 \in \mathcal{P}$ then $T^*T = P_2P_1P_2$ and $T - T^*T = (I - P_2)P_1P_2$. Thus, $R(T^*T) \perp R(T - T^*T)$.

Conversely, suppose that $R(T^*T) \perp R(T - T^*T)$. Then $\overline{R(T - T^*T)} \subseteq N(P_{\overline{R(T^*T)}})$ and so $P_{\overline{R(T^*T)}}T = P_{\overline{R(T^*T)}}(T - T^*T + T^*T) = T^*T$; since conditions 1 and 2 are equivalent it follows that $T \in \mathcal{PP}$. □

The set $\mathcal{QQ}$ can be also characterized in terms of the generalized Wiener–Hopf operators, i.e., operators of the form $P_{\mathcal{M}}T|_{\mathcal{M}}$ where $T \in L(\mathcal{H})$. For this, we state the next result:

Lemma 2.10. *Let $T \in L(\mathcal{H})$, then $T \in \mathcal{Q}$ if and only if $T = P_{\overline{R(T)}}A$ for some $A \in Gl(\mathcal{H})^+$ and $P_{\overline{R(T)}}AP_{\overline{R(T)}} = P_{\overline{R(T)}}$.*

Proof. If $T \in \mathcal{Q}$ then the existence of $A \in Gl(\mathcal{H})^+$ such that $T = P_{\overline{R(T)}}A$ is guaranteed because of [17, Theorem 1] (see also [4, Theorem 3.3]) and then, trivially, $P_{\overline{R(T)}}AP_{\overline{R(T)}} = P_{\overline{R(T)}}$. The converse is obvious. □

Now, applying the previous lemma we get the following:

Proposition 2.11. *Let $T \in L(\mathcal{H})$. Therefore $T \in \mathcal{QQ}$ if and only if $T = P_{\overline{R(T)}}ABP_{N(T)^\perp}$ for some $A, B \in Gl(\mathcal{H})^+$ such that $P_{\overline{R(T)}}A|_{\overline{R(T)}} = I|_{\overline{R(T)}}$ and $P_{N(T)^\perp}B|_{N(T)^\perp} = I|_{N(T)^\perp}$.*

2.1. Some Examples

Lauzon and Treil [19] parametrized the set $\mathcal{X}$ of pairs of closed subspaces of a Hilbert space $\mathcal{H}$ which admit a common direct complement, in symbols, $\mathcal{X} = \{(\mathcal{M}, \mathcal{N}) : \mathcal{M}, \mathcal{N} \in Gr(\mathcal{H}),\ \exists\, \mathcal{S} \in Gr(\mathcal{H})\ s.t.\ \mathcal{M} \dot{+} \mathcal{S} = \mathcal{N} \dot{+} \mathcal{S} = \mathcal{H}\}$. We show now that any $T \in L(\mathcal{H})$ such that $(\overline{R(T)}, N(T)) \in \mathcal{X}$ belongs to $\mathcal{QQ}$. We also characterize $\mathcal{X}$ by proving that if $\mathcal{M}, \mathcal{N} \in Gr(\mathcal{H})$ then $(\mathcal{M}, \mathcal{N}) \in \mathcal{X}$ if and only if there exists $T \in \mathcal{PQ}$ such that $R(T) = \mathcal{N}^\perp$ and $N(T) = \mathcal{M}$.

Proposition 2.12. *Let $T \in L(\mathcal{H})$. If $\overline{R(T)}$ and $N(T)$ have a common topological complement then $T \in \mathcal{QQ}$.*

Proof. Let $\mathcal{S} \in Gr(\mathcal{H})$ such that $\mathcal{H} = \overline{R(T)} \dot{+} \mathcal{S} = N(T) \dot{+} \mathcal{S}$ and define $E = Q_{\overline{R(T)}//\mathcal{S}}$. Hence, $R(T(I-E)) = T(\mathcal{S}) = R(T)$ where the last equality holds because $N(T) \dot{+} \mathcal{S} = \mathcal{H}$. Thus, as $R(T-T^2) \subseteq R(T)$ we have that $R(T-T^2) \subseteq R(T(I-E))$. Therefore, by Proposition 2.3, $T \in \mathcal{QQ}$. □

The converse of the above corollary is false, in general. For example, consider $E \in \mathcal{Q}$ with $\dim(R(E)) \neq \dim(N(E))$; trivially, $E \in \mathcal{QQ}$ and $R(E)$ and $N(E)$ may not have a common complement.

Proposition 2.13. *Let $\mathcal{S}, \mathcal{T}$ be two closed subspaces of $\mathcal{H}$. Then, $\mathcal{S}, \mathcal{T}$ have a common topological complement in $\mathcal{H}$ if and only if there exists $T \in \mathcal{PQ}$ with $R(T) = \mathcal{T}^\perp$ and $N(T) = \mathcal{S}$.*

Proof. Suppose that there exists a closed subspace $\mathcal{W}$ such that $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{W} = \mathcal{T} \dot{+} \mathcal{W}$. Define $E = E_{\mathcal{W}//\mathcal{S}}$ and $T = P_{\mathcal{T}^\perp} E \in \mathcal{PQ}$. We claim that $R(T) = \mathcal{T}^\perp$ and $N(T) = \mathcal{S}$. In fact, $R(T) = P_{\mathcal{T}^\perp}(\mathcal{W}) = R(P_{\mathcal{T}^\perp}) = \mathcal{T}^\perp$ because $\mathcal{H} = \mathcal{T} \dot{+} \mathcal{W}$ and $N(T) = N(E) + R(E) \cap N(P_{\mathcal{T}^\perp}) = \mathcal{S} + \mathcal{W} \cap \mathcal{T} = \mathcal{S}$ because $\mathcal{W} \cap \mathcal{T} = \{0\}$.

