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A Note on the Application of Wazewski's Topological Method to an Integro-Differential Equation of Volterra Type

Juan E. Nápoles Valdes[1,2], José R. Velázquez C[†], Luciano M. Lugo Motta Bittencurt[1], Paulo M. Guzmán[1]
[1]UNNE, FACENA, Av. Libertad 5450, (3400) Corrientes, Argentina.
[2]UTN, FRRE, French 414, (3500) Resistencia, Chaco, Argentina.

***Corresponding Author:**
Juan E. Nápoles Valdes
Email: jnapoles@frre.utn.edu.ar

Abstract: The purpose of this note is to generalize the *Wazewski's Topological Method* [11], originally stated for ordinary differential equations, to the integro – differential equation of Volterra type (1), under suitable conditions on the functions involved.
Keywords: Boundedness, stability, Wazewski's Topological Method.

INTRODUCTION

Consider the following integro – differential equation of Volterra type:

$$\begin{cases} x'(t) = A(t)x(t) + \int_a^t B(t,s)x(s)\,ds + x(t)^T H(t,\sigma(t),x(t)) & (1) \\ x(0) = x_0 & \\ \sigma(t) = f(t) + \int_a^t k(t,s,x(t))\,ds & (2) \end{cases}$$

where: $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ is the unknown vector function ; $A : R^+ \to R^{n\times n}$; $B : R^+ \times R^+ \to R^{n\times n}$ and $H : R^+ \times R \times R^n \to R^{n\times n}$ are continuous functions with $R^+ = [0, +\infty)$, σ, f, k are continuous functions, the number a is arbitrary, and we suppose also that functions involved in equation (1) satisfies some uniqueness condition of solutions.

The purpose of this note is to study the asymptotic behavior of equation (1), under suitable conditions. There are some results in that direction. In general, all asymptotic results in question are obtained under continuity assumptions on functions considered therein, while no assumptions the continuity in the whole for f and g is made here.

Many qualitative results for equation of type (1) have been obtained by constructing Lyapunov's functionals. So, our results here are more general and apply (1) wether A is stable, identically zero or completely unstable.

In [1] Burton considered (1) with A constant and B(t, s) = B(t – s) and showed that the theory of existence, uniqueness, dimensionality of solutions space and the variation of parameters formula are virtually indistinguishable from the corresponding elementary theory of ordinary differential equations.
Mahfoud in [8] gave sufficient conditions to ensure that (1) has bounded solutions, the method used is new and the main results unifies, improves and extends earlier results.

The equations (1) and (2) has been considered in [15] taking B (t, s) = B(t – s), but the techniques used there are quite different. In particular, that paper can be considered as a sequel of [10 – 13].

The Wazewski Topological Method, originally stated for ordinary differential equations, had received a considerable amount of attention in the last decades (see [3–5], [7], [9], [16 – 18]), but in the case of integro – differential equations, that situation is quite different (see [14]).

In the next section of the present note, we give the basic theory, the concept of generalized polifacial set and Wazewski's theorem for equation (1). For references see [4], [14] and [19] concerning the applicability of Topological Method of Wazewski.
Now, we give the notation and some basic lemmas.

Let (τ, x_0) be a point in E. The initial value problem (i.v.p.) for (1) is to find an intreval $I \subset R^+$, a differentiable function $\varphi : I \to R^n$ satisfying (1) for every $t \in [a, +\infty)$ and such that $\varphi(t) = x_0$ for $t \in [0, a]$

The i.v.p. for (1) is equivalent to the integral equation (using the variation of parameters formula):

$$x'(t) = R(t)x_0 + \int_a^t R(t,s)x(s)H(s,\sigma(s),x(s))ds \quad ; \quad t \geq a \qquad (3)$$

where R(t, s) is an $n \times n$ matrix which is the unique solution of the equation:

$$\frac{\partial R(t,s)}{\partial t} = A(t)R(t,s) + \int_a^t B(u,s)R(t,u)du \quad ; \quad R(s, s) = I \ \text{(identity matrix)}$$

(see Grossman and Miller [6] for details)

So that we shall refer to (3) as the (a) – ivp for equation (1). For given $t \geq 0$ we call $E(t) = \{(t, x) / x \in E\}$ the section of E by t, or more briefly the t – section of E.

We remind that if $a \in R^+$ it makes sense to consider de (a) – ivp and indeed we consider $a \in R^+$.
In the following, we assume the hypothesis below, which is not the most general possible but is sufficient for our purposes, and avoid superfluous technicalities.

Hypothesis (A): For each compact $J \subset R^+$ and each compact $P \subset R^n$ there exist in correspondence a continuous function k : $J \to R^+$ such that

$$\left|H(t,x_1(t),\sigma_1(t)) - H(t,x_2(t),\sigma_2(t))\right| \leq k(t)\left|x_1(t) - x_2(t)\right|$$

for every $t \in J$, $\sigma_1(t)$ and $\sigma_2(t)$ and every $x_1, x_2 \in P$.

Local Existence, Continuation and Continuous Dependence upon Initial Data of the Solutions of Equation (1). The following results for the (a) – ivp can be proved as the case of ordinary differential equations, with natural adaptations (see, for example, Coppel [2]). We make below some remarks about the existence and continuation of solutions outside an a – section of E, wich have some peculiarities proper of integro – differential equations of type (1).

Lemma 1: Under hypothesis (A), exists $\delta > 0$ such that the (a) – ivp has a unique solution in $[a, a + \delta]$ ($[a - \delta, a + \delta]$ if $a - \delta \geq 0$).

The proof can be done by using Banach's Contraction Principle in the space of continuous functions C(J, Q), $Q \subset E$ with the supreme norm.

We say that a solution φ of the (a) – ivp defined on the interval J is continuable if there exists a solution ϑ of the (a) – ivp defined on some interval J^* which contains J properly and such that $\varphi(t) = \vartheta(t)$ for all $t \in J$. If it is not continuable we say that φ is a *maximal solution* and its interval of definition the maximal interval of φ.

Lemma 2: Under hypothesis (A), if φ is a solution of the (a) – ivp in [a, b) (a < b) then φ can be extended as the unique solution of the (a) – ivp to the right of b, that is to an interval [a, c] with b < c.

Lemma 3: If φ is a solution of the (a) – ivp defined and bounded in [a, b] then φ can be extended as the unique solution of the (a) – ivp to the interval [a, b].

Let P = (a, x) be a point of E. If the solution of the (a) – ivp through P is unique we say that P is an ordinary point and put $O = \{P \in E : P \text{ is ordinary}\}$.

The above results imply that $E(a) \subset O$ if hypothesis (A) is satisfied.

We indicate $\varphi(t, P)$ the solution of (1) through P and by $\Delta(P)$ its maximal interval of definition. Also we put:

$L(t, P) = (t, \varphi(t, P))$; $L(J, P) = \{(t, \varphi(t, P)) : t \in J\}$; $L(\Delta(P), P) = L(P)$

and $L^+(P) = \{(t, \varphi(t, P)) : t \geq \tau\}$ when $P = (\tau, x) \in E$.

Lemma 4: Let $\{P_n\}$ be a sequence of ordinary points $P_n = (t_n, x_n)$ such that $\lim_{n\to\infty} P_n = P_0 \in O$. If $[t_0, b] \subset \Delta(P_0)$, then there exists N such that for every $n \geq N$ $[t_0, b] \subset \Delta(P_0)$ and $\varphi(. , P_n)$ converges uniformly to $\varphi(. , P_0)$ in $[t_0, b]$.