Conversely, let $T \in \mathcal{PQ}$ with $R(T) = \mathcal{T}^\perp$ and $N(T) = \mathcal{S}$. Then, $T = P_{\mathcal{T}^\perp} Q_{\mathcal{W}//\mathcal{S}}$ for some complement $\mathcal{W}$ of $\mathcal{S}$. Now, as $R(T) = \mathcal{T}^\perp$ then $\mathcal{H} = \mathcal{W} + \mathcal{T}$. On the other hand, as $\mathcal{S} = N(T) = \mathcal{S} \dot{+} \mathcal{W} \cap \mathcal{T}$ we have that $\mathcal{W} \cap \mathcal{T} = \{0\}$, i.e., $\mathcal{H} = \mathcal{W} \dot{+} \mathcal{T}$. Therefore, $\mathcal{W}$ is a common complement of $\mathcal{S}$ and $\mathcal{T}$. □

Example. Applying Theorem 2.3 and Proposition 2.12 the following examples of operators in $\mathcal{QQ}$ can be easily obtained:

1. If $\dim(\overline{R(T)} \cap \overline{R(T^*)}) = \dim(N(T) \cap N(T^*))$ then, by Lauzon and Treil [19, Remark 0.4], $\overline{R(T)}$ and $N(T)$ have a common topological complement. Hence, by the previous corollary $T \in \mathcal{QQ}$. In particular, if T is a normal operator with $\dim(\overline{R(T)}) = \dim N(T)$ then $T \in \mathcal{QQ}$. On the other hand, notice that if $T \in \mathcal{PP}$ is normal then $T \in \mathcal{P}$. In fact, if $T \in \mathcal{PP}$ then $T = P_{\overline{R(T)}} P_{N(T)^\perp}$, but as T is normal then $\overline{R(T)} = N(T)^\perp$ and so $T = P_{N(T)^\perp} \in \mathcal{P}$.
2. If $T^2 = 0$ then $T \in \mathcal{QQ}$. In fact, $R(T - T^2) = R(T) = R(T(I - P_{\overline{R(T)}}))$ where the last equality holds because $R(T) \subseteq N(T)$. Then, by Theorem 2.3, $T \in \mathcal{QQ}$ (moreover, $T \in \mathcal{PQ}$). See also [1, Theorem 6.1]. On the other side, notice that if $T^2 = 0$ and $T \in \mathcal{PP}$ then $T = 0$. Indeed, if $T^2 = 0$ then $R(T) \subseteq N(T)$ and so $T = P_{\overline{R(T)}} P_{N(T)^\perp} = 0$.

3. The Sets $(\mathcal{QQ})_T$ and $[\mathcal{QQ}]_T$

This section is devoted to study the sets $(\mathcal{QQ})_T$ and $[\mathcal{QQ}]_T$ for $T \in \mathcal{QQ}$. For this aim, we start by establishing the relationship between $(\mathcal{QQ})_T$ and $[\mathcal{QQ}]_T$:

Proposition 3.1. *Let $T \in \mathcal{QQ}$. Then,*

$$(\mathcal{QQ})_T = \{(E, F) \in \mathcal{Q} \times \mathcal{Q} : E = E_0 + E_1, F = F_0 + F_1 \text{ with } E_1, F_1 \in \mathcal{Q}, \\ (E_0, F_0) \in [\mathcal{QQ}]_T, \text{ and } E_0 F_1 = E_1 F_0 = E_1 F_1 = 0\}.$$

Proof. Let $(E, F) \in (\mathcal{QQ})_T$ and define $E_0 := P_{\overline{R(T)}}E$ and $F_= := FP_{N(T)^\perp}$. By the proof of Lemma 2.1, we have that $(E_0, F_0) \in [\mathcal{QQ}]_T$. Denote by $E_1 = E - E_0 = (I - P_{\overline{R(T)}})E$ and $F_1 = F - F_0 = F(I - P_{N(T)^\perp})$. Hence, $E_1^2 = (I - P_{\overline{R(T)}})E(I - P_{\overline{R(T)}})E = (I - P_{\overline{R(T)}})(E - EP_{\overline{R(T)}})E = (I - P_{\overline{R(T)}})(E - P_{\overline{R(T)}})E = (I - P_{\overline{R(T)}})(I - P_{\overline{R(T)}})E^2 = (I - P_{\overline{R(T)}})E = E_1$, where the third equality holds because $\overline{R(T)} \subseteq R(E)$ since $T = EF$. Thus, $E_1 \in \mathcal{Q}$. Analogously, since $N(F) \subseteq N(T)$ because $T = EF$, we get that $F_1^2 = F(I - P_{N(T)^\perp})F(I - P_{N(T)^\perp}) = F(F - P_{N(T)^\perp}F)(I - P_{N(T)^\perp}) = F(F - P_{N(T)^\perp})(I - P_{N(T)^\perp}) = F(I - P_{N(T)^\perp}) = F_1$, i.e., $F_1 \in \mathcal{Q}$. Finally, $E_0F_1 = P_{\overline{R(T)}}EF(I - P_{N(T)^\perp}) = P_{\overline{R(T)}}T(I - P_{N(T)^\perp}) = 0$, $E_1F_1 = (I - P_{\overline{R(T)}})EF(I - P_{N(T)^\perp}) = (I - P_{\overline{R(T)}})T(I - P_{N(T)^\perp}) = 0$ and $E_1F_0 = (I - P_{\overline{R(T)}})EFP_{N(T)^\perp} = (I - P_{\overline{R(T)}})TP_{N(T)^\perp} = 0$; as desired.

For the other inclusion, let $(E, F) \in \mathcal{Q} \times \mathcal{Q}$ with the stated properties. Let us see that $(E, F) \in (\mathcal{QQ})_T$. For this, we only need to prove that $T = EF$. Now, $EF = (E_0 + E_1)(F_0 + F_1) = E_0F_0 + E_0F_1 + E_1F_0 + E_1F_1 = E_0F_0 = T$. □

Proposition 3.2. *Let $T \in \mathcal{QQ}$, then*

$$\begin{aligned}[\mathcal{QQ}]_T = \{&(E, F) \in \mathcal{Q} \times \mathcal{Q} : R(E) = \overline{R(T)},\ R(T - T^2) \subseteq R(T(I - E)) \\ &\text{and } F = T + (I - E)XP_{N(T)^\perp} \quad \text{with } X \text{ a solution of} \\ &T - T^2 = T(I - E)X\}. \\ = \{&(E, F) \in \mathcal{Q} \times \mathcal{Q} : R((T - T^2)^*) \subseteq R((T(I - F))^*), \\ &N(F) = N(T), \text{and } E = T + P_{\overline{R(T)}}X(I - F) \\ &\text{with } X \text{ a solution of } T - T^2 = X(I - F)T\}.\end{aligned}$$

Proof. Let $(E, F) \in [\mathcal{QQ}]_T$ then, clearly, $R(E) = \overline{R(T)}$. Moreover, $F = EF + (I - E)F = T + (I - E)FP_{N(T)^\perp}$ because $N(F) = N(T)$ and it is straightforward that $T - T^2 = T(I - E)F$. Conversely, let $(E, F) \in \mathcal{Q} \times \mathcal{Q}$ with $R(E) = \overline{R(T)}$ and $F = T + (I - E)XP_{N(T)^\perp}$ for some $X \in L(\mathcal{H})$ such that $T - T^2 = T(I - E)X$. Notice that the existence of X is guaranteed because $R(T - T^2) \subseteq R(T(I - E))$. Clearly, $EF = ET = T$ and $N(F) = N(T)$. It remains to show that $F \in \mathcal{Q}$. Fist, observe that as $T - T^2 = T(I - E)X$ then $(I - E)X = I - T + Z$ for some $Z \in L(\mathcal{H})$ with $R(Z) \subseteq N(T)$. Now,

$$\begin{aligned}F^2 &= T^2 + T(I - E)XP_{N(T)^\perp} + (I - E)XP_{N(T)^\perp}(T + (I - E)XP_{N(T)^\perp}) \\ &= T^2 + (T - T^2) + (I - E)XP_{N(T)^\perp}(T + (I - E)XP_{N(T)^\perp}) \\ &= T + (I - E)XP_{N(T)^\perp}(T + P_{N(T)^\perp} - T + ZP_{N(T)^\perp}) \\ &= T + (I - E)XP_{N(T)^\perp} = F\end{aligned}$$

Therefore $(E, F) \in [\mathcal{QQ}]_T$ and the first equality is proved.

Analogously, but working with $T^* \in \mathcal{QQ}$, we get the second equality. □

Given $T \in \mathcal{QQ}$ every pair $(E, F) \in [\mathcal{QQ}]_T$ can be associated to the pair of subspaces $(R(F), N(E))$. The next result gives a necessary and sufficient condition that these subspaces must fulfill in order that $(E, F) \in [\mathcal{QQ}]_T$.

Lemma 3.3. *Let $T \in \mathcal{QQ}$ and $(E,F) \in (\mathcal{QQ})_T$. Then $(E,F) \in [\mathcal{QQ}]_T$ if and only if $\overline{R(F) \dotplus N(E)} = \mathcal{H}$.*

Proof. Let $(E,F) \in [\mathcal{QQ}]_T$, i.e., $T = EF$, $R(E) = \overline{R(T)}$ and $N(F) = N(T)$. We claim that $R(F) \cap N(E) = \{0\}$. In fact, if $y \in R(F) \cap N(E)$ then $y = Fy$ and $0 = Ey = EFy = Ty$, i.e., $y \in N(T) = N(F)$ and so $y = Fy = 0$. Analogously, since $(F^*, E^*) \in [\mathcal{QQ}]_{T^*}$, we get that $R(E^*) \cap N(F^*) = \{0\}$ or, equivalently, $\overline{R(F) + N(E)} = \mathcal{H}$. Therefore, $\overline{R(F) \dotplus N(E)} = \mathcal{H}$ as claimed.

Conversely, let $(E,F) \in \mathcal{Q} \times \mathcal{Q}$ such that $T = EF$ and $\overline{R(F) \dotplus N(E)} = \mathcal{H}$. Let us prove that $N(F) = N(T)$. Clearly, as $T = EF$ then $N(F) \subseteq N(T)$. On the other hand, if $x \in N(T)$ then $0 = Tx = EFx$, so $Fx \in R(F) \cap N(E) = \{0\}$, i.e., $x \in N(F)$. Hence, $N(F) = N(T)$. Analogously, since $T^* = F^*E^*$ and $R(E^*) \cap N(F^*) = \{0\}$ (because $\overline{R(F) + N(E)} = \mathcal{H}$) we have that $N(E^*) = N(T^*)$ or, equivalently, $R(E) = \overline{R(T)}$. Therefore, $(E,F) \in [\mathcal{QQ}]_T$. □