Lemma 5: Let $\{P_n\}$ be a sequence of ordinary points $P_n = (t_n, x_n)$ such that $\lim_{n\to\infty} P_n = P_0 \in O$. If X is an open subset of E such that:

$[c, d] \subset \Delta(P_0)$ and $L([c, d], P_0) \subset X$,

then there exists N such that for every $n \geq N$

$[c, d] \subset \Delta(P_n)$ and $L([c, d], P_n) \subset X$

Lemma 6: Let $\{P_n\}$ be a sequence of ordinary points and X an open subset of E. If $\lim_{n\to\infty} P_n = P_0 \in O$, $\tau \in \Delta(P_0)$ and $L(t, P_0) \in X$, then there exists N such that for $n \geq N$ $L(t, P_n)$ converges to $L(t, P_0)$ when $n \to \infty$.

Lemma 7: Assume hypothesis (A). If P(t, x) is in E, then there exists a solution of the (τ) – ivp in the interval $[\tau, \tau + \delta]$, where δ is small enough.

The following hypothesis will be taken in the next section.

Hypothesis (B)

1. For every point $P \in E$, there exists at least one integral L (P) of (1).
2. Every integral L(P) of (1) is continuable to the right up to the boundary of E
3. E(O) = O

RESULTS

Let $W \subset E$ be an open set for which $E(O) \cap W \neq \emptyset$, we put $W(O) = E(O) \cap W$ and $\partial W = \overline{W} \cap \overline{C_E W}$ the boundary of W with respect to E.

If $P = (t, x) \in W$ and if L(P) is not asymptotic with respect to W then there exists one first point $Q = L(\tau, P)$ on which L(P) reaches the boundary of W. Q is called the consequent of P and we write Q = C(P). To a generic point P there corresponds, in general, an infinity of consequents, but if $P \in O$, then C(P) is unique.
The set of all points $P \in W(O)$ for which there exists C(P) is called the left shadow of (1) with respect to W, and it is indicated by G(O).

In the sequel, S is the set of egress points and S^* is the set of strict egress points.
The following theorem can be easily proved with the help of Lemma 4.

Theorem 1. If $P \in G(O)$ and $C(P) \in S^*$ then the consequence mapping

$C : G(O) \to S$

is continuous at P.

Theorem 2. If $S \subset S^*$, the mapping

$$K : G(O) \cup S \to S \,/\, K(P) = \begin{cases} C(P) & \text{if } P \in G(O) \\ P & \text{if } P \in S \end{cases}$$

is continuous.

Proof: If $P \in G(O)$ the result is true by Theorem 1. Let $P \in S$ and $P_n = (O, x_n)$ $\forall \in N$ be a sequence in G(O) converging to P, otherwise the theorem is also true, because K is the identity mapping on S. The P is a limiting point of G(O), that is $P = (O, x)$. Therefore, $P \in E(O)$ and L(P) is unique. As $P \in S^*$, $L((O, \delta), P) \in W^*$ for δ small enough. By Lemma 4 we have for n large $[O, \delta] \subset \Delta(P_n)$ and by Lemma 6, $L(\delta, P_n) \in W^*$. Given $\varepsilon > 0$ so small that

$$\left| L(t,P) - P \right| < \frac{\varepsilon}{2} \quad \text{for every } t \in [0, \delta] \tag{5}$$

As we have that $P_n = L(O, P_n) \in W$ and $L(\tau, P_n) \in W^*$ there exists τ_n, $0 < \tau_n < \tau$ such that $L(\tau_n, P_n) = C(P_n) = K(P_n)$.
By Lemma 4, we have for large n:

$$\left| L(\tau_n, P) - L(\tau_n, P_n) \right| < \frac{\varepsilon}{2} \tag{6}$$

From (5) and (6) we obtain:

$$\left| K(P_n) - P \right| = \left| L(\tau_n, P_n) - P \right| \le \left| L(\tau_n, P) - L(\tau_n, P_n) \right| + \left| L(\tau_n, O) - P \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Theorem 3 (Wazewski). If $S \subset V \subset S^*$ and there exists $Z \subset W(O) \cup V$ such that $Z \cap V$ is a retract of V and $Z \cap V$ is not a retract of Z, then there exists at least one point $P \in Z - V$ for which the integral L(P) is asymptotic with respect to W.