Corollary 3.4. *Let $T \in \mathcal{QQ}$ and $(E,F) \in [\mathcal{QQ}]_T$. Then, T has closed range if and only if $N(E) \dotplus R(F) = \mathcal{H}$.*

Proof. It follows by Lemma 3.3 and the fact that if $A, B \in L(\mathcal{H})$ have closed ranges then AB has closed range if and only if $N(A) + R(B)$ is closed, see [11, Theorem 22]. □

In order to get another description of $\mathcal{QQ}$ we need the concept of (not necessarily bounded) closed projection. A densely defined operator H is a projection if $R(H) \subseteq \mathcal{D}(H)$ and $H(Hx) = Hx$ for all $x \in \mathcal{D}(H)$. In this case, it holds that $\mathcal{D}(H) = R(H) \dotplus N(H)$. Moreover, H is a closed operator if and only if $R(H)$ and $N(H)$ are closed subspaces of $\mathcal{H}$; and H is bounded if and only if it is closed and $\mathcal{D}(H) = \mathcal{H}$. We refer the reader to Ota's paper [20] for a treatment of unbounded projections. In addition, given two closed subspaces $\mathcal{S}, \mathcal{T}$ such that $\mathcal{S} \cap \mathcal{T} = \{0\}$ and $\mathcal{S} + \mathcal{T}$ is dense we denote by $H_{\mathcal{S}//\mathcal{T}}$ the closed projection with range $\mathcal{S}$ and kernel $\mathcal{T}$ (here, $\mathcal{D}(H_{\mathcal{S}//\mathcal{T}}) = \mathcal{S} \dotplus \mathcal{T}$). Recall that we denote by $\tilde{\mathcal{Q}}$ the set of all (not necessarily bounded) closed projections in $\mathcal{H}$. In the sequel given two operators A, B the symbol $B \subseteq A$ means that A is an extension of B.

Remark 3.5. Let $T = EF \in \mathcal{QQ}$ where $E = Q_{\overline{R(T)}//\mathcal{S}}$ and $F = Q_{\mathcal{W}//N(T)}$. By Lemma 3.3, $H_{\mathcal{W}//\mathcal{S}}$ is a closed projection. Moreover, by Corollary 3.4, $H_{\mathcal{W}//\mathcal{S}}$ is bounded if and only if T has closed range. In what follows, given $(E,F) \in [\mathcal{QQ}]_T$ we denote

$$H_{F,E} := H_{R(F)//N(E)}.$$

Lemma 3.6. *Let $T \in \mathcal{QQ}$ and $(E,F) \in [\mathcal{QQ}]_T$, the next conditions hold:*

1. *$R(T) \subseteq \mathcal{D}(H_{F,E})$.*
2. *$N(H_{F,E}T) = N(T)$.*

Proof. 1. Let $y = Tx \in R(T)$ then $y = Tx = EFx = EFx - Fx + Fx = -(I - E)Fx + Fx \in N(E) + R(F) = \mathcal{D}(H_{F,E})$.

2. By the previous item $H_{F,E}T$ is well-defined and it is clear that $N(T) \subseteq N(H_{F,E}T)$. On the other hand, if $H_{F,E}Tx = 0$ then $Tx \in R(T) \cap N(E) \subseteq R(E) \cap N(E) = \{0\}$, i.e. $x \in N(T)$ and so $N(H_{F,E}T) = N(T)$. □

Recall the concept of inner inverses of a bounded linear operator. Given $T \in L(\mathcal{H})$, the **Moore–Penrose inverse** of T, $T^\dagger$, is the unique linear extension of $(T|_{N(T)^\perp})^{-1}$ to $R(T) \dotplus R(T)^\perp$ such that $N(T^\dagger) = R(T)^\perp$. The densely defined operator $T^\dagger$ fulfills the following equations, which could also be used as a definition of $T^\dagger$ if we take as the domain the maximal domain for which these equations have a solution, namely $\mathcal{D}(T^\dagger) = R(T) \dotplus R(T)^\perp$:

1. $TXT = T$.
2. $XTX = X$.
3. $TX \subseteq P_{\overline{R(T)}}$.
4. $XT = P_{N(T)^\perp}$.

Observe that $T^\dagger$ is bounded if and only if $R(T)$ is closed. We denote by $T[i, j, k, l]$ the set of densely defined operators that satisfy equations i, j, k, l with $i, j, k, l \in \{1, \ldots, 4\}$. The elements of $T[1]$ are usually called **inner inverses** of T. The reader is referred to [7] and [18] for a complete treatment on generalized inverses.

Penrose [21] and Greville [16] proved that the Moore–Penrose inverse of the product of two orthogonal projections in $\mathbb{C}^{n\times n}$ is an idempotent matrix, and conversely. Extensions to bounded linear operators can be found in [9] and [8]. Here, we analyze the case for operators in $\mathcal{QQ}$.

Theorem 3.7. *Let $T \in L(\mathcal{H})$. The next conditions are equivalent:*

1. *$T \in \mathcal{QQ}$.*
2. *there exists $H \in \tilde{\mathcal{Q}}$ such that $THT = T$ and $T^*H^*T^* = T^*$.*

Proof. 1 ⇒ 2. Suppose that $T \in \mathcal{QQ}$ and for $(E, F) \in [\mathcal{QQ}]_T$ consider the closed projection $H = H_{F,E}$ (see Remark 3.5). We claim that $THT = T$. First observe that THT is well-defined because of Lemma 3.6. Now, $THT = EFHEF = EHEF = E|_{\mathcal{D}(H)}EF = EF = T$. Similarly, since $(F^*, E^*) \in (\mathcal{QQ})_{T^*}$ and $H_{E^*,F^*} = (H_{F,E})^* = H^*$, we have that $T^*H^*T^* = T^*$. Therefore item 2 holds.

2 ⇒ 1. Suppose that there exists a closed projection H such that $THT = T$ and $T^*H^*T^* = T^*$. Then, $HTHT = HT$, i.e., $(HT)^2 = HT$ and since $T \in L(\mathcal{H})$ and H is closed, then HT is also closed. Moreover, as $\mathcal{D}(HT) = \mathcal{D}(T) = \mathcal{H}$ then $HT \in \mathcal{Q}$. Similarly, from $T^* = T^*H^*T^*$ we get that $H^*T^* \in \mathcal{Q}$. Hence, $(H^*T^*)^* \in \mathcal{Q}$. Now, $(H^*T^*)^* = ((TH)^*)^* = \overline{TH}$ where the overline stands for the closure of TH. Therefore, $T = THT = (TH)(HT) = (\overline{TH})(HT) \in \mathcal{QQ}$. □

From now on, L_{cr} stands for the set of closed range operators of $L(\mathcal{H})$.

Corollary 3.8. *Let $T \in L_{cr}$. The next conditions are equivalent:*

1. *$T \in \mathcal{QQ}$.*
2. *$T[1] \cap \mathcal{Q} \neq \emptyset$.*
3. *$T^\dagger \in \mathcal{PQP}$.*

Proof. 1 ⇔ 2. Follows from Theorem 3.7.

2 ⇒ 3. If $Q \in T[1] \cap \mathcal{Q}$ then an easy computation shows that $T^\dagger = P_{N(T)^\perp} Q P_{R(T)}$, i.e., $T^\dagger \in \mathcal{PQP}$.

3 ⇒ 2. If $T^\dagger \in \mathcal{PQP}$ then $T^\dagger = P_{N(T)^\perp} Q P_{R(T)}$, for some $Q \in \mathcal{Q}$. Then, $T = TT^\dagger T = TP_{N(T)^\perp} Q P_{R(T)} T = TQT$, i.e, $Q \in T[1]$. □

Notice that the previous corollary states that the Moore–Penrose inverse maps bijectively $\mathcal{QQ} \cap L_{cr}$ onto $\mathcal{PQP}$.

Corollary 3.9. *Let* $T \in L(\mathcal{H})$.

1. *The following conditions are equivalent:*
   (a) $T \in \mathcal{QQ}$.
   (b) *There exists* $H \in \tilde{\mathcal{Q}}$ *such that* $THT = T$, $HTH = H$ *and* $T^*H^*T^* = T^*$.
2. *The following conditions are equivalent:*
   (a) $T \in \mathcal{PQ}$.
   (b) *There exists* $H \in \tilde{\mathcal{Q}}$ *such that* $THT = T$ *and* $TH \subseteq P_{\overline{R(T)}}$.
   (c) *There exists* $H \in \tilde{\mathcal{Q}}$ *such that* $THT = T$, $HTH = H$ *and* $TH \subseteq P_{\overline{R(T)}}$.

   In particular, $T \in \mathcal{PQ} \cap L_{cr}$ *if and only if* $\mathcal{Q} \cap T[1, 2, 3] \neq \emptyset$.
3. *The following conditions are equivalent:*
   (a) $T \in \mathcal{PP}$.
   (b) $T^\dagger \in \tilde{\mathcal{Q}}$.

Proof. 1. $(a) \Leftrightarrow (b)$. Suppose that $T \in \mathcal{QQ}$ and for $(E, F) \in [\mathcal{QQ}]_T$ consider the closed projection $H = H_{F,E}$. Clearly, $HTH = HEFH = HEH = H$. Moreover, by the proof of Theorem 3.7, $THT = T$ and $T^*H^*T^* = T^*$ and so item (b) holds. The converse follows by Theorem 3.7.

2. $(a) \Rightarrow (c)$. Let $T \in \mathcal{PQ}$. Then, $T = P_{\overline{R(T)}} F$ for some $F \in \mathcal{Q}$ with $N(F) = N(T)$, i.e., $(P_{\overline{R(T)}}, F) \in [\mathcal{QQ}]_T$. Let $H := H_{F, P_{\overline{R(T)}}}$. Now, by Theorem 3.7, $THT = T$ and $HTH = H$. Moreover, $TH = P_{\overline{R(T)}} FH = P_{\overline{R(T)}} H = P_{\overline{R(T)}}|_{\mathcal{D}(H)} \subseteq P_{\overline{R(T)}}$. Thus, item (c) holds.
   $(c) \Rightarrow (b)$. It is trivial.
   $(b) \Rightarrow (a)$. Let $H \in \tilde{\mathcal{Q}}$ such that $THT = T$ and $TH \subseteq P_{\overline{R(T)}}$. By the proof of Theorem 3.7, $HT \in \mathcal{Q}$. Thus, $T = THT = THHT = P_{\overline{R(T)}} HT \in \mathcal{PQ}$.
3. See [9, Theorem 6.2]. □

By the above corollary, if $T \in \mathcal{PP} \cap L_{cr}$ then $T^\dagger \in T[1] \cap \mathcal{Q}$. However $T^\dagger$ is not, in general, the unique element in $T[1] \cap \mathcal{Q}$ if $T \in \mathcal{PP}$. For example, an easy computation shows that $T^\dagger + P_{R(T)^\perp \cap N(T)}$ is also in $T[1] \cap \mathcal{Q}$. Observe that $R(T)^\perp \cap N(T) = \{0\}$ if and only if T admits a unique factorization in $\mathcal{PP}$ (see [9, Corollary 3.8]).