The proof is the same given by Wazewski.

Remark 1. The condition $S \subset S^*$ is essential for the realization of Wazewski's Theorem. The other condition, v.g. the existence of the set Z depends upon the character of equation (1).

Also we assume that functions involved in equation (1) are functions of class C^k with $k \ge 1$, and we consider that $H : R^{n+1} \to R$ is a function of class C^{k+1}. If $P = (\tau, x) \in E$ and L(P) is an integral of (1) through P we put $h(t) = H(L(t, P))$, is clear that h is of class C^{k+1} in $\Delta(P)$. The derivative of order q ($q \le k + 1$) of h(t) at t = T will be denoted by $D^q_{L(P)}H(P)$.
Now consider the sets:

$$W = \left\{ P \in E : H_i(P) < 0 ; i = 1, 2, \ldots \right\}$$

$$\Gamma_i = \left\{ P \in E : H_i(P) = 0 \wedge H_k(P) \le 0 \wedge i \ne k ; i = 1, 2, \ldots \right\}$$

for i = 1, 2, . . . , m , where $H_i : R^{n+1} \to R$ are m functions of class C^k.

If $W \cap E(O) \neq \emptyset$, W will be called generalised polifacial set (with respect to (1)) and the Γ_i will be called the faces of W provided that one of the following conditions is satisfied for each y and each $P \in \Gamma_i$:

(1) For each L(P), the least $q \leq k + 1$ for which $D^q_{L(P)} H(P) \neq 0$ is odd and the corresponding derivative is positive.

(2) P is not an egress point.

Let L_i be the set of points of Γ_i for which condition (1) is satisfied and M_i the set of points of Γ_i that satisfies condition (2). To verify if a point $P = (\tau, x) \in \Gamma_i$is in M_i, it is sufficient that one of the following conditions be satisfied:

(a) For each L(P), the least value $q \leq k + 1$ for which $D^q_{L(P)} H(P) \neq 0$ is, either odd with corresponding derivative negative, or even with a positive value of the derivative.

(b) There exist for each L(P) and a = a(L(P)), b = b(L(P)) such that $[a, b] \subset \Delta(P)$, $L([a, b), P) \subset \Gamma_i$ and $a < t \leq b$.

Theorem 4. If W is a generalised polifacial set, then $S \subset V \subset S^*$ where V = $= \left(\bigcup_i L_i\right) - \left(\bigcup_i M_i\right)$.

Proof. It is obvious that $\partial W = \bigcup_i \Gamma_i$ and $S \cap M_i = \emptyset$. Then:

$$S \subset \bigcup_i \Gamma_i - M_i = \bigcup_i \left(L_i \cup M_i\right) - M_i = \left(\bigcup_i L_i\right) - \left(\bigcup_i M_i\right)$$

If $P = (\tau, x) \in \left(\bigcup_i L_i\right) - \left(\bigcup_i M_i\right)$, then $H_i(P) \leq 0$ for i = 1, 2, . . . , m and $H_k(P) = 0$ if $P \in L_k$. By condition (1), for each L(P) we have that $H_k(L(t, P)) > 0$ for $\tau < t < \tau + \delta^+$, where $\delta = \delta(L(P)) > 0$ is sufficient small. Then $L((\tau, \tau + \delta], P) \subset W^*$ and $P \in S^*$, so that:

$$\left(\bigcup_i L_i\right) - \left(\bigcup_i M_i\right) \subset S^*,$$

and the proof is complete.

Remark 2. The above theorem shows that a generalised polifacial set W is an open subset of E for which there exist V such that $S \subset V \subset S^*$ (see remark 1).

Remark 3. Our results are consistent with those of [1], [8] and [15] in the referent to asymptotic behavior of some "closed" examples to equation (1).

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