Corollary 3.10. *Let* $T \in L(\mathcal{H})$ *with closed range. If there exists* $T' \in T[1]$ *such that* $(T')^2 = I$ *then* $T^2 \in \mathcal{QQ}$.

Proof. If $T = TT'T$ then $E := TT' \in \mathcal{Q}$ and $F := T'T \in \mathcal{Q}$. Therefore, as $(T')^2 = I$, $T^2 = EF \in \mathcal{QQ}$. □

Corollary 3.11. *Let $T \in L(\mathcal{H})$ with closed range. If $R(T) = R(T^*)$ and $\dim R(T) \leq \dim N(T)$ then $T \in \mathcal{QQ}$.*

Proof. By Corollary 3.9, it suffices to prove that $T^\dagger = P_{R(T)}EP_{R(T)}$ for some $E \in \mathcal{Q}$. Now, as $\dim R(T) \leq \dim N(T) = \dim R(T)^\perp$ then there exists $J : R(T) \to R(T)^\perp$ such that $J^*J = P_{R(T)}$. Therefore, considering the matrix representation induced by the Hilbert space decomposition $\mathcal{H} = R(T) \oplus R(T)^\perp$ we can define $E := \begin{pmatrix} T^\dagger & (T^\dagger - (T^\dagger)^2)J^* \\ J & J(I - T^\dagger)J^* \end{pmatrix} \begin{matrix} R(T) \\ R(T)^\perp \end{matrix}$. It is easy to show that $E = E^2$, i.e., $E \in \mathcal{Q}$ and, clearly, $T^\dagger = P_{R(T)}EP_{R(T)} \in \mathcal{PQP}$. Hence, by Corollary 3.9, $T \in \mathcal{QQ}$. □

By the previous corollary, if $\mathcal{H}$ is separable then every closed range normal operator $T \in L(\mathcal{H})$ with infinite dimensional kernel belongs to $\mathcal{QQ}$.

From the proof of Corollary 3.9 it follows that, for $T \in \mathcal{QQ}$ and $(E, F) \in [\mathcal{QQ}]_T$ it holds that $H_{F,E} \in \left\{H \in \tilde{\mathcal{Q}} : H \in T[1,2] \text{ and } H^* \in T^*[1]\right\}$. The next result shows that this property fully describes $[\mathcal{QQ}]_T$. For this, given $T \in \mathcal{QQ}$ define the mapping

$$\Phi : [\mathcal{QQ}]_T \to \tilde{\mathcal{Q}}, \ \Phi((E, F)) = H_{F,E}.$$

Theorem 3.12. *Let $T \in \mathcal{QQ}$, then*

$$\Phi([\mathcal{QQ}]_T) = \left\{H \in \tilde{\mathcal{Q}} : H \in T[1,2] \text{ and } H^* \in T^*[1]\right\}.$$

Proof. If $(E, F) \in [\mathcal{QQ}]_T$ then, by the proof of Corollary 3.9, we have that $H := H_{F,E} \in \tilde{\mathcal{Q}} \cap T[1,2]$ and $H^* \in T^*[1]$.

Conversely, let $H := H_{\mathcal{W}//\mathcal{S}} \in \tilde{\mathcal{Q}}$ such that $H \in T[1,2]$ and $H^* \in T^*[1]$. Let us define $E := \overline{TH}$ and $F := HT$. By the proof of the implication $2 \Rightarrow 1$ in Theorem 3.7, we have that $E, F \in \mathcal{Q}$ and $T = EF$. Let us prove that $(E, F) \in [\mathcal{QQ}]_T$ and $H_{F,E} = H$ or, equivalently, that $E = Q_{\overline{R(T)}//\mathcal{S}}$ and $F = Q_{\mathcal{W}//N(T)}$.

First, as $THT = T$ then $N(T) \subseteq N(HT) = N(F) \subseteq N(T)$, i.e., $N(F) = N(T)$. On the other hand, from $HTH = H$, we have that $R(F) = R(HT) \subseteq R(H) = R(HTH) \subseteq R(HT) = R(F)$, i.e., $R(F) = R(H) = \mathcal{W}$. Thus, $F = HT = H_{R(H)//N(T)} = Q_{\mathcal{W}//N(T)}$. Similarly, as $T^*H^*T^* = T^*$ and $H^*T^*H^* = H^*$ then $H^*T^* = H_{R(H^*)//N(T^*)}$. Notice that $H^*T^*H^* = H^*$ since $H = HTH$ and $R(T^*) \subseteq \mathcal{D}(H^*)$ (because $T^* = T^*H^*T^*$). Therefore, $E = \overline{TH} = (H^*T^*)^* = Q^*_{R(H^*)//N(T^*)} = Q_{\overline{R(T)}//N(H)} = Q_{\overline{R(T)}//\mathcal{S}}$ as desired. □

Corollary 3.13. *Let $T \in \mathcal{QQ}$ with closed range. Then*

$$\Phi([\mathcal{QQ}]_T) = \{Q \in \mathcal{Q} : Q \in T[1,2]\}.$$

4. Split Operators in $\mathcal{QQ}$

If $T \in \mathcal{PP}$ then $\overline{R(T) \dot{+} N(T)} = \mathcal{H}$, see [9, Theorem 3.2]. Moreover, $T \in \mathcal{PP}$ has closed range if and only if $\overline{R(T)} \dot{+} N(T) = \mathcal{H}$. However, these properties do not hold, in general, for operators in $\mathcal{QQ}$. For instance, $T = \frac{1}{2}\begin{pmatrix} -1 & 1 & 2 \\ -1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{3}\begin{pmatrix} -1 & 1 & 2 \\ -2 & 2 & 4 \\ -1 & 1 & 2 \end{pmatrix} \in \mathcal{QQ}$ and $R(T) \cap N(T) = R(T) = \text{gen}\{(1,1,0)^T\}$. Thus, $\overline{R(T) \dot{+} N(T)} \neq \mathcal{H}$. On the other hand, consider a non-closed range positive operator T with $\dim N(T) = \dim \overline{R(T)}$. Then, by Examples 2.1, $T \in \mathcal{QQ}$ and $\overline{R(T)} \dot{+} N(T) = \mathcal{H}$, but $R(T)$ is not closed. The aim of this section is to study the operators $T \in \mathcal{QQ}$ such that $\overline{R(T) \dot{+} N(T)} = \mathcal{H}$.

Proposition 4.1. *Let $T \in \mathcal{QQ}$ and $(E, F) \in [\mathcal{QQ}]_T$. Then, $N(E + F - I) = \overline{R(T)} \cap N(T)$ and $\overline{R(E + F - I)} = \overline{\overline{R(T)} + N(T)}$.*

Proof. An easy computation shows that, $N(E-F) = N(E) \cap N(F) \dot{+} R(E) \cap R(F)$ for all $E, F \in \mathcal{Q}$. Therefore, if $(E, F) \in [\mathcal{QQ}]_T$ then, by Lemma 3.3, $N(E) \cap R(F) = \{0\}$ and so $N(E+F-I) = N(E-(I-F)) = N(E) \cap R(F) \dot{+} R(E) \cap N(F) = R(E) \cap N(F) = \overline{R(T)} \cap N(T)$. Analogously, but considering $(F^*, E^*) \in [\mathcal{QQ}]_{T^*}$, we have that $N(E^* + F^* - I) = \overline{R(T^*)} \cap N(T^*)$ or, equivalently, $\overline{R(E + F - I)} = \overline{\overline{R(T)} + N(T)}$. □

Corollary 4.2. *Let $T \in \mathcal{QQ}$ and $(E, F) \in [\mathcal{QQ}]_T$. Then,*

1. *$\overline{R(T)} \cap N(T) = \{0\}$ if and only if $E + F - I$ is injective.*
2. *$\overline{R(T) \dot{+} N(T)} = \mathcal{H}$ if and only if $E + F - I$ is an injective operator with dense range.*
3. *$\overline{R(T)} \dot{+} N(T) = \mathcal{H}$ if and only if $E + F - I$ is injective and $R(E) + R(I - F) = \mathcal{H}$.*
4. *$R(T) \dot{+} N(T) = \mathcal{H}$ if and only $E + F - I$ is invertible.*

Proof. Items 1, 2 and 3 follow by Proposition 4.1. Let us prove item 4. Assume that $R(T) \dot{+} N(T) = \mathcal{H}$. Notice that this implies that $R(T)$ is closed. Now, as $R(T) \cap N(T) = \{0\}$ then, by item 1, $E + F - I$ is injective. It remains to show that $R(E + F - I) = \mathcal{H}$. Now, since $R(E) \cap R(F - I) = R(T) \cap N(T) = \{0\}$ and $N(E) + N(F - I) = N(E) + R(F) = \mathcal{H}$ because of Corollary 3.4 then, by Arias and Corach [5, Theorem 2.10], $R(E + F - I) = R(E) + R(F - I) = R(T) + N(T) = \mathcal{H}$.

Conversely, if $E + F - I$ is invertible then, by item 1, $R(E) \cap R(I - F) = R(T) \cap N(T) = \{0\}$. Moreover, as $R(E + F - I) = \mathcal{H}$ then $R(E) + R(I - F) = \mathcal{H}$. Thus, $\overline{R(T)} \dot{+} N(T) = \mathcal{H}$. It remains to show that $R(T)$ is closed. For this, as $\mathcal{H} = R(E + F - I) = R(E) + R(I - F)$, applying again [5, Theorem 2.10], we have that $N(E) + N(I - F) = \mathcal{H}$. Therefore, by Corollary 3.4, T has closed range as desired. □

As we highlighted previously, there is an identity which characterizes $\mathcal{PP}$, namely $TT^*T = T^2$. We wonder if there exist a corresponding identity for $\mathcal{QQ}$. A first approach in this direction is the next result:

Proposition 4.3. *If $T \in \mathcal{QQ}$ then there exists $X \in L(\mathcal{H})$ such that $TXT = T^2$ and $XTX = X^2$.*

Proof. Let $T = EF \in \mathcal{QQ}$. Define $X := FE$. Then $TXT = EFFEEF = EFEF = T^2$ and $XTX = FEEFFE = FEFE = X^2$. □

Our next step is to investigate whether the converse of Proposition 4.3 holds. In the next result we show that this happens if T satisfies that $\overline{R(T)} \dotplus N(T) = \mathcal{H}$.

Proposition 4.4. *Let $T \in L(\mathcal{H})$ such that $\overline{R(T)} \dotplus N(T) = \mathcal{H}$. Then, $T \in \mathcal{QQ} \cap L_{cr}$ if and only if there exists $X \in L(\mathcal{H})$ such that $TXT = T^2$, $XTX = X^2$ and $\overline{R(X)} \dotplus N(X) = \mathcal{H}$.*

Proof. Let $T \in \mathcal{QQ} \cap L_{cr}$ and write $T = EF$ for some $(E, F) \in [\mathcal{QQ}]_T$. Define $X = FE$. It follows from Proposition 4.3 that $TXT = T^2$ and $XTX = X^2$. We claim that $R(X) = R(F)$ and $N(X) = N(E)$ and so, by Corollary 3.4, $\overline{R(X)} \dotplus N(X) = \mathcal{H}$. In fact, $R(X) = R(FE) = FR(E) = F(R(E) + N(F)) = F(R(T) + N(T)) = F(\mathcal{H}) = R(F)$ and $N(X) = N(FE) = N(E) + E^{-1}(N(F)) = N(E) + E^{-1}(N(F) \cap R(E)) = N(E) + E^{-1}(\{0\}) = N(E)$.

Conversely, let $X \in L(\mathcal{H})$ such that $TXT = T^2$, $XTX = X^2$ and $\overline{R(X)} \dotplus N(X) = \mathcal{H}$. First, let us prove that $T \in \mathcal{QQ}$. For this, notice that an easy computation on $XTX = X^2$ implies that $P_{N(X)^\perp} T P_{\overline{R(X)}} = P_{N(X)^\perp} P_{\overline{R(X)}} \in \mathcal{PP}$. From this, and since $\overline{R(X)} \dotplus N(X) = \mathcal{H}$ we have that $N(X)^\perp = R(P_{N(X)^\perp} P_{\overline{R(X)}}) = R(P_{N(X)^\perp} T P_{\overline{R(X)}}) = P_{N(X)^\perp} R(T P_{\overline{R(X)}})$. Therefore, $\mathcal{H} = R(T P_{\overline{R(X)}}) + N(X)$ and so $\mathcal{H} = R(T) + N(X)$. Moreover, $R(T) \cap N(X) = \{0\}$. Indeed, if $y = Tx \in R(T) \cap N(X)$ then $0 = TXTx = T^2x$, i.e., $y = Tx \in R(T) \cap N(T) = \{0\}$. Therefore, $\mathcal{H} = R(T) \dotplus N(X)$. Notice that this implies that $T \in L_{cr}$. Similarly, since $TXT = T^2$ and $R(T) \dotplus N(T) = \mathcal{H}$ we obtain that $\mathcal{H} = R(X) \dotplus N(T)$ (hence, $X \in L_{cr}$). Summarizing, we have that $P_{N(X)^\perp} T P_{R(X)} = P_{N(X)^\perp} P_{R(X)} \in \mathcal{PP}$, $\mathcal{H} = R(T) \dotplus N(X)$ and $\mathcal{H} = R(X) \dotplus N(T)$. Therefore, by Proposition 2.5, $T \in \mathcal{QQ}$. □

Finally, we present a complement to the characterization of $\mathcal{QQ}$ for matrices due to Ballantine. In fact, he proved the next result:

Theorem 4.5. *Let $A \in \mathbb{C}^{n\times n}$. Then, A is a product of k idempotent matrices if and only if $\dim R(A - I) \leq k \dim N(A)$.*

By Ballantine's result we obtain the following:

Proposition 4.6. *Let $T \in \mathbb{C}^{n\times n}$. If there exists $X \in \mathbb{C}^{n\times n}$ such that $TXT = T^2$, $XTX = X^2$ then T is a product of 4 idempotent matrices.*

Proof. By Theorem 4.5, it suffices to prove that $\dim R(T-I) \leq 4 \dim N(T)$. First, if $XTX = X^2$ then $T = I + Z_1 + Z_2$ for some $Z_1, Z_2 \in \mathbb{C}^{n\times n}$ such that $XZ_1 = Z_2X = 0$. Thus, $R(T-I) = R(Z_1 + Z_2) \subseteq R(Z_1) + R(Z_2)$. Now, $R(Z_1) \subseteq N(X)$, so $\dim R(Z_1) \leq \dim N(X)$, and $R(Z_2^*) \subseteq N(X^*)$, so $\dim R(Z_2) = \dim R(Z_2^*) \leq \dim N(X^*) = \dim N(X)$. Therefore,

$$\dim R(T-I) \leq \dim R(Z_1) + \dim R(Z_2) \leq 2 \dim N(X). \tag{4.1}$$

On the other hand, as $TXT = T^2$ then $X = I + W_1 + W_2$ for some $W_1, W_2 \in \mathbb{C}^{n\times n}$ such that $TW_1 = W_2T = 0$. Hence, notice that $N(X) \subseteq R(W_1 + W_2)$. Therefore,

$$\dim N(X) \leq \dim R(W_1 + W_2) \leq \dim R(W_1) + \dim R(W_2) \leq 2 \dim N(T), \tag{4.2}$$

where the last inequality follows since $\dim R(W_1), \dim R(W_2) \leq \dim N(T)$ because $TW_1 = W_2T = 0$. Finally, from (4.1) and (4.2) we get that $\dim R(T-I) \leq 4 \dim N(T)$, as desired. □

Corollary 4.7. *Let* $T \in \mathbb{C}^{n\times n}$. *If there exists* $X \in \mathbb{C}^{n\times n}$ *such that* $XTX = X^2$ *and* $\dim N(X) \leq \dim N(T)$ *then* $T \in \mathcal{QQ}$.

Proof. Following the proof of Proposition 4.6 we get inequality (4.1), i.e., $\dim R(T-I) \leq 2 \dim N(X)$. Now, since $\dim N(X) \leq \dim N(T)$, we obtain that $\dim R(T-I) \leq 2 \dim N(T)$ and so $T \in \mathcal{QQ}$. □

